

RT-MAT 90-04

The theory of manifolds from the
non-deterministic point of view.

R. G. LINTZ

Julho 1990

THE THEORY OF MANIFOLDS FROM THE NON-DETERMINISTIC POINT OF VIEW

by R.G. LINTZ

Summary

Using the methods of non-deterministic analysis it is possible to define in a differentiable manifold a special structure called Gauss structure which allows one to study differentiable maps from one manifold into another without the use of local charts. The main theorem in this paper shows how it is possible to locally recover the jacobian of such maps by using the Gauss structures defined in the manifolds in question.

§I - Introduction

1. We start by discussing the general philosophy behind that we call non-deterministic mathematics, which for a number of years has been applied to several questions in mathematics and physics. As the subject is not well known we try to render this paper as selfsufficient as possible by recalling briefly the basic ideas of non-deterministic analysis which we intend to use here. More details can be seen at [1].

Traditionally the concept of function in mathematics is deterministic, in the sense that a function $f : X \rightarrow Y$ is defined at a "sharp" point $x \in X$ and has a "sharp" value $f(x)$ in Y . On the other hand, in physics sometimes when we want to represent facts of the real world we feel the need for non-deterministic concepts, namely, we would like a function whose value is not sharp and which is defined in a space whose points are not localized sharply. In other words we would like to associate open sets in X to open sets in Y , instead of point to point. But it is in mathematics where there are situations when the concept of a non-deterministic function is interesting.

More precisely, this concept was suggested by the theory of generalized manifolds in the sense of Wilder [2] where the concept of deformation retract cannot be used because a continuous function, in general, cannot map a point x of a space X in another point of a space Y if x has a base of neighbourhoods of cardinality smaller than that of $f(x)$. Again for examples and remarks one can see [1].

2. Let us discuss now the concept of derivative. When in the turn of the century M. Fréchet among other started the vast program of generalizing the concepts of analysis from R^n to abstract sets it was felt that to talk about continuity in abstract spaces we needed the notion of proximity, namely, we have to talk about a point close to another point, the value of a function to be close to a certain value etc. Certainly the first idea coming to our minds is that of introducing a "distance" in abstract space. After the great success of such an enterprise the next step was a further generalization leading to the concept of topological space. In other words, topology was invented for the purpose of talking about continuity in abstract spaces.

Now what happened to the concept of derivative? Following the same line of development of continuity we would expect that some kind of structure would be finally introduced in abstract spaces so that we could speak about derivative of a function in such spaces. However, that was not exactly what happened. Indeed, perhaps due to the strong influence of Leibniz in the creation of Calculus, somehow the idea of linearity became forever attached to the concept of derivative. For instance, if we just look to the usual definition of differentiable manifold we realize that what is really done it is to "glue" Euclidean spaces on the manifold and then to define differentiability of a map from one manifold into another we have to come down to the attached euclidean spaces (local charts) and define differentiability in terms of their linear structures. In contrast for continuity we only use the euclidean spaces to define the topological structure of the manifolds; after that, we do not have to come down to local charts to talk about continuity. If the concept of

derivatives were to be used only when some linear structure were available then we should not have to worry too much about that dependence, but the trouble is that as soon as we connect the concept of derivative with the concept of velocity, which lies at the roots of the historical evolution of analysis, we have situations where we need a concept of derivative when we do not have a linear structure available. That is the case of a particle describing a path which is a continuous curve without tangent in any point, as in Wiener's approach to the Brownian motion. The lack of a concept of derivative independent of linear structures forced people, in such situations, to look to other approaches to the concept of velocity. All this suggests that perhaps we could try to introduce the concept of derivative along the same lines as we did for continuity, namely, by defining a certain structure in abstract spaces so that it becomes possible to talk about derivatives and in such a way that for the cases when a linear structure is available the more abstract structure would in some sense lead to the usual concept of derivative. This is exactly one of the important points of the program of non-deterministic analysis and indeed such a structure has been introduced in topological spaces with the name of Gauss structure and after the work of several people in that direction it became clear that the basic requirements for a theory of derivatives are fulfilled so that we have a reasonable way of talking about derivative, velocity, etc. in situations where no linear or even metric structure is available. In particular this is appealing to an important branch of topology which is developing very fast these days; the theory of generalized manifolds. Indeed, since Wilder published his book in 1949 (see [2]) the subject has attracted the attention of many people. However eventhough from the algebraic point of view, Cech homology theory was the right tool to use, the same could not be said of topological methods connected with homotopy theory and the general theory of continuous deformation. The difficulty relied on the fact that a generalized manifold is in general a Hausdorff space without even having a metric structure. This creates an uncomfortable situation concerning homotopy and its connection with Cech homology theory in the sense explained in details at [1]. With the introduction and development of

non-deterministic mathematics it became possible an unification between Cech homology and the theory of continuous deformation in a sense which provides new tools to handle problems impossible to deal with in the classical theory of deformations as can be seen at [1]. Therefore, it becomes possible to extend the methods and scope of Wilder's approach to generalized manifolds and the implementation of this program will be developed by us in future publications. Right now, our aim in this paper is more modest: we only consider the investigation of what happens when we frame Gauss structure on usual differentiable manifolds. We are going to prove one main theorem which we believe will suggest further developments on the use of the methods of non-deterministic mathematics in the theory of manifolds, which we try to summarize at the end. For other applications of the concept of Gauss structure one can see for instance [3].

§II

Basic Notions of Non-deterministic Mathematics

1. A pair is a topological space X together with a family of collections of open sets of X and will be denoted by (X, ν) . So the elements $\sigma \in \nu$ are collections of open sets of X .

A n -function f from the pair (X, ν) in to the pair (Y, ν') , denoted by

$$f : (X, \nu) \rightarrow (Y, \nu')$$

is

1) a function

$$f_\nu : \nu \rightarrow \nu',$$

2) for every $\sigma \in \nu$, a function

$$f_\nu : \sigma \rightarrow \sigma' = f_\nu(\sigma).$$

Let us introduce in ν an order relation given by refinements, i.e., $\sigma < \tau$ or $\tau > \sigma$ means that τ refines σ in the sense that every set of τ is contained in some set of σ .

In many previous papers what we call a n -function was called a g -function and therefore the reader should be aware of this change in nomenclature when looking to older references in the area.

A n -function f is continuous if

$$1) \sigma < \tau \Rightarrow f_\nu(\tau) > f_\nu(\sigma),$$

$$2) \text{ if } B \subset A, B \in \tau, A \in \sigma \text{ and } \sigma < \tau, \text{ then } f_\tau(B) \subset f_\sigma(A).$$

Later on, when we introduce the concept of derivative of a n -function we shall need to consider the set of real numbers R together with a family ν_R of collections of subsets of R which are open intervals or points namely if σ_R is a collection of the family ν_R then the elements of σ_R are either open sets or "singletons", namely subsets of R with only one point. We shall use the notation $[R, \nu_R]$ and call it a real pair. In the same way as we did for n -functions we define a real n -function, as

$$f : (X, \nu) \rightarrow [R, \nu_R]$$

given by

I) a function

$$f_\nu : \nu \rightarrow \nu_R$$

II) for each $\sigma \in \nu$ a function

$$f_\sigma : \sigma \rightarrow \sigma_R = f_\nu(\sigma).$$

In previous papers a real n -function was called special g -function.

A real n -function

$$f : (X, \nu) \rightarrow [R, \nu_R]$$

is continuous if

1) $\tau > \sigma \Rightarrow f_\nu(\tau) > f_\nu(\sigma)$

2) if $B \subset A$, $B \in \tau$, $A \in \sigma$, $\sigma < \tau$ then

a) $f_\tau(B) \subset f_\sigma(A)$, if $f_\tau(B)$ and $f_\sigma(A)$ are both open sets, or $f_\sigma(A)$ is a point.

b) $f_\tau(B) \subset \overline{f_\sigma(A)}$, if $f_\tau(B)$ is a singleton and $f_\sigma(A)$ is an open set.

Let $\phi : X \rightarrow Y$ be an usual continuous function and $f : (X, \nu) \rightarrow (Y, \nu')$ a continuous n -function.

We say that f generates ϕ if for any $x \in X$ and any neighbourhood V of $\phi(x)$, there exists $\sigma \in \nu$ and $A \in \sigma$ such that $x \in A$ and

a) $f_\sigma(A) \subset V$

b) $\phi(x) \in \overline{f_\sigma(A)}$.

The case when f is a real n -function is defined in the same way and we only remark that when $f_\sigma(A)$ is a point condition b) above reduces to $\phi(x) = f_\sigma(A)$.

The question of deciding when a given n -function generates an usual function was studied by A. Jansen in [4] and to state one of his results needed by us later we recall a few more concepts due to him.

A n -function, real case included,

$$f : (X, \nu) \rightarrow (Y, \nu')$$

is:

cofinal, if $f_\nu(\nu)$ is cofinal in the set of all open covering of Y , in the sense that if Γ is an arbitrary open coverings of Y , then there is $\sigma \in \nu$ such that any element of $f_\nu(\sigma)$ is contained in some set of Γ ;

point-wise cofinal, if for any $x \in X$ and any open covering Γ of Y , there is $\sigma \in \nu$ and $A \in \sigma$ such that $x \in A$ and $f_\sigma(A)$ is contained in some set of Γ ;

fully regular, if for any $\sigma, \tau \in \nu$ and $A \in \sigma$, $B \in \tau$ with $A \cap B \neq \emptyset$ then $f_\sigma(A) \cap f_\tau(B) \neq \emptyset$. Here, for the case of a real n -function, we must add that if $f_\sigma(A)$, for instance, is an open set and $f_\tau(B)$ is a singleton, then $f_\tau(B) \subset \overline{f_\sigma(A)}$.

