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Stability theory for two-lobe states on the tadpole graph for the NLS equation

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Abstract

The aim of this work is to present new spectral tools for studying the orbital stability of standing waves solutions for the nonlinear Schrödinger equation (NLS) with power nonlinearity on a tadpole graph, namely, a graph consisting of a circle with a half-line attached at a single vertex. By considering δ-type boundary conditions at the junction and bound states with a positive two-lobe profile, the main novelty of this paper is at least twofold. Via a splitting eigenvalue method developed by the author, we identify the Morse index and the nullity index of a specific linearized operator around of an *a priori* positive two-lobe state profile for every positive power; and we also obtain new results about the existence and the orbital stability of positive two-lobe states at least in the cubic NLS case. To our knowledge, the results contained in this paper are the first in studying positive bound states for the NLS on a tadpole graph by non-variational techniques. In particular, our approach has prospect of being extended to study stability properties of other bound states for the NLS on a tadpole graph or on other non-compact metric graph such as a looping edge graph, as well as, for other nonlinear evolution models on a tadpole graph.

Keywords: nonlinear Schrödinger equation, quantum graphs, standing wave solutions, stability, extension theory of symmetric operators, Sturm comparison theorem, analytic perturbation theory

Mathematics Subject Classification numbers: Primary 35Q51, 35Q55, 81Q35, 35R02; Secondary 47E05

1. Introduction

In this paper we study the stability of specific standing wave profiles for the following nonlinear vectorial Schrödinger equation (NLS)

$$i\mathbf{U}_t + \Delta\mathbf{U} + (p+1)|\mathbf{U}|^{2p}\mathbf{U} = 0, \quad p > 0, \tag{1.1}$$

on a tadpole graph or lasso graph, namely, a graph given by a ring with one half-line attached at one vertex point as is shown in figure 1 below.

We recall that, a metric graph $\mathcal{G}$ is a structure represented by a finite number of *vertices* $V = \{v_i\}$ and a set of adjacent edges at the vertices $E = \{e_j\}$ (for further details see [21]). Each edge e_j can be identified with a finite or infinite interval of the real line, I_e. Thus, we can see the edges of $\mathcal{G}$ not as abstract relations between vertices, but rather as physical '*wires*' or '*networks*' connecting them. Now, the notation $e \in E$ will be used to mean that e is an edge of $\mathcal{G}$. This identification introduces the coordinate x_e along the edge e. Thus, if in the tadpole graph, the ring is identified by the interval $[-L, L]$ and the semi-infinite line with $[L, +\infty)$, we obtain a metric graph $\mathcal{G}$ with a structure represented by the set $E = \{e_0, e_1\}$ where $e_0 = [-L, L]$ and $e_1 = [L, +\infty)$, which are the edges of $\mathcal{G}$ and they are connected at the unique vertex $\nu = L$. $\mathcal{G}$ is also called a tadpole graph or lasso graph, which appears in quantum mechanics and it is associated to the study of a nonrelativistic particle by a homogeneous magnetic field perpendicular to the loop plane (see Exner [30] and references therein).

We identify any function $\mathbf{U}$ on $\mathcal{G}$ (the *wave functions*) with a collection $\mathbf{U} = (u_e)_{e\in E}$ of functions u_e defined on the edge e of $\mathcal{G}$. Thus, each u_e will be considered as a complex-valued function on the interval I_e. In the case of the NLS in (1.1), we have $\mathbf{U}(x_e, t) = (u_e(x_e, t))_{e\in E}$ and the nonlinearity $|\mathbf{U}|^{2p}\mathbf{U}$, $p > 0$, acting componentwise, i.e. for instance $(|\mathbf{U}|^{2p}\mathbf{U})_e = |u_e|^{2p}u_e$. The action of the Laplacian operator Δ on $\mathcal{G}$ is given by

$$-\Delta : (u_e)_{e\in E} \to (-u_e'')_{e\in E}. \tag{1.2}$$

Here, we will consider $-\Delta$ as being a self-adjoint operator on a specific domain of $L^2(\mathcal{G})$ which will give the coupling conditions in the vertex $\nu = L$. There are several domains that make the Laplacian operator to be self-adjoint on a tadpole graph. Here, we will consider a domain of interest in quantum mechanics. Thus, if we denote a wave function $\mathbf{U}$ on the tadpole graph $\mathcal{G}$ as $\mathbf{U} = (\phi, \psi)$, where $\phi : [-L, L] \to \mathbb{C}$ and $\psi : [L, +\infty) \to \mathbb{C}$, we will be considering the following domains ($Z \in \mathbb{R}$) for $-\Delta$:

$$D_Z = \left\{\mathbf{U} \in H^2(\mathcal{G}) : \quad \phi(L) = \phi(-L) = \psi(L), \text{ and, } \phi'(L) - \phi'(-L) = \psi'(L+) + Z\psi(L)\right\}, \tag{1.3}$$

where for any $n \geqq 0$, $n \in \mathbb{N}$,

$$H^n(\mathcal{G}) = H^n(-L, L) \oplus H^n(L, +\infty).$$

The boundary conditions in (1.3) are called of δ-type if $Z \neq 0$, and of Neuman–Kirchhoff type if $Z = 0$. By using the extension theory for symmetric operators (see theorem A.6 in appendix below), it follows that $(-\Delta, D_Z)_{Z\in\mathbb{R}}$ represents a one-parameter family of self-adjoint operators on the tadpole graph $\mathcal{G}$. The parameter Z is a coupling constant between the disconnected loop and the half-line. The choice of the coupling at the vertex $\nu = L$ corresponds to a conceivable quantum-wire experiment (see [30, 31] and reference therein).

Nonlinear evolution models on metric graph arise as models in wave propagation, for instance, in a quasi one-dimensional (e.g. meso- or nanoscale) system that looks like a thin

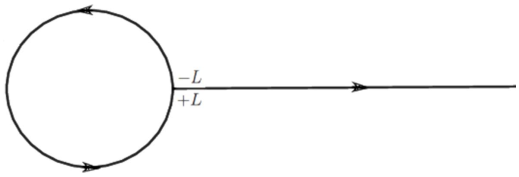

Figure 1. Tadpole graph.

neighbourhood of a graph. These models provide a convenient means to study various physical effects in the real world, both from the theoretical and experimental points of view, and the freedom in setting the geometry of the graph configuration allows us to create different dynamics. Just to mention a few examples of bifurcated systems, we have the Josephson junction structures, network of planar optical waveguides and fibre, branched structure associated to the DNA, blood pressure waves in large arteries, nerve impulses in complex arrays of neurons, conducting polymers, and Bose–Einstein condensation (see [21, 23, 24, 28–30, 32, 43, 45, 47, 48, 54] and reference therein). About non-linear models we have the NLS, the sine-Gordon model, and the Korteweg–de Vries model (see [2, 3, 11, 12, 14–18, 32, 48] and references therein). We recall that, from a mathematical viewpoint, the nature of an evolution model on a graph is equivalent to a system of PDEs defined on the edges (intervals) in which the coupling is given exclusively through the boundary conditions at the vertices (known as the 'topology of the graph'), which determines the dynamic on the network.

In the past years, evolution models on networks have attracted much attention in the context of soliton transport. Soliton and other nonlinear waves in branched systems provide in-depth informations about the dynamic of the model. To address these issues, in general the problem is difficult to tackle because both the equations of motion and the topology of the graph can be complex. Moreover, a central point that makes this analysis a delicate problem is the presence of a vertex (or several vertices) where a soliton-profile coming into the vertex along one of the bonds shows a complicated motion around the vertex such as reflection and emergence of the radiation there. In particular, sometimes one cannot see easily how the energy travels across the network. Thus, the study of soliton propagation through networks can become a challenge. Results about the existence and stability (or instability) mechanism of soliton profiles are still unclear for many type of graphs and models.

In the case of the NLS model (1.1) many dynamic issues have been studied for a variety of metric graphs (see the review manuscript [40]), by instance, in a star graph ([1, 2, 14, 15, 38, 48] and reference therein), a looping edge graph ([8, 26, 49, 50] and reference therein), flower graph, dumbbell graphs, double-bridge graphs and periodic ring graphs ([26, 39, 44, 49, 51] and reference therein). One of the objectives of this work is to shed light on the existence and stability of specific standing waves profiles (bound states) so called of *two single-lobe states* in the case of the NLS model on a tadpole graph.

Here, we will consider standing wave solutions for NLS model in (1.1) posed on a tadpole graph $\mathcal{G}$ and given by $\mathbf{U}(x,t) = e^{-i\omega t}\Theta(x)$, with $\omega < 0$, $\Theta = (\Phi, \Psi) \in D_Z$ (real-valued components), and satisfying the stationary NLS vectorial equation

$$-\Delta\Theta - \omega\Theta - (p+1)\,|\Theta|^{2p}\Theta = 0. \tag{1.4}$$

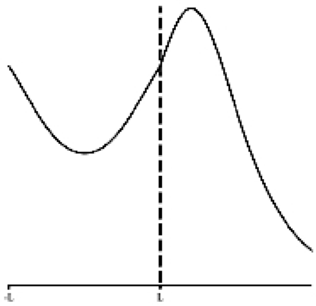

Figure 2. A positive two-lobe state profile on a tadpole graph.

More explicitly, Φ and Ψ satisfy the following system, one on the ring and the other one on the half-line,

$$\begin{cases} -\Phi''(x) - \omega\Phi(x) - (p+1)\,|\Phi(x)|^{2p}\Phi(x) = 0, & x \in (-L, L), \\ -\Psi''(x) - \omega\Psi(x) - (p+1)\,|\Psi(x)|^{2p}\Psi(x) = 0, & x \in (L, +\infty), \\ \Phi(L) = \Phi(-L) = \Psi(L), & \\ \Phi'(L) - \Phi'(-L) = \Psi'(L+) + Z\Psi(L), & Z \in \mathbb{R}. \end{cases} \tag{1.5}$$

It is clear that the delicate point in the existence of solutions for (1.5) is given for the component Φ on $(-L, L)$. The component Ψ will have obviously a soliton profile of the form

$$Q_\omega(x) = (-\omega)^{1/2p}\, sech^{1/p}\left(p\sqrt{-\omega}x\right), \qquad \text{modulo translation and sign.} \tag{1.6}$$

Among all possible solutions for (1.5) (see [8, 26, 49, 50]), we are interested here in profiles that we will call of *positive two-lobe states* (see figures 2 and 5). More exactly, we have the following definition.

Definition 1.1. The standing wave $\Theta = (\Phi, \Psi) \in D_Z$ is said to be a positive two-lobe state for (1.5) if $\Theta(x) > 0$ for every $x \in \mathcal{G}$, Φ is symmetric on $[-L, L]$ with a single minimum at 0, monotonically increasing on $[0, L]$ and Ψ is of bump-type on $[L, +\infty)$.

We recall that a real-profile $\varphi \in H^2(L, +\infty)$ has a bump-profile if there is a unique $\zeta \in (L, +\infty)$ such that $\varphi'(\zeta) = 0$, $\varphi'(x) > 0$ for $x \in (L, \zeta)$ and $\varphi'(x) < 0$ for $x \in (\zeta, +\infty)$.

For $\Theta = (\Phi, \Psi)$ being a positive two-lobe state, we have that every profile Ψ has the bump profile representation

$$\Psi(x) = (-\omega)^{1/2p}\,\psi_0\left(\sqrt{-\omega}(x - L) + a\right), \quad x \geqq L, \;\; a < 0, \tag{1.7}$$

with $\psi_0 = Q_{-1}$ giving by

$$\psi_0(y) = sech^{1/p}(py), \quad \psi_0(0) = 1, \;\; \psi_0'(0) = 0, \;\; y \in \mathbb{R}. \tag{1.8}$$

For Z fixed, the shift parameter $a = a(\omega, Z) < 0$ will be uniquely determined by the δ-interaction boundary conditions in (1.5). Moreover, since $\Phi'(L) > 0$, it follows *a priori* the following restriction about ω, $Z > \sqrt{-\omega}\tanh(pa) > -\sqrt{-\omega}$.

The existence and dynamics of positive two-lobe state for (1.5) on a tadpole graph have been studied very little in the literature, especially because a variational characterization of these as a minimizer of a specific variational problem seems unlikely (see [5]). In general, positive two-lobe state profiles turn out to be of bound-state type (or excited states), namely, they are solutions to the stationary equation (1.5) without being ground state (see [5, 26] and the review in [40] for previous studies on specific excited states), that is, functions with a prescribed mass such that are constrained *critical points* of the NLS energy functional and possibly without being absolute minimizers.

Now, for $Z = 0$ and $p = 1$, in [26] was already proved the existence of positive two-lobe states (Φ, Ψ) for (1.5) with a dnoidal-profile for Φ on the loop (see remark 4.5 below). This type of profiles will be a main part of our study here. Indeed, based on the dynamical system theory for orbits on the plane and from the period map associated to second-order differential equations, we get a C^1-mapping $\omega \in (-\infty, \omega_0) \to (\Phi_\omega, \Psi_\omega)$, $\omega_0 < 0$, of positive two-lobe states solutions for (1.5), and such that do not exist profiles of this type for (1.5) with $\omega > \omega_0$ (see figure 5, lemma 4.1 and remark 4.5). Some references on the strategy of the period map are contained in [33, 39, 40, 50].

Our main focus in this paper is at least twofold. To provide new spectral tools for studying the orbital stability of *a priori positive two-lobe states* for every positive power p on a tadpole graph $\mathcal{G}$; and to obtain new results about the existence and orbital stability of positive two-lobe states at least in the cubic NLS case, $p = 1$ and $Z = 0$ (we note that in [26] the stability issue of these profiles was not studied). We are not aware of previous studies for NLS models on our framework. Moreover, our analysis is not of variational type and so the results to be established in this paper, can be seen as a first step towards studying stability properties of other bound states for the NLS model on $\mathcal{G}$ (among which could be the multi-lobe profiles in figure 3, see [26]), or on other non-compact metric graphs such as a looping edge graphs (a graph consisting of a circle with N half-lines attached at a single vertex), as well as, for other nonlinear evolution models such as the sine-Gordon and Korteweg– de Vries equations.

We note that the existence and stability of bound states to the NLS model (1.1) were considered via a energy minimization problem in the limit of large mass (large negative ω) in the case of subcritical nonlinearities ($p \in (0,2)$) and $Z = 0$, for non-compact metric graphs in [5] (see also [22, 37] and [40]-section 6), nonetheless, these results do not apply to our case because the positive bound states obtained have a absolute maximum on a bounded edge-localized (single-pulse). Additionally, in Angulo [8] was recently extended our approach to the case of the NLS model (1.1) on looping edge graphs for positive bound states with a single pulse. In this case, we know that for $N \geqq 3$, $Z = 0$ and $p \in (0,2]$, we can not obtain these profiles as being the ground states associated to a constraint variational problem (see theorem 2.5 of [3] and theorem 3.2 of [4]). As far as we know, the results in [8] are the first to establish the existence and stability of positive single-lobe profiles for any $N \geqq 1$, $p > 0$, $Z \leqq 0$ and $-\omega$ small enough (to compare with the results in [5]).

Next, by convenience of the reader, we establish the main points in the stability study of standing waves solutions for NLS models on metric graphs. After that, we give the main results of our work. So, by starting, we note that the basic symmetry associated to the NLS model (1.1) on a tadpole graph is the phase invariance, that means, if $\mathbf{U}$ is a solution of (1.1) then $e^{i\theta}\mathbf{U}$ is also a solution for any $\theta \in [0, 2\pi)$. Thus, it is reasonable to define orbital stability for the model (1.1) as follows (see [34, 35]).

