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# Global normalizations for centers of planar vector fields

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**Abstract**

This paper addresses a question posed by Carmen Chicone and proves that an analytic vector field with a non-degenerate global center can be transformed into a classical Newtonian equation $\ddot{x} = -V'(x)$.

Additionally, we establish a global Poincaré normal form for planar centers. We also demonstrate the global analytic integrability of the equation $\ddot{x} = F(u, \dot{u})$, where $F(u, v) = F(u, -v)$, under some additional conditions.


## 1. Introduction

In this paper we give an answer to the following question in the last paragraph of [4]:
*If the differential equation*

$$\dot{u} = P(u, v)$$
$$\dot{v} = Q(u, v)$$

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*has a center at the origin surrounded by a region $R$ consisting of periodic orbits, when is there a change of coordinates defined on $R$ such that in the new coordinates the differential equation is in Hamiltonian form with Hamiltonian*

$$H(x,y) = \frac{y^2}{2} + V(x)\,?$$

The region $R$ is termed as the period annulus of the center. A center is non-degenerate if the vector field's linearization at equilibrium has a non-zero determinant. We always assume that the center is located at the origin. Theorems 1.1 and 1.2 provide an answer to Chicone's question when the center is non-degenerate.

**Theorem 1.1.** *Suppose the real analytic vector field $X(u,v) = P(u,v)\partial_u + Q(u,v)\partial_v$ has a non-degenerate center. For a periodic orbit $\gamma$ in the center's period annulus, let $\mathcal{D}_\gamma$ denote the open disk bounded by $\gamma$. There exists an analytic change of variables $\Lambda(u,v) = (x,y)$ for $(u,v) \in \mathcal{D}_\gamma$, so that in the new variables, $X(x,y) = y\partial_x - V'(x)\partial_y$.*

**Theorem 1.2.** *Suppose that $X(u,v)$ (real analytic) has a non-degenerate global center, meaning the period annulus is $\mathbb{R}^2 - (0,0)$. An analytic diffeomorphism exists,*

$$\Lambda : \mathbb{R}^2 \to (-a,a) \times \mathbb{R} \quad \textit{with} \quad 0 < a \leq \infty \quad \textit{and} \quad \Lambda(u,v) = (x,y),$$

*so that in the new variables,*

$$X(x,y) = y\partial_x - V'(x)\partial_y.$$

Before proving Theorems 1.1 and 1.2, we establish an auxiliary theorem of interest: a global version of Poincaré's local normal form [20].

**Theorem 1.3** *(Global Poincaré normal form). Assume that the real analytic vector field $X(u,v) = P(u,v)\partial_u + Q(u,v)\partial_v$ has a non-degenerate center. An analytic change of variables exists from the period annulus to a disk such that in the new variables, $X(q,p) = \Omega\big(\frac{q^2+p^2}{2}\big)\big(p\partial_q - q\partial_q\big)$, with $\Omega > 0$. If the period annulus is $\mathbb{R}^2 - (0,0)$, the change of variables is from $\mathbb{R}^2$ to $\mathbb{R}^2$.*

We could not locate a written proof of this theorem, although it is an anticipated result, and a proof might already exist. If the vector field $X$ is expressed in Hamiltonian form, which is possible after identifying a global integrating factor, then the theorem results from the construction of "action-angle variables" (see [8], Section 6).

Consider the second-order equation

$$\ddot{u} = F(u,\dot{u}), \quad \text{where} \quad F(u,v) = F(u,-v),$$

and $F : \mathbb{R}^2 \to \mathbb{R}$ is analytic. $F$ can be expressed as $F(u,v) = f(u, v^2/2)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ (see [11], Theorem 6.1.3). Consider the following system of ordinary differential equations:

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= f(u, v^2/2),\end{aligned} \tag{1.1}$$

and the associated vector field

$$X(u, v) = v\partial_u + f(u, v^2/2)\partial_v. \tag{1.2}$$

Dividing the two components of $X$ yields

$$\frac{\dot{v}}{\dot{u}} = \frac{f(u, v^2/2)}{v} \Rightarrow \frac{dv^2/2}{du} = f(u, v^2/2),$$

which leads to the definition of another vector field

$$W(u, z) = \partial_u + f(u, z)\partial_z, \tag{1.3}$$

where $z \in \mathbb{R}$ replaces the variable $v^2/2$.

The flow of $W$ is complete if all solutions of the corresponding differential equation are defined for all $t \in \mathbb{R}$. The following theorem, which is a minor extension of Theorem 1.1 in [21], is presented:

**Theorem 1.4.** *Let $X(u, v) = v\partial_u + f(u, v^2/2)\partial_u$ be an analytic vector field. Assuming that the flow of $W(u, z) = \partial_u + f(u, z)\partial_z$ is complete, there exists an analytic change of variables $\Phi(u, v) = (u, p)$ onto $\mathbb{R}^2$ such that in the new variables:*

$$X(u, p) = a(u, p)\big(p\partial_u - G'(u)\partial_p\big), \quad \textit{with} \quad a(u, -p) = a(u, p). \tag{1.4}$$

Theorem 1.4 asserts that, barring time parameterization, the vector field $X$ is equivalent to the same type of vector field $p\partial_u - G'(u)\partial_p$ as presented in Theorem 1.2.

These findings build on a substantial body of prior work. The following references are recommended for further reading. For results on global centers for polynomial systems, see [14] and references therein. For the relationship between non-degenerate centers and reversibility, refer to [26] and for degenerate centers and reversibility, see [9]. For the intricate relationship between degenerate centers and analytic integrability, consult [3] and references therein. For isochronous centers, see [16], [2], and references therein. For the period function and its relation to partial differential equations, refer to [25], and for the variation of the period function, see [24] and references therein. Some of the questions examined in this paper are also explored in the Ph.D. Thesis (in Portuguese) of one of the authors (FJSN) using different mathematical tools, particularly a theorem in [10]. The thesis contains many examples but the transformations of variables between the several vector fields are less regular than here.

The organization of this paper is as follows:

In Section 2, we provide a proof for Theorem 1.3.

In Section 3, we present a proof for Theorem 1.2.

Most of the mathematical concepts utilized in this paper are standard, with the exception of the part extending from Proposition 3.1 to the end of Section 3. This section is dedicated to the construction of a diffeomorphism that displaces the periodic orbits of the vector field $X$, which is our key idea. The principal contribution of this paper is Theorem 1.2.

In Section 4, we sketch the proof of Theorem 1.4. The proof resembles that of Theorem 1.1 in [21], thus we only highlight some points pertaining to the analyticity and the domain of the transformation of variables.

Section 5 concludes the paper, where we discuss the motivations for Ragazzo [21] and propose questions.

## 2. Proof of Theorem 1.3

We will prove Theorem 1.3 in the case where the period annulus is $\mathbb{R}^2 - (0,0)$. The proof for other cases is essentially the same.

After a linear change of variables, we can assume that

$$X(u,v) = P(u,v)\partial_u + Q(u,v)\partial_v = \big(\omega v + \dots\big)\partial_u - (\omega u + \dots\big)\partial_v\,, \tag{2.5}$$

where $\omega > 0$ and the dots represent terms of quadratic order.

We denote the flow operator of the vector field $X$ as $\phi_t(u,v) = \Big(\phi_{1t}(u,v), \phi_{2t}(u,v)\Big)$. The functions $\phi_1$ and $\phi_2$ are analytic for all variables $(t,u,v)$.

Let $T(u,v)$ represent the period of the orbit that begins at $(u,v) \in \mathbb{R}^2 - (0,0)$. We extend the period function to the origin defining $T(0,0) = 2\pi/\omega$. As a consequence of a Theorem due to Poincaré [20], this extension is analytic everywhere (see also [27], Proposition 3.1). Note that $T$ is a first integral of $X$.