Theorem 1 (A. Jansen). If $f : (X, \nu) \rightarrow (Y, \nu')$ is fully regular, pointwise cofinal and if Y is regular T_1 space, then there exists a unique continuous function $\phi : X \rightarrow Y$ generated by f .

Proof: See [4], p.5, Prop.1.

Remark: A. Jansen proves the theorem above only for n -functions, but with slight modifications his proof remains true for the case where f is a real n -function. Actually, we shall face this situation later on in this paper.

It is also sometime of interest to consider the inverse problem, i.e., given $\phi : X \rightarrow Y$ continuous to find a n -function or a real n -function which generates ϕ . This problem was also studied by A. Jansen in his Ph.D. thesis, referred above, but here we shall be interested in a particular situation analysed by V. Buonomano [5], which is the following: let $\phi : X \rightarrow Y$ be such that the interior of $\phi(A)$, denoted by $\overset{\circ}{\phi}(A)$ is not empty for all non empty open sets A of X . In this case if ν is an arbitrary family of open coverings of X we define for each $\sigma \in \nu$, $f_\nu(\sigma)$ as

$$f_\nu(\sigma) = \{\overset{\circ}{\phi}(A); A \in \sigma\}$$

and

$$\nu' = \{f_\nu(\sigma); \sigma \in \nu\}$$

finally for every $\sigma \in \nu$ we define

$$f_\sigma : \sigma \rightarrow \sigma' = f_\nu(\sigma)$$

by

$$A \in \sigma, f_\sigma(A) = \overset{\circ}{\phi}(A).$$

In the general case where ϕ is an arbitrary continuous function with $Y = R$ (reals) we define for any $\sigma \in \nu$ and $A \in \sigma$

$$f_\sigma(A) = \begin{cases} \overset{\circ}{\phi}(A), & \text{if it is not empty} \\ \{\phi(A)\}, & \text{if it is one point.} \end{cases}$$

In this case we obtain a real n -function. In both cases we say that f is defined by the image method.

It is easy to see that f as defined above generates ϕ , in the sense established before and moreover it is continuous either as a n -function or a real n -function.

2. Definition I. A standard family of coverings, F in a topological space X is a family of collections α , of subsets of X such that:

- a) Any set A of $\alpha \in F$ is the closure of an open set of X .
- b) Given $\alpha \in F$ and two distinct sets $A_1, A_2 \in \alpha$, then $\overset{\circ}{A}_1 \cap \overset{\circ}{A}_2 = \emptyset$
- c) Any $\alpha \in F$ is a covering of X .
- d) Given any point $x \in X$ there is a neighbourhood N of x such that any $\alpha \in F$ has only a finite number of sets intersecting N (each $\alpha \in F$ is locally finite).
- e) Given any open set O of X there is a covering $\alpha \in F$ such that α has a set $A \subset O$.
- f) Ordered by refinements, F is a directed set.

Definition II. We will also call the standard family of coverings F a Gauss structure on X .

Notation: (X, F) . If there is a Gauss structure F on X we call (X, F) a Gauss space.

Proposition I. Each space X satisfying the T_3 axiom is a Gauss space. See [6].

Proposition II. Each T_2 paracompact topological space is a Gauss space. See [6].

Corollary: Each differentiable manifold is a Gauss space.

The reason for above nomenclature is due to the fact that a standard family of coverings is a generalization of a system of Gauss coordinates on a surface.

Definition III. Let (X, F) and (Y, F') be two Gauss spaces. A Gauss transformation is a function $G : F \rightarrow F'$ compatible with the order of refinement of F and F' i.e., $\alpha, \beta \in F$, $\alpha < \beta$ then $G(\alpha) < G(\beta)$.

Definition IV. A continuous n -function $f : (X, \nu) \rightarrow (Y, \nu')$ is called n -differentiable relative to the Gauss transformation: $G : F \rightarrow F'$ and the standard family of coverings F, F' of X and Y , respectively, if for any $\mu \in \nu$, $\alpha \in F$, $A \in \mu$, the number of sets of α which intersect A is finite and the same for $\alpha' = G(\alpha)$, $\mu' = f_\nu(\mu)$, $A' = f_\mu(A)$; we denote these numbers by $n(A, \alpha)$ and $n(A', \alpha')$ respectively.

Definition V: Let the n -function: $f : (X, \nu) \rightarrow (Y, \nu')$ be n -differentiable relative to F, F' and G . We define the n -derivative of f as a real n -function,

$$Df : (X, \nu) \rightarrow [R, \nu_R]$$

constructed as indicated below.

To construct Df we must have the following:

- a) $F : (X, \nu) \rightarrow (Y, \nu')$ a n -differentiable n -function or real n -function.
- b) The Gauss spaces (X, F) and (Y, F') .
- c) The Gauss transformation $G : (X, F) \rightarrow (Y, F')$. Then the construction procedure is:
- d) Let $n(A, \alpha)$ denote the number of sets of α that intersect the set A , which is an element of $\mu \in \nu$.
- e) Let $Df_\mu(A, \alpha) = \frac{n(A', \alpha')}{n(A, \alpha)}$, where $f_\nu(\mu) = \mu'$, $f_\mu(A) = A'$ and $G(\alpha) = \alpha'$.
- f) Let $\overline{Df}_\mu(A) = \overline{\lim}_{\alpha \in F} Df_\mu(A, \alpha)$, $\underline{Df}_\mu(A) = \underline{\lim}_{\alpha \in F} Df_\mu(A, \alpha)$, where: $\overline{\lim}_{\alpha \in F}$ is the upper limit over the net F (directed set) and $\underline{\lim}_{\alpha \in F}$ = lower limit.
- g) So for each $B \in \mu$ and $\mu \in \nu$ we have two real numbers i.e. $\overline{Df}_\mu(B)$ and $\underline{Df}_\mu(B)$. Let us call $\overline{Df}_\mu(A)$ the set of all such numbers for $B \subset A$, with

$B \in \lambda$ and $\lambda > \mu$, i.e.:

$$\underline{Df}_\mu(A) = \{ \underline{Df}_\lambda(B), \overline{Df}_\lambda(B) :$$

$$B \subset A, B \in \lambda, \lambda > \mu, \lambda, \mu \in \nu \}$$

h) Finally we define $Df_\mu(A)$ to be the open interval or singleton in R : $Df_\mu(A) = (\inf \underline{Df}_\mu(A), \sup \overline{Df}_\mu(A)) = A_R$. If the inf and sup are equal we understand the above to be the set consisting of that point.

i) So for each $A \in \mu$ we get a point or an interval in R . We denote this collection $\{Df_\mu(A)\}$ by μ_R . As μ runs through ν we get a family ν_R of such collections:

$$\nu_R = \{\mu_R\}.$$

This gives the required real n -functions which we call the n -derivative, Df of f . We note that by the definition of n -derivative, Df is always a continuous real n -function, even though f might not be continuous.

3. Now we recall some results of V. Buonomano to be used later and for more details see [5].

Theorem 2. (V. Buonomano) Let $\varphi : U \rightarrow R^n$, $n \geq 1$, U open in R^n , be a continuously differentiable map whose Jacobian matrix has a determinant $J_\varphi \neq 0$ for all $x \in U$. Then there exists a continuous n -function generating φ , whose derivative generates a real function $\psi(x) = |J_\varphi(x)|$.

Proof. See [5].

V. Buonomano proves his theorem using a Gauss structure in R^n called canonical and defined as follows: define a covering α_1 of R^n by considering all $(n-1)$ -hyperplanes, $x_i = 0, \pm 1, \pm 2, \dots$, $i = 1, 2, \dots, n$, which gives rise to a covering of R^n by n -cubes of edges 1. Then, for α_2 consider hyperplanes $x_i = 0, \pm \frac{1}{2}, \pm(1 + \frac{1}{2}), \dots$, $i = 1, 2, \dots, n$, and so on. The family F of all coverings $\alpha_1, \alpha_2, \dots$ defines the required Gauss structure

in R^n . However, the theorem above remains true if we use a Gauss structure, called compatible with Jordan measure and defined as follows:

Definition VI. A Gauss space (X, F) is said to have a Gauss structure F compatible with Jordan measure if:

- 1) X is an open set or the closure of an open set in R^n .
- 2) For each $\alpha \in F$ all $A \in \alpha$ have the same Jordan measure.
- 3) If $\|\alpha\|$ is the maximum of the diameters of the elements of α , then given $\varepsilon > 0$ there is $\alpha \in F$ such that $\|\alpha\| < \varepsilon$.

From the definition of Gauss space it is easy to show that if F is compatible with Jordan measure then F has a countable, cofinal set of coverings in F , such that

$$\alpha_1 < \alpha_2 < \alpha_3 \dots \alpha_i < \alpha_{i+1} < \dots$$

Hence we might always assume, without loss of generality, that F is countable with $\alpha_i < \alpha_j$ if $i < j$.

§III

1. In this section we study the main problem of this paper and to clarify our ideas we discuss a few preliminary concepts.