Definition 1.2. The standing wave $\mathbf{U}(x,t) = e^{-i\omega t}(\mathbf{\Phi}(x), \mathbf{\Psi}(x))$ is said to be *orbitally stable* in a Banach space X if for any $\varepsilon > 0$ there exists $\eta > 0$ with the following property: if $\mathbf{U}_0 \in X$ satisfies $||\mathbf{U}_0 - (\mathbf{\Phi}, \mathbf{\Psi})||_X < \eta$, then the solution $\mathbf{U}(t)$ of (1.1) with $\mathbf{U}(0) = \mathbf{U}_0$ exists for any $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} ||\mathbf{U}(t) - e^{i\theta}(\mathbf{\Phi}, \mathbf{\Psi})||_X < \varepsilon.$$

Otherwise, the standing wave $\mathbf{U}(x,t) = e^{-i\omega t}(\mathbf{\Phi}(x), \mathbf{\Psi}(x))$ is said to be *orbitally unstable* in X.

The space X in definition 1.2 for the model (1.1) and with the action of $-\Delta$ on D_Z, will be the continuous energy-space $\mathcal{E}(\mathcal{G})$ defined by

$$\mathcal{E}(\mathcal{G}) = \left\{(f,g) \in H^1(\mathcal{G}) : f(-L) = f(L) = g(L)\right\}.$$

For other domain of $-\Delta$ on $\mathcal{G}$, the energy space X in definition 1.2 can be different (see [11, 13, 14, 18]).

Next, we consider the following two conserved functionals for (1.1) defined in the energy space $\mathcal{E}(\mathcal{G})$,

$$E_Z(\mathbf{U}) = \|\nabla \mathbf{U}\|^2_{L^2(\mathcal{G})} - \|\mathbf{U}\|^{2p+2}_{L^{2p+2}(\mathcal{G})} - Z|u(L)|^2, \qquad \text{(energy)} \tag{1.9}$$

and

$$Q(\mathbf{U}) = \|\mathbf{U}\|^2_{L^2(\mathcal{G})}, \qquad \text{(mass)} \tag{1.10}$$

where $\mathbf{U} = (u,v)$. We observe that $E_Z, Q \in C^2(\mathcal{E}(\mathcal{G}), \mathbb{R})$ because $p > 0$. Now, for a fixed $\omega < 0$, let $\mathbf{U}(x,t) = e^{-i\omega t}(\mathbf{\Phi}_\omega(x), \mathbf{\Psi}_\omega(x))$ be a standing wave solution for (1.1) with $(\Phi_\omega, \Psi_\omega) \in D_Z$ being a positive two-lobe state. Then, for the action functional

$$\mathbf{S}(\mathbf{U}) = E_Z(\mathbf{U}) - \omega Q(\mathbf{U}), \qquad \mathbf{U} \in \mathcal{E}(\mathcal{G}), \tag{1.11}$$

we have $\mathbf{S}'(\Phi_\omega, \Psi_\omega) = 0$. Next, for $\mathbf{U} = \mathbf{U}_1 + i\mathbf{U}_2$ and $\mathbf{W} = \mathbf{W}_1 + i\mathbf{W}_2$, where the vector functions $\mathbf{U}_j$, $\mathbf{W}_j$, $j = 1,2$, are assumed to have real components, it is not difficult to see that the second variation of $\mathbf{S}$ in $(\Phi_\omega, \Psi_\omega)$ is given by

$$\mathbf{S}''(\Phi_\omega, \Psi_\omega)(\mathbf{U}, \mathbf{W}) = \langle \mathcal{L}_{+,Z}\mathbf{U}_1, \mathbf{W}_1\rangle + \langle \mathcal{L}_{-,Z}\mathbf{U}_2, \mathbf{W}_2\rangle, \tag{1.12}$$

where the two 2×2-diagonal operators $\mathcal{L}_{\pm,Z}$ are given by

$$\mathcal{L}_{+,Z} = \text{diag}\left(-\partial_x^2 - \omega - (p+1)(2p+1)\Phi_\omega^{2p}, -\partial_x^2 - \omega - (p+1)(2p+1)\Psi_\omega^{2p}\right)$$

$$\mathcal{L}_{-,Z} = \text{diag}\left(-\partial_x^2 - \omega - (p+1)\Phi_\omega^{2p}, -\partial_x^2 - \omega - (p+1)\Psi_\omega^{2p}\right). \tag{1.13}$$

We note that these two last diagonal operators are self-adjoint with domain $D(\mathcal{L}_{\pm,Z}) \equiv D_Z$ (see A.6 in appendix). We also have that since $(\Phi_\omega, \Psi_\omega) \in D_Z$ and satisfies system (1.5), $\mathcal{L}_{-,Z}(\Phi_\omega, \Psi_\omega)^t = 0$ and so the kernel of $\mathcal{L}_{-,Z}$ is non-trivial.

Now, from [34, 35] we know that the Morse index and the nullity index of the operators $\mathcal{L}_{\pm,Z}$ are fundamental to decide on orbital stability of standing wave solutions for NLS models (see theorem A.8 in appendix). Thus, in the following we study these indices in the case of the profile $(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)$ to be an *a priori* positive two-lobe state. Indeed, the profile $\mathbf{\Psi}_\omega$ will be determined by an unique member of the family of soliton-profiles $a \in (-\infty, 0) \to \psi_a$, namely,

$$\psi_a(x) = (-\omega)^{1/2p}\psi_0\left(\sqrt{-\omega}(x-L) + a\right), \quad x \geqq L, \tag{1.14}$$

with ω fixed and $a = a(\omega)$. Moreover, from (1.14) we get the existence of a unique $a^* < 0$ such that $\psi''_{a^*}(L+) = 0$ (namely, a^* such that $(p+1)sech^2(pa^*) = 1$). Thus, $\psi''_a(L+) < 0$ if and only if $a \in (a^*, 0)$ (ψ_a is concave close to L), and $\psi_a''(L+) > 0$ if and only if $a \in (-\infty, a^*)$ (ψ_a is convex close to L). As we will see below, the position of the shift $a = a(\omega, Z, p)$ with regard to a^* will induce the values of the Morse and nullity indices of operator $\mathcal{L}_{+,Z}$.

Our main results in this paper are the following.

Theorem 1.3. *Let us consider the self-adjoint operator $(\mathcal{L}_{+,Z}, D_Z)$ in (1.13), for Z fixed, determined by a positive two-lobe state $(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)$ where $\mathbf{\Psi}_\omega = \psi_a$, with ψ_a in (1.14) and $a = a(\omega, Z) < 0$. Then,*

(1) *Perron-Frobenius property: let $\beta_0 < 0$ be the smallest eigenvalue for $\mathcal{L}_{+,Z}$ with associated eigenfunction $(f_{\beta_0}, g_{\beta_0})$. Then, f_{β_0} is positive and even on $[-L, L]$, and $g_{\beta_0}(x) > 0$ with $x \in [L, +\infty)$. Moreover, β_0 is simple.*
(2) *We consider the mapping*

$$\gamma(a) \equiv \frac{\psi_a''(L+)}{\psi_a'(L+)}. \tag{1.15}$$

Then, for $\gamma \equiv \gamma(a)$ we have the following:
(a) *if $-Z \geqq \gamma$ then $n(\mathcal{L}_{+,Z}) = 1$,*
(b) *if $-Z < \gamma$ then $1 \leqq n(\mathcal{L}_{+,Z}) \leqq 2$. Moreover, for the operator*

$$\mathcal{L}_0 = -\partial_x^2 - \omega - (p+1)(2p+1)\Phi_\omega^{2p}$$

with associated periodic boundary conditions (the only one possible in our framework), we have $n(\mathcal{L}_0) \leqq 2$ and,
(i) *$n(\mathcal{L}_{+,Z}) = 1$ if and only if $n(\mathcal{L}_0) = 1$.*
(ii) *$n(\mathcal{L}_{+,Z}) = 2$ if and only if $n(\mathcal{L}_0) = 2$.*

Theorem 1.4. *Let us consider the self-adjoint operator $(\mathcal{L}_{+,Z}, D_Z)$ in (1.13), for Z fixed, determined by a positive two-lobe state $(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)$ where $\mathbf{\Psi}_\omega = \psi_a$, with ψ_a in (1.14) and $a = a(\omega, Z) < 0$, and the function $\gamma = \gamma(a)$ in (1.15). Then, we have $Ker(\mathcal{L}_{+,Z}) = \{\mathbf{0}\}$ in the following cases:*

(1) *if $a \neq a^*$ and $-Z \neq \gamma$.*
(2) *if $a = a^*$ and Z arbitrary.*
(3) *if $a < a^*$ and $Z = -\gamma < 0$.*

Remark 1.5. From theorems 1.3 and 1.4 we have the following comments which will be useful in our stability theory for the case $p = 1$ and $Z = 0$:

(a) In the following we give a naive argument for showing that the Morse index of $\mathcal{L}_{+,Z}$ is one, in the case of $Z \leqq 0$ and $p > 0$. Indeed, it supposes that for $Z \leqq 0$ fixed, there is an open interval I and a smooth-diffeomorphism mapping (real-analytic) $\omega \in I \to a(\omega, Z) \in J$ of shift-parameters, with $a^* \in J \subset (-\infty, 0)$, which guarantees the existence of positive two-lobe state solutions for (1.5). Then, for any $\omega \in I$, we have that $n(\mathcal{L}_{+,Z}) = 1$. In fact, for $a(\omega, Z) \geqq a^*$ we have $\gamma(a(\omega, Z)) \leqq 0 \leqq -Z$, and so by theorem 1.3, $n(\mathcal{L}_{+,Z}) = 1$. Next, for any shift-parameter $a(\omega, Z)$, by theorem 1.4 we have always the non-degeneracy of the kernel of $\mathcal{L}_{+,Z}$. Then, by using analytic perturbation theory around a^* and a continuation argument, we get that $n(\mathcal{L}_{+,Z}) = 1$ for any $\omega \in I$ (the former argument has been applied for proving theorem 1.7 below in section 4).

(b) The possible 'threshold value' $\gamma(a)$ in (1.15) associated to the Morse index of $\mathcal{L}_{+,Z}$ is given explicitly in (3.36) (see figure 4). The only remains open case in theorem 1.4 is exactly when for some $Z>0$, the shift-parameter a ($a>a^*$) satisfies $-Z=\gamma(a(\omega,Z))$ (see remark 3.10).

Next, we have the spectral informations for the operator $\mathcal{L}_{-,Z}$.

Theorem 1.6. *Let us consider the self-adjoint operator* $(\mathcal{L}_{-,Z},D_Z)$ *in (1.13), for Z fixed, determined by a positive two-lobe state* $(\boldsymbol{\Phi}_\omega,\boldsymbol{\Psi}_\omega)$ *with* $\boldsymbol{\Psi}_\omega=\psi_a$*,* ψ_a *as in (1.14) and* $a=a(\omega,Z)<0$*. Then,*

(1) *the kernel of* $\mathcal{L}_{-,Z}$*, ker*$(\mathcal{L}_{-,Z})$*, satisfies ker*$(\mathcal{L}_{-,Z})=$ *span*$\{(\boldsymbol{\Phi}_\omega,\boldsymbol{\Psi}_\omega)\}$*.*
(2) $\mathcal{L}_{-,Z}$ *is a non-negative operator, namely,* $\mathcal{L}_{-,Z}\geqq 0$*.*

The strategy to show theorems 1.3 and 1.4 will be based in the *splitting eigenvalue method* introduced by Angulo in [8] and it applied to $\mathcal{L}_{+,Z}\equiv(\mathcal{L}_0,\mathcal{L}_1)$ on tadpole graphs (see lemma 3.4 below). More exactly, we reduce the eigenvalue problem associated to $\mathcal{L}_{+,Z}$ with domain D_Z into two classes of eigenvalue problems, one for $\mathcal{L}_0$ with periodic boundary conditions on $[-L,L]$ and the second one for $\mathcal{L}_1$ with δ-type boundary condition on the half-line $[L,+\infty)$. Thus, the use of tools of the extension theory of Krein & von Neumann for symmetric operators and the use of the theory of real coupled self-adjoint boundary conditions on $[-L,L]$, of the Sturm Comparison Theorem and the analytic perturbation theory will imply our spectral results associated to $\mathcal{L}_{+,Z}$.

We would like to point out that our splitting eigenvalue strategy was recently established in Angulo [8] for studying the stability/instability of standing waves for the NLS model (1.1) on looping edge graphs. In this case, the profiles of the standing wave were positive single-lobe states satisfying (1.5) with $-\omega$ small enough.

Lastly, we give our results about the existence and stability of positive two-lobe states on a tadpole graph in the case of the cubic NLS and $Z=0$. As far as we know, our results in this case are the first to be established in the literature.

Theorem 1.7. *We consider* $p=1$ *and* $Z=0$ *in (1.5). Then,*

(1) *there is a* C^1*-mapping* $\omega\in(-\infty,-\omega_0)\to(\Phi_\omega,\Psi_\omega)\in D_0$*, with* $\omega_0>0$*, of positive two-lobe states for the cubic-NLS on a tadpole graph.*
(2) *For every admissible value of* ω*,* $e^{-i\omega t}(\boldsymbol{\Phi}_\omega,\boldsymbol{\Psi}_\omega)$ *is orbitally stable in* $\mathcal{E}(\mathcal{G})$*.*

The proof of the existence of a C^1-mapping of positive two-lobe states in theorem 1.7 is based in the periodic map (of truncated-type) for second-order differential equations (see also [33, 39, 40, 50] for other applications of the period map in the case of graphs). The statement of the orbital stability of these profiles follows from theorems 1.3, 1.4 and 1.6, theorem 2.1, lemma 4.3, and from the abstract stability framework established by Grillakis *et al* in [34, 35]. By convenience of the reader (and for futures studies), we establish in theorem A.8 (appendix) an adaptation of the abstract results in [35] for the cases of a tadpole graph and standing waves with a profile giving by a positive two-lobe state. We believe that some arguments used for studying the case $p=1$ can be applied to other values of p, as well as, for $Z\neq 0$.

The paper is organized as follows. In section 2, we establish a local and global well-posedness results for the NLS model (1.1) on a tadpole graph. In section 3, we show theorems 1.3 and 1.4 via our splitting eigenvalue method (lemma 3.4). In section 4, we show theorem

1.7. Lastly, in appendix we briefly establish some tools and applications of the extension theory of Krein and von Neumann used in our study, as well as, we establish the orbital stability criterion from Grillakis *et al* in [35] adapted to our interests.

Notation. Let $-\infty \leqslant a < b \leqslant \infty$. We denote by $L^2(a,b)$ the Hilbert space equipped with the inner product $(u,v) = \int_a^b u(x)\overline{v(x)}\mathrm{d}x$. By $H^n(\Omega)$ we denote the classical Sobolev spaces on $\Omega \subset \mathbb{R}$ with the usual norm. We denote by $\mathcal{G}$ a tadpole graph parametrized by the set of edges $\mathbf{E} = E_0 \cup E_1$ with $E_0 = \{e_0\}$, $e_0 = [-L,L]$, $E_1 = \{e_1\}$, $e_1 = [L,+\infty)$, and attached to the common vertex $\nu = L$. On the graph $\mathcal{G}$ we define the spaces

$$L^p(\mathcal{G}) = L^p(-L,L) \oplus L^p(L,+\infty), \quad p > 1,$$

with the natural norms. Also, for $\mathbf{U} = (u_1,v_1), \mathbf{V} = (u_2,v_2) \in L^2(\mathcal{G})$, the inner product on $L^2(\mathcal{G})$ is defined by

$$\langle \mathbf{U},\mathbf{V}\rangle = \int_{-L}^{L} u_1(x)\overline{u_2(x)}\mathrm{d}x + \int_L^{\infty} v_1(x)\overline{v_2(x)}\mathrm{d}x.$$

Let A be a closed densely defined symmetric operator in the Hilbert space H. The domain of A is denoted by $D(A)$. The deficiency indices of A are denoted by $n_\pm(A) := \dim ker(A^* \mp iI)$, with A^* denoting the adjoint operator of A. The number of negative eigenvalues counting multiplicities (or Morse index) of A is denoted by $n(A)$.