We denote $I(u,v)$ as the Euclidean area inside the periodic orbit that begins at $(u,v)$, i.e.,

$$I(u,v) = \frac{1}{2\pi}\oint u dv = -\frac{1}{2\pi}\int_0^{T(u,v)} \phi_{1t}(u,v) Q\big(\phi_{1t}(u,v), \phi_{2t}(u,v)\big) dt\,. \tag{2.6}$$

The negative sign in front of the integral results from the clockwise orientation of the periodic orbits, which opposes the usual orientation of the boundary of a disk. Given the analyticity of the functions used in the definition of $I$, $I$ is also analytic on $\mathbb{R}^2$. As the orbits of $X$ are almost circular near the origin, we obtain

$$I(u,v) = \frac{u^2+v^2}{2} + \text{higher order terms.} \tag{2.7}$$

Note that $I$ is not the "action variable" Arnol'd [1] associated with the vector field $X$, because in general, $d\big(\iota_X du \wedge dv\big)$ is not zero, i.e., there is no Hamiltonian function $H$ such that $\iota_X du \wedge dv = dH$. However, the area function $I$ is a first integral of $X$.

Let $Y(u,v) = \nabla I = (\partial_u I)\partial_u + (\partial_v I)\partial_v$ be the Euclidean gradient vector field of $I$. Each integral curve of $Y$ is asymptotic to the origin as $t \to -\infty$, and intersects each periodic orbit of $X$ exactly once. Furthermore, there always exist regular analytic curves through the origin such that each open half-branch of the curve is a gradient trajectory. This holds even when the equilibrium is degenerate [19]. After applying a rigid rotation to the coordinates $(u,v)$, any integral curve of $Y$ close to the origin can be described by a graph $u \to v$ tangent to the $u$-axis:

$$u \to v = h(u) = a_0 u^k + a_1 u^{k+1} \dots = u^k \chi(u)\,,\;\; 2 \le k \le \infty\,, \tag{2.8}$$

where $\chi(u) = a_0 + a_3 u + \dots$ is analytic.

We denote by $\gamma_+$ the curve given by the trajectory of $Y$, for $t \in (-\infty, \infty)$, that starts at $(u, h(u))$ with $u > 0$. Similarly, we define $\gamma_-$ as the trajectory that starts at $(u, h(u))$ with $u < 0$. We denote $\overline{\gamma} = \gamma_- \cup (0,0) \cup \gamma_+$ as the analytic arc that extends infinitely in both directions.

**Proposition 2.1.** *The curve $\overline{\gamma} = \gamma_- \cup (0,0) \cup \gamma_+$ can be parameterized by an analytic function $\beta : (-\infty, \infty) \to \mathbb{R}^2$ such that: $\beta(0) = (0,0)$, $\dot{\beta}(0) = (1,0)$, and*

$$I \circ \beta(r) = \frac{r^2}{2}\,. \tag{2.9}$$

*Let $\gamma(r)$ be the periodic orbit corresponding to the parameter $r$. The definition of $I$ in equation (2.6) implies*

$$I \circ \beta(r) = \frac{1}{2\pi} \int\limits_{\mathcal{D}_{\gamma(r)}} du \wedge dv = \frac{r^2}{2} \Rightarrow \int\limits_{\mathcal{D}_{\gamma(r)}} du \wedge dv = \pi r^2\,. \tag{2.10}$$

*Therefore, $r$ is the radius of a circular disk with the same area as the topological disk bounded by the periodic orbit $\gamma(r)$.*

**Proof.** To prove this proposition, we rescale the vector field $Y$ restricted to $\overline{\gamma}$ as follows:

$$\overline{Y}(u,v) = \begin{cases} \sqrt{2I}\,\frac{\nabla I}{|\nabla I|^2} & \text{if} \quad (u,v) \in \gamma_+ \\ -\sqrt{2I}\,\frac{\nabla I}{|\nabla I|^2} & \text{if} \quad (u,v) \in \gamma_- \end{cases} \tag{2.11}$$

We denote by $r$ the parameter of the integral curves of $\overline{Y}$. On $\gamma_+$,

$$\frac{d}{dr}\sqrt{2I} = \frac{\frac{d}{dr} I}{\sqrt{2I}} = \frac{\nabla I \cdot \overline{Y}}{\sqrt{2I}} = 1 \tag{2.12}$$

and on $\gamma_-$, $\frac{d}{dr}\sqrt{2I} = -1$.

The vector field $\overline{Y}$ in the coordinates $u$ determined by the graph $u \to v$ in equation (2.8) is given by $\frac{du}{dr}\partial_u$ with

$$\frac{d}{dr}u = \frac{u}{|u|}\sqrt{2I(u,v)}\,\frac{\partial_u I(u,v)}{|\nabla I(u,v)|^2}\Big|_{v=u^k\chi(u)}, \quad \text{for } u \neq 0 \text{ small.} \tag{2.13}$$

Using that $k \geq 2$ and $I(u,v) = \frac{u^2+v^2}{2} + \dots > 0$ for $u \neq 0$, we obtain

$$I(u, u^k\chi(u)) = \frac{u^2}{2}\big(1 + R_1(u)\big), \; |\nabla I(u, u^k\chi(u))|^2 = u^2\big(1 + R_2(u)\big),$$
$$\text{and } \partial_u I(u, u^k\chi(u)) = u\big(1 + R_3(u)\big),$$

where $R_1$, $R_2$, and $R_3$ are analytic functions of order $\mathcal{O}(|u|)$. The substitution of these relations into equation (2.13) gives

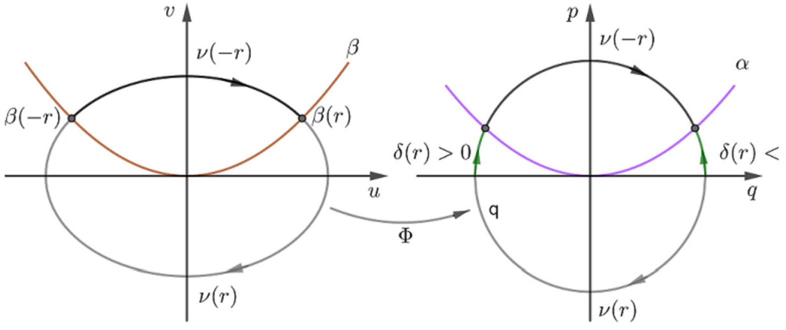


Fig. 1. The elements used in the construction of the curve $\alpha(r)$.

$$\frac{du}{dr} = \frac{1+R_3(u)}{1+R_2(u)}\sqrt{\big(1+R_1(u)\big)}. \tag{2.14}$$

Therefore, the vector field $\overline{Y}$ on $\overline{\gamma}$ extends analytically to $u=0$ with the value $\dot{u}=1$. If $\beta:\mathbb{R}\to\mathbb{R}^2$ is the solution to $\dot{\beta}=\overline{Y}\circ\beta$ with $\beta(0)=(0,0)$, then $\dot{\beta}(0)=(1,0)$ and equation (2.12) implies $I\circ\beta(r)=r^2/2$, as we aimed to demonstrate. □

Let $T\circ\beta(r)$ represent the period of the orbit of $X(u,v)$ through $\beta(r)$. The function $T\circ\beta$ is even because $T$ is constant on the level sets of $I$ and $I\circ\beta(r)=\frac{r^2}{2}$. As $T\circ\beta(r)$ is analytic, we can define an analytic function $\tau$ such that

$$\tau(r) := T\circ\beta(r). \tag{2.15}$$

Theorem 1.3 follows from the following lemma:

**Lemma 2.1.** *An analytic change of variables* $\Phi(u,v)=(q,p)$ *exists that transforms* $X(u,v)=P(u,v)\partial_u+Q(u,v)\partial_v$ *into* $\Omega\big(\frac{q^2+p^2}{2}\big)\big(p\partial_q-q\partial_p\big)$*, where* $\Omega\big(\frac{q^2+p^2}{2}\big):=2\pi/\tau\big(\frac{q^2+p^2}{2}\big)$.