Let M_n, M'_n be differentiable manifolds of dimension n and let

$$\phi : M_n \rightarrow M'_n$$

be a differentiable map. If $x \in M_n$ is an arbitrary point we can select local charts (U, h) and (U', h') at x and $\phi(x)$ respectively and look to the map $h' \circ \phi \circ h^{-1}$ from $h(U)$ into $h'(U')$, both regarded as open subsets of R^n . Look to the determinant of the Jacobian matrix of the transformation $h' \circ \phi \circ h^{-1}$ at the point $h(x)$ which will give

us a certain real number. In this way we can associate to each $x \in U$ a real number by looking to $h(x)$ and then to the determinant of the Jacobian indicated above, thus obtaining a real continuous function from U into the reals.

Let us use the notation

$$J_{UU'}^\phi : U \rightarrow R$$

calling it simply the Jacobian of ϕ relative to the local charts U, U' .

Now our main problem can be stated as follows:

Main problem: given a differentiable map

$$\phi : M_n \rightarrow M'_n$$

where M_n, M'_n are differentiable manifolds of dimension n to find a continuous n -function

$$f : (M_n, \nu) \rightarrow (M'_n, \nu')$$

and convenient Gauss structures in M_n and M'_n and Gauss transformations such that:

- i) f generates ϕ .
- ii) Df generates a continuous function $\psi : M_n \rightarrow R$ such that for every $x \in M_n$ there exists local charts (U, h) at x and (U', h') at $\phi(x)$ with

$$\psi(x) = |J_{UU'}^\phi(x)|.$$

Under restrictive assumptions a solution for this problem was found by R. Mirabal [7]. We shall give here a solution for this problem without those restrictions by using entirely different methods.

Before going into the proof of the main theorem of this paper, theorem 3 below, we shall discuss a few questions concerning the triangulations of a differentiable manifold.

A triangulation of a n -manifold M_n is a pair (K, t) where K is a locally finite simplicial n -complex (not necessarily finite) and t a homeomorphism of K onto M_n .

As usual the topology of K is the weak topology and due to locally finite property of K it coincides also with the linear metric topology.

The triangulation (K, t) of M_n is smooth if M_n is a differentiable manifold and for each n -simplex s of K there is a linear map $\ell : s \rightarrow R^n$ such that $t \circ \ell^{-1}$ is an imbedding in the sense of differentiable maps, namely $\lambda = t \circ \ell^{-1}$ has an extension $\bar{\lambda}$ to an open set L in R^n containing $\ell(s)$ such that $\tilde{\lambda} : L \rightarrow M_n$ is a differentiable map with rank n at each point of L , which is also a homeomorphism. Of course, the pair $(\tilde{\lambda}(L), \tilde{\lambda}^{-1})$ is a local chart of M_n , i.e., belongs to the atlas defining its differentiable structure.

It is well known that any differentiable manifold has a smooth triangulation and moreover according to [9] we can assume that the star of each vertex of this triangulation is simplicially isomorphic with a certain decomposition of a n -simplex. Such a triangulation is sometimes called a Brouwer's triangulation and so, from now on, we always suppose that we are dealing with Brouwer's triangulations.

2. Theorem 3. The main problem has a solution when ϕ is a local diffeomorphism.
Proof. Our first step is to define in M_n and M'_n suitable Gauss structures F and F' and corresponding Gauss transformation $G : F \rightarrow F'$. Let (K, t) be a triangulation (smooth and Brouwer) of M_n and (K', t') a triangulation of M'_n . For simplicity we refer to the elements of the triangulations of M_n and M'_n either as belonging to K, K' or to M_n, M'_n , i.e., when we consider, say, the star $St(a)$ of a vertex a we might consider it as a subset of K or of M_n whichever is more convenient and the context will make clear which situation we are considering. As both triangulations have at most a countable number of vertices we can write them as $a_1, a_2, \dots, a_i, \dots$ and $a'_1, a'_2, \dots, a'_i, \dots$. Look to a_1 and consider the star $St(a_1)$ with vertex a_1 of K . It is simplicially isomorphic with a certain decomposition of a n -simplex Δ_n . Due to [10], theorem 1, there is a stellar decomposition of Δ_n which is a subdivision of the previous decomposition of Δ_n .

But any stellar subdivision has a subdivision isomorphic with a barycentric subdivision, so we can assume that a convenient decomposition of $St(a_1)$ is simplicially isomorphic with some iterated barycentric subdivision of a n -simplex Δ_n giving rise to a n -complex in R^n which we shall denote by $B_n(a_1)$. As barycentric subdivisions of a simplex provide n -simplices of same n -area we finally conclude that by a convenient subdivision the star of a_1 becomes simplicially isomorphic with a n -complex $B_n(a_1)$ all of whose n -simplices have equal n -area. Observe that the triangulation K with $St(a_1)$ so subdivided is no more in general a triangulation of M_n , but it is a collection of parts of K which qualifies for belonging to a Gauss structure. Let us call α_1 the image by t of such decomposition of K which is a covering of M_n by closed sets. In the same way we build α'_1 in M'_n . One point we should be careful is that the $B_n(a_1)$ and $B_n(a'_1)$ are not necessarily isomorphic because we might need to perform in Δ_n different numbers of iterated barycentric subdivisions in each case. However by further iterations of barycentric subdivisions in one or the other we can get them to be isomorphic. In conclusion we get a covering α_1 of M_n and another one α'_1 in M'_n such that the stars $St(a_1)$ and $St(a'_1)$ are both isomorphic to the same n -complex $B_n(a_1)$ or $B_n(a'_1)$ of R^n having all simplices with same n -area.

Now let us look to a_2 and $St(a_2)$. If $St(a_2)$ has no n -simplex in common with $St(a_1)$ we proceed exactly as before and define α_2 as α_1 with $St(a_2)$ with the convenient decomposition and the same for α'_2 , getting $\alpha_1 < \alpha_2$ and $\alpha'_1 < \alpha'_2$. We shall need later on that α_1 and α_2 do not have common sets, so to avoid this we perform an additional barycentric subdivision on all simplices so far obtained and then we shall have α_2 and α'_2 as desired. If $St(a_2)$ has n -simplices in common with $St(a_1)$ then it might happen that the convenient decompositions for each of them do not coincide for the common simplices. But as they are always barycentric subdivisions we can subdivide more one of these stars to get subdivisions which agree on common simplices. Then α_2 will be α_1 after the right decompositions for $St(a_1)$ and $St(a_2)$ and additional barycentric subdivisions to

ensure that $\alpha_1 < \alpha_2$. We do the same for $St(a'_1)$ and $St(a'_2)$. As we did for α_1 and α'_1 it might be necessary further subdivisions to get $St(a_2)$ and $St(a'_2)$ both isomorphic with the same n -complex of R^n having all simplices of same n -area and no common sets with α_1, α'_1 , respectively.

We can do the same with a_3, a'_3, a_4, a'_4 , etc., obtaining two sequences of coverings of M_n and M'_n

$$\alpha_1 < \alpha_2 \dots \alpha_i < \alpha_{i+1} < \dots$$

$$\alpha'_1 < \alpha'_2 \dots \alpha'_i < \alpha'_{i+1} < \dots$$

As barycentric subdivisions give rise to n -simplices of equal n -area we can further assume that each n -simplex of α_{i+1} belongs to a barycentric subdivision of some simplex of α_i , namely, everytime we go from one covering to the next we perform barycentric subdivisions even if that were not required by the process of building $\alpha_1, \alpha_2, \dots$ etc. This precaution is necessary to guarantee that both families $\{\alpha_i\}$ and $\{\alpha'_i\}$, $i = 1, 2, \dots$, define Gauss structures F and F' in M_n and M'_n , what is now easy to check, because, as $i \rightarrow \infty$, $\|\alpha_i\| \rightarrow 0$.

The Gauss transformation $G : F \rightarrow F'$ is defined by

$$\forall \alpha_i \in F, G(\alpha_i) = \alpha'_i.$$

Our second step is the definition of

$$f : (M_n, \nu) \rightarrow (M'_n, \nu')$$

generating ϕ . This is done as follows: first of all we assume without loss of generality that M_n and M'_n are differentiably imbedded in some euclidean space R^p due to classical Whitney's imbedding theorem. This will allow us to define a measure in M_n and M'_n induced by the usual Jordan measure in R^p .

To define ν and ν' we proceed as follows: as seen above, call an open set $A \subset M^n$, Jordan-measurable, if it is the image through some local chart homeomorphism of an open

set in R^n which is Jordan-measurable and $m(A)$ is its n -area or its measure in M_n . The same holds for M'_n . Now let ν and ν' be the families of all coverings of M_n and M'_n made up of open, Jordan-measurable sets in M_n and M'_n .

Let $\sigma \in \nu$ and let $A \in \sigma$. As φ is a local diffeomorphism we can use, in defining σ , only local charts (U, h) such that $\varphi|U$ is a diffeomorphism and hence $\varphi(A)$ is also open and Jordan-measurable in M'_n , due to Brouwer's theorem of invariance of open sets and the fact that diffeomorphisms preserve Jordan measurability. This gives an open covering $\sigma' = \varphi(\sigma) \in \nu'$.

Following our standard notations for n -functions define

$$f_\nu : \nu \rightarrow \nu'$$

by

$$\forall \sigma \in \nu \leftrightarrow f_\nu(\sigma) = \varphi(\sigma).$$

In the same way we define for each $\sigma \in \nu$

$$f_\sigma : \sigma \rightarrow \sigma' = f_\nu(\sigma)$$

by

$$\forall A \in \sigma, f_\sigma(A) = \varphi(A) \in \sigma'.$$

That $f : (M_n, \nu) \rightarrow (M'_n, \nu')$ is continuous and generates ϕ it is trivial.