2. Local and global well-posedness in $\mathcal{E}(\mathcal{G})$

In this section we give informations about the well-posedness problem associated to the NLS model (1.1) in the energy-space $\mathcal{E}(\mathcal{G})$,

$$\mathcal{E}(\mathcal{G}) = \left\{(f,g) \in H^1(\mathcal{G}) : f(-L) = f(L) = g(L)\right\}. \tag{2.1}$$

We emphasize that our results about the Cauchy problem for the NLS model play a main role in the subsequent stability study of the positive two-lobe state profiles (see definition 1.2). In particular, for possible future stability issues, we establish the C^2-regularity property of the mapping data-solution associated to (1.1) (see the comments following theorem A.8 in appendix).

Theorem 2.1. (1) *(Local well-posedness in $\mathcal{E}(\mathcal{G})$) Let $p > 0$. For any $\mathbf{U}_0 \in \mathcal{E}(\mathcal{G})$, there exists $T > 0$ such that the model in (1.1) has a unique solution $\mathbf{U} \in C([0,T] : \mathcal{E}(\mathcal{G})) \cap C^1([0,T] : \mathcal{E}(\mathcal{G})^*)$ satisfying $\mathbf{U}(0) = \mathbf{U}_0$. Moreover, the mapping*

$$\mathbf{U}_0 \in \mathcal{E}(\mathcal{G}) \to \mathbf{U} \in C([0,T] : \mathcal{E}(\mathcal{G})),$$

is at least of class C^2 for $2p > 1$.

(2) *(Global well-posedness in $\mathcal{E}(\mathcal{G})$) Let $p \in (0,2)$. For any $\mathbf{U}_0 \in \mathcal{E}(\mathcal{G})$, the model in (1.1) has a unique solution $\mathbf{U} \in C([0,+\infty) : \mathcal{E}(\mathcal{G})) \cap C^1([0,+\infty) : \mathcal{E}(\mathcal{G})^*)$.*

Proof. Local well-posedness statement was established in [27] via an application of Banach fixed point theorem. The global result is an immediate consequence of the conservation laws in (1.9)–(1.10) and from Gagliardo-Nirenberg inequality on tadpole graphs.

Next, we give some highlights of the regularity property proof of the mapping data-solution. We consider the mapping $J_{\mathbf{U}_0}: C([0,T]:\mathcal{E}(\mathcal{G}))\longrightarrow C([0,T];\mathcal{E}(\mathcal{G}))$ given by

$$J_{\mathbf{U}_0}[\mathbf{U}](t)=e^{i\Delta t}\mathbf{U}_0+i\int_0^t e^{i\Delta(t-s)}F(\mathbf{U}(s))\,\mathrm{d}s,$$

where $F(\mathbf{U}(s))=(p+1)|\mathbf{U}(s)|^{2p}\mathbf{U}(s)$ and $\{e^{i\Delta t}\}_{t\in\mathbb{R}}$ is the unitary group determined by $(-\Delta,D_Z)$. We recall that the argument based on the contraction mapping principle applied to $J_{\mathbf{U}_0}$, for obtaining the local well-posedness statement above, has the advantage that if $F(\mathbf{U})$ has a specific regularity, then it is inherited by the mapping data-solution. Indeed, following the ideas in [15, 27], we consider for $(\mathbf{U_0},\mathbf{V})\in B(\mathbf{U_0};\epsilon)\times C([0,T],\mathcal{E}(\mathcal{G}))$ the mapping

$$\Gamma(\mathbf{U_0},\mathbf{V})(t)=\mathbf{V}(t)-J_{\mathbf{U_0}}[\mathbf{V}](t),\qquad t\in[0,T].$$

Then by item (1) (local well-posedness statement) we have $\Gamma(\mathbf{U_0},\mathbf{U})(t)=0$ for all $t\in[0,T]$. Now, for $2p$ being even integer, $F(\mathbf{U})=(p+1)|\mathbf{U}|^{2p}\mathbf{U}$ is smooth, and for $2p$ not being even integer and $2p>1$, $F(\mathbf{U})=(p+1)|\mathbf{U}|^{2p}\mathbf{U}$ is $C^{[2p+1]}$-function. Therefore, we obtain that Γ is at least of class C^2 for any p such that $2p>1$. Hence, using the arguments applied for obtaining the local well-posedness in $\mathcal{E}(\mathcal{G})$ (see proposition 2.3 in [27]), we can show that the operator $\partial_{\mathbf{V}}\Gamma(\mathbf{U}_0,\mathbf{U})$ is one-to-one and onto. Thus, by the Implicit Function Theorem there exists a mapping $\mathbf{\Lambda}: B(\mathbf{U}_0;\delta)\to C([0,T],\mathcal{E}(\mathcal{G}))$ at least of class C^2, such that $\Gamma(\mathbf{V}_0,\mathbf{\Lambda}(\mathbf{V}_0))=0$ for all $\mathbf{V}_0\in B(\mathbf{U}_0;\delta)$. This argument establishes the regularity property of the mapping data-solution associated to the NLS model in (1.1). This finishes the proof. □

3. Spectral study associated to positive two-lobe states

In this section we show theorems 1.3, 1.4 and 1.6. In fact, for Z fixed, we consider $\omega<0$ and an associated positive two-lobe state $(\Phi_\omega,\Psi_\omega)\equiv(\Phi,\Psi)$ of (1.5). Let $(\mathcal{L}_{+,Z},D_Z)$ the associated linearized operator in (1.13). In the following, we make a change of referential. For $(f,g)\in D_Z$, let us consider $h(x)=g(x+L)$ for $x>0$. Then $h(0)=g(L)$ and $h'(0)=g'(L)$. So, the eigenvalue problem $\mathcal{L}_{+,Z}(f,g)^t=\lambda(f,g)^t$ will be equivalent to the following one

$$\begin{cases}\mathcal{L}_{0,+}f(x)=\lambda f(x), & x\in(-L,L),\\ \mathcal{L}_{1,+}h(x)=\lambda h(x), & x\in(0,+\infty),\\ (f,h)\in D_{Z,0},\end{cases}\tag{3.1}$$

where

$$\mathcal{L}_{0,+}=-\partial_x^2-\omega-(p+1)(2p+1)\Phi^{2p},\quad \mathcal{L}_{1,+}\equiv-\partial_x^2-\omega-(p+1)(2p+1)\psi_{0,a}^{2p},\tag{3.2}$$

with $\psi_{0,a}$ being the bump-soliton profile

$$\psi_{0,a}(x)=(-\omega)^{1/2p}\psi_0\left(\sqrt{-\omega}x+a\right),\ \ x>0,\ \ a<0,$$

and ψ_0 as defined in (1.8). The domain $D_{Z,0}$ is given by

$$D_{Z,0}=\left\{(f,h)\in X^2(-L,L): f(L)=f(-L)=h(0)\ \text{and}\ f'(L)-f'(-L)=h'(0)+Zh(0)\right\},\tag{3.3}$$

with $X^n(-L,L)\equiv H^n(-L,L)\oplus H^n(0,+\infty)$, $n\in\mathbb{N}$. We note immediately that $(\Phi,\psi_{0,a})\in D_{Z,0}$.

For convenience of notation, we will consider $\mathcal{L}_0 \equiv \mathcal{L}_{0,+}$, $\mathcal{L}_1 \equiv \mathcal{L}_{1,+}$, and $\psi_a \equiv \psi_{0,a}$. Thus, we define $\mathcal{L}_+ \equiv diag(\mathcal{L}_0, \mathcal{L}_1)$ with domain $D_{Z,0}$. Therefore, the statements in theorems 1.3 and 1.4, for $(\mathcal{L}_{+,Z}, D_Z)$, will be sufficient to be proved for $(\mathcal{L}_+, D_{Z,0})$.

3.1. Perron–Frobenius property for $(\mathcal{L}_+, D_{Z,0})$

Initially, we will see that $n(\mathcal{L}_+) \geqq 1$. Indeed, since $(\Phi, \psi_a) \in D_{Z,0}$ and

$$\langle \mathcal{L}_+ (\Phi, \psi_a)^t, (\Phi, \psi_a)^t \rangle \quad = -2p\,(p+1) \left[\int_{-L}^{L} \Phi^{2p+1}(x)\,\mathrm{d}x + \int_0^{+\infty} \psi_a^{2p+1}(x)\,\mathrm{d}x \right] < 0, \tag{3.4}$$

we obtain from the mini-max principle that $n(\mathcal{L}_+) \geqq 1$.

In the following we prove the Perron-Frobenius property associated to the eigenvalue problem in (3.1) on $D_{Z,0}$, for any $Z \in \mathbb{R}$ fixed. The use of the Sturm–Liouville theory for real coupled self-adjoint boundary conditions associated to $\mathcal{L}_0$ together with oscillations results for δ-interactions conditions on whole the line, will be essential in our analysis (see [19, 27, 31] for other strategies in obtaining this Perron-Frobenius property for the case $Z \geqq 0$).

In this form, we consider the quadratic form $\mathcal{Q}_Z$ associated to $\mathcal{L}_+ = \mathrm{diag}(\mathcal{L}_0, \mathcal{L}_1)$ on $D_{Z,0}$, namely, $\mathcal{Q}_Z : D(\mathcal{Q}_Z) \to \mathbb{R}$ defined by

$$\mathcal{Q}_Z(\phi, \zeta) = \int_{-L}^{L} (\phi')^2 + V_\Phi \phi^2 \mathrm{d}x + \int_0^{+\infty} (\zeta')^2 + W_a \zeta^2 \mathrm{d}x - Z|\zeta(0)|^2, \tag{3.5}$$

with $V_\Phi = -\omega - (p+1)(2p+1)\Phi^{2p}$, $W_a = -\omega - (p+1)(2p+1)\psi_a^{2p}$ and $D(\mathcal{Q}_Z)$ is defined by

$$D(\mathcal{Q}_Z) = \left\{ (\phi, \zeta) \in X^1(-L, L) : \phi(L) = \phi(-L) = \zeta(0) \right\}. \tag{3.6}$$

Theorem 3.1 (Perron–Frobenius property). *Fix $Z \in \mathbb{R}$. Let $\lambda_0 < 0$ be the smallest eigenvalue for $(\mathcal{L}_+, D_{Z,0})$ in (3.1) with associated eigenfunction $(\phi_{\lambda_0}, \zeta_{\lambda_0})$. Then, $\phi_{\lambda_0}, \zeta_{\lambda_0}$ are positive functions. Moreover, ϕ_{λ_0} is even.*

Proof. Based on some ideas laid down by Angulo in [8], we split the proof in several steps.

(1) The profile ζ_{λ_0} is not identically zero: Indeed, suppose $\zeta_{\lambda_0} \equiv 0$, then ϕ_{λ_0} satisfies

$$\begin{cases} \mathcal{L}_0 \phi_{\lambda_0}(x) = \lambda_0 \phi_{\lambda_0}(x), & x \in (-L, L), \\ \phi_{\lambda_0}(L) = \phi_{\lambda_0}(-L) = 0 \\ \phi'_{\lambda_0}(L) = \phi'_{\lambda_0}(-L). \end{cases} \tag{3.7}$$

Next, from the Dirichlet condition (which implies that the eigenvalue λ_0 is simple) and from the evenness of the potential V_Φ we need to have that ϕ_{λ_0} is odd or even. Now, from oscillations theorems of the Floquet theory (which implies that the number of zeros of ϕ_{λ_0} is even on $[-L, L)$), we need to have that ϕ_{λ_0} is odd. Then, by Sturm–Liouville theory, there is an eigenvalue θ for $\mathcal{L}_0$ such that $\theta < \lambda_0$, with associated eigenfunction $\xi > 0$ on $(-L, L)$, and with $\xi(-L) = \xi(L) = 0$.

Now, let $\mathcal{Q}_{Dir}$ be the quadratic form associated to $\mathcal{L}_0$ with Dirichlet domain, namely, $\mathcal{Q}_{Dir} : H_0^1(-L, L) \to \mathbb{R}$ given by

$$\mathcal{Q}_{Dir}(f) = \int_{-L}^{L} (f')^2 + V_\Phi f^2 \mathrm{d}x. \tag{3.8}$$

Then, $\mathcal{Q}_{Dir}(\xi) = \mathcal{Q}_Z(\xi,0) \geqq \lambda_0\|\xi\|^2$ and so, $\theta \geqq \lambda_0$. This implies a contradiction.

(2) $\zeta_{\lambda_0}(0) \neq 0$: suppose $\zeta_{\lambda_0}(0) = 0$ and we consider the odd-extension ζ_{odd} for ζ_{λ_0}, and the even-extension ψ_{even} of the bum-profile ψ_a on whole the line. Then, $\zeta_{\text{odd}} \in H^2(\mathbb{R})$ and $\psi_{even} \in H^2(\mathbb{R}-\{0\}) \cap H^1(\mathbb{R})$. Next, we consider the unfold operator $\widetilde{\mathcal{L}}$ associated to $\mathcal{L}_1$,

$$\widetilde{\mathcal{L}} = -\partial_x^2 - \omega - (p+1)(2p+1)\psi_{\text{even}}^{2p} \tag{3.9}$$

on the δ-interaction type domain

$$D_{\delta,\gamma} = \left\{f \in H^2(\mathbb{R}-\{0\}) \cap H^1(\mathbb{R}) : f'(0+) - f'(0-) = \gamma f(0)\right\}, \quad \gamma \in \mathbb{R}. \tag{3.10}$$

Thus, we get immediately that $\zeta_{\text{odd}} \in D_{\delta,\gamma}$ for any γ and $\widetilde{\mathcal{L}}\zeta_{\text{odd}} = \lambda_0\zeta_{\text{odd}}$ on $\mathbb{R}$. Next, for $\beta_\gamma = \inf\sigma(\widetilde{\mathcal{L}})$ being the smallest eigenvalue for $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$, we have that β_γ is simple and its corresponding eigenfunction ζ_{β_γ} can be chosen positive and even (see proposition A.4 in appendix). Therefore, we need to have $\lambda_0 > \beta_\gamma$.
Next, ϕ_{λ_0} satisfies

$$\begin{cases} \mathcal{L}_0\phi_{\lambda_0}(x) = \lambda_0\phi_{\lambda_0}(x), & x \in (-L,L), \\ \phi_{\lambda_0}(L) = \phi_{\lambda_0}(-L) = 0 \\ \phi'_{\lambda_0}(L) - \phi'_{\lambda_0}(-L) = \zeta'_{\lambda_0}(0), \end{cases} \tag{3.11}$$

and so from step 1) above, we need to have $\zeta'_{\lambda_0}(0) \neq 0$. Besides, we have that ϕ_{λ_0} is even and so $2\phi'_{\lambda_0}(L) = \zeta'_{\lambda_0}(0)$. Now, let μ_0 be the smallest eigenvalue for $\mathcal{L}_0$ with Dirichlet conditions. Then, its corresponding eigenfunction ϕ_{μ_0} satisfies $\phi_{\mu_0}(x) > 0$ for $x \in (-L,L)$. Furthermore, $\mu_0 \leqq \lambda_0$. Suppose $\mu_0 < \lambda_0$, then by the analysis in step 1) we get a contradiction. So, we need to have $\mu_0 = \lambda_0$.