**Proof.** Let $\phi_t$ and $\psi_t$ be the flow functions of the vector fields $X(u,v)$ and $\Omega\big(\frac{q^2+p^2}{2}\big)\big(p\partial_q-q\partial_p\big)$, respectively. Let $\nu(r)>0$, $r\neq 0$, be the time to transition from $\beta(r)$ to $\beta(-r)$, namely $\phi_{\nu(r)}\circ\beta(r)=\beta(-r)$, with $\nu(r)+\nu(-r)=\tau(r)$, see Fig. 1. We define

$$\delta(r) := \frac{\frac{\tau(r)}{2}-\nu(r)}{2} = \frac{\nu(-r)-\nu(r)}{4}. \tag{2.16}$$

As $\nu$ is analytic for $r\neq 0$, $\delta(r)$ is also analytic for $r\neq 0$. The function $\delta(r)=-\delta(-r)$ is odd and $|\delta(r)|<\frac{\tau(r)}{4}$.

Let $\alpha:\mathbb{R}\to\mathbb{R}^2$ be the curve on the $(q,p)$-plane, given by (see Fig. 1):

$$\big(q(r),p(r)\big)=\alpha(r):=\psi_{\delta(r)}(r,0). \tag{2.17}$$

This curve is analytic for $r\neq 0$.

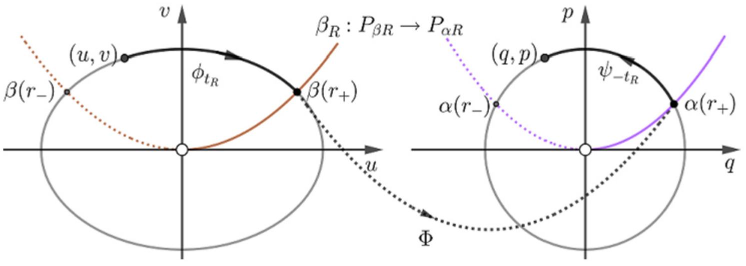


Fig. 2. Construction of $\beta_R$.

Considering the flow expression

$$\psi_t(q,p) = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}, \quad \Omega = \Omega\left(\frac{q^2+p^2}{2}\right), \tag{2.18}$$

and the properties of $\delta(r)$, we have:

$$q(r) = -q(-r), \;\; p(r) = p(-r), \;\; \text{and } \frac{q^2(r)+p^2(r)}{2} = \frac{r^2}{2}. \quad \square \tag{2.19}$$

We begin the construction of $\Phi$ by imposing

$$\Phi \circ \beta(r) := \alpha(r)\,. \tag{2.20}$$

To extend this definition to the plane, we define $P_{\beta R}$ as $\{(u,v) \in \mathbb{R}^2\} - \{\beta(r) : r \le 0\}$ and $P_{\beta L}$ as $\{(u,v) \in \mathbb{R}^2\} - \{\beta(r) : r \ge 0\}$.

For a specific point $(u,v)$ in $P_{\beta R}$, we determine $t_R(u,v)$ as the value of time at which $\phi_{t_R}(u,v) = \beta(r_+)$, $r_+ > 0$, and $\phi_t(u,v) \in P_{\beta R}$ for all $t \in (0, t_R)$ (see Fig. 2). Note that $t_R : P_{\beta R} \to \mathbb{R}$ is $C^\omega$ and

$$-\nu(r_+) < t_R(u,v) < \nu(-r_+)\,. \tag{2.21}$$

Similarly, for a specific point $(u,v)$ in $P_{\beta L}$, we determine $t_L(u,v)$ as the value of time at which $\phi_{t_L}(u,v) = \beta(r_-)$, $r_- < 0$, and $\phi_t(u,v) \in P_{\beta L}$ for all $t \in (0, t_L)$ (see Fig. 3). Note that $t_L : P_{\beta L} \to \mathbb{R}$ is $C^\omega$ and

$$-\nu(r_-) < t_L(u,v) < \nu(-r_-)\,. \tag{2.22}$$

We then define the analytic map $\beta_R : P_R \to \mathbb{R}^2$ as (see Fig. 2)

$$(q,p) = \beta_R(u,v) := \psi_{-t_R(u,v)} \circ \Phi \circ \phi_{t_R(u,v)}(u,v)\,. \tag{2.23}$$

The definition of $\alpha(r)$ implies that $\psi_t \circ \alpha(r) \ne \alpha(-r)$ for all $-\nu(-r_+) < t < \nu(r_+)$. Given this, and considering equation (2.21), we can infer that $\beta_R(u,v)$ does not intersect the curve $\{\alpha(r) : r \le 0\}$ for any value of $(u,v) \in P_{\beta L}$. By defining $P_{\alpha R}$ as $\{(q,p) \in \mathbb{R}^2\} - \{\alpha(r) : r \le 0\}$, we can conclude that $\beta_R : P_{\beta R} \to P_{\alpha R}$.

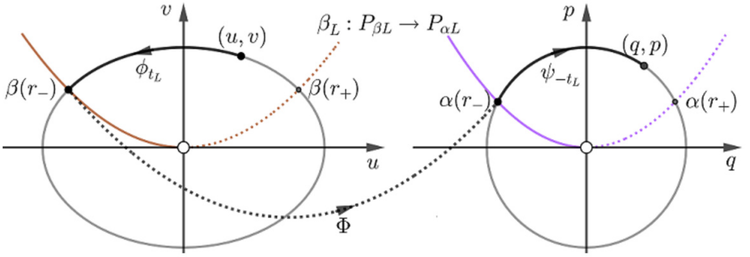


Fig. 3. Construction of $\beta_L$.

To construct the inverse of $\beta_R$, we let $(q, p) \in P_{\alpha R}$ and denote $\tilde{t}_R(q, p)$ as the time value for which $\psi_{\tilde{t}_R}(q, p) = \alpha(r_+)$, $r_+ > 0$, and $\psi_t(q, p) \in P_{\alpha R}$ for all $t \in (0, \tilde{t}_R)$. Given that $t_R(u, v) = \tilde{t}_R \circ \beta_R(u, v)$, we derive:

$$\phi_{-\tilde{t}_R \circ \beta_R(u,v)} \circ \Phi^{-1} \circ \psi_{\tilde{t}_R \circ \beta_R(u,v)} \circ \beta_R(u, v) = (u, v)\,,$$

implying that $\beta_R^{-1}(q, p) = \phi_{-\tilde{t}_R(q,p)} \circ \Phi^{-1} \circ \psi_{\tilde{t}_R(q,p)}(q, p)$.

Given that both $\beta_R$ and its inverse are combinations of analytic functions, we can conclude that $\beta_R : P_R \to P_R$ is an analytic diffeomorphism.

For a specific point $(u, v)$ in $P_{\beta L}$, we define $t_L(u, v)$ as the time value such that $\phi_{t_L}(u, v) = \beta(r_-)$, $r_- < 0$, and $\phi_t(u, v) \in P_{\beta L}$ for all $t \in (0, t_L)$. We then define $P_{\alpha L}$ as $\{(q, p) \in \mathbb{R}^2\} - \{\alpha(r) : r \geq 0\}$ and similarly establish the analytic diffeomorphism $\beta_L : P_{\beta L} \to P_{\alpha L}$ as depicted in Fig. 3:

$$(q, p) = \beta_L(w, p) := \psi_{-t_L(u,v)} \circ \Phi \circ \phi_{t_L(u,v)}(u, v)\,.$$

We claim that the two mappings $\beta_R$ and $\beta_L$ agree in $P_{\beta L} \cap P_{\beta R}$. Indeed, if $(u, v)$ is a point where $t_R(u, v) > 0$, then $t_R(u, v) - t_L(u, v) = \nu(-r_+) = \nu(r_-)$, and we deduce that:

$$\begin{aligned} \beta_R(u, v) &= \psi_{-t_R(u,v)} \circ \Phi \circ \phi_{t_R(u,v)}(u, v) \\ &= \psi_{-t_L(u,v)} \circ \psi_{-\nu(r_-)} \circ \Phi \circ \phi_{\nu(r_-)} \circ \phi_{t_L(u,v)}(u, v) \\ &= \psi_{-t_L(u,v)} \circ \Phi \circ \phi_{t_L(u,v)}(u, v) = \beta_L(u, v)\,. \end{aligned}$$

The same reasoning applies to a point $(u, v)$ with $t_R(u, v) < 0$, verifying that $\beta_R$ and $\beta_L$ coincide in $P_{\beta L} \cap P_{\beta R}$.