Our final step is to prove that Df generates a continuous function $\Psi : M_n \rightarrow R$ with the required property. Using A. Jansen's result, quoted in §II, 1, all we have to do is to prove that Df is pointwise cofinal. Let then $x \in M_n$ and Γ an open covering of the reals be given. Suppose that $\mathring{S}t(a_i)$ is an open star of the tringulation of M_n containing x . According to the definition of F , $St(a_i)$ is simplicially isomorphic to a n -complex $B_n(a_i)$ of R^n all of whose simplices have the same n -area. Of course, we are considering $St(a_i)$ with the subdivision induced by α_i , as defined before. So there

is a map

$$h : \overset{\circ}{S} t(a_i) - \overset{\circ}{B}_n(a_i)$$

which is a isomorphism and due to the fact that our original triangulation (K, t) is smooth the pair $(\overset{\circ}{S} t(a_i), h)$ is a local chart. Therefore if we consider the corresponding pair $(\overset{\circ}{S} t(a'_i), h')$ for M'_n we have a differentiable map

$$\tilde{\varphi} = h' \circ \varphi \circ h^{-1} : \overset{\circ}{B}_n(a_i) \rightarrow \overset{\circ}{B}_n(a'_i).$$

Let us now define a n -function

$$\tilde{f} : (\overset{\circ}{B}_n(a_i), \nu) \rightarrow \overset{\circ}{B}_n(a'_i), \tilde{\nu}$$

generating $\tilde{\varphi}$, as follows: let $\tilde{\nu}$ contain all open coverings of $\overset{\circ}{B}_n(a_i)$ made up of Jordan measurable sets and as $\tilde{\varphi}$ is an open map (indeed a diffeomorphism) by considering the images of coverings $\tilde{\sigma} \in \tilde{\nu}$ by $\tilde{\varphi}$ we get a family $\tilde{\nu}'$ of open covering $\tilde{\sigma}' = \tilde{\varphi}(\tilde{\sigma})$ in $\overset{\circ}{B}_n(a'_i)$. So we define

$$\tilde{f}_{\tilde{\nu}} : \nu \rightarrow \nu'$$

by

$$f_{\tilde{\nu}}(\tilde{\sigma}) = \tilde{\varphi}(\tilde{\sigma})$$

and for each $\tilde{\sigma} \in \tilde{\nu}$

$$f_{\tilde{\sigma}} : \tilde{\sigma} \rightarrow \tilde{\sigma}'$$

by

$$f_{\tilde{\sigma}}(\tilde{A}) = \tilde{\varphi}(\tilde{A}), \tilde{A} \in \tilde{\sigma}.$$

This defines a continuous n -function generating $\tilde{\varphi}$, as desired. Define now Gauss structures in $\overset{\circ}{B}_n(a_i)$ and $\overset{\circ}{B}_n(a'_i)$ by using the homeomorphisms h, h' and the corresponding coverings $\alpha_j \in F$ and $\alpha'_j \in F'$. More precisely, taking α_j with $j \geq i$ we look to its restriction to $\overset{\circ}{S} t(a_i)$ then apply h to get

$$\tilde{\alpha}_j = h(\alpha_j \text{ restricted to } \overset{\circ}{S} t(a_i)).$$

Do the same for $\mathring{B}_n(a'_i)$. We finally get \tilde{F} and \tilde{F}' and the Gauss transformation

$$\tilde{F} : \tilde{F} \rightarrow \tilde{F}'$$

defined by

$$\tilde{G}(\tilde{\alpha}_j) = \tilde{\alpha}'_j.$$

Now as \tilde{F} and \tilde{F}' are compatible with Jordan measure in the sense of definition VI we can obtain by V. Buonomano's result (Th.2), that $D\tilde{f}$ generates a continuous function Ψ and is therefore pointwise cofinal. So, there is $\tilde{\sigma} \in \nu$ and $\tilde{A} \in \tilde{\sigma}$ with $h(x) \in \tilde{A}$ and

$$D_{\tilde{f}^{-1}}(\tilde{A}) \subset D \in \Gamma.$$

But by the definition of f , $A = h^{-1}(\tilde{A})$ will belong to some $\sigma \in \nu$ and recalling the fact that f was defined by the image method using φ and the definitions of $\tilde{\varphi}$ and \tilde{f} we get

$$Df_\sigma(A) = D\tilde{f}'_\sigma(\tilde{A}) \subset D \in \Gamma$$

and as $x \in A$ this proves that Df is pointwise cofinal and therefore generates a continuous function $\psi : M_n \rightarrow R$ which is such that $\tilde{\psi} = \psi \circ h^{-1}$. Therefore calling $(U, h) = (\mathring{S} t(a_i), h)$ and $(U', h') = (\mathring{S} t(a'_i), h')$ we have by Buonomano's result indicated above

$$\psi(x) = |J_{UU'}\varphi(x)|$$

completing the proof of the theorem.

Remarks. a) If we analyse the last part of the proof we observe that it fits in a more general situation as follows: suppose we give two homeomorphisms

$$h_1 : X \rightarrow \tilde{X}$$

$$h_2 : Y \rightarrow \tilde{Y}$$

and a continuous n -function

$$f : (X, \nu) \rightarrow (Y, \nu').$$

Then using h_1 and h_2 it is easy to build another continuous n -function

$$\tilde{f} : (\tilde{X}, \tilde{\nu}) \rightarrow (\tilde{Y}, \tilde{\nu}')$$

where

$$\tilde{\nu} = \{\tilde{\sigma} : \tilde{\sigma} = h_1(\sigma), \sigma \in \nu\}$$

$$\tilde{\nu}' = \{\tilde{\sigma}' : \tilde{\sigma}' = h_2(\sigma'), \sigma' \in \nu'\}.$$

In the same way if we have Gauss structures F and F' in X and Y respectively we can define Gauss structures \tilde{F} and \tilde{F}' in \tilde{X} and \tilde{Y} as

$$\tilde{F} = \{\tilde{\alpha} : \tilde{\alpha} = h_1(\alpha), \alpha \in F\}$$

$$\tilde{F}' = \{\tilde{\alpha}' : \tilde{\alpha}' = h_2(\alpha'), \alpha' \in F'\};$$

also a Gauss transformation $G : F \rightarrow F'$ induces another one $\tilde{G} : \tilde{F} \rightarrow \tilde{F}'$ through h_1 and h_2 . We say that \tilde{f} , \tilde{F} , \tilde{F}' and \tilde{G} are induced by the pair (h_1, h_2) . Immediately we see that if f or Df and moreover if $A \in \sigma \in \nu$, then $h_1(A) \in \tilde{\sigma} \in \tilde{\nu}$ by construction and

$$\tilde{f}_{\tilde{\sigma}}[h_1(A)] = h_2\{f_{\sigma}(A)\}$$

$$D\tilde{f}_{\tilde{\sigma}}[h_1(A)] = Df_{\sigma}(A),$$

what shows that the derivative of n -functions is invariant by homeomorphisms. Actually in [11] it is shown that derivative of n -functions are invariant by morphisms in the category of Gauss spaces, which is a more general situation.

b) Another point to notice in the previous proof is that in a certain sense "pointwise cofinal" is a local property because all reduces to analyse the situation inside an open set without references to the whole space.

In the next section, we discuss a series of questions suggested by what we have done so far and which, we believe, are either interesting by themselves or may be taken as starting points for future developments in the same line of work.

§IV

1. An important consequence from theorem 3, §III it is to open the possibility of an intrinsic study of differentiable manifolds without appeal to local charts. More precisely, the theorem just quoted shows that it is possible to find a Gauss structure in M_n which reproduces the original differentiable structure in the sense that locally we have the classical situation.

However, we do not need necessarily to restrict ourselves to this case; we might just start the study of manifolds directly with n -functions and Gauss structures. More precisely, instead of looking to the category D of differentiable manifolds and differentiable maps, we look at the category P of pairs (M, ν) , where M is a topological manifold with Gauss structure and the morphisms are continuous n -functions.

From this point of view, our theorem merely states conditions under which objects and morphisms of P can be identified with objects and morphisms of D .

The advantage of starting with P is that most of the concepts defined will be topological invariant, and this avoids one of the serious problems with manifolds, namely, everytime we define something using local charts it is not usually easy to verify the topological invariance of the concept introduced. In few words, we have always to worry about local charts which were really introduced because we did not have any other means of speaking about derivatives. Now that an intrinsic theory of derivatives does exist, it seems to us that the natural thing to do is to use Gauss structures instead of local charts. We agree that it is perhaps too early to decide which philosophy to adopt, but the idea of reconstructing all

of differential topology and differential geometry in terms of Gauss structure seems to us a fascinating enterprise.

Anyway, when we are dealing with differentiable manifold we can always define a certain function $\psi : M_n \rightarrow R$ as shown before which we could call intrinsic derivative of ϕ . What happens in situations when no differentiable structures is available?

2. We only sketch here a first attempt concerning the study of this question because a deeper analysis of this problem will be published later. If (X, F) is a Gauss space denote by $\bar{k}(\alpha, \beta)$ for $\alpha, \beta \in F$ with $\alpha < \beta$, the supremum of the number of sets of β contained in one set of α . In the same way we define $\underline{k}(\alpha, \beta)$ using the infimum. It follows from the definition of Gauss space that for compact spaces these numbers are always finite. If $\underline{k}(\alpha, \beta) = \bar{k}(\alpha, \beta)$ finite for any pair $\alpha, \beta \in F$ with $\alpha < \beta$ we say that F is equitable and we denote by $k(\alpha, \beta)$ the common value of the numbers above. If (Y, F') is another equitable Gauss space, a Gauss transformation $G : F \rightarrow F'$ is called equitable if

$$\alpha < \beta, k(\alpha, \beta) = k[G(\alpha), G(\beta)].$$

Observe that in theorem 3 we used equitable Gauss structure and equitable Gauss transformations.