Now, we consider the following eigenvalue problem for $\mathcal{L}_0$ with real coupled self-adjoint boundary conditions determined by $\alpha \in \mathbb{R}$,

$$(RC_\alpha): \begin{cases} \mathcal{L}_0 y(x) = \eta y(x), & x \in (-L,L), \\ y(L) = y(-L), \\ y'(L) - y'(-L) = \alpha y(L). \end{cases} \tag{3.12}$$

Also, by following the notations in [55], we consider the 2×2-matrix associated to (3.12), $K_\alpha = [k_{ij}]$ given by $k_{11} = 1$, $k_{12} = 0$, $k_{21} = \alpha$ and $k_{22} = 1$ ($\det(K_\alpha) = 1$), and the eigenvalues for the (RC_α)-problem in (3.12) denoted by $\eta_n = \eta_n(K_\alpha)$, $n \in \mathbb{N}_0$. We also denote by μ_n, the eigenvalues in (3.12) with only the Dirichlet condition $y(-L) = y(L) = 0$. Thus, from theorem 1.35 in Kong *et al* [42] or theorem 4.8.1 in Zettl [55], we obtain that for every α (fixed), $\eta_0(K_\alpha)$ is simple and in particular $\eta_0(K_\alpha) < \mu_0$. So, for $\alpha \equiv 2Z$ we have clearly from the former paragraph that $\eta_0 = \eta_0(K_{2Z})$ satisfies $\eta_0 < \lambda_0 = \mu_0$. Moreover, the corresponding eigenfunction y_{η_0} is even and strictly positive on $[-L,L]$ (see theorem 4.8.5 in [55]).

Next, let $\gamma = 2Z$ in (3.10) and $\beta \equiv \beta_{2Z}$ being the smallest eigenvalue of $\widetilde{\mathcal{L}}$ with corresponding eigenfunction ζ_β positive. Thus, by the analysis above we have $\lambda_0 > \beta$. Define $g \equiv a y_{\eta_0}$ with $a \in \mathbb{R}$ chosen such that $a y_{\eta_0}(L) = \zeta_\beta(0)$. Then, $(g, \widetilde{\zeta}_\beta) \in D(\mathcal{Q}_Z)$ for $\widetilde{\zeta}_\beta \equiv \zeta_\beta|_{[0,+\infty)}$, and

$$\mathcal{Q}_Z\left(g, \widetilde{\zeta}_\beta\right) \geqq \lambda_0\left(\|g\|^2 + \|\widetilde{\zeta}_\beta\|^2\right) > \eta_0\|g\|^2 + \lambda_0\|\widetilde{\zeta}_\beta\|^2. \tag{3.13}$$

Now,

$$\mathcal{Q}_Z\left(g,\widetilde{\zeta}_\beta\right) = 2Zg^2(L) - \zeta_\beta(0)\zeta_\beta'(0+) + \eta_0\|g\|^2 + \beta\|\widetilde{\zeta}_\beta\|^2 - Z|\zeta_\beta(0)|^2$$
$$= 2Z\zeta_\beta^2(0) - Z\zeta_\beta^2(0) - Z|\zeta_\beta(0)|^2 + \eta_0\|g\|^2 + \beta\|\widetilde{\zeta}_\beta\|^2 = \eta_0\|g\|^2 + \beta\|\widetilde{\zeta}_\beta\|^2. \tag{3.14}$$

Therefore, from (3.13) follows $\beta > \lambda_0$, which is a contradiction. Then, $\zeta_{\lambda_0}(0) \neq 0$.

(3) $\zeta_{\lambda_0} : [0,+\infty) \to \mathbb{R}$ can be chosen strictly positive: without loss of generality we suppose $\zeta_{\lambda_0}(0) > 0$. Then $\phi'_{\lambda_0}(L) - \phi'_{\lambda_0}(-L) = \zeta'_{\lambda_0}(0) + Z\zeta_{\lambda_0}(0)$ implies

$$\zeta'_{\lambda_0}(0) = \left[\frac{\phi'_{\lambda_0}(L) - \phi'_{\lambda_0}(-L)}{\zeta_{\lambda_0}(0)} - Z\right]\zeta_{\lambda_0}(0) \equiv \gamma_0\zeta_{\lambda_0}(0).$$

Next, by considering ζ_{even} as being the even-extension of ζ_{λ_0} on whole the line, we obtain $\zeta_{\text{even}} \in D_{\delta,2\gamma_0}$ (see (3.10)) and $\widetilde{\mathcal{L}}\zeta_{\text{even}} = \lambda_0\zeta_{\text{even}}$. Suppose that there is $s > 0$ such that $\zeta_{\lambda_0}(s) = 0$, then ζ_{even} has at least two zeros ($\pm s$). Thus, by extending classical oscillation results of the Sturm–Liouville theory to δ-interactions boundary conditions (see lemma 5.3 in [15], proposition 3.2 and theorem 3.5 in [20]), we obtain that for $\beta_{2\gamma_0} = \inf\sigma(\widetilde{\mathcal{L}})$ on $D_{\delta,2\gamma_0}$, there is an eigenvalue $\kappa \in (\beta_{2\gamma_0}, \lambda_0)$ for $\widetilde{\mathcal{L}}$ on $D_{\delta,2\gamma_0}$, with corresponding odd-eigenfunction χ_κ ($\chi_\kappa(x) = 0$ if and only if $x = 0$). Then, for $g_\kappa = \chi_\kappa|_{[0,+\infty)}$ we have that $(0, g_\kappa) \in D(\mathcal{Q}_Z)$ in (3.6) and

$$\mathcal{Q}_Z(0, g_\kappa) = \kappa\|g_\kappa\|^2 \geqq \lambda_0\|g_\kappa\|^2.$$

Therefore, $\kappa \geqq \lambda_0$ which gives a contradiction. Thus, $\zeta_{\lambda_0} > 0$ on $[0,+\infty)$.

(4) $\phi_{\lambda_0} : [-L, L] \to \mathbb{R}$ can be chosen strictly positive: initially we have that ϕ_{λ_0} satisfies the following relation

$$\phi'_{\lambda_0}(L) - \phi'_{\lambda_0}(-L) = \left[\frac{\zeta'_{\lambda_0}(0)}{\zeta_{\lambda_0}(0)} + Z\right]\zeta_{\lambda_0}(0) \equiv \alpha_0\zeta_{\lambda_0}(0) = \alpha_0\phi_{\lambda_0}(L). \tag{3.15}$$

Now, we consider the eigenvalue problem in (3.12) with $\alpha \equiv \alpha_0$. Then, the first eigenvalue η_0 for (3.12) is simple (see [42, 55]). Moreover, since the pair $(\phi_{\lambda_0}, \lambda_0)$ is a solution for (3.12), we have $\lambda_0 \geqq \eta_0$.

In the following we show $\lambda_0 = \eta_0$. Indeed, we consider the quadratic form, $\mathcal{Q}_{RC}$, associated to the (RC_{α_0})-problem in (3.12),

$$\mathcal{Q}_{RC}(h) = \int_{-L}^{L} (h')^2 + V_\Phi h^2 \mathrm{d}x - \alpha_0|h(L)|^2, \tag{3.16}$$

where for $h \in H^1(-L, L)$ we have $h(L) = h(-L)$. Next, for h fixed as before, define $\xi = \nu\zeta_{\lambda_0}$ with $\nu \in \mathbb{R}$ being chosen such that $\xi(0) = \nu\zeta_{\lambda_0}(0) = h(L)$. Then, $(h, \xi) \in D(\mathcal{Q}_Z)$ in (3.6). Now, by using that $\mathcal{L}_1\zeta_{\lambda_0} = \lambda_0\zeta_{\lambda_0}$ on $(0,+\infty)$, we obtain

$$\mathcal{Q}_{RC}(h) = \mathcal{Q}_Z(h,\xi) - \alpha_0 h^2(L) - \int_0^{+\infty} (\xi')^2 + W_a\xi^2 \mathrm{d}x + Z|h(L)|^2$$
$$= \mathcal{Q}_Z(h,\xi) - \alpha_0 h^2(L) + \nu^2\zeta'_{\lambda_0}(0)\zeta_{\lambda_0}(0) - \lambda_0\nu^2\|\zeta_{\lambda_0}\|^2 + Z|h(L)|^2$$
$$= \mathcal{Q}_Z(h,\xi) - \left[\zeta'_{\lambda_0}(0)h(L) + Zh^2(L)\right] + h(L)\zeta'_{\lambda_0}(0) + Zh^2(L) - \lambda_0\|\xi\|^2$$
$$= \mathcal{Q}_Z(h,\xi) - \lambda_0\|\xi\|^2 \geqq \lambda_0\left[\|h\|^2 + \|\xi\|^2\right] - \lambda_0\|\xi\|^2 = \lambda_0\|h\|^2.$$

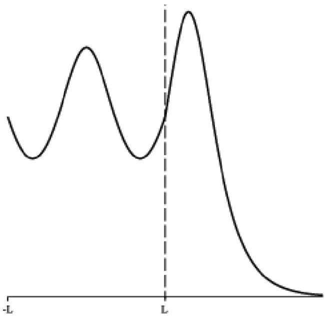

Figure 3. Multi-lobe profile solution for (1.5).

Then, $\eta_0 \geqq \lambda_0$ and so $\eta_0 = \lambda_0$.

Therefore, λ_0 *is the first eigenvalue for the problem* (RC_{α_0}) in (3.12). Then, ϕ_{λ_0} is odd or even. If ϕ_{λ_0} is odd, then the condition $\phi_{\lambda_0}(L) = \phi_{\lambda_0}(-L)$ implies $\phi_{\lambda_0}(L) = 0$. But, $\phi_{\lambda_0}(L) = \zeta_{\lambda_0}(0) > 0$. So, we need to have that ϕ_{λ_0} is even. Now, from Oscillation Theorems for the (RC_{α_0})-problem, the number of zeros of ϕ_{λ_0} on $[-L,L)$ is 0 or 1 (see theorem 4.8.5 in [55]). Since $\phi_{\lambda_0}(-L) > 0$ and ϕ_{λ_0} is even, we obtain necessarily that $\phi_{\lambda_0} > 0$ on $[-L,L]$. This finishes the proof.

□

Corollary 3.2. *Let $\lambda_0 < 0$ be the smallest eigenvalue for $\mathcal{L}_+$ on $D_{Z,0}$. Then, λ_0 is simple.*

Proof. The proof is immediate. Suppose λ_0 is double. Then, there is a (f_0, g_0)-eigenfunction associated to λ_0 orthogonal to $(\phi_{\lambda_0}, \zeta_{\lambda_0})$. By theorem 3.1 we have that $f_0, g_0 > 0$. So, we arrive to a contradiction from the orthogonality property of the eigenfunctions. □

Remark 3.3. From the splitting eigenvalue result in lemma 3.4 below, follows that $\alpha_0 = 0$ in (3.15) and so $\phi'_{\lambda_0}(L) = \phi'_{\lambda_0}(-L)$ (ϕ_{λ_0} will satisfy periodic boundary conditions and so λ_0 coincides with the first eigenvalue for $\mathcal{L}_0$ with periodic conditions) and $\zeta'_{\lambda_0}(0) = -Z\zeta_{\lambda_0}(0)$ (ζ_{λ_0} will satisfy δ-interaction boundary conditions on $(0,+\infty)$ and so λ_0 coincides with the first eigenvalue for $\mathcal{L}_1$ on this domain). Lemma 3.1 and corollary 3.2 can be extended to the case of Φ having 'multiples lobes' on $[-L,L]$ (see figure 3 and [26]), or extended to other metric graphs, such as, a looping edge graph (see Angulo [8]).

3.2. Splitting eigenvalue method on a tadpole graph

In the following we establish our main strategy for studying eigenvalue problems on a tadpole graph $\mathcal{G}$. More exactly, we reduce our eigenvalue problem for $\mathcal{L}_+ \equiv (\mathcal{L}_0, \mathcal{L}_1)$ in (3.1) to two classes of eigenvalue problems, one for $\mathcal{L}_0$ with periodic boundary conditions on $[-L,L]$ and the other one for the operator $\mathcal{L}_1$ with δ-type boundary conditions on $(0,+\infty)$.

Lemma 3.4. *Let us consider the self-adjoint operator $(\mathcal{L}_+, D_{Z,0})$ in (3.1) with Z fixed. Suppose $(f,g) \in D_{Z,0}$ with $g(0) \neq 0$ and $\mathcal{L}_+(f,g)^t = \gamma(f,g)^t$, for $\gamma \in \mathbb{R}$. Then, we obtain the following two eigenvalue problems:*

$$\begin{cases} \mathcal{L}_0 f(x) = \gamma f(x), \; x \in (-L,L), \\ f(L) = (-L), \; f'(L) = f'(-L), \end{cases} \qquad \begin{cases} \mathcal{L}_1 g(x) = \gamma g(x), \; x > 0, \\ g'(0+) = -Zg(0+). \end{cases}$$

Proof. For $(f,g) \in D_{Z,0}$ and $g(0) \neq 0$, one has

$$f(-L) = f(L), \; f'(L) - f'(-L) = \left[\frac{1}{g(0+)} g'(0+) + Z\right] f(L) \equiv \theta f(L),$$

and so f satisfies the real coupled problem (RC_α) in (3.12) with $\alpha = \theta$ and $\eta = \gamma$. In the following, we will see that $\theta = 0$ which proves the lemma. We consider $K_\theta = [k_{ij}]$ the 2×2-matrix associated to (3.12) given by $k_{11} = 1$, $k_{12} = 0$, $k_{21} = \theta$ and $k_{22} = 1$ ($\det(K_\theta) = 1$), and by $\eta_n = \eta_n(K_\theta)$, $n \in \mathbb{N}_0$, the eigenvalues for the (RC_θ)-problem in (3.12). We also consider $\mu_n = \mu_n(K_\theta)$ and $\nu_n = \nu_n(K_\theta)$, $n \in \mathbb{N}_0$, the eigenvalues problems in (3.12) induced by K_θ with the following boundary conditions

$$\begin{array}{lr} y(-L) = y(L) = 0, & \text{(Dirichlet condition)}, \\ y'(-L) = 0, \;\; \theta y(L) - y'(L) = 0, & \text{(Neumann-type condition)}, \end{array} \tag{3.17}$$

respectively. We recall that if y_n is an eigenfunction of μ_n, then y_n is unique up to constant multiples and it has exactly n zeros in $(-L,L)$, $n \in \mathbb{N}_0$ (a similar result is obtained for u_n an eigenfunction of ν_n, $n \in \mathbb{N}_0$). Next, by theorem 4.8.1 in [55], we have that ν_0 and η_0 are simple eigenvalues and in particular, we have the following partial distribution of eigenvalues

$$\nu_0 \leqq \eta_0 < \{\mu_0, \nu_1\} < \eta_1 \leqq \{\mu_1, \nu_2\} \leqq \eta_2, \tag{3.18}$$

where the notation $\{\mu_n, \nu_m\}$ is used to indicate either μ_n or ν_m but non-comparison is made between μ_n and ν_m. In the following, we will see $u_0'(L) = 0$ and so $\theta = 0$ (because by (3.18) we have $\nu_0 < \mu_0$ and so $u_0(\pm L) \neq 0$). Indeed, since ν_0 is simple and the profile-solution Φ is even, we need to have that u_0 is even or odd, but as u_0 has not zeros in $[-L,L]$ we get that u_0 even. Thus u_0' is odd and therefore $u_0'(\pm L) = 0$. This finishes the proof. □

Remark 3.5. (a) Lemma 3.4 holds for Φ even with multiple bumps on $(-L,L)$ (see figure 3 above), as well as, for ψ_a, $a > 0$, in (1.14) being a tail-profile on $(0,+\infty)$ (see figure 3 in [26]). Thus, our splitting eigenvalue strategy can be a first step for studying stability properties of multiple bumps profiles on tadpole graphs.
(b) Lemma 3.4 has been extended to the case of looping edge graphs in [8] for the stability study of positive single-lobe states (see also [3–5]) and section 6 in [40] for additional results).