We define the global homeomorphism $\Phi$ onto $\mathbb{R}^2$ as follows:

$$\Phi(u, v) = \begin{cases} \beta_R(u, v) & \text{if} \quad (u, v) \in P_{\beta R} \\ \beta_L(u, v) & \text{if} \quad (u, v) \in P_{\beta L} \\ (0, 0) & \text{if} \quad (u, v) = (0, 0)\,. \end{cases}$$

This is the aimed extension of $\Phi$ from the curve $\beta$, as given in equation (2.20), to the whole plane.

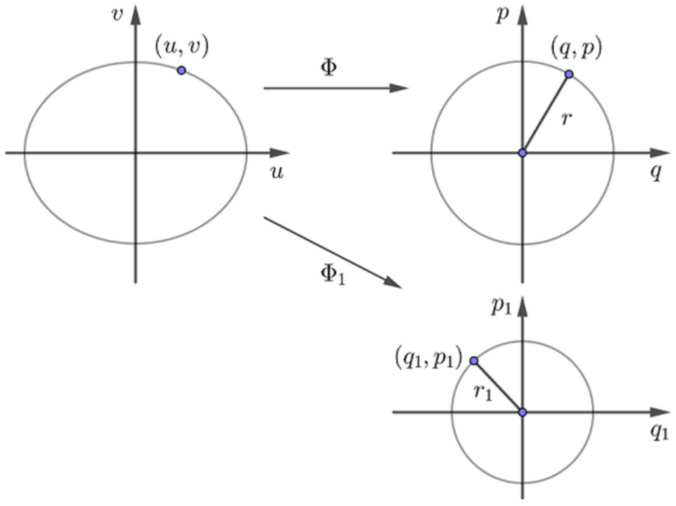


Fig. 4. In general $r \neq r_1$, which shows that $\Phi \neq \Phi_1$.

For $(u, p) \in P_{\beta R}$ and sufficiently small times $t$, $t_R \circ \phi_t(u, v) = t_R(w, p) - t$ and we have

$$\beta_R \circ \phi_t(u, v) = \psi_{-t_R \circ \phi_t(u,v)} \circ \phi_{t_R \circ \phi_t(u,v)} \circ \phi_t(u, v) = \psi_{t - t_R(u,v)} \circ \phi_{t_R}(u, v) = \psi_t \circ \beta_R \, .$$

The same is valid in $P_{\beta L}$. This leads to the following equation:

$$\Omega\left(\frac{q^2+p^2}{2}\right)\left(p\partial_q - q\partial_p\right) = (D\Phi)\, X \circ \Phi^{-1} \, ,$$

which is applicable on $\mathbb{R}^2 - (0, 0)$.

Our next step is to demonstrate the analyticity of $\Phi$ at the origin. A theorem by Poincaré [20] and Liapounoff [13], see e.g. Manosas and Villadelprat [15], asserts that there exists an analytic change of variables $(q_1, p_1) = \Phi_1(u, v)$, defined near the origin, that transforms $\phi_t$ into the normal form:

$$\phi_{1t}(q_1, p_1) = \begin{bmatrix} \cos(\Omega_1 t) & \sin(\Omega_1 t) \\ -\sin(\Omega_1 t) & \cos(\Omega_1 t) \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix}, \tag{2.24}$$

where $\Omega_1 = \Omega_1(\frac{q_1^2+p_1^2}{2})$. Generally, $\Phi_1$ does not coincide with $\Phi$ (see Fig. 4). We will subsequently show that if $\Phi_1 \neq \Phi$, it is possible to transform $\Phi_1$ into another normalizer that matches $\Phi$.

The function $\frac{q_1^2+p_1^2}{2} \circ \Phi_1 \circ \beta(r)$ is even because $I \circ \beta(r) = r^2/2$, and $\Phi_1$ maps level sets of $I$ into level sets of $\frac{q_1^2+p_1^2}{2}$. Therefore, there exists an analytic function $\tilde{\xi}$ such that $\frac{q_1^2+p_1^2}{2} \circ \Phi_1 \circ \beta(r) = \tilde{\xi}(r^2/2)$. As $\tilde{\xi}(0) = 0$, $\tilde{\xi}(r^2/2) > 0$ for $r \neq 0$, and $D\Phi_1$ is non-singular at the origin, there exists an analytic function $\xi$ near zero such that $\frac{q_1^2+p_1^2}{2} \circ \Phi_1 \circ \beta(r) = \frac{r^2}{2}\xi(r^2/2)$ and $\xi(r^2/2) > 0$.

Equation (2.19) implies that $\frac{q^2+p^2}{2} \circ \Phi \circ \beta(r) = \frac{r^2}{2}$. If $\frac{r^2}{2}\xi(r^2/2) \neq r^2/2$, then $\Phi_1$ cannot coincide with $\Phi$ near zero.

We start by defining a change of variables as follows:

$$h_1(q_2, p_2) = \begin{bmatrix} q_1 \\ p_1 \end{bmatrix} = \left( \xi \left( \frac{q_2^2 + p_2^2}{2} \right) \right)^{1/2} \begin{bmatrix} q_2 \\ p_2 \end{bmatrix} \tag{2.25}$$

This change leaves the Poincaré normal form (2.24) invariant, resulting in:

$$\phi_{2t}(q_2, p_2) := h_1^{-1} \circ \phi_{1t} \circ h_1(q_2, p_2) = \begin{bmatrix} \cos(\Omega_2 t) & \sin(\Omega_2 t) \\ -\sin(\Omega_2 t) & \cos(\Omega_2 t) \end{bmatrix} \begin{bmatrix} q_2 \\ p_2 \end{bmatrix}, \tag{2.26}$$

where $\Omega_2 = \Omega_2 \left( \frac{q_2^2+p_2^2}{2} \right)$ is analytic. Equation (2.25) implies

$$\frac{q_1^2 + p_1^2}{2} = \left( \xi \left( \frac{q_2^2 + p_2^2}{2} \right) \frac{q_2^2 + p_2^2}{2} \right) \circ h^{-1}(q_1, p_1) \tag{2.27}$$

We then define the local change of variables $\Phi_2(u, v) = (q_2, p_2)$ by $\Phi_2 = h_1^{-1} \circ \Phi_1$. With these variables, the flow $\phi_t$ is presented in equation (2.26).

The curve $\beta$ in the new coordinates is

$$\big(q_2(r), p_2(r)\big) = \Phi_2 \circ \beta(r) = h_1^{-1} \circ \Phi_1 \circ \beta(r). \tag{2.28}$$

Equation (2.27) then leads to

$$\frac{q_2^2(r) + p_2^2(r)}{2} \xi \left( \frac{q_2^2(r) + p_2^2(r)}{2} \right) = \left( \frac{q_2^2 + p_2^2}{2} \xi \left( \frac{q_2^2 + p_2^2}{2} \right) \right) \circ h_1^{-1} \circ \Phi_1 \circ \beta(r)$$
$$= \frac{q_1^2 + p_1^2}{2} \circ \Phi_1 \circ \beta(r) = \frac{r^2}{2} \xi(r^2/2) \, .$$

Because $r \to \frac{r^2}{2} \xi(r^2/2)$ increases for $r > 0$, we get

$$\frac{q_2^2(r) + p_2^2(r)}{2} = \frac{q_2^2 + p_2^2}{2} \circ \Phi_2 \circ \beta(r) = \frac{r^2}{2} \, . \tag{2.29}$$

So, $\Phi$ and $\Phi_2$ map the periodic orbit $\gamma(r)$, starting at $\beta(r)$, to circles of radius $r$ in the $(q, p)$ and $(q_2, p_2)$ planes, respectively (see Fig. 5). Therefore, $\Omega_2 \left(r^2/2\right) = \Omega \left(r^2/2\right)$, making the expressions of $\phi_{2t}$ and $\psi_t$ identical.