Definition VII. Let $\varphi : X \rightarrow Y$ be a continuous open function and suppose (X, F) and (Y, F') are equitable Gauss space and $G : F \rightarrow F'$ is an equitable Gauss transformation. We say that φ is differentiable in the intrinsic sense is Df is pointwise cofinal where f is defined by the image method. The function $\psi : X \rightarrow Y$ generated by Df is called the intrinsic derivative of φ , relative to F, F' and G .

According to this definition we see that theorem 3 just shows that for differentiable mappings between differentiable manifolds we can always define intrinsic derivatives which locally reproduce the Jacobian for convenient local charts. But it is interesting to study also

more general cases, for instance, the case of open maps and try to find out when intrinsic derivatives do exist, etc. In particular it might be interesting to study the relations between intrinsic derivatives and generalized Jacobians as defined by L. Cesari in [12].

Problem I. Let $\varphi : X \rightarrow Y$ be a continuous open function, with X and Y open subsets of R^n , having an intrinsic derivative. Does it imply that φ is differentiable in the usual sense?

Connected with this question we have the following one:

Problema II. Let $\varphi : R^n \rightarrow R^n$ be a continuous function satisfying the conditions:

- a) If $A \subset R^n$ has non-empty interior and finite Jordan measure than $\varphi(A)$ has also finite Jordan measure and with non-empty interior.
- b) There exists a continuous function $\psi : R^n \rightarrow R$ such that for any $A \subset R^n$ as in a) we have

$$\psi(x) = \frac{m[\varphi(A)]}{m(A)}$$

for some $x \in A$, where $m(A)$ stands for the Jordan measure of A .

Under these assumptions is it φ differentiable in the usual sense?

Observe that conditions a) and b) are true if φ is differentiable with ψ equal to the determinant of Jacobian of φ , so that they are necessary conditions for differentiability. The problem is to know if they are sufficient.

It can be shown that, under conditions of Problem II and considering Gauss structures and Gauss transformations in R^n compatible with Jordan measure in the sense of definition VI, the function ψ is the intrinsic derivative of φ in the sense of definition VII. Therefore if problem I has positive solution the same is true for problem II.

3. Let us consider now the case when we are dealing with functions $\varphi : X \rightarrow Y$ which are continuous but not necessarily open. If X and Y are compact metric spaces,

using results of A. Jansen ([4], p.23, Prop.7), we can build a continuous n -function $f : (X, \nu) \rightarrow (Y, \nu')$ generating φ , which is cofinal in the sense that $f_\nu(\nu)$ is cofinal in the set of all open coverings of Y . Also ν is cofinal in the sense of all open coverings of X . Suppose that only X is compact and define f as follows: consider the sequence $1, \frac{1}{2} \dots \frac{1}{2}n, \dots$ and let $\sigma_i, i = 0, 1, 2, \dots, n, \dots$, be defined as the covering of X by balls with radius $[\frac{1}{2}]^i$. To have all σ_i non-empty we might assume, for simplicity, that diameter of X , $\text{diam } X$, is greater than 1. Define ν as the family of all coverings σ_i , which is then cofinal in X . To each σ_i define σ'_i , an open collection of sets in Y not necessarily a covering as follows: if $A \in \sigma_i \in \nu$ define $f_{\sigma_i}(A)$ as an open ball with center $\varphi(a)$, where a is the center of A , and radius $2r_i$, where r_i is the radius of A . Define σ'_i as the set of all $f_{\sigma_i}(A)$ for $A \in \sigma_i$ and ν' as the family of all σ'_i . Then

$$F : (X, \nu) \rightarrow (Y, \nu')$$

is well defined and it can be proved that it is continuous, cofinal and generates φ . Also ν is clearly cofinal in X , because X is compact. The proof is the same as the one given by A. Jansen if we remember that $\varphi(X)$ is compact and all σ'_i are coverings of $\varphi(X)$ even though they are not necessarily coverings of Y .

We call f defined as above canonical generator of φ .

Definition VIII. Let $\varphi : X \rightarrow Y$ be a continuous function, with X and Y compact metric spaces, with equitable Gauss structures and Gauss transformations. We say that φ is differentiable in the intrinsic sense if the canonical generator f of φ has a derivative Df which is pointwise cofinal. The function $\varphi : X \rightarrow R$ generated by Df is called the intrinsic derivative of φ , relative to the Gauss structures in question.

We should remark that even when Df is not pointwise cofinal, i.e., when the intrinsic derivative of φ does not exist, it has still interest in many problems. For instance, if X is a closed interval in the reals, we can look to $\varphi : X \rightarrow Y$ as the path, in the classical sense of a particle in Y taking $t \in X$ as the time. The funny situation is when φ has no

derivative in classical sense in any point $t \in X$. What is then the velocity of the particle in question? No answer can be given in the framework of classical mechanics but it might be very well that φ has an intrinsic derivative ψ and then why not take ψ as its velocity! Even when ψ does not exist we can take Df as the velocity of φ which has in the case a non-deterministic characteristic as considered in [3]. A typical example of this case is the Brownian motion. In an unpublished paper on non-deterministic dynamical systems this question is treated in all detail.

4. To prove theorem 3 we defined in M_n particular Gauss structures starting from triangulations compatible with the given differentiable structures. But it is possible that Gauss structures built in other ways, without using triangulations, might give similar results. The theorem below is an example of this situation.

Theorem 4. (R. Mirabal) The main problem of §III has a solution under following hypothesis:

- 1) M_n and M'_n are differentially imbedded in R^p with Gauss structures F and F' , and a differentiable map $\varphi : M_n \rightarrow M'_n$ satisfying the conditions:
 - 1-a) for any $\alpha \in F$ all sets of α have the same n -area denoted by m_α and the same is true for any $\alpha' \in F'$;
 - 1-b) the Gauss transformation $G : F \rightarrow F'$ is such that for any $\alpha \in F$,

$$m_\alpha = m_{\alpha'}, \quad \alpha' = G(\alpha);$$

- 1-c) F and F' are countable and the coverings are totally ordered by refinements, i.e., if $i > j$, then $\alpha_i > \alpha_j$; ($\alpha'_i > \alpha'_j$).
- 2) For any $x \in M_n$ there is a local chart (U, h) at x such that $h(x)$ belongs to the closure of an open set $W \subset h(U)$ whose Gauss structure induced by h is compatible with Jordan measure. The same for M'_n and local chart (U', h') at $\varphi(x)$. Moreover,

if G is the Gauss transformation induced by the pair (h, h') , then all sets α in W and those of $G(\alpha)$ in $W' \subset h'(U')$ have all the same n -area.

3) φ is a local diffeomorphism, i.e., $J_\varphi(x) \neq 0$, for any $x \in M_n$.

Proof: We assume that φ is surjective, this being no restriction at all because $\varphi(M_n)$ is open in M'_n due to 3). The construction of $f : (M_n, \nu) \rightarrow (M'_n, \nu')$ is exactly the same as for theorem 3 and so all reduces to show that Df is pointwise cofinal, i.e., given an arbitrary open covering Γ of the reals R and any point $x \in M_n$ we must find a $\sigma \in \nu$ and $A \in \sigma$, with $x \in A$ and $Df_\sigma(A)$ contained in some open set of Γ . Suppose then that Γ and $x \in M_n$ are given. Select a local chart at x , (U, h) such that $\varphi|U$ is a diffeomorphism and such that the coordinates of $h(x)$ in R^n are bounded then x runs in U . This is easy to have because M_n is locally compact and calling h^{-1} by $\Phi : h(U) \rightarrow U$ the derivatives

$$\frac{\partial h^{-1}}{\partial x_i} = \frac{\partial \Phi}{\partial x_i} \quad i = 1, \dots, n$$

are supposed to be continuous in $h(U)$, where Φ is differentiable as a function from $h(V)$ into R^n .

Take an arbitrary $\sigma \in \nu$ and an arbitrary $B \in \sigma$ with $B \subset U$. If for simplicity we call

$$T(U) = \sqrt{\det \left(\frac{\partial \Phi}{\partial x_i} \mid \frac{\partial \Phi}{\partial x_j} \right)}$$

the area of B which is equal to $m(B)$ in the present case is given by the integral of $T(U)$ extended to $h(B)$ and the mean value theorem gives

$$m(B) = T(\bar{P})m[h(B)]$$

with $\bar{P} \in h(B)$. Looking to $B' = f_\sigma(B) = \varphi(B)$ we get in the same way relative to the local chart (U', h') , $U' = \varphi(U)$,

$$m(B') = T'(\bar{Q})m[h'(B')], \quad \bar{Q} \in h'(B')$$

where

$$T' = \sqrt{\det\left(\frac{\partial\phi'}{\partial y_i} \mid \frac{\partial\phi'}{\partial y_j}\right)}$$

with $\Phi' = h'^{-1}$ and $(y_1, y_2, \dots, y_n) \in R^n$. Finally,

$$\frac{m(B')}{m(B)} = \frac{T'(\bar{Q})}{T(\bar{P})} \cdot \frac{m[h(B')]}{m[h(B)]} = \frac{T'(\bar{Q})}{T(\bar{P})} \cdot J_{\bar{\varphi}}(\bar{P})$$

where $J_{\bar{\varphi}}(\bar{P})$ is the determinant of the Jacobian matrix of the map $h \circ \phi \circ h^{-1} = \bar{\varphi}$ computed in a point $\bar{P} \in h(B)$. This is a well-known result from calculus. Also as remarked before we have

$$\bar{D}f_{\sigma}(B) = Df_{\sigma}(B) = \frac{T'(\bar{Q})}{T(\bar{P})} \cdot J_{\bar{\varphi}}(\bar{P}).$$

Now according to our hypothesis when \bar{P}, \bar{P} runs in $h(U)$ and \bar{Q} runs in $h'(U')$ the functions T, T' and $J_{\bar{\varphi}}$ are continuous, bounded and T in particular is always greater than a certain positive number. This implies that when $B \in \sigma$ is arbitrarily selected for any $\sigma \in \nu$ such that $B \subset U$ we have that all numbers $\bar{D}f_{\sigma}(B)$ and $Df_{\sigma}(B)$ belong to a finite closed interval I of the reals.