3.3. *Spectral analysis with δ-interactions on* $(0,+\infty)$

From the splitting eigenvalue result in lemma 3.4, we need to obtain spectral informations for the family of self-adjoint operators $(\mathcal{L}_1, S_\gamma)_{\gamma\in\mathbb{R}}$ with

$$\mathcal{L}_1 = -\partial_x^2 + 1 - (p+1)(2p+1)\psi_a^{2p}, \quad S_\gamma = \left\{f \in H^2(0,+\infty) : f'(0+) = \gamma f(0+)\right\}, \tag{3.19}$$

where we are using $\omega = -1$ in (3.2) (without loss of generality). Thus, in this subsection, we will consider the soliton-bump profiles $\psi_a(x) = \psi_0(x+a)$, $a < 0$, $x \in (0,+\infty)$, without initially having any relationship with some *a priori* positive two-lobe state. Also, we will use the notation $\mathcal{L}_{1,a}$ for $\mathcal{L}_1$. We note that operators of type $(-\Delta + V(x), S_\gamma)$ appear in the study of boundary conditions for Schrödinger operators on $(0,+\infty)$ in the quantum mechanic (see appendix D in [6] and references therein). In our case, a more detailed study on the Morse and nullity indeces of operators in (3.19) depending on the shift-parameter a and the strength γ, are main in our study of $(\mathcal{L}_{+,Z}, D_Z)$. As the classical Sturm–Liouville oscillation theory for

$(\mathcal{L}_{1,a}, S_\gamma)$ is not enough for a complete spectral information, we decide to give some deeper details of this study.

From proposition A.5 in appendix, we have initially that the Morse index for $\mathcal{L}_{1,a}$ on S_γ, satisfies $n(\mathcal{L}_{1,a}) \leqq 2$ for any value of γ. Next, we begin our analysis by determining the exact values of the Morse and nullity indices for $\mathcal{L}_{1,a}$ on S_0 (Neumann-condition) depending of the shift-value a.

Lemma 3.6. *Let $a^* < 0$ be the unique value such that $\psi''_{a^*}(0+) = 0$. Then, by considering $(\mathcal{L}_{1,a}, S_0)$, $a < 0$, we have:*

(a) $n(\mathcal{L}_{1,a^*}) = 1$ and $ker(\mathcal{L}_{1,a^*}) = span\{\psi'_{a^*}\}$,
(b) for $a \neq a^*$, $ker(\mathcal{L}_{1,a}) = \{0\}$,
(c) for $a \in (a^*, 0)$, $n(\mathcal{L}_{1,a}) = 1$, and for $a \in (-\infty, a^*)$, $n(\mathcal{L}_{1,a}) = 2$.

Proof. Items (a)–(b) are an immediate consequence of Sturm–Liouville theory on half-lines (see [20]). Hence, item (c) will follow from the analytic perturbation theory.

(a) Let $f \in S_0$ such that $\mathcal{L}_{1,a^*} f = 0$. Since $\mathcal{L}_{1,a^*}\psi'_{a^*} = 0$ and $\psi'_{a^*}(x) \to 0$ as $x \to +\infty$, it follows from Sturm–Liouville theory on half-lines that zero will be simple and so $f = \beta\psi'_{a^*}$. Moreover, since ψ'_{a^*} has exactly one zero on $(0, +\infty)$, it follows from oscillations theory that $\mathcal{L}_{1,a^*}$ has exactly one negative eigenvalue and its corresponding eigenfunction can be chosen as being positive.
(b) Let $a \neq a^*$ and $f \in S_0$ such that $\mathcal{L}_{1,a} f = 0$. Then, there is $s \in \mathbb{R}$ such that $f = s\psi'_a$. From $f'(0) = 0$ we get that $s\psi''_a(0) = 0$ and so $s = 0$.
(c) By items (a)–(b) and by the mapping $a \to \psi_a$ to be real-analytic, next we will see that the zero eigenvalue for $\mathcal{L}_{1,a^*}$ will jump to the right or to the left depending of the position of $a \approx a^*$. To decide this movement we will use an argument based in the analytic perturbation theory (see [11, 13, 15] for similar situations). Thus, the proof will be only sketched for sake of brevity. Therefore, by using the spectrum's structure of $\mathcal{L}_{1,a^*}$ given in item (a) and since $\mathcal{L}_{1,a}$ converges to $\mathcal{L}_{1,a^*}$ as $a \to a^*$ in the generalized sense (Kato [41]), we can conclude that the non-positive spectrum of $\mathcal{L}_{1,a^*}$ (which is discrete) moves continuously with the parameter a. Thus, from Kato-Rellich Theorem (see theorem XII.8 in [52]) we obtain the existence of two analytic functions $\eta : (a^* - \epsilon, a^* + \epsilon) \to \mathbb{R}$ and $\Pi : (a^* - \epsilon, a^* + \epsilon) \to L^2(0, +\infty)$, for $\epsilon > 0$ small enough, such that
   (i) $\eta(a^*) = 0$ and $\Pi(a^*) = \psi'_{a^*}$.
   (ii) For all $a \in (a^* - \epsilon, a^* + \epsilon)$, $\eta(a)$ is the simple isolated second eigenvalue of $\mathcal{L}_{1,a}$ on S_0, and $\Pi(a)$ is the associated eigenvector for $\eta(a)$.
   (iii) ϵ can be chosen small enough to ensure that for $a \in (a^* - \epsilon, a^* + \epsilon)$ the spectrum of $\mathcal{L}_{1,a}$ is positive, except at most the first two eigenvalues.

   Now we investigate how the perturbed second eigenvalue moves depending on the position of a with regard to a^*. From Taylor's theorem we have the following expansions

$$\eta(a) = \mu(a - a^*) + O\left(|a - a^*|^2\right) \quad \text{and}$$
$$\Pi(a) = \psi'_{a^*} + \Pi'(a^*)(a - a^*) + O\left(|a - a^*|^2\right), \tag{3.20}$$

   where $\mu \equiv \eta'(a^*) \in \mathbb{R}$ and $\Pi'(a^*) \equiv \frac{d}{da}\Pi(a)|_{a=a^*} \in L^2(0, +\infty)$. The desired result in item (c) will follow if we show that $\mu > 0$. Thus, we will compute $\langle \mathcal{L}_{1,a}\Pi(a), \psi'_{a^*}\rangle$ in two different ways. The first one is obtained from (3.20),

$$\langle \mathcal{L}_{1,a}\Pi(a), \psi'_{a^*}\rangle = \eta(a)\langle \Pi(a), \psi'_{a^*}\rangle = \mu\|\psi'_{a^*}\|^2(a - a^*) + O\left(|a - a^*|^2\right). \tag{3.21}$$

Next, by $\mathcal{L}_{1,a^*}\psi'_{a^*}=0$ and $C(p)=(p+1)(2p+1)$ we get

$$\begin{aligned}\mathcal{L}_{1,a}\psi'_{a^*} &= \mathcal{L}_{1,a^*}\psi'_{a^*} - C(p)\left[\psi_a^{2p}-\psi_{a^*}^{2p}\right]\psi'_{a^*}\\ &= -2pC(p)(a-a^*)\psi_{a^*}^{2p-1}\psi'_{a^*}\Omega + O\left(|a-a^*|^2\right)\end{aligned} \tag{3.22}$$

with $\Omega=\frac{\partial\psi_a}{\partial a}|_{a=a^*}=\psi'_{a^*}$. Then, as $\psi'_{a^*}\in S_0$ we get from the self-adjoint property of $\mathcal{L}_{1,a}$ and from (3.20)–(3.22) that

$$\begin{aligned}\langle\mathcal{L}_{1,a}\Pi(a),\psi'_{a^*}\rangle &= \langle\Pi(a),\mathcal{L}_{1,a}\psi'_{a^*}\rangle = \langle\Pi(a),-2pC(p)(a-a^*)\psi_{a^*}^{2p-1}(\psi'_{a^*})^2\\ &\quad + O\left(|a-a^*|^2\right)\rangle\\ &= -2pC(p)(a-a^*)\langle\psi'_{a^*},\psi_{a^*}^{2p-1}(\psi'_{a^*})^2\rangle + O\left(|a-a^*|^2\right).\end{aligned} \tag{3.23}$$

Then, from (3.21) and (3.22) follow

$$\mu = -\frac{2pC(p)}{\|\psi'_{a^*}\|^2}\int_0^{+\infty}(\psi'_{a^*})^3(x)\,\psi_{a^*}^{2p-1}(x)\,\mathrm{d}x + O(|a-a^*|). \tag{3.24}$$

In the following, we will see that the sign of the integral in (3.24) is negative, and therefore we prove item (c) at least for a close to a^*. Indeed, by using that ψ'_{a^*} satisfies $(\psi'_{a^*})^2=\psi_{a^*}^2-\psi_{a^*}^{2p+2}$ and $(p+1)\psi_{a^*}^{2p}(0)=1$ we obtain

$$\begin{aligned}\int_0^{+\infty}(\psi'_{a^*})^3\psi_{a^*}^{2p-1}\mathrm{d}x &= \int_0^{+\infty}\psi'_{a^*}\left[\psi_{a^*}^2-\psi_{a^*}^{2p+2}\right]\psi_{a^*}^{2p-1}\mathrm{d}x\\ &= \int_0^{+\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2(p+1)}\psi_{a^*}^{2p+2}-\frac{1}{4p+2}\psi_{a^*}^{4p+2}\right]\\ dx &= \psi_{a^*}^{2p+2}(0)\left[\frac{1}{4p+2}\psi_{a^*}^{2p}(0)-\frac{1}{2(p+1)}\right]\\ &= \frac{1}{2(p+1)}\psi_{a^*}^{2p+2}(0)\left[\frac{1}{2p+1}-1\right]<0.\end{aligned} \tag{3.25}$$

Next, we obtain the Morse index of $\mathcal{L}_{1,a}$ on S_0 for any $a\neq a^*$. Here, the strategy is to use a classical continuation argument based on the Riesz projection and that for any $a\neq a^*$ we have $ker(\mathcal{L}_{1,a})=\{0\}$ (see [11, 13, 15] for similar situations). So, without loss of generality we consider only the case of $a<a^*$ (the other case, $a\in(a^*,0)$, is similar). Define $a_{-\infty}$ by

$$a_{-\infty}=\inf\{\tilde{a}<a^*:\mathcal{L}_{1,a}\text{ has exactly two negative eigenvalues for all }a\in(\tilde{a},a^*)\}. \tag{3.26}$$

By the former analysis with $a\approx a^*$, $a_{-\infty}$ is well defined and $a_{-\infty}\in[-\infty,a^*)$. We claim that $a_{-\infty}=-\infty$. Indeed, suppose that $a_{-\infty}>-\infty$. Let $M=n(\mathcal{L}_{1,a_{-\infty}})$ and let Γ be a closed curve (for instance, a circle or a rectangle) such that $0\in\Gamma\subset\rho(\mathcal{L}_{1,a_{-\infty}})$, and such that all the negative eigenvalues of $\mathcal{L}_{1,a_{-\infty}}$ belong to the inner domain of Γ. The existence of such Γ can be deduced from the lower semi-boundedness of the quadratic form associated to $\mathcal{L}_{1,a_{-\infty}}$. Next, since $\mathcal{L}_{1,a}$ converges to $\mathcal{L}_{1,a_{-\infty}}$ as $a\to a_{-\infty}$ in the generalized sense, it follows that there is $\epsilon > 0$ such that for $a \in [a_{-\infty} - \epsilon, a_{-\infty} + \epsilon]$ we have $\Gamma \subset \rho(\mathcal{L}_{1,a})$ and for $\xi \in \Gamma$, $a \to (\mathcal{L}_{1,a} - \xi)^{-1}$ is analytic (see theorem XII.7 in [52]). Therefore, we have the existence of an analytic family of Riesz-projections $a \to P(a)$ given by

$$P(a) = -\frac{1}{2\pi i}\oint_{\Gamma} (\mathcal{L}_{1,a} - \xi)^{-1}\, \mathrm{d}\xi.$$

Therefore,

$$\dim (Ran\, P(a)) = \dim (Ran\, P(a_{-\infty})) = M, \text{ for all } a \in [a_{-\infty} - \epsilon, a_{-\infty} + \epsilon].$$

Next, by definition of $a_{-\infty}$, $\mathcal{L}_{1,a_{-\infty}+\epsilon}$ has two negative eigenvalues and therefore $M = 2$, hence $\mathcal{L}_{1,a}$ has two negative eigenvalues for $a \in (a_{-\infty} - \epsilon, a^*)$, which contradicts the definition of $a_{-\infty}$. Therefore, $a_{-\infty} = -\infty$. This finishes the proof.

□

In the next two lemmas, we determine the exact values of the Morse and nullity indexes for $\mathcal{L}_{1,a}$ on S_γ, with $\gamma \neq 0$. We will see that these values depend of a threshold value of γ and from the position of the shift-value a with regard to a^*.

Lemma 3.7. *Let $a^* < 0$ be the unique value such that $\psi''_{a^*}(0+) = 0$. We consider the family of self-adjoint operators $(\mathcal{L}_{1,a}, S_\gamma)_{\gamma \in \mathbb{R}}$ defined in (3.19) with $a \in (a^*, 0)$ fixed. Define*

$$\gamma^* = \frac{\psi''_a(0+)}{\psi'_a(0+)} < 0.$$

Then,

(*a*) *for $\gamma = \gamma^*$, $ker(\mathcal{L}_{1,a}) = span\{\psi'_a\}$ and $n(\mathcal{L}_{1,a}) = 1$ on S_{γ^*},*
(*b*) *for $\gamma \neq \gamma^*$, $ker(\mathcal{L}_{1,a}) = \{0\}$ on S_γ,*
(*c*) *for $\gamma \in (\gamma^*, +\infty)$, $n(\mathcal{L}_{1,a}) = 1$, and for $\gamma \in (-\infty, \gamma^*)$, $n(\mathcal{L}_{1,a}) = 2$, on S_γ.*

Proof. (*a*) It is immediate that $\psi'_a \in S_{\gamma^*}$ and $\mathcal{L}_{1,a}\psi'_a = 0$. Next, suppose $f \in ker(\mathcal{L}_{1,a})$ and $f \in S_{\gamma^*}$. Then, since $\psi'_a(x) \to 0$ as $x \to +\infty$, there is $s \in \mathbb{R}$ such that $f = s\psi'_a$. Thus, $ker(\mathcal{L}_{1,a}) = span\{\psi'_a\}$. Next, since ψ'_a has exactly one zero in $(0, +\infty)$, the oscillation theory implies that $\mathcal{L}_{1,a}$ will have exactly one negative eigenvalue on S_{γ^*}.

(*b*) For $\gamma = 0$, lemma 3.6-item (*b*) implies $ker(\mathcal{L}_{1,a}) = \{0\}$ on S_0. Next, we consider $\gamma > 0$. Then for $f \in ker(\mathcal{L}_{1,a})$ and $f \in S_\gamma$, we get that $f = r\psi'_a$ and so

$$r\gamma\psi'_a(0) = \gamma f(0) = f'(0) = r\psi''_a(0).$$

If $r \neq 0$, then $\gamma = \psi''_a(0)/\psi'_a(0) < 0$ (we recall that for $a \in (a^*, 0)$, $\psi''_a(0) < 0$). Then, $r = 0$ and so $ker(\mathcal{L}_{1,a}) = \{0\}$ on S_γ. Lastly, let $\gamma < 0$ and $\gamma \neq \gamma^*$. Then, by the former analysis we get immediately $\gamma = \psi''_a(0)/\psi'_a(0) = \gamma^*$, which is false.