Even though $\alpha_2(r) := \big(q_2(r), p_2(r)\big) = \Phi_2 \circ \beta(r)$ shares with $\alpha(r)$ the same property expressed in equation (2.29), these two curves are generally different (see Fig. 6).

Equation (2.29) implies

$$\frac{d^2}{dr^2} \frac{q_2^2(r) + p_2^2(r)}{2} \bigg|_{r=0} = \dot{q}_2^2(0) + \dot{p}_2^2(0) = 1$$

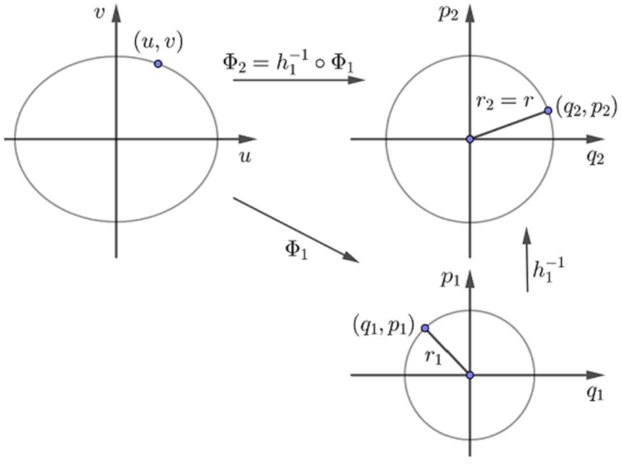


Fig. 5. $h_1^{-1}$ maps a circle of radius $r_1$ to a circle of radius $r$, such that $\Phi_2 = h_1^{-1} \circ \Phi_1$ maps a circle of radius $r$ onto a circle of radius $r$.

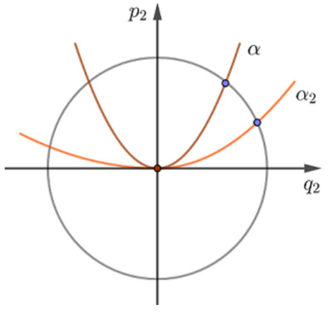


Fig. 6. In general the curves $r \to \alpha(r) = \Phi \circ \beta(r)$ and $r \to \alpha_2(r) = \Phi_2 \circ \beta(r)$ do not coincide, which implies $\Phi \neq \Phi_2$.

Thus, by using a rigid rotation of the coordinate system $(q_2, p_2)$, we can always make $\frac{d}{dr}\alpha_2(r)\big|_{r=0} = (1, 0)$. We will assume that this is the case.

Define

$$\theta(r) = \arctan\left(\frac{p_2(r)}{q_2(r)}\right)$$

as the angle that the point $\big(q_2(r), p_2(r)\big)$ makes with the horizontal axis. This function $\theta(r)$ is analytic in a neighborhood of the origin, and $\theta(0) = 0$.

We decompose $\theta(r)$ into even and odd parts

$$\theta(r) = \underbrace{\frac{\theta(r) + \theta(-r)}{2}}_{\theta_e(r^2/2)} + \underbrace{\frac{\theta(r) - \theta(-r)}{2}}_{\theta_o(r)} .$$

Both $\theta_e$ and $\theta_o$ are analytic.

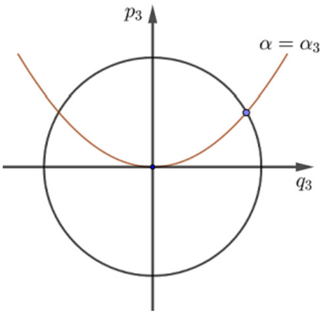


Fig. 7. The function $h_2$ was defined to eliminate the discrepancy observed in Fig. 6 between curves $\alpha$ and $\alpha_2$, i.e., $\alpha_3 = h_2^{-1} \circ \alpha_2$ results in $\alpha_3 = \alpha$.

Next, we define an analytic change of coordinates

$$h_2(q_3, p_3) = \begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_e & -\sin\theta_e \\ \sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} q_3 \\ p_3 \end{bmatrix}, \tag{2.30}$$

where $\theta_e = \theta_e\left(\frac{q_3^2+p_3^2}{2}\right)$. Since $\frac{q_3^2+p_3^2}{2} = \frac{q_2^2+p_2^2}{2}$, the map $h_2$ commutes with the flow operator $\phi_{2t} = \psi_t$.

Define the change of variables $(q_3, p_3) = \Phi_3(u, v) = h_2^{-1} \circ \Phi_2(u, v)$ for $(u, v)$ in a neighborhood of the origin. $\Phi_3$ is analytic. In the new variables, the flow operator is represented by $\psi_t$ and $\frac{q_3^2+p_3^2}{2} \circ \Phi_3 \circ \beta(r) = \frac{r^2}{2}$.

Let $\big(q_3(r), p_3(r)\big) = \alpha_3(r) := \Phi_3 \circ \beta(r) = h_2^{-1} \circ \alpha_2(r)$. Using that

$$\begin{bmatrix} q_2(r) \\ p_2(r) \end{bmatrix} = \begin{bmatrix} \cos\theta(r) & -\sin\theta(r) \\ \sin\theta(r) & \cos\theta(r) \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} \Rightarrow \frac{q_2^2(r) + p_2^2(r)}{2} = \frac{r^2}{2}$$

and

$$\begin{aligned}
\begin{bmatrix} q_3(r) \\ p_3(r) \end{bmatrix} &= h_2^{-1}\big(q_2(r), p_2(r)\big) = \begin{bmatrix} \cos\theta_e(r^2/2) & \sin\theta_e(r^2/2) \\ -\sin\theta_e(r^2/2) & \cos\theta_e(r^2/2) \end{bmatrix} \begin{bmatrix} q_2(r) \\ p_2(r) \end{bmatrix} \\
&= \begin{bmatrix} \cos\big(\theta(r) - \theta_e(r^2/2)\big) & -\sin\big(\theta(r) - \theta_e(r^2/2)\big) \\ \sin\big(\theta(r) - \theta_e(r^2/2)\big) & \cos\big(\theta(r) - \theta_e(r^2/2)\big) \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\big(\theta_o(r)\big) & -\sin\big(\theta_o(r)\big) \\ \sin\big(\theta_o(r)\big) & \cos\big(\theta_o(r)\big) \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} = r \begin{bmatrix} \cos\big(\theta_o(r)\big) \\ \sin\big(\theta_o(r)\big) \end{bmatrix},
\end{aligned}$$

we find

$$q_3(r) = -q_3(-r) \;\text{ and }\; p_3(r) = p_3(-r). \tag{2.31}$$

Therefore, $\alpha_3(r)$ satisfies the same properties (2.19) as $\alpha(r)$. Since in the coordinates $(q_3, p_3)$ the flow $\phi_t$ is given by $\psi_t$, the definitions of $\nu(r)$, and equations (2.16) and (2.17) imply that $\alpha_3(r) = \alpha(r)$ (see Fig. 7) after the identification $(q_3, p_3) = (q_2, p_2)$.

Finally, the identity $\Phi_3 \circ \phi_t \circ \Phi_3^{-1} = \psi_t = \Phi \circ \phi_t \circ \Phi^{-1}$ and the definition of $\Phi$ restricted to $P_R$ in equation (2.23) imply that $\Phi_3 = \Phi$ in a neighborhood of the origin. Since $\Phi_3$ is analytic, $\Phi$ is analytic at the origin. □

## 3. Proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 is similar to, and easier than, that of Theorem 1.2. In the following, we present only the proof of Theorem 1.2.