Look now to our given covering Γ and restrict it to I . As I is compact call η the Lebesgue number of the restriction of Γ to I . Again but the continuity of T, T' and $J_{\bar{\varphi}}$ in U it is possible to find a neighbourhood V of x in U such that for all points in V and $h(V)$ the possible values taken by T, T' and J belong to intervals in the reals whose length is less than η . This being so select a $\sigma \in \nu$ and $A \in \sigma$ with $x \in A$ and $A \subset V$. As seen above for any $\tau > \sigma$ and any $B \in \tau$ with $B \subset A$ we have that the set of all possible values $\bar{D}f_{\tau}(B)$ and $Df_{\tau}(B)$ so obtained have diameter less than η and therefore are contained in some set of Γ . But by definition this set is precisely $Df_{\sigma}(A)$, what proves that Df is pointwise cofinal.

Now to prove that Df generates a unique continuous function $\psi : M_n \rightarrow R$ we shall use a result of A. Jansen [4] and as M'_n is a regular T_1 space we have only to verify

that Df is fully regular, namely, that for any $A \in \sigma \in \nu$ if $A \cap B \neq \emptyset$ then

$$Df_\sigma(A) \cap Df_\tau(B) \neq \emptyset.$$

Indeed, if $A \cap B \neq \emptyset$ there is some open set $C \in \gamma \in \nu$ with $\gamma > \sigma, \tau$ and so $\overline{Df}_\gamma(C), \underline{Df}_\gamma(C)$ will belong to $Df_\sigma(A)$ and to $Df_\tau(B)$ as follows from the definition of derivative of n -function.

Therefore there exists a unique continuous function $\psi : M_n \rightarrow R$ generated by Df ([4], p.5, Prop.1).

Our final step is to prove that for each $x \in M_n$ we have

$$\psi(x) = |J_{UU'}\varphi(x)|$$

for convenient local charts U, U' .

Let $x \in M_n$ be given and select local charts (U, h) at x and (U', h') at $\varphi(x)$ satisfying conditions stated in hypothesis 2) of our theorem.

The function ψ determines a function $\tilde{\psi} : h(U) \rightarrow R$, given by $\tilde{\psi} = \psi \circ h^{-1}$ and also $J_{UU'}\varphi$ determines a function $J_{\tilde{\varphi}} : h(U) \rightarrow R$ which is the determinant of the Jacobian matrix of at transformation $h' \circ \varphi \circ h^{-1} = \tilde{\varphi}$.

Due to a result of V. Buonomano [5] $\tilde{\varphi}$ coincides with $|J_{\tilde{\varphi}}|$ in W given by the hypothesis 2) and as they are both continuous functions defined in $h(U)$ they also coincide in $h(x) \in \overline{W}$. Therefore looking to ψ and $J_{UU'}\varphi$ they also coincide in x , i.e.

$$\psi(x) = |J_{UU'}\varphi(x)|.$$

This completes the proof.

Analysing the Gauss structures defined in Theorem 3 and 4 above we are led to the definition:

Definition IX: A Gauss structure F on a differentiable manifold M^n is called canonical if for each $x \in M^n$ there is a local chart (U, h) such that for each $\alpha \in F$, the image by h of the sets of α contained in U have equal measures as subsets of R^n .

This type of Gauss structure will play a central role in the future development of the non-deterministic theory of manifolds.

§V

1. In this last paragraph we want to indicate a series of questions which show that our approach to manifolds do not finish with the theorems proved in this paper but on the contrary it has potentially many possibilities open for later developments. As a matter of fact some of the problems to be discussed below are already under investigation and will be published in the near future.

2. Let us to back to theorem 3 of §III, 2. There we defined particular Gauss structures called canonical in the manifolds M and M' and a particular Gauss transformation between those structures. Later we used the assumption of differentiability of the map $\phi : M \rightarrow M'$ to prove that the derivative Df of f generating ϕ is pointwise cofinal and therefore produces a function $\psi : M \rightarrow R$ which has been called the intrinsic derivative of ϕ relative to the Gauss structures and transformation involved. Now we come to the question: suppose that ϕ is open continuous but not necessarily differentiable and assume that we can prove that the function f generating ϕ in the same way as in theorem 3 has a derivative Df which is cofinal relative to the same Gauss structures and transformation as in that theorem. In this case we shall have again an intrinsic derivative ψ , having no connection with Jacobians, because these do not exist now. However, we can ask the same question, namely, suppose that in a certain point $x_0 \in M$ we have $\psi(x_0) \neq 0$, is it true that ϕ has a local inverse? More precise we ask if there is a neighbourhood $V(x_0)$ and a neighbourhood $W(\phi(x_0))$ such that, restricted to these, ϕ has an inverse. In few words we are looking for an extension of the inverse function

theorem when no differentiable structure is available. Even more general situations can be considered when instead of manifolds we have more general spaces, but in this case, we have to consider deeper results in the theory of Gauss spaces and transformations which we do not discuss here.

3. Consider now the notions of tangent vector and tangent space attached to a differentiable manifold. Let $\phi : R \rightarrow M$ be a curve in the manifold M and let θ be a real differentiable function defined in a neighbourhood of $\phi(t_0)$ where $t_0 \in R$ is a given real number. Assuming that we define in M a canonical Gauss structure F as considered before we can build a family of coverings V on M by considering the open stars of coverings forming the Gauss structure. This family is called associated to F and has been used before in other areas of non-deterministic mathematics. In R we consider the family of coverings U associated to the canonical Gauss structure of R as considered in §II, 3. Now it can be proved that there are non-deterministic functions f and g generating ϕ and θ respectively such that the composite n -function $g \circ f$ has a n -derivative which generates a function $\psi : R \rightarrow R$ with the property that the value of ψ at t_0 , $\psi(t_0)$ is equal to the derivative of θ at $\phi(t_0)$ in the direction given by ϕ . Of course, we have to use particular local charts as in theorem 3 above. After ψ is defined when t runs in R or in an interval therein we have for each t the value $\psi(t)$ which is equal to the derivative of θ at $\phi(t)$ in the direction given by ϕ . Therefore we have an intrinsic definition of derivative of θ on M in the direction ϕ . It can be shown as well that when M reduces, say, to the plane we have the usual derivative of θ in a certain direction given by ψ at a certain point. Intuitively speaking, a fellow living in M could compute the derivative of θ in a given direction, once ψ is known, without worrying about the space where M could be eventually embedded. That is indeed the old idea of Gauss on his research on the intrinsic study of surfaces.

4. Now let us talk about geodesics. Let F be a Gauss structure on a manifold M , say the canonical Gauss structure and let $\alpha \in F$. An open α -chain in M is a collection C_α of sets $F_i \in \alpha$, $i \in I$ with I the set of all ordinal number from 1 up to an ordinal n (finite or infinite) such that

$$\begin{cases} F_i \cap F_j \neq \emptyset & 0 \leq |i - j| \leq 1 \\ F_i \cap F_j = \emptyset & \text{otherwise.} \end{cases}$$

If $F_1 \cap F_n \neq \emptyset$ we call C_α a closed α -chain. The number n is called the length of C_α , denoted by $|C_\alpha|$ and if C_α is open F_1 and F_n are called its extremities. An open α -chain C_α has minimum length in α if for any other α -chain C'_α with same extremities we have $|C_\alpha| \leq |C'_\alpha|$.

If $F_1 \cap F_n \neq \emptyset$ we call C_α a closed α -chain. The number n is called the length of C_α , denoted by $|C_\alpha|$ and if C_α is open F_1 and F_n are called its extremities. An open α -chain C_α has minimum length in α if for any other α -chain C'_α with same extremities we have $|C_\alpha| \leq |C'_\alpha|$.

An α -geodesic C_α^g is a collection of sets of α such that given any two sets $F_1, F_2 \in C_\alpha^g$, $F_1 \neq F_2$, there is a unique α -chain with minimum length in α having F_1 and F_2 as extremities and contained in C_α^g . A closed α -geodesic C_α^g is a collection of sets of α such that by deleting any two sets $F_1 \neq F_2$ of C_α^g we get two α -geodesics.

A geodesic in M is a connected subset E of M such that for every $\sigma \in F$, E can be covered by some α -geodesic C_α^g .