(*c*) Consider the quadratic form $\mathcal{Q}_\gamma : H^1(0, +\infty) \to \mathbb{R}$, given by

$$\mathcal{Q}_\gamma(f) = \int_0^{+\infty} (f')^2 + W_a f^2 \mathrm{d}x + \gamma |f(0)|^2, \quad \gamma \in \mathbb{R}, \tag{3.27}$$

with $W_a = 1 - (p+1)(2p+1)\psi_a^{2p}$. We denote by $n_\gamma(\mathcal{L}_{1,a})$ the Morse index of $\mathcal{L}_{1,a}$ on S_γ. We divide our analysis in several cases:

(i) Let $\gamma \in (\gamma^*, 0]$. Then $\mathcal{Q}_\gamma(f) \geqq \mathcal{Q}_{\gamma^*}(f)$ for $f \in H^1(0,+\infty)$. Therefore, $n_\gamma(\mathcal{L}_{1,a}) \leqq n_{\gamma^*}(\mathcal{L}_{1,a}) = 1$ (by using item (a) above). Next, as $\mathcal{Q}_\gamma(f) \leqq \mathcal{Q}_0(f)$ for $f \in H^1(0,+\infty)$, lemma 3.6-item (c) implies that there is $\chi \in S_0$ such that $\mathcal{Q}_0(\chi) < 0$. Hence, by the min-max theorem we get that $n_\gamma(\mathcal{L}_{1,a}) = 1$.

(ii) Let $\gamma > 0$. It is not difficult to see that $(\mathcal{L}_{1,a}, S_\gamma)$ as a function of $\gamma \in \mathbb{R}$ is a real-analytic family of type (B) in the sense of Kato (see [41], theorem VII-4.2). Moreover, for $\gamma = 0$ and $a \in (a^*, 0)$, from lemma 3.6 we have $ker(\mathcal{L}_{1,a}) = \{0\}$ and $n_0(\mathcal{L}_{1,a}) = 1$ on S_0. Thus, since $ker(\mathcal{L}_{1,a}) = \{0\}$ on S_γ for any $\gamma \in (-\epsilon, \epsilon)$ (by item (b) above), we have the right framework to apply the analytic perturbation theory and the Kato-Rellich Theorem (see proof of item (c) of lemma 3.6) to obtain that

$$\gamma_\infty = sup\{\tilde{\gamma} > 0 : \mathcal{L}_{1,a} \text{ has exactly one negative eigenvalue on } S_\gamma \text{ for all } \gamma \in (0, \tilde{\gamma})\}$$

is well defined and that $\gamma_\infty \in (0, +\infty]$. Thus, by using a classical continuation argument based on the Riesz-projection, we get that $\gamma_\infty = +\infty$. Then, $n_\gamma(\mathcal{L}_{1,a}) = 1$ for all $\gamma > 0$.

(iii) Let $\gamma \in (-\infty, \gamma^*)$. By item ($a$) above, we consider $\lambda^* < 0$ and the positive eigenfunction $\chi^* \in S_{\gamma^*}$ such that $\mathcal{L}_{1,a}\chi^* = \lambda^*\chi^*$. Next, we consider the unfold operator $\tilde{\mathcal{L}}$ in (3.9) defined on the δ-interaction domain $D_{\delta,2\gamma^*}$ (see (3.10)). Then, for $\psi'_{a,\mathrm{even}}$ and χ^*_{even} being, respectively, the even-extension of ψ'_a and χ^* on all the line, we obtain $\psi'_{a,\mathrm{even}}, \chi^*_{\mathrm{even}} \in D_{\delta,2\gamma^*}$, and

$$\tilde{\mathcal{L}}\psi'_{a,\mathrm{even}} = 0, \quad \tilde{\mathcal{L}}\chi^*_{\mathrm{even}} = \lambda^*\chi^*_{\mathrm{even}}, \quad \text{for } x \neq 0.$$

Thus, as $\psi'_{a,\mathrm{even}}$ has exactly two different zeros, we obtain from Sturm–Liouville oscillations theory for δ-interaction boundary conditions, that there are $\theta \in (\lambda^*, 0)$ and $\Theta \in D_{\delta,2\gamma^*}$, such that Θ is odd and $\Theta(x) = 0$ if and only if $x = 0$ (therefore, $n(\tilde{\mathcal{L}}) = 2$). We note that as $\Theta \in H^2(\mathbb{R})$, then $\Theta \in D_{\delta,2\gamma}$ for any γ. Now, we consider the quadratic form associated to $\tilde{\mathcal{L}}$ on $D_{\delta,2\gamma}$,

$$\mathcal{Q}_{\delta,\gamma}(f) = \int_{-\infty}^{+\infty} \left(f'\right)^2 + W_{\mathrm{even}} f^2 \mathrm{d}x + 2\gamma |f(0)|^2, \tag{3.28}$$

where $W_{\mathrm{even}} = 1 - (p+1)(2p+1)\psi^{2p}_{a,\mathrm{even}}$. Then for any $\gamma \in (-\infty, \gamma^*)$ we get

$$\mathcal{Q}_{\delta,\gamma}(\chi^*_{\mathrm{even}}) \leqq \mathcal{Q}_{\delta,\gamma^*}(\chi^*_{\mathrm{even}}) < 0 \text{ and } \mathcal{Q}_{\delta,\gamma}(\Theta) < 0.$$

Moreover,

$$\mathcal{Q}_{\delta,\gamma}\left(\psi'_{a,\mathrm{even}}\right) = -2\psi''_a(0+)\psi'_a(0+) + 2\gamma|\psi'_a(0+)|^2 = 2(\gamma - \gamma^*)|\psi'_a(0+)|^2 < 0.$$

Therefore, $\mathcal{Q}_{\delta,\gamma}$ is a negative quadratic form on the three-dimensional subspace $M = span\{\chi^*_{\mathrm{even}}, \Theta, \psi'_{a,\mathrm{even}}\}$ where $\chi^*_{\mathrm{even}}, \Theta$ and $\psi'_{a,\mathrm{even}}$ are orthogonal two by two. Then, $n(\tilde{\mathcal{L}}) \geqq 3$ for $\tilde{\mathcal{L}}$ with domain $D_{\delta,2\gamma}$. But, via extension theory we know that $n(\tilde{\mathcal{L}}) \leqq 3$ (see proposition A.4 in appendix), and so $n(\tilde{\mathcal{L}}) = 3$. Next, let $\lambda_{0,2\gamma}, \theta, \lambda_{2,2\gamma}$ be the three simple negative eigenvalues for $\tilde{\mathcal{L}}$ on $D_{\delta,2\gamma}$. Then, by oscillation theory we have that the eigenfunctions associated to $\lambda_{0,2\gamma}$ and $\lambda_{2,2\gamma}$ are even, so they are the only ones that will survive when we restrict our problem to the half-line $(0,+\infty)$. Then, $n_\gamma(\mathcal{L}_{1,a}) = 2$. This finishes the Lemma.

□

Lemma 3.8. *Let $a^* < 0$ be the unique value such that $\psi''_{0,a^*}(0+) = 0$. We consider the family of self-adjoint operators $(\mathcal{L}_{1,a}, S_\gamma)_{\gamma\in\mathbb{R}}$ defined in (3.19) with $a \in (-\infty, a^*)$ fixed. Define $\gamma_* = \psi''_{0,a}(0+)/\psi'_{0,a}(0+) > 0$. Then,*

(a) for $\gamma = \gamma_*$, $ker(\mathcal{L}_{1,a}) = span\{\psi'_{0,a}\}$ and $n(\mathcal{L}_{1,a}) = 1$ on S_{γ_*},
(b) for $\gamma \neq \gamma_*$, $ker(\mathcal{L}_{1,a}) = \{0\}$ on S_γ,
(c) for $\gamma \in (\gamma_*, +\infty)$, $n(\mathcal{L}_{1,a}) = 1$, and for $\gamma \in (-\infty, \gamma_*)$, $n(\mathcal{L}_{1,a}) = 2$, on S_γ.

Proof. The proof of items (a)–(b) are similar to items (a)–(b) in lemma 3.7. The proof of item (c) will require an approach based in the analytic perturbation theory. We denote by $n_\gamma(\mathcal{L}_{1,a})$ the Morse index of $\mathcal{L}_{1,a}$ on S_γ.

(c) Consider the quadratic form $\mathcal{Q}_\gamma$ in (3.27). Then, for $\gamma \leqq 0$ we obtain $\mathcal{Q}_\gamma \leqq \mathcal{Q}_0$ and so $n_\gamma(\mathcal{L}_{1,a}) \geqq n_0(\mathcal{L}_{1,a}) = 2$ (see item (c) in lemma 3.6). But, by extension theory we have always $n_\gamma(\mathcal{L}_{1,a}) \leqq 2$ for any $\gamma \in \mathbb{R}$ (see appendix), then $n_\gamma(\mathcal{L}_{1,a}) = 2$ for $\gamma \leqq 0$.

Let $\gamma > \gamma_*$. Then, $\mathcal{Q}_\gamma \geqq \mathcal{Q}_{\gamma_*}$. Hence, $n_{\gamma_*}(\mathcal{L}_{1,a}) = 1$ implies $n_\gamma(\mathcal{L}_{1,a}) \leqq 1$. Next, let $\lambda_* < 0$ be the unique negative eigenvalue for $\mathcal{L}_{1,a}$ on S_{γ_*} (by former item (a)). Then, by following a similar argument as in the proof of item (c) in lemma 3.7, we get the following: there are $\theta \in (\lambda_*, 0)$ and $\Theta \in H^2(\mathbb{R})$, with Θ odd and with an unique zero on all $\mathbb{R}$ in $x = 0$, such that $\mathcal{L}_{1,a}\Theta = \theta\Theta$ on $(0,+\infty)$. Then, $\mathcal{Q}_\gamma(\Theta) = \theta\|\Theta\|^2 < 0$. Hence, by the min-max theorem we get $n_\gamma(\mathcal{L}_{1,a}) = 1$.

Let $\gamma \in (0, \gamma_*)$. By the analysis above, we have for $\gamma = 0$ that $ker(\mathcal{L}_{1,a}) = \{0\}$ and $n_0(\mathcal{L}_{1,a}) = 2$. Then, by analytic perturbation theory, we obtain $n_\gamma(\mathcal{L}_{1,a}) = 2$ for $\gamma \approx 0$. Define σ_* by

$$\sigma_* = sup\{\tilde{\gamma} > 0 : n_\gamma(\mathcal{L}_{1,a}) = 2 \text{ for all } \gamma \in (0, \tilde{\gamma})\}. \tag{3.29}$$

Then, by the former statement, there is $\epsilon_0 > 0$ such that for all $\gamma \in (0, \epsilon_0)$ we have $n_\gamma(\mathcal{L}_{1,a}) = 2$. Then σ_* is well defined and $\sigma_* \in [\epsilon_0, +\infty]$. But, since for all $\gamma > \gamma_*$, $n_\gamma(\mathcal{L}_{1,a}) = 1$, then $\sigma_* \leqq \gamma_*$.

Next, we show that $\sigma_* = \gamma_*$. Suppose $\sigma_* < \gamma_*$ and define $M = n_{\sigma_*}(\mathcal{L}_{1,a})$. Since $ker(\mathcal{L}_{1,a}) = \{0\}$ on S_{σ_*} we can consider Γ as being a closed curve (for example, a circle or a rectangle) such that $0 \in \Gamma \subset \rho(\mathcal{L}_{1,a})$, and all the negative eigenvalues of $\mathcal{L}_{1,a}$ on S_{σ_*} belong to the inner domain of Γ. Next, it is convenient to use the notation $\mathcal{L}_{1,a,\gamma} \equiv \mathcal{L}_{1,a}$ for drawing the attention that $D(\mathcal{L}_{1,a,\gamma}) = S_\gamma$. Thus, since as a function of γ, $(\mathcal{L}_{1,a,\gamma}, S_\gamma)_{\gamma\in\mathbb{R}}$ is a real-analytic family of self-adjoint operators of type (B) in the sense of Kato (see theorem VII-4.2 in [41] and lemma 3.11 in [15]), it follows that there is $\delta > 0$ small enough such that for $\gamma \in [\sigma_* - \delta, \sigma_* + \delta]$ we have $\Gamma \subset \rho(\mathcal{L}_{1,a})$ ($\mathcal{L}_{1,a}$ on S_γ) and for $\xi \in \Gamma$, $\gamma \to (\mathcal{L}_{1,a,\gamma} - \xi)^{-1}$ is analytic. In this point we note that from formula (2.8) in [16] with convenient constants and from the Krein's Formula for deficiency indices $(1,1)$ (see theorem A.2 and appendix D in [6]), we can obtain an expression that shows the analytic-dependence of the resolvent

$(\mathcal{L}_{1,a,\gamma}-\xi)^{-1}$ on γ (see also the proof of theorem A.7 in appendix). Therefore, a similar argument as in the proof of lemma 3.6 we obtain that $\sigma_* < \gamma_*$ is not possible and so $\sigma_* = \gamma_*$, which implies $n_\gamma(\mathcal{L}_{1,a}) = 2$ for all $\gamma \in (0,\gamma_*)$. This finishes the Lemma.

□

3.4. Morse index

In this subsection we show theorem 1.3 about the Morse index of $(\mathcal{L}_{+,Z}, D_Z)$. Thus, by using the notation at the beginning of this section, it is sufficient to study $\mathcal{L}_+ = \mathrm{diag}(\mathcal{L}_0,\mathcal{L}_1)$ on $D_{Z,0}$.

Proof (theorem 1.3). Theorem 3.1 shows item (1) in theorem 1.3. In the following, we then focus to prove item (2). In fact, we consider the family of soliton-profiles $a \in (-\infty,0) \to \psi_{0,a}$ and $a^* < 0$ the unique value such that $\psi''_{0,a^*}(0+) = 0$. We will divide our analysis in several steps.

(1) Let $a \in (a^*,0)$ and $\gamma^* = \psi''_{0,a}(0+)/\psi'_{0,a}(0+) < 0$ (we note that $\gamma^* = \gamma(a)$ in (1.15)).
   (a) Let $-Z \geqq \gamma^*$. Suppose $n(\mathcal{L}_+) \geqq 2$ and it considers λ_1 as being the second eigenvalue such that $0 > \lambda_1 > \lambda_0$, and with associated eigenfunction $(f,g) \in D_{Z,0}$. Thus, we have the relations

$$\begin{cases} \mathcal{L}_0 f(x) = \lambda_1 f(x), & x \in (-L,L), \\ \mathcal{L}_1 g(x) = \lambda_1 g(x), & x > 0, \\ f(L) = f(-L) = g(0), \\ f'(L) - f'(-L) = g'(0) + Z g(0). \end{cases} \tag{3.30}$$

   Next, we consider the following cases:
   (i) Suppose $g \equiv 0$: then, f satisfies Dirichlet boundary conditions and so λ_1 is a simple eigenvalue. Therefore, f is even or odd. Next, as $f \perp \phi_{\lambda_0}$ then f changes of sign. By using Floquet theory follows that f is odd (see step 1) in the proof of theorem 3.1). Now, $\mathcal{L}_0\Phi' = 0$, Φ' is odd and $\Phi'(x) < 0$ on $(-L,0)$. Then, since $f(-L) = f(0) = 0$ and $\lambda_1 < 0$ we obtain from the Sturm comparison theorem that there is $r \in (-L,0)$ such that $\Phi'(r) = 0$, which is false. Then, the component g is non-trivial.
   (ii) For $g \neq 0$, suppose $g(0) = 0$ (so $g'(0) \neq 0$ by item (i)): let η be the smallest eigenvalue for $\mathcal{L}_0$ with Dirichlet conditions, then $\eta \leqq \lambda_1$. Moreover, by the analysis in item (i), f is even (if f is odd then f' is even and so from (3.30) we obtain $g'(0) = 0$). Suppose $\eta < \lambda_1$, then f has at least one zero s with $s \in (-L,0)$. Therefore, from the Sturm comparison theorem, there is $w \in (-L,s)$ such that $\Phi'(w) = 0$, which is false. Then, $\eta = \lambda_1$ and so $f > 0$ on $(-L,L)$.