Consider a Hamiltonian system with Hamiltonian function $H = (y^2 + \rho^2)/2$ and symplectic form $\sigma = \Psi(\rho) d\rho \wedge dy$. Let $X_\Psi = \frac{1}{\Psi(\rho)}\left(y\partial_\rho - \rho\partial_y\right)$ denote the vector field associated with $H$ and $\sigma$. Given that $\Psi(\rho)\dot{\rho} = y$ and, for an orbit with energy $H = \eta$, $y = \pm\sqrt{2\eta - \rho^2}$, the period of the orbit is

$$T_\Psi(\eta) = 2 \int_{-\sqrt{2\eta}}^{\sqrt{2\eta}} \frac{\Psi(\tilde{\rho})}{\sqrt{2\eta - \tilde{\rho}^2}} d\tilde{\rho} = 4 \int_0^{\sqrt{2\eta}} \frac{\Psi(\tilde{\rho})}{\sqrt{2\eta - \tilde{\rho}^2}} d\tilde{\rho}\,. \tag{3.32}$$

Multiplying both sides of this equation by $1/\sqrt{2(E-\eta)}$ and integrating over $\eta$ from zero to $E$, then changing the order of integration in the double integral (first integrating on $\eta$), and using the fact that $\int_{\tilde{\rho}^2/2}^{E} [(E-\eta)(\eta - \tilde{\rho}^2/2)]^{-1/2} d\eta = \pi$, we obtain

$$\int_0^{\sqrt{2E}} \Psi(\tilde{\rho}) d\tilde{\rho} = \frac{1}{2\pi} \int_0^{E} \frac{T_\Psi(\eta)}{\sqrt{2(E-\eta)}} d\eta\,. \tag{3.33}$$

Equation (3.32), where $\Psi$ is unknown, is referred to as the Abel equation in honor of N. H. Abel, who solved it in 1823.

Given that $H = (y^2 + \rho^2)/2$, the periodic orbits in the plane $(\rho, y)$ are circular. In this case, the parameter $r$ defined in Proposition 2.1 is the radius of an orbit and $\eta = r^2/2$. The function $\tau = \tau_\Psi$ that appears in Proposition 2.1 satisfies

$$\tau_\Psi(r) = T_\Psi(r^2/2)\,. \tag{3.34}$$

Substituting $\rho = \sqrt{2E}$ and $\eta = s^2/2$ into equation (3.33), we derive

$$\int_0^{\rho} \Psi(\tilde{\rho}) d\tilde{\rho} = \frac{1}{2\pi} \int_0^{\rho} \frac{T_\Psi(s^2/2)s}{\sqrt{\rho^2 - s^2}} ds = \frac{1}{2\pi} \int_0^{\rho} \frac{\tau_\Psi(s)s}{\sqrt{\rho^2 - s^2}} ds\,. \tag{3.35}$$

As in [25] (equation 4-2-11), we change variables to $s = \rho\sin\theta$, yielding

$$\int_0^{\rho} \Psi(\tilde{\rho}) d\tilde{\rho} = \frac{1}{2\pi} \int_0^{\pi/2} \left(s\tau_\Psi(s)\right)_{s=\rho\sin\theta} d\theta\,. \tag{3.36}$$

Differentiating equation (3.36) with respect to $\rho$, we obtain

$$\begin{aligned}\Psi(\rho) &= \frac{1}{2\pi}\int_0^{\pi/2}\Big(s\tau_\Psi(s)\Big)'_{s=\rho\sin\theta}\sin\theta d\theta\\ &= \frac{1}{2\pi}\int_0^{\pi/2}\Big(\tau_\Psi(s)+s\tau_\Psi'(s)\Big)_{s=\rho\sin\theta}\sin\theta d\theta\,.\end{aligned} \tag{3.37}$$

The proof of Theorem 1.2 is essentially elucidated in the following two paragraphs.

Given a vector field $X(u,v)$ with a non-degenerate global center, an analytic change of variables $\Phi(u,v)=(q,p)$ exists that transforms it into the Poincaré normal form according to Theorem 1.3. Associated with this normal form is a period function $\tau(r):=2\pi/\Omega\big(r^2/2\big)$. Initially, we assume $\tau'\geq 0$. Under this assumption, we impose $\tau_\Psi(r)=\tau(r)$. Consequently, equation (3.37) yields $\Psi(\rho)=\Psi(-\rho)>0$ for all $\rho\geq 0$. Theorem 1.3, with $\beta(r)=(r,0)=(\rho,y)$, implies the existence of an analytic change of variables $\Phi_\Psi$ that transforms $X_\Psi$ into the same normal form as $X$. Therefore, $(y,\rho)=\Phi_\psi^{-1}\circ\Phi(u,v)$ is an analytic change of variables that transforms $X$ into $X_\Psi$.

Subsequently, we define a new variable:

$$x=\lambda(\rho)=\int_0^{\rho}\Psi(\tilde\rho)d\tilde\rho\,,\quad\text{with}\ \ x\in(-a,a)\,, \tag{3.38}$$

where:

$$a=\int_0^{\infty}\Psi(\tilde\rho)d\tilde\rho\,,\quad\text{possibly}\ \ a=\infty. \tag{3.39}$$

The map $(x,y)=\Phi_\lambda(\rho,y):=\big(\lambda(\rho),y\big)$ is an analytic diffeomorphism from $\mathbb{R}^2$ to $(-a,a)\times\mathbb{R}$. In these new variables, $\sigma=dx\wedge dy$ and $H=\frac{y^2}{2}+V(x)$, where $V(x)=\big(\lambda^{-1}(x)\big)^2/2$. The change of variables $\Lambda=\Phi_\lambda\circ\Phi_\psi^{-1}\circ\Phi$ is as stated in Theorem 1.2.

The above proof is incomplete due to the additional hypothesis $\tau'\geq 0$. If $\tau'(r)<0$ for certain values of $r$ and we maintain $\tau_\Psi=\tau$, then $\Psi(\rho)$ can become negative. To overcome this, we utilize the fact that the Poincaré normal form is not unique and $\tau(r)$ can be altered through a change of variables as in equation (2.25).

**Proposition 3.1.** *Let $\zeta:\mathbb{R}\to(-\bar r,\bar r)$, $0<\bar r\leq\infty$, be an analytic diffeomorphism of the form:*

$$s=\zeta(r)=rb\big(r^2/2\big)\,,$$

*where $b$ is an analytic function. Then, there exists an analytic diffeomorphism $\Phi_\zeta:\mathbb{R}^2\to\mathcal{D}_{\bar r}$, where $\mathcal{D}_{\bar r}$ is the open disk of radius $\bar r$, that transforms the vector field $X=2\pi\big/\tau\big(\sqrt{q^2+p^2}\big)\big(p\partial_q$*

$- q\partial_q\big)$ *into* $\tilde{X} = 2\pi/\tilde{\tau}\big(\sqrt{\tilde{q}^2+\tilde{p}^2}\big)\big(\tilde{p}\partial_{\tilde{q}} - \tilde{q}\partial_{\tilde{q}}\big)$, *i.e.* $\Phi_\zeta$ *preserves the Poincaré normal form, such that:*

$$\tau(r) = \tilde{\tau}\big(\zeta(r)\big)\,. \tag{3.40}$$

**Proof.** Consider the analytic diffeomorphism defined as follows:

$$\begin{aligned}&\begin{bmatrix} q \\ p \end{bmatrix} \to b\left(\frac{p^2+q^2}{2}\right)\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix} \Rightarrow \\ &\sqrt{\tilde{q}^2+\tilde{p}^2} = \sqrt{q^2+p^2}\, b\left(\frac{p^2+q^2}{2}\right) = \zeta\left(\sqrt{p^2+q^2}\right).\end{aligned} \tag{3.41}$$