A n -function

$$g : (R, U) \rightarrow (M, V)$$

where U is the family of coverings in R associated to the canonical Gauss structure in R is called a geodesic in (M, V) if there is a geodesic E in M such that for any $\sigma_u \in U$ and $A_u \in \sigma_u$ we have

$$g_{\sigma_u}(A_u) \cap E \neq \emptyset.$$

It can be shown with simple examples that for arbitrary Gauss structure the concept of geodesic introduced here does not coincide with the usual one. However, in important cases it does. For instance that is the case for R^n with the canonical Gauss structure. It has not yet been studied for what Gauss structure the concept of geodesic introduced in here coincides with the usual ones. By another side, as can be seen in [11], in some applications to physics the interesting situation is exactly that when those two concepts of geodesic do not coincide. So there are many variations to the same theme and the question is actually deeper than it looks at a first moment.

Now having tangent vectors and geodesics a reasonable theory of manifolds starting with non-deterministic concepts can be developed and some results reproduce classical ones but some don't and they give rise to new situations impossible to describe in the usual manner.

We hope that this brief survey of future developments might give some confidence to the reader that a systematic study of manifolds with the methods of non-deterministic mathematics might have some relevance to the area.

References

- [1] Lintz, R.G. *On the Foundations of Topology*, Port. Math., 40, 1, 1981, pp. 1-39.
- [2] Wilder, R.L. *Topology of Manifolds*, AMS Publ., Vol. XXXII, 1949.
- [3] R. Lintz and V. Buonomano. *The concept of differential equation in topological spaces and generalized mechanics*. Journal für die Reine und Angewandte Mathematik, Bd. 265, 1974 Seite 31-60.
- [4] A. Jansen. *Some Mapping and Homological Properties of g -functions*. Ph.D. Thesis, McMaster Univ., 1970.
- [5] V. Buonomano. *g -functions and g -derivatives in Euclidean space*. Annali di Mat. Pura ed Applicata (4), 90 (1971), pp. 59-74.

- [6] O. Alas. *On Gauss Spaces*. Annali di Mat. Pura ed Applicata (4) 82 (1969), pp. 24-34.
- [7] R. Mirabal. *g-derivatives and Gauss structures in differentiable manifolds*. Ph. D. thesis, McMaster University, 1973.
- [8] L. Schwartz. *Cours d'Analyse*, vol. I, Hermann Cie, Paris (1967).
- [9] J.H.C. Whitehead. *Note on Manifolds*. Quart. Jour. Math. (2) 12 (1941), pp. 26-29.
- [10] J.H.C. Whitehead. *On subdivision of complexes*. Proc. Camb. Phil. Soc. 31 (1935), pp. 67-75.
- [11] Lintz, R.G. *Non deterministic foundations of Mechanics*, McMaster Univ., Hamilton, Ontario, 1985.
- [12] L. Cesari. *Surface Area*, Annals of Math. Study no. 35, Princeton, U.S.A. , (1956).

MW

TRABALHOS DO DEPARTAMENTO DE MATEMATICA

TITULOS PUBLICADOS

- 80-01 PLETCH, A. Local freeness of profinite groups. 10 p.
 80-02 PLETCH, A. Strong completeness in profinite groups. 8 p.
 80-03 CARNIELLI, W.A. & ALCANTARA, L.P. de Transfinite induction on ordinal configurations. 22 p.
 80-04 JONES RODRIGUES, A.R. Integral representations of cyclic p-groups. 13 p.
 80-05 CORRADA, M. & ALCANTARA, L.P. de Notes on many-sorted systems. 25 p.
 80-06 POLCINO MILIES, F.C. & SEHGAL, S.K. FC-elements in a group ring. 10 p.
 80-07 CHEN, C.C. On the Ricci condition and minimal surfaces with constantly curved Gauss map. 10 p.
 80-08 CHEN, C.C. Total curvature and topological structure of complete minimal surfaces. 21 p.
 80-09 CHEN, C.C. On the image of the generalized Gauss map of a complete minimal surface in R^4 . 8 p.
 81-10 JONES RODRIGUEZ, A.R. Units of ZC_p^n . 7 p.
 81-11 KOTAS, J. & COSTA, N.C.A. da Problems of model and discursive logics. 35 p.
 81-12 BRITO, F.B. & GONCALVES, D.L. Algebras não associativas, sistemas diferenciais polinomiais homogêneos e classes características. 7 p.
 81-13 POLCINO MILIES, F.C. Group rings whose torsion units form a subgroup II. Iv. (não paginado).
 81-14 CHEN, C.C. An elementary proof of Calabi's theorems on holomorphic curves. 5 p.
 81-15 COSTA, N.C.A. da & ALVES, E.H. Relations between paraconsistent logic and many-valued logic. 8 p.
 81-16 CASTILLA, M.S.A.C. On Przymusiński's theorem. 6 p.
 81-17 CHEN, C.C. & GOES, C.C. Degenerate minimal surfaces in R^4 . 21 p.
 81-18 CASTILLA, M.S.A.C. Imagens inversas de algumas aplicações fechadas. 11 p.
 81-19 ARAGONA VALLEJO, A.J. & EXEL FILHO, R. An infinite dimensional version of Hartogs' extension theorem. 9 p.
 81-20 GONCALVES, J.Z. Groups rings with solvable unit groups. 15p.
 81-21 CARNIELLI, W.A. & ALCANTARA, L.P. de Paraconsistent algebras. 16 p.
 81-22 GONCALVES, D.L. Nilpotent actions. 10 p.
 81-23 COELHO, S.P. Group rings with units of bounded exponent over the center. 25 p.
 81-24 PARMENTER, M.M. & POLCINO MILIES, F.C. A note on isomorphic group rings. 4 p.
 81-25 MERKLEN GOLOSCHMIDT, H.A. Hereditary algebras with maximum spectra are of finite type. 10 p.
 81-26 POLCINO MILIES, F.C. Units of group rings: a short survey. 32 p.

- 81-27 CHEN, C.C. & GACKSTATTER, F. Elliptic and hyper-elliptic functions and complete minimal surfaces with handles. 14 p.
- 81-28 POLCINO MILIES, F.C. A glance at the early history of group rings. 22 p.
- 81-29 FERRER SANTOS, W.R. Reductive actions of algebraic groups on affine varieties. 52 p.
- 81-30 COSTA, N.C.A. da The philosophical import of paraconsistent logic. 26 p.
- 81-31 GONCALVES, D.L. Generalized classes of groups, spaces c-nilpotent and "the Hurewicz theorem". 30 p.
- 81-32 COSTA, N.C.A. da & MORTENSEN, Chris. Notes on the theory of variable binding term operators. 18 p.
- 81-33 MERKLEN GOLDSCHMIDT, H.A. Homogenous 1-hereditary algebras with maximum spectra. 32 p.
- 81-34 PERESI, L.A. A note on semiprime generalized alternative algebras. 10 p.
- 81-35 MIRAGLIA NETO, F. On the preservation of elementary equivalence and embedding by filtered powers and structures of stable continuous functions. 9 p.
- 81-36 FIGUEIREDO, G.V.R. Catastrophe theory: some global theory a full proof. 91 p.
- 82-37 COSTA, R.C.F. On the derivations of gemetic algebras. 17 p.
- 82-38 FIGUEIREDO, G.V.R. A shorter proof of the Thom-Zeeman global theorem for catastrophes of cod ≤ 5 . 7 p.
- 82-39 VELOSO, J.M.M. Lie equations and Lie algebras: the intransitive case. 97 p.
- 82-40 GOES, C.C. Some results about minimal immersions having flat normal bundle. 37 p.
- 82-41 FERRER SANTOS, W.R. Cohomology of comodules II. 15 p.
- 82-42 SOUZA, V.H.G. Classification of closed sets and diffeos of one-dimensional manifolds. 15 p.
- 82-43 GOES, C.C. The stability of minimal cones of codimension greater than one in R^n . 27 p.
- 82-44 PERESI, L.A. On automorphisms of gemetic algebras. 27 p.
- 82-45 POLCINO MILIES, F.C. & SEHGAL, S.K. Torsion units in integral group rings of metacyclic groups. 18 p.
- 82-46 GONCALVES, J.Z. Free subgroups of units in group rings. 8 p.
- 82-47 VELOSO, J.M.M. New classes of intransitive simple Lie pseudogroups. 8 p.
- 82-48 CHEN, C.C. The generalized curvature ellipses and minimal surfaces. 10 p.
- 82-49 COSTA, R.C.F. On the derivation algebra of zygotic algebras for polyploidy with multiple alleles. 24 p.
- 83-50 GONCALVES, J.Z. Free subgroups in the group of units of group rings over algebraic integers. 3 p.