   In the following we show that $g > 0$ on $(0,+\infty)$. Suppose that there is $b \in (0,+\infty)$ such that $g(b) = 0$. Then, by considering the odd-extension $g_{\mathrm{odd}} \in H^2(\mathbb{R})$ of g, g_{odd} will be an eigenfunction associated to the eigenvalue λ_1 for the extension-operator $\widetilde{\mathcal{L}}$ in (3.9) ($\omega = -1$) on the δ-interaction domain $D_{\delta,\gamma}$ in (3.10) for any γ. Thus, since g_{odd} has at least three zeros on $\mathbb{R}$, it follows from the Sturm–Liouville oscillation theory for $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ (see lemma 5.3 in [15]) that λ_1 is at least the fourth negative eigenvalue for $\widetilde{\mathcal{L}}$ on $D_{\delta,\gamma}$, which is a contradiction because $n(\widetilde{\mathcal{L}}) \leqq 3$ (see proposition A.4 in appendix). Therefore, we have $f,g > 0$. Now, since $(f,g) \perp (\phi_{\lambda_0}, \zeta_{\lambda_0})$ we get a contradiction by lemma 3.1.

(iii) Let $g \neq 0$ with $g(0) > 0$ (without loss of generality): Then, by the splitting eigenvalue lemma (lemma 3.4) and by the Perron–Frobenius property for $\mathcal{L}_+$, we get that the pairs (g, λ_1) and $(\zeta_{\lambda_0}, \lambda_0)$ satisfy the eigenvalue problem,

$$\begin{cases} \mathcal{L}_1 h(x) = \tau h(x), & x > 0, \\ h'(0) = -Zh(0). \end{cases} \tag{3.31}$$

Therefore, the Morse index of $\mathcal{L}_1$ on $S_{-Z} = \{h \in H^2(0, +\infty) : h'(0) = -Zh(0)\}$, satisfies $n(\mathcal{L}_1) \geqq 2$. But, by the condition $-Z \geqq \gamma^*$ and lemma 3.7, we obtain a contradiction. Then, we need to have $n(\mathcal{L}_+) = 1$.

(b) Suppose the relation $-Z < \gamma^*$ and we will show $1 \leqq n(\mathcal{L}_+) \leqq 2$. Suppose $n(\mathcal{L}_+) = 3$ (we note that by using theorem A.6 and a similar analysis as in proposition A.4 in appendix, we can obtain the estimative $n(\mathcal{L}_+) \leqq 3$). Let λ_i be the negative eigenvalues for the problem in (3.1) such that $0 > \lambda_2 \geqq \lambda_1 > \lambda_0$. Then, if $\lambda_2 \neq \lambda_1$ we obtain by the analysis in the former item (a)–(ii)–(iii) that the eigenvalue problem in (3.31) will have at least 3 different negative eigenvalues, which is a contradiction because $n(\mathcal{L}_1) = 2$ on S_{-Z} (see lemma 3.7). Thus, we obtain that λ_1 is a double eigenvalue for $\mathcal{L}_+$. Then, there are eigenfunctions (ϕ_i, g_i), $i = 1, 2$, associated to the eigenvalue λ_1 such that $\phi_i(L) = \phi_i(-L) = g_i(0)$. By the analysis in the former item (a)–(ii), we can suppose that $g_i(0) > 0$ (we note that g_1 and g_2 can be linearly dependent and so since $n(\mathcal{L}_1) = 2$ on S_{-Z}, by lemma 3.7 *a priori* we do not get a contradiction). Then, by splitting eigenvalue lemma, every ϕ_i satisfies also periodic boundary conditions. Now, if some ϕ_i satisfies $\phi_i(0) = 0$ then from Floquet's oscillation theory there is a zero for ϕ_i in $(-L, 0)$ or $(0, L)$ (we recall that the number of zeros of any periodic eigenfunction for $\mathcal{L}_0$ is even in $[-L, L)$). Therefore, the Sturm comparison theorem implies that the profile Φ' will have one zero in $(-L, 0)$ or $(0, L)$, which is false. Thus, we need to have $\phi_i(0) \neq 0$ for $i = 1, 2$. Next, we consider the following periodic eigenfunction associated to λ_1

$$\Upsilon(x) = -\frac{\phi_2(0)}{\phi_1(0)} \phi_1(x) + \phi_2(x).$$

Then, $\Upsilon(0) = 0$ and so Υ needs to have a zero in $[-L, 0)$ or $(0, L)$. Hence, we get again a contradiction by the profile-structure of Φ'. Therefore, we need to have $1 \leqq n(\mathcal{L}_+) \leqq 2$. In the following we show items (i)–(ii) in the theorem's statements.

(i) $n(\mathcal{L}_+) = 1$ if and only if $n(\mathcal{L}_0) = 1$: suppose $n(\mathcal{L}_+) = 1$, then by the splitting eigenvalue lemma $n(\mathcal{L}_0) \geqq 1$ ($\mathcal{L}_0$ with periodic boundary conditions). Next, we assume $n(\mathcal{L}_0) \geqq 2$ and we take $r \in (\lambda_0, 0)$ being the second eigenvalue for $\mathcal{L}_0$ (it can be double) with associated eigenfunction $f_r \perp \phi_{\lambda_0}$. Then, by Floquet theory, f_r needs to have two-zeros in $[-L, L)$ and f_r can be chosen even or odd. If f_r is odd then by the Sturm comparison theorem Φ' has a zero in $(-L, 0)$, which is false. Suppose f_r even and so $f_r(-L) = f_r(L) \neq 0$.

Next, for $-Z < \gamma^*$ we know that $n(\mathcal{L}_1) = 2$ on S_{-Z} by lemma 3.7. Thus, let $\lambda_1 \in (\lambda_0, 0)$ be the second eigenvalue for $\mathcal{L}_1$ with associated eigenfunction g_{λ_1}, then by oscillation theory, g_{λ_1} has exactly one zero on $(0, +\infty)$ with $g_{\lambda_1}(0) \neq 0$ (see item (ii) in the proof of step a) above). Moreover, $g_{\lambda_1} \perp \zeta_{\lambda_0}$ and we can choose $g_{\lambda_1}(0) = f_r(L) = f_r(-L)$. Therefore $(f_r, g_{\lambda_1}) \in D(\mathcal{Q}_Z)$ and $(\phi_{\lambda_0}, \zeta_{\lambda_0}) \perp (f_r, g_{\lambda_1})$. Then, we obtain from (3.5) and $n(\mathcal{L}_+) = 1$ that

$$0 \leqq \mathcal{Q}_Z(f_r, g_{\lambda_1}) = r\|f_r\|^2 + \lambda_1 \|g_{\lambda_1}\|^2 < 0.$$

Hence, $n(\mathcal{L}_0)=1$.

In the following, suppose $n(\mathcal{L}_0)=1$ and $n(\mathcal{L}_+)=2$. Then, there is $\lambda_1\in(\lambda_0,0)$ (we recall that λ_0 is simple) and $(\phi_1,g_1)\in D_{Z,0}$ such that $\mathcal{L}_0\phi_1=\lambda_1\phi_1$, and by the analysis in the former paragraph $\phi_1(-L)=\phi_1(L)=g_1(0)\neq 0$. Then, by the splitting eigenvalue lemma $\phi_1'(-L)=\phi_1'(L)$. Therefore, we conclude $n(\mathcal{L}_0)\geqq 2$. Thus, we need to have necessary $n(\mathcal{L}_+)=1$.

(ii) $n(\mathcal{L}_0)=2$ if and only if $n(\mathcal{L}_+)=2$: the necessary condition follows from the former item (*i*). Now, suppose $n(\mathcal{L}_+)=2$. Then again by item (*i*) we get $n(\mathcal{L}_0)\geqq 2$ ($\mathcal{L}_0$ with periodic boundary conditions).

In the following, we will see that $n(\mathcal{L}_0)\leqq 2$ in general. So, without loss of generality, suppose $n(\mathcal{L}_0)=3$ and consider the eigenvalue distribution $\lambda_0<\lambda_1\leqq\lambda_2<0$. For the case, $\lambda_1<\lambda_2$ with associated eigenfunctions f_1,f_2, respectively, we have that f_i is even or odd (because $\lambda_i's$ are simple). Next, if some f_i is odd we obtain by comparison's theory that Φ' has one zero in $(-L,0)$, which is false. Then, both f_i are even and by oscillation theory each f_i has exactly 2 zeros in $(-L,L)$, so $f_i(L)\neq 0$. Suppose, $f_i(L)>0$ and b such that $f_1(\pm b)=0$, $-L<-b<0<b<L$. Then, since $\lambda_1<\lambda_2$, it follows from oscillation theory that there are points $\pm a$ with $-b<-a<0<a<b$ and $f_2(\pm a)=0$. Thus, from $f_i(x)>0$ for $x\in(b,L)$, $f_i'(L)=0, f_1'(b)>0$ and $f_2(b)>0$, we obtain the following contradiction

$$
\begin{aligned}
0 &= \int_b^L (f_1f_2''-f_1''f_2)\,\mathrm{d}x+(\lambda_2-\lambda_1)\int_b^L f_1f_2\mathrm{d}x\\
&=(f_2'f_1-f_1'f_2)\,|_b^L+(\lambda_2-\lambda_1)\int_b^L f_1f_2\mathrm{d}x\\
&=f_1'(b)f_2(b)+(\lambda_2-\lambda_1)\int_b^L f_1f_2\mathrm{d}x>0.
\end{aligned}
$$

The case $\lambda_1=\lambda_2$ also leads a contradiction by oscillation and comparison theories. Therefore, $n(\mathcal{L}_0)\leqq 2$.

(2) For $a<a^*$ and $\gamma_*\equiv\psi_{0,a}''(0+)/\psi_{0,a}'(0+)>0$, we have that a similar analysis as the one in item (1), now using lemma 3.8 and the splitting eigenvalue lemma, it shows the statement.

(3) For $a=a^*$ and $\gamma_0\equiv\psi_{0,a^*}''(0+)/\psi_{0,a^*}'(0+)=0$, we have that a similar analysis as the one in item (1), now using lemma 3.6-item (*a*) and the splitting eigenvalue lemma, it shows the statement.

□

Remark 3.9. Some comments about theorem 1.3:

(a) The smallest eigenvalue, λ_0, for $\mathcal{L}_{+,Z}\equiv(\mathcal{L}_0,\mathcal{L}_{1,\omega})$ on D_Z in (1.13), coincides with the smallest eigenvalue for both $\mathcal{L}_0$ and $\mathcal{L}_{1,\omega}$, with periodic boundary conditions and δ-boundary conditions at zero, respectively (which follows from the proof of theorem 3.1, the splitting eigenvalue lemma, and from oscillation theory).

(b) The arguments in item (*b*)–(*ii*) of the proof of theorem 1.3, show that in general we have $1\leqq n(\mathcal{L}_0)\leqq 2$.

3.5. Nullity index

In this subsection we show theorem 1.4 about the kernel of $(\mathcal{L}_{+,Z}, D_Z)$. Thus, by using the notation at the beginning of this section, it is sufficient to study $\mathcal{L}_+ = diag(\mathcal{L}_0, \mathcal{L}_1)$ on $D_{Z,0}$.

Proof (theorem 1.4). Suppose $(f,h) \in D_{Z,0}$ such that $\mathcal{L}_+(f,h)^t = (0,0)^t$, then we obtain the eigenvalue problem

$$\begin{cases} \mathcal{L}_0 f(x) = 0, & x \in (-L, L), \\ \mathcal{L}_1 h(x) = 0, & x \in (0, +\infty), \\ f(-L) = f(L) = h(0) \\ f'(L) - f'(-L) = h'(0) + Zh(0). \end{cases} \tag{3.32}$$

Now, since $\mathcal{L}_1 \psi'_{0,a} = 0$ it follows immediately $h = c\psi'_{0,a}$ on $(0, +\infty)$ (see theorem 3.3 in [20]).

(a) Suppose $c = 0$: Then $h \equiv 0$ and f satisfies Dirichlet-periodic boundary conditions $f(L) = f(-L) = 0$ and $f'(L) = f'(-L)$. We consider $f \neq 0$, then from oscillation theory and remark 3.9 we have that zero is not the first eigenvalue for $\mathcal{L}_0$ and so f needs to change of sign. Now, from Sturm–Liouville theory for Dirichlet conditions we have that f is even or odd (because 0 will be a simple eigenvalue). Next, since $\mathcal{L}_0 \Phi' = 0$, Φ' is odd and $\Phi'(L) \neq 0$, we get that f is even (indeed, from classical ODE's theory $f = a\Phi' + bP$, where P is the even-solution of $\mathcal{L}_0 P = 0$ with $P(0) = 1$, $P'(0) = 0$, and so we need to have $a = 0$). Now, from Floquet's oscillation theory, f has an *even* number of zeros on $[-L, L)$. But, as $f(-L) = 0$ we deduce that f has an odd number of zeros on $(-L, L)$ which implies $f(0) = 0$. Thus, since $f'(0) = 0$ we get a contradiction. Therefore $f \equiv 0$.

(b) Suppose $c \neq 0$, then $h(0) \neq 0$ ($h(0) > 0$ without loss of generality for $c > 0$). Then, by the splitting eigenvalue lemma we get the following relations:

$$f(-L) = f(L) = h(0) > 0, \; f'(L) = f'(-L), \; h'(0) = -Zh(0). \tag{3.33}$$

Then, in particular we have the relation

$$\psi''_{0,a}(0) = -Z\psi'_{0,a}(0). \tag{3.34}$$

Now, for $a^* < 0$ being the unique value of a such that $\psi''_{0,a}(0+) = 0$, we consider the following cases:

(i) Suppose $a \neq a^*$ and $-Z \neq \psi''_{0,a}(0)/\psi'_{0,a}(0)$: then obviously we can not have (3.34) and so $c = 0$.

(ii) Suppose $a = a^*$: for $Z \neq 0$ we obtain obviously that (3.34) can not happen and so $c = 0$. For $Z = 0$, we have $ker(\mathcal{L}_1) = span\{\psi'_{0,a^*}\}$ on S_0. In the following we show that the second eigenvalue for $\mathcal{L}_0$, with periodic-boundary conditions, is exactly zero (simple) and with f being even. Indeed, from oscillation theory we have the following distribution of eigenvalues for $\mathcal{L}_0$ (see theorem 4.8.1 in [55])

$$\eta_0 < \mu_0 < \eta_1 \leqq \mu_1 \leqq \eta_2 < \mu_2 < \eta_3 < \cdots$$

where μ_i and η_i are the eigenvalues for $\mathcal{L}_0$ with Dirichlet and periodic conditions, respectively. We will see that $\eta_1 = 0$ and simple. Suppose $0 > \mu_1$. Then, since the eigenfunction associated to μ_1 is odd, it follows by Sturm comparison theorem that Φ' has one zero on $(-L, 0)$, which is impossible. So, $0 \leqq \mu_1$. Next, by supposing $\mu_1 = 0$, we have the existence of a odd-eigenfunction χ_1 with $\mathcal{L}_0 \chi_1 = 0$ and $\chi_1(-L) =$

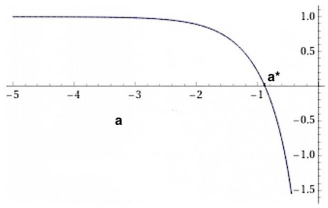

Figure 4. Profile of γ in (3.36) with $\omega = -1$ and $p = 1$.