The transformation from variables $(q, p)$ to $(\tilde{q}, \tilde{p})$ is analytic and changes the vector field $X(q, p)$ into $\tilde{X}(\tilde{q}, \tilde{p})$. □

If $\tau'$ is not positive, we then enforce $\tau_\Psi$ to adopt the following form:

$$\tau_\Psi(s) := \tau \circ \zeta^{-1}(s)\,.$$

Here, $\zeta$ represents an analytic diffeomorphism that awaits construction. We necessitate that $\zeta$ satisfies the following:

$$\begin{aligned}0 &< \frac{d}{ds}\big(s\tau_\Psi(s)\big) = \frac{d}{ds}\Big(s\tau \circ \zeta^{-1}(s)\Big) = \frac{d}{ds}\Big(\zeta(r)\tau(r)\Big)_{r=\zeta^{-1}(s)} \\ &= \frac{d}{dr}\Big(\zeta(r)\tau(r)\Big)\Big|_{r=\zeta^{-1}(s)} \frac{d}{ds}\zeta^{-1}(s)\,,\end{aligned} \tag{3.42}$$

$s > 0$. This is equivalent to the following:

$$\frac{d}{dr}\Big(\zeta(r)\tau(r)\Big) > 0\,, \quad r > 0\,.$$

By defining

$$A(r) := \zeta(r)\tau(r) \Longrightarrow \zeta(r) = \frac{A(r)}{\tau(r)}, \tag{3.43}$$

we are able to recast the four conditions:

$$\zeta(r) = -\zeta(-r), \quad \zeta(r) \geq 0, \quad \zeta'(r) > 0, \quad \text{and} \quad \frac{d}{dr}\Big(\zeta(r)\tau(r)\Big) > 0,$$

for $r > 0$, in the following manner:

$$A(r) = -A(-r), \quad A(r) \geq 0, \quad A'(r)\tau(r) - A(r)\tau'(r) > 0, \quad \text{and} \quad A'(r) > 0. \tag{3.44}$$

To fulfill the four conditions, we impose the following:

$$A'(r)\tau(r) - A(r)\tau'(r) = M(r)A(r) + \tau(r), \tag{3.45}$$

where $M : \mathbb{R} \to \mathbb{R}$ is an analytic function satisfying the below conditions for $r > 0$:

$$M(r) = -M(-r), \quad M(r) > 0, \quad \text{and} \quad \tau'(r) > -M(r). \tag{3.46}$$

Here, $-M(r)$ is a lower bound of $\tau'(r)$.

If we can find a function $M$ that meets the required properties and a solution to equation (3.45) with $A(0) = 0$ and $A(r) > 0$ for $r > 0$, then the right-hand side of equation (3.45) will be positive. Subsequently,

$$A'(r)\tau(r) = \big(M(r) + \tau'(r)\big)A(r) + \tau(r) > 0, \tag{3.47}$$

since $M(r) + \tau'(r) \geq 0$.

The solution to equation (3.45) with $A(0) = 0$ is given as:

$$A(r) = \tau(r) \int\limits_0^r \frac{\exp\left(\int_\xi^r \frac{M(\eta)}{\tau(\eta)} d\eta\right)}{\tau(\xi)} d\xi > 0\,, \tag{3.48}$$

for $r > 0$. Utilizing the fact that $M$ is odd, we find that $A$ is also odd.

In the following Proposition, which adapts Theorem 2 in [12], we construct a function $M$ that possesses the properties (3.46).

**Proposition 3.2.** *Let $m(r) = \sup_{0\leq\eta\leq r} |\tau'(r)|$. There exists an entire function $M(z)$, $z \in \mathbb{C}$, with nonnegative coefficients such that*

$$M(r) > m(r) \geq |\tau'(r)| \text{ for all } r > 0$$

*and $M(-z) = -M(z)$.*

**Proof.** Since $\tau'(0) = 0$ and $\tau'$ is analytic, there exists a constant $c > 0$ such that $m(r) < cr$ for $0 < r \leq 2$.

Consider $\{\lambda_k\}$, an increasing sequence of positive odd integers such that for every positive integer $k$, we have

$$\left(\frac{k+1}{k}\right)^{\lambda_k} > m(k+2).$$

We then define

$$M(z) = cz + \sum_{k=1}^{\infty} \left(\frac{z}{k}\right)^{\lambda_k},$$

which is an entire function that satisfies $M(-z) = -M(z)$. For $r > 2$, let $j$ be the integer such that $j + 1 \leq r < j + 2$. Then

$$M(r) > \left(\frac{r}{j}\right)^{\lambda_j} \geq \left(\frac{j+1}{j}\right)^{\lambda_j} > m(j+2) > m(r),$$

while for $0 < r \leq 2$, $M(r) > cr > m(r)$. □

We still need to demonstrate that $\lim_{r\to\infty} \zeta(r) = \infty$ to ensure that $\Phi_\zeta$ in Proposition 3.1 maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Equations (3.43) and (3.48) lead to

$$\zeta(r) = \frac{A(r)}{\tau(r)} = \int_0^r \frac{\exp\left(\int_\xi^r \frac{M(\eta)}{\tau(\eta)} d\eta\right)}{\tau(\xi)} d\xi > \int_0^r \frac{1}{\tau(\xi)} d\xi,$$

for $r > 0$. If $\int_0^\infty \frac{1}{\tau(\xi)} d\xi = \infty$, then $\lim_{r\to\infty} \zeta(r) = \infty$.

Otherwise, if $\int_0^\infty \frac{1}{\tau(\xi)} d\xi = c < \infty$, there must exist a sequence $r_1 < r_2 < \dots$ with $\lim_{k\to\infty} r_k = \infty$ such that $\tau(r_1) < \tau(r_2) < \dots$ and $\lim_{k\to\infty} \tau(r_k) = \infty$. From equations (3.43) and (3.45), we infer

$$\frac{d}{dr}\zeta(r) = \frac{M(r)}{\tau(r)}\zeta(r) + \frac{1}{\tau(r)} > 0.$$

Upon integrating and using the fact that $\zeta(r)$ is increasing, we deduce

$$\begin{aligned} \zeta(r) &= \int_0^r \frac{M(s)}{\tau(s)}\zeta(s) ds + \int_0^r \frac{1}{\tau(s)} ds > \int_0^r \frac{M(s)}{\tau(s)}\zeta(s) ds \\ &= \int_0^1 \frac{M(s)}{\tau(s)}\zeta(s) ds + \int_1^r \frac{M(s)}{\tau(s)}\zeta(s) ds > \zeta(1) \int_1^r \frac{M(s)}{\tau(s)} ds. \end{aligned}$$

In this last integral, by using $M(r) > \tau'(r)$, we find

$$\zeta(r_k) > \zeta(1) \int_1^{r_k} \frac{M(s)}{\tau(s)} ds > \zeta(1) \int_1^{r_k} \frac{\tau'(s)}{\tau(s)} ds = \log \frac{\tau(r_k)}{\tau(1)},$$

which implies $\lim_{r_k\to\infty} \zeta(r_k) = \lim_{r\to\infty} \zeta(r) = \infty$ as $\zeta'(r) > 0$.

The remainder of the proof aligns with the case when $\tau' \geq 0$.

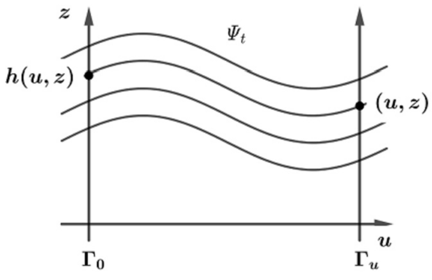


Fig. 8. Construction of $h$.

## 4. Proof of Theorem 1.4

Consider the vector field $W(u,z) = \partial_u + f(u,z)\partial_z$, as defined in Equation (1.3). If the flow of $W$ is complete, an integral curve of $W$ passes through every point $(u,z)$, intersecting the $z$-axis exactly once at a value $z_0$. We define the function $h(u,z) := z_0$, which is an analytic first integral of $W$, that is,

$$\partial_u h(u,z) + f(u,z)\partial_z h(u,z) = 0\,. \tag{4.49}$$

This first integral meets the condition $\partial_z h(u,z) > 0$, as illustrated in Fig. 8 and discussed in detail on page 283 of [21].