- 83-51 MANDEL, A. & GONÇALVES, J.Z. Free k -triples in linear groups. 7 p.
- 83-52 BRITO, F.G.B. A remark on closed minimal hypersurfaces of S^4 with second fundamental form of constant length. 12 p.
- 83-53 KIIHL, J.C.S. U -structures and sphere bundles. 8 p.
- 83-54 COSTA, R.C.F. On genetic algebras with prescribed derivations. 23 p.
- 83-55 SALVITTI, R. Integrabilidade das distribuições dadas por subálgebras de Lie de codimensão finita no $gh(n,C)$. 4 p.
- 83-56 MANDEL, A. & GONÇALVES, J.Z. Construction of open sets of free k -Tuples of matrices. 18 p.
- 83-57 BRITO, F.G.B. A remark on minimal foliations of codimension two. 24 p.
- 83-58 GONÇALVES, J.Z. Free groups in subnormal subgroups and the residual nilpotence of the group of units of group rings. 9 p.
- 83-59 BELOQUI, J.A. Modulus of stability for vector fields on 3-manifolds. 40 p.
- 83-60 GONÇALVES, J.Z. Some groups not subnormal in the group of units of its integral group ring. 8 p.
- 84-61 GOES, C.C. & SIMOES, P.A.Q. Imersões mínimas nos espaços hiperbólicos. 15 p.
- 84-62 GIANBRUNO, A.; MISSO, P. & POLCINO MÍLIES, F.C. Derivations with invertible values in rings with involution. 12 p.
- 84-63 FERRER SANTOS, W.R. A note on affine quotients. 6 p.
- 84-64 GONÇALVES, J.Z. Free-subgroups and the residual nilpotence of the group of units of modular and p -adic group rings. 12 p.
- 84-65 GONÇALVES, D.L. Fixed points of S^1 -fibrations. 18 p.
- 84-66 RODRIGUES, A.A.M. Contact and equivalence of submanifolds of homogenous spaces. 15 p.
- 84-67 LOURENCO, M.L. A projective limit representation of (DFC)-spaces with the approximation property. 20 p.
- 84-68 FORNARI, S. Total absolute curvature of surfaces with boundary. 25 p.
- 84-69 BRITO, F.G.B. & WALCZAK, P.G. Totally geodesic foliations with integral normal bundles. 6 p.
- 84-70 LANGEVIN, R. & POSSANI, C. Quase-folheações e integrais de curvatura no plano. 26 p.
- 84-71 OLIVEIRA, M.E.G.G. de Non-orientable minimal surfaces in RN . 41 p.
- 84-72 PERESI, L.A. On baric algebras with prescribed automorphisms. 42 p.
- 84-73 MIRAGLIA NETO, F. & ROCHA FILHO, G.C. The measurability of Riemann integrable-function with values in Banach spaces and applications. 27 p.
- 84-74 MERKLEN GOLDSCHMIDT, H.A. Artin algebras wích are equivalent to a hereditary algebra modulo pre-projectives. 38 p.
- 84-75 GOES, C.C. & SIMOES, P.A.Q. The generalized Gauss map of minimal surfaces in H^3 and H^4 . 16 p.

- 84-76 GONÇALVES, J.Z. Normal and subnormal subgroups in the group of units of a group rings. 13 p.
- 85-77 ARAGONA_VALLEJO, A.J. On existence theorems for the ∂ -operator on generalized differential forms. 13 p.
- 85-78 POLCINO MILIES, C.; RITTER, J. & SEHGAL, S.K. On a conjecture of Zassenhaus on torsion units in integral group rings II. 14 p.
- 85-79 JONES RODRIGUEZ, A.R. & MICHLER, G.O. On the structure of the integral Green ring of a cyclic group of order p^2 . The Jacobson radical of the integral Green ring of a cyclic group of order p^2 . 26 p.
- 85-80 VELOSO, J.M.M. & VERDERESI, J.A. Three dimensional Cauchy-Riemann manifolds. 19 p.
- 85-81 PERESI, L.A. On baric algebras with prescribed automorphisms II. 18 p.
- 85-82 KNUDSEN, C.A. O impasse aritmo-geométrico e a evolução do conceito de número na Grécia antiga. 43p.
- 85-83 VELOSO, J.M.M. & VERDERESI, J.A. La géométrie, le problème d'équivalence et le classification des CR-varietés homogenes en dimension 3. 30 p.
- 85-84 GONÇALVES, J.Z. Integral group rings whose group is solvable, an elementary proof. 11 p.
- 85-85 LUCIANO, O.O. Nebuleuses infinitesimement fibrées. 5 p.
- 85-86 ASPERTI, A.C. & DAJCZER, M. Conformally flat Riemannian manifolds as hypersurfaces of the lighth cone. 8 p.
- 85-87 BELOQUI, J.A. A quasi-transversal Hopf bifurcations. 11 p.
- 85-88 POLCINO MILIES, F.C. & RAPHAEL, D.M. A note on derivations with power central values in prime rings. 7 p.
- 85-89 POLCINO MILIES, F.C. Torsion units in group rings and a conjecture of H.J.Zassenhaus. 14 p.
- 86-90 LOURENÇO, M.L. Riemann domains over (DFC) spaces. 32 p.
- 86-91 ARAGONA VALLEJO, A.J. & FERNANDES, J.C.D. The Hartogs extension theorem for holomorphic generalized functions. 9 p.
- 86-92 CARRARA ZANETIC, V.L. Extensions of immersions in dimension two. 27 p.
- 86-93 PERESI, L.A. The derivation algebra of gametic and zygotic algebras for linked loci. 25 p.
- 86-94 COELHO, S.P. A note on central idempotents in group ring. 5 p.
- 86-95 PERESI, L.A. On derivations of baric algebras with prescribed automorphisms. 21 p.
- 86-96 COELHO, F.U. A generalization of a theorem of Todorov on preprojectives partitions. 18 p.
- 86-97 ASPERTI, A.C. A note on the minimal immersions of the two-sphere. 11 p.
- 86-98 COELHO, S.P. & POLCINO MILIES, F.C. A note on central idempotents in group rings II. 8 p.

- 86-99 EXEL FILHO, R. Hankel matrices over right ordered amenable groups. 18 p.

NOVA SERIE

- 86-01 GOODAIRE, E.G. & POLCINO MILIES, F.C. Isomorphisms of integral alternative loop rings. 11 p.
- 86-02 WALCZAK, P.G. Foliations which admit the most mean curvature functions. 11 p.
- 86-03 OLIVEIRA, M.E.G.G. Minimal Klein bottles with one end in R^3 and R^4 . 12 p.
- 86-04 MICALI, A. & VILLAMAYOR, O.E. Homologie de Hochschild de certaines algebres de groupes. 11 p.
- 86-05 OLIVEIRA, M.E.G.G. Minimal Klein bottles in R^3 with finite total curvature. 9 p.
- 86-06 CARRARA ZANETIC, V.L. Classification of stable maps between 2-manifolds with given singular set image. 22 p.
- 87-01 BRITO, F.G.B. & WALCZAK, P.G. Total curvature of orthogonal vector fields on three-manifolds. 4 p.
- 87-02 BRITO, F.G.B. & LEITE, M.L. A remark on rotational hypersurfaces of S_n . 13 p.
- 87-03 GONÇALVES, J., RITTER, J. & SEHGAL, S. Subnormal subgroups in $U(ZG)$. 13 p.
- 87-04 ARAGONA VALLEJO, A.J. & COLOMBEAU, J.F. The interpolation theorem for holomorphic generalized functions. 12 p.
- 87-05 ALMEIDA, S.C. de & BRITO, F.G.B. Immersed hypersurfaces of a space form with distinct principal curvatures. 9 p.
- 87-06 ASPERTI, A.C. Generic minimal surfaces. 21 p.
- 87-07 GOODAIRE, E.G. & POLCINO MILIES, F.C. Torsion units in alternative group rings. 17 p.
- 87-08 MERKLEN GOLDSCHMIDT, H.A. Algebras which are equivalent to a hereditary algebra modulo preprojectives II. 27 p.
- 87-09 REYNOL FILHO, A. P-localization of some classes of groups. 30 p.
- 88-01 CARLSON, J.F. & JONES, A. An exponential property of lattices over group rings. 22 p.
- 88-02 CARLSON, J.F. & JONES, A. Wild categories of periodic modules. 6 p.
- 88-03 SALLUM, E.M. The nonwandering set of flows on a Reeb foliation. 13 p.
- 88-04 ALMEIDA, R. Cohomologie des suites d'Atiyah. 14 p.
- 88-05 GUZZO JR., H. Alguns teoremas de caracterização para álgebras alternativas à direita. 16 p.
- 88-06 HARLE, C.E. Subvariedades isoparamétricas homogêneas. 5 p.

- 88-07 LANGEVIN, R. Vers une classification des difféomorphismes Morse-Smale d'une surface. 18 p.
- 88-08 ARAGONA VALLEJO, A. J. & BIAGIONI, H. A. An intrinsic definition of the Colombeau algebra of generalized functions. 48 p.
- 88-09 BORSARI, H.D. A Cohomological characterization of reductive algebraic groups. 20 p.
- 88-10 COELHO, S.P. & POLCINO MILIES, F.C. Finite conjugacy in group rings. 20 p.
- 88-11 FERNANDEZ, R. Characterization of the dual of an Orlicz Space. 19 p.
- 89-01 GONÇALVES, J.Z. & MANDEL, A. Embedding the Free Group Ring into Formal Series. 4 p.
- 89-02 REYNOLDS, A. Nilpotent spaces: some inequalities on nilpotency degrees. 17 p.
- 89-03 GOUVÊA, F.Q. Deforming Galois Representations: Controlling the Conductor. 22 p.
- 89-04 CORDARO, P. & TREVES, F. Homology & Cohomology in Hypo-Analytic Structures of the Hypersurfaces Type. 108p.
- 89-05 ARAGONA VALLEJO, A. J. & VILLARREAL ALVARADO, F. Colombeau's theory and shock Waves in a problem of Hydrodynamics. 48 p.
- 90-01 MANDEL, A. & GONÇALVES, J.Z. Semigroup laws and free semigroups in the group of units of group rings. 11 p.
- 90-02 LINTZ, R.G. Generation of Quantum-fields by non-deterministic fields. 32p.
- 90-03 LINTZ, R.G. A critical study on the foundations of Geometry. 41p.
- 90-04 LINTZ, R.G. The theory of manifolds from the non-deterministic point of view. 32p.