Introducing the notation $\partial_2 h(u, v^2/2) := \partial_z h(u,z)\big|_{z=v^2/2}$, we express the differential of $h$ as

$$\begin{aligned} dh(u, v^2/2) &= \partial_u h(u, v^2/2)du + v\partial_2 h(u, v^2/2)dv \\ &\overset{(4.49)}{=} \partial_2 h(u, v^2/2)\big(-f(u, v^2/2)du + vdv\big). \end{aligned}$$

Given that $\partial_2 h(u, v^2/2) > 0$, we infer that the vector field $X = v\partial_u + f(u, v^2/2)\partial_v$ is Hamiltonian:

$$\iota_X\big\{\partial_2 h(u, v^2/2)du \wedge dv\big\} = dh(u, v^2/2)\,,$$

with symplectic form $\mu$ and Hamiltonian function $h(u, v^2/2)$.

Theorem 1.4 ensues from the following proposition:

**Proposition 4.1.** *We define*

$$p(u,v) = \frac{v}{a(u,v)} := v\left\{\int_0^1 \partial_2 h(u, sv^2/2)ds\right\}^{1/2}. \tag{4.50}$$

*The analytic transformation $\Phi(u,v) = \big(u, p(u,v)\big)$ maps $\mathbb{R}^2$ onto itself such that:*

$$H(u,p) = \frac{p^2}{2} + G(u) = h(u, v^2/2) \quad \textit{with} \quad G(u) = h(u,0)\,,$$

$$\mu(u,p) = \frac{1}{a(u,p)} du \wedge dp \quad where \quad a(u,p) := a\big(u, v(u,p)\big), \tag{4.51}$$
$$X = a(u,p)\Big(p\partial_u - G'(u)\partial p\Big) \quad with \quad a(u,-p) = a(u,p).$$

**Proof.** The fundamental theorem of calculus combined with $\partial_z h(u,z) > 0$ results in

$$h(u, v^2/2) = \underbrace{h(u,0)}_{\equiv G(u)} + \underbrace{\int_0^{v^2/2} \partial_2 h(u,z)dz}_{\equiv p^2/2} \equiv H(u,p). \tag{4.52}$$

Subsequently, we define the new variable $p$ according to equation (4.50).

Differentiating the equation $h(u, v^2/2) = G(u) + p^2(u,v)/2$ with respect to $v$ results in

$$\partial_v p(u,v) = \frac{v}{p(u,v)} \partial_2 h(u, v^2/2) = \frac{\partial_2 h(u, v^2/2)}{\left\{\int_0^1 \partial_2 h(u, sv^2/2) ds\right\}^{1/2}} > 0.$$

This equation implies that $\Phi(u,v) = \big(u, p(u,v)\big)$ is a local analytic diffeomorphism.

We claim that $p \to \infty$ as $v \to \infty$ for any fixed $u$. Given that $h(u, v^2/2) = G(u) + p^2(u,v)/2$, it is sufficient to demonstrate that $\lim_{z\to\infty} h(u,z) = \infty$. If this was not the case, $\hat{u}$ would exist such that $\lim_{z\to\infty} h(\hat{u}, z) = \hat{z}_0 < \infty$ and all integral curves starting at $u = 0$ with $z > z_0$ would diverge in the interval $(0, \hat{u})$. This would contradict the completeness of the flow of $W$.

The symplectic form in the proposition statement follows from

$$p du \wedge dp = du \wedge d\frac{p^2}{2} = du \wedge dh(u, v^2/2) = v\partial_2 h(u, v^2/2) du \wedge dv = v\mu$$

and the relationship $p/v = 1/a(p,v)$. □

**Remark a:** The class of equations $\ddot{u} + a(u)\dot{u}^2 + b(u) = 0$ has been analyzed in [23]. In this case, $W = \partial_u + \big(a(u)z + b(u)\big)\partial_z$ can be integrated (see [21], Section 2.1, for the computations).

**Remark b:** The completeness of the flow $W$, as assumed in Theorem 1.4, can be replaced by the existence of a first integral $h$ on the upper-half plane. The example below illustrates this.

The vector field $X = v\partial_u - \frac{2u}{1+v^2}\partial_v$ possesses a global center at the origin. This vector field corresponds to $W(u,z) = \partial_u - \left(\frac{2u}{1+2z}\right)\partial_z$, which is undefined at $z = -\frac{1}{2}$ and is not complete. However, for any $(u,z) \in \mathbb{R}^2_{z\geq 0}$, there is a trajectory of $W$ that intersects the $z$-axis only at $z_0 \geq 0$, as shown in Fig. 9. This permits the definition of a first integral $z_0 = h(u,z)$ of $W$ on the upper half plane that satisfies $\partial_z h(u,z) > 0$.

See [17] and [18] (pg. 131) for the existence of global first integrals of $W$.

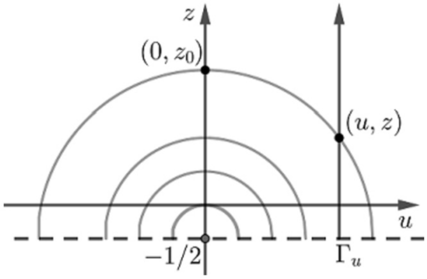


Fig. 9. The integral curves of $W$ in the case where $X$ has a global center.

## 5. Conclusion

The primary motivation for Ragazzo [21] was a query posed by C. Rocha relating to partial differential equations of the parabolic type $u_t = \partial_{xx}u - F_0(u, u_x)$, $x \in \mathbb{R}/\mathbb{Z}$ (the notation used in this paragraph differs from that used in the remainder of the paper). Under certain conditions, these equations exhibit global attractors composed of equilibria and connecting orbits. Periodic solutions of $u_{xx} = F_0(u, u_x)$ with period one are the equilibria for the parabolic equation. As noted in [22], "When the global attractor is Morse-Smale, there exists a smooth homotopy $F_\tau(u, u_x)$, $\tau \in [0, 1]$, which preserves the hyperbolicity of all the equilibria and periodic orbits, and reduces $u_{xx} = F_0(u, u_x)$ to a problem $u_{xx} = F_1(u, u_x)$, where $F_1(u, u_x)$ is an even function of the second variable" (see [5], [6], and [7]). The understanding of the 'period map' of the equation $u'' = F_1(u, u')$ is crucial for the description of the global attractor of the parabolic problem. This paper's findings show that if $u'' = F_1(u, u')$ has a non-degenerate global center, then the period map of $u'' = F_1(u, u')$ is always equivalent to the period map of an equation in the form $u'' = -V'(u)$.

Theorem 1.2 asserts that if $X(u, v) = P(u, v)\partial_u + Q(u, v)\partial_v$ possesses a global center, it can be transformed into $y\partial_x - V'(x)\partial_y$. Theorem 1.4 stipulates that if $X(u, v) = v\partial_u + f(u, v^2/2)\partial_v$ and the flow of the associated vector field $W(u, z) = \partial_u + f(u, z)\partial_z$ is complete, then $X$ can be converted into $a(u, p)\Big(p\partial_u - G'(u)\partial p\Big)$ without any assumptions on the singularities of $X$. Aside from time parameterization, this vector field is equivalent to $p\partial_u - G'(u)\partial p$.

Two natural questions arise from the results presented in this paper:

1) Does the question posed by C. Chicone have a positive answer when the center is degenerate?
2) Is it possible to analytically transform the vector field $X(u, v) = v\partial_u + f(u, v^2/2)\partial_v$ into $y\partial_x - V'(x)\partial_y$ when $X(u, v)$ has not only a center but also finitely many non-degenerate centers and saddles?

## Data availability

No data was used for the research described in the article.

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