$\chi_1(L) = 0$. Thus, the Wronskiano $W(\chi_1, \Phi')$ of χ_1 and Φ' is constant and satisfies for every $x \in [-L, L]$

$$A \equiv W(\chi_1, \Phi')(x) = \chi_1(x)\Phi''(x) - \chi_1'(x)\Phi'(x).$$

Then, for $x = 0$ we obtain that $A = 0$ and so there is a β such that $\chi_1 = \beta\Phi'$. Therefore, as $\Phi'(L) \neq 0$ we obtain $\beta = 0$, and so we get a contradiction. Then, $0 < \mu_1$ and so $\eta_1 = 0$ is simple with eigenfunction f being obviously even ($f(L) \neq 0$). Moreover, $f'(L) = 0$ and $f(0) < 0$ (because f will have exactly two zeros in $(-L, L)$). Thus, we get that

$$B \equiv W(f, \Phi')(x) < 0, \quad \text{for } x \in [-L, L]. \tag{3.35}$$

But, $W(f, \Phi')(L) = f(L)\Phi''(L) = f(L)\psi''_{0,a^*}(0) = 0$. Therefore, we need to have again that $c = 0$.

(iii) Suppose $a < a^*$ and $-Z = \psi''_{0,a}(0)/\psi'_{0,a}(0)$ (hence $Z < 0$, see figure 4 below): similarly as in the former case ii), we have $ker(\mathcal{L}_1) = span\{\psi'_{0,a}\}$ on S_{-Z} and zero will be exactly the second eigenvalue for $\mathcal{L}_0$ (with periodic-boundary conditions), simple, and with f being even. Recall, we also have the relation in (3.35). Thus,

$$B = W(f, \Phi')(L) = f(L)\Phi''(L) = h(0)\psi''_{0,a}(0) = -cZ\left[\psi'_{0,a}(0)\right]^2 > 0.$$

Therefore $c = 0$. This finishes the proof.

□

Remark 3.10. In the next, we have some comments about the proof of theorem 1.4): we consider the profile of the following mapping γ, for $\omega < 0$ and with p fixed values (see (1.15)), namely,

$$\gamma(a) = \frac{\psi''_{0,a}(0)}{\psi'_{0,a}(0)} = \sqrt{-\omega}\,\frac{(p+1)\operatorname{sech}^2(pa) - 1}{\tanh(pa)}, \qquad a < 0. \tag{3.36}$$

Then, we have that for any $Z > 0$ (fixed), the equation $-Z = \gamma(a)$ has a solution (and unique) if and only if $a \in (a^*, 0)$ with $\gamma(a^*) = 0$ (see figure 4). Thus, if for the shift-parameter $a(\omega, Z)$ we have $-Z = \gamma(a(\omega, Z)))$, then the strategy for showing item (3) of theorem 1.4 does not work for obtaining some information about $ker(\mathcal{L}_{+,Z})$ in the case of $a > a^*$ and $Z = -\gamma((a(\omega, Z)) > 0$.

3.6. Non-negative spectrum for $\mathcal{L}_{-,Z}$

In the following, we show theorem 1.6. In fact, we consider the operator $\mathcal{L}_{-,Z} = \mathrm{diag}(\mathcal{L}_{0,-}, \mathcal{L}_{1,-})$ on D_Z in (1.3), where

$$\mathcal{L}_{0,-} = -\partial_x^2 - \omega - (p+1)\,\Phi_\omega^{2p}, \quad \mathcal{L}_{1,-} = -\partial_x^2 - \omega - (p+1)\,\psi_a^{2p},$$

with ψ_a defined in (1.14).

Proof. It is clear that $(\Phi_\omega, \psi_a) \in ker(\mathcal{L}_{-,Z})$ by (1.5). Next, we note that for any $\mathbf{V} = (f,g) \in D_Z$

$$\begin{aligned} \mathcal{L}_{0,-}(f) &= -\frac{1}{\Phi_\omega}\frac{\mathrm{d}}{\mathrm{d}x}\left[\Phi_\omega^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{\Phi_\omega}\right)\right], \quad x \in (-L, L) \\ \mathcal{L}_{1,-}(g) &= -\frac{1}{\psi_a}\frac{\mathrm{d}}{\mathrm{d}x}\left[\psi_a^2 \frac{d}{dx}\left(\frac{g}{\psi_a}\right)\right], \quad x > L. \end{aligned} \tag{3.37}$$

Thus, we obtain immediately that

$$\langle \mathcal{L}_{-,Z}\mathbf{V}, \mathbf{V}\rangle = \int_{-L}^{L} \Phi_\omega^2 \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{\Phi_\omega}\right)\right)^2 \mathrm{d}x + \int_{L}^{+\infty} \psi_a^2 \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g}{\psi_a}\right)\right)^2 \mathrm{d}x \geqq 0.$$

Therefore, since $\langle \mathcal{L}_{-,Z}\mathbf{V}, \mathbf{V}\rangle = 0$ if and only if $f = c\Phi_\omega$ and $g = d\psi_a$ (with $c = d$ because $f(L) = g(L)$), we obtain $ker(\mathcal{L}_{-,Z}) = span\{(\Phi_\omega, \psi_a)\}$. This finishes the proof. □

4. Applications and proof of theorem 1.7

In this section, we prove initially the existence of a C^1-mapping of positive two-lobe state profiles $\omega \in I \to (\Phi_\omega, \Psi_\omega) \in D_0$ for the cubic-NLS in (1.5), with $p = 1$. The existence of these profiles will be based on the dynamical system theory for orbits on the plane via the period function for second-order differential equations (see [33, 39, 40, 50]). We note that in [50], this period function strategy was already applied for the case $p = 2$ for obtaining the existence of positive single-lobe state profiles of (1.5) with $Z = 0$. We believe that the periodic function can also be applied for obtaining positive two-lobe states with other values of p and it will be pursued in a future work. In our analysis, we prove that $I = (-\infty, \omega_0)$, with $\omega_0 < 0$, and that positive two-lobe states do not exist for $\omega \in (\omega_0, 0)$ (see lemma 4.1 and remark 4.5). After this, we also prove the monotonicity of the mass-map $\omega \in I \to Q(\Phi_\omega, \Psi_\omega)$ and so, by using theorems 1.3, 1.4 and 1.6, and the orbital criterion in theorem A.8 (appendix), we show theorem 1.7.

We begin by considering $L = \pi$, $\psi(x) = \Psi(x+\pi)$ for $x > 0$. Then, the scaling transformation for $\omega \equiv -\epsilon^2$ with $\epsilon > 0$; namely,

$$\psi(x) = \epsilon u_0(\epsilon x), \quad \Phi(x) = \epsilon \varphi(\epsilon x), \tag{4.1}$$

implies that system (1.5) is transformed in the following system of differential equations:

$$\begin{cases} -\varphi''(x) + \varphi(x) - 2\varphi^3(x) = 0, & x \in (-\epsilon\pi, \epsilon\pi), \\ -u_0''(x) + u_0(x) - 2u_0^3(x) = 0, & x \in (0, +\infty), \\ \varphi(-\epsilon\pi) = \varphi(\epsilon\pi) = u_0(0), \\ \varphi'(\epsilon\pi) - \varphi'(-\epsilon\pi) = u_0'(0). \end{cases} \tag{4.2}$$

The only positive decaying solution to the second equation in (4.2) on the half-line, is given by the shifted classical NLS soliton ψ_0 in (1.8):

$$u_0(x) = \psi_0(x+a) = sech(x+a), \quad x > 0, \tag{4.3}$$

where $a \in \mathbb{R}$ will be considered $a < 0$ for obtaining a non-monotone u_0 on $[0,\infty)$ (a bump-profile).

It is immediate that each second-order differential equation in system (4.2) is integrable with the first-order invariant:

$$E(u,v) = v^2 - A(u), \quad v = \frac{\mathrm{d}u}{\mathrm{d}x}, \quad A(u) = u^2 - u^4 \tag{4.4}$$

where the value $E(u,v) = E$ is independent of x. For $E = 0$ we obtain the well-know two homoclinic orbits on the phase plane (u,u'), one corresponds to positive u (which is represented by the curve $x \in \mathbb{R} \to (\psi_0(x), \psi_0'(x))$, the green colour in figure 5), and the other one corresponds to negative $-u$. Now, since $A(u)$ for $u > 0$, has only one critical point $p_* \in (0,1)$ ($p_* = \frac{1}{\sqrt{2}}$), we have that periodic orbits exist inside each of the two homoclinic loops and correspond to $E \in (E_*, 0)$, where $E_* = -A(p_*) = -\frac{1}{4}$. Here we are interested in strictly positive profile u (the blue colour in figure 5). Note that

$$E + A(p_*) > 0, \quad E \in (E_*, 0). \tag{4.5}$$

In the following we define $p_0 \equiv u_0(0) = sech(a) \in (0,1)$, and so $u_0'(0) = \sqrt{A(p_0)}$ by (4.4), $E = 0$ and $a < 0$. In our analysis, p_0 will be a free parameter such that $p_0(a) \to 1$ when $a \to 0^-$ and $p_0(a) \to 0$ when $a \to -\infty$.

Hence, our first positive single-lobe state will be found from the following over-determined boundary-value problem:

$$\begin{cases} -\varphi''(x) + \varphi(x) - 2\varphi^3(x) = 0, \quad x \in (-\epsilon\pi, \epsilon\pi), \\ \varphi(-\epsilon\pi) = \varphi(\epsilon\pi) = p_0, \\ \varphi'(-\epsilon\pi) = -\varphi'(\epsilon\pi) = -\frac{1}{2}\sqrt{A(p_0)} \equiv q_0, \end{cases} \tag{4.6}$$

where $p_0 \in (0,1)$ is a free parameter of the problem. Figure 5 shows a geometric construction of solutions to (4.6) on the plane (u,v). The red solid line plots $q_0(p_0) = -\frac{1}{2}\sqrt{A(p_0)}$, $p_0 \in (0,1)$. The dashed-dotted vertical line depicts the value of $p_0 = u_0(0) = sech(a)$ and so the green dashed line represents the homoclinic orbit at $E = 0$ with the solid part depicting the shifted NLS soliton (4.3). Therefore, the level curve $E(u,v) = E(p_0,q_0)$, at $p_0 = sech(a)$ and $q_0 = -\frac{1}{2}\sqrt{A(p_0)}$, is shown by the blue dashed line, whereas the solid part depicts a suitable solution to the boundary-value problem (4.6). Thus, our positive-even solution φ for (4.6) induces the planar curve $x \to \gamma(x) = (\varphi(x), \varphi'(x))$ which corresponds to *a part of the level curve* $E(u,v) = E(p_0,q_0) \equiv E_0(p_0) = -\frac{3}{4}A(p_0) \in (E_*, 0)$, which intersects the line $u \equiv p_0$ only twice at the ends of the interval $[-\pi\epsilon, \pi\epsilon]$, with $\gamma(-\epsilon\pi) = (p_0, q_0)$ and $\gamma(0) = (\varphi(0), 0)$. Therefore, the green and blue solid parts in figure 5 depict a suitable positive two-lobe state profile solution to the boundary-value problem (4.2).

Our formal analysis of the existence of φ in (4.6) will be based in the following period function (see [33, 39, 50]) defined for (p_0, q_0) as

$$T_-(p_0,q_0) = \int_{p_-}^{p_0} \frac{\mathrm{d}u}{\sqrt{E + A(u)}}, \tag{4.7}$$

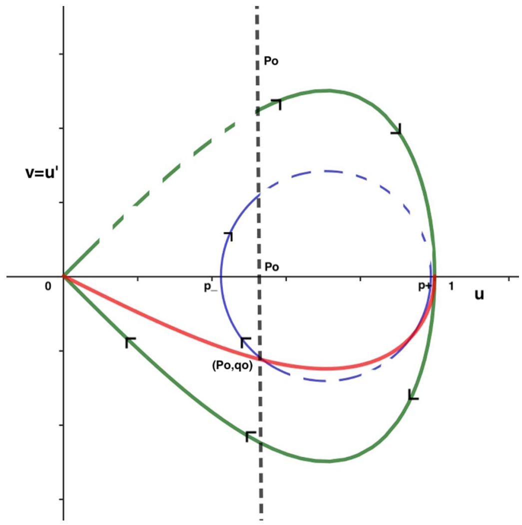

Figure 5. Representation of the solution to the boundary-value problem (4.6) on the phase plane.

where the fixed value $-E \in (0, A(p_*))$ and the point p_-, are defined from (p_0, q_0) by

$$E(p_0) = q_0^2 - A(p_0) = -A(p_-) \tag{4.8}$$

with $0 < p_- < p_* < 1$ and $p_- = p_-(p_0)$ (see figure 5). Note that since φ is even, $\varphi'(x) < 0$ for all $x \in (-\epsilon\pi, 0)$ and $[\varphi'(0)]^2 = E + A(\varphi(0))$, we get $A(\varphi(0)) = -E = A(p_-)$. Then, since $0 < \varphi(0) < p_0$ we need to have $\varphi(0) = p_-$ (the minimum value for φ on $[-\epsilon\pi, \epsilon\pi]$). Thus, the planar solution curve $x \in (-\epsilon\pi, 0] \to \gamma(x) = (\varphi(x), \varphi'(x))$ starts at (p_0, q_0) and ends at $(p_-, 0)$.

From the classical theory of dynamical system at the plane, φ is a positive single-lobe profile (sometimes it is called a depression profile) of problem (4.6) if, and only if, $p_0 \in (0,1)$ is a solution of the equation

$$\mathcal{T}(p_0) = \pi\epsilon, \text{ with } \mathcal{T}(p_0) \equiv T_-\left(p_0, -\frac{1}{2}\sqrt{A(p_0)}\right). \tag{4.9}$$

The following result shows that problem in (4.9) has a solution for a restricted range of values of ϵ and so for ω.

Lemma 4.1. *The mapping $p_0 \in (0,1) \to \mathcal{T}(p_0)$ is C^1 and monotonically increasing. Moreover, $\mathcal{T}(p_0) \to \alpha_0 > 0$ as $p_0 \to 0$ and $\mathcal{T}(p_0) \to +\infty$ as $p_0 \to 1$. Therefore, problem in (4.6) has a unique solution for $\pi\epsilon > \alpha_0$. Thus, by the scaling transformation (4.1) and (4.3), we obtain at least a C^1-mapping $\omega \in (-\infty, \omega_0) \to \Theta(\omega) = (\Phi_\omega, \Psi_\omega) \in D_0$, $\omega_0 < 0$, of two-lobe state for the cubic-NLS on a tadpole graph. Also, we get that the shift-mapping $\omega \to a(\omega) \in (-\infty, 0)$ (a diffeomorphism-mapping) is smooth with sech$(a(\omega)) = p_0$ such that $\mathcal{T}(p_0) = \pi\sqrt{-\omega}$.*

Remark 4.2. In remark 4.5, we give a formula for the profile Φ_ω in terms of the Jacobi elliptic function of dnoidal-type. Therefore, we obtain that ω_0 in lemma 4.1 is giving by $\omega_0 = -\alpha_0^2/\pi^2$ with α_0 being the positive root of $sech^2(\alpha_0) = 3/4$.

Proof. To prove the lemma we use the technique developed in [39] for a flower graph in the cubic-NLS case (see also [33]). We start by considering the following C^1-function

$$W(u,v) = \frac{2(A(u) - A(p_*))v}{A'(u)},$$

defined inside the open region limited by the homoclinic orbit (see figure 5), so $u \in (0,1)$. Moreover, the differential of W is given by

$$\mathrm{d}W = \left[2 - \frac{2(A(u) - A(p_*))A''(u)}{[A'(u)]^2}\right] v\mathrm{d}u + \frac{2(A(u) - A(p_*))}{A'(u)} dv. \tag{4.10}$$

Now, we consider the level curve $E(u,v) = v^2 - A(u) = E$, with $E \in (E_*, 0)$ ($E_* = -\frac{1}{4}$), inside the homoclinic orbit and where $v = -\sqrt{E(u,v) + A(u)}$. Then, in this level curve we have $2vdv = A'(u)du$ and so

$$\frac{A(u) - A(p_*)}{v} du = -\left[2 - \frac{2(A(u) - A(p_*))A''(u)}{[A'(u)]^2}\right] v\mathrm{d}u + \mathrm{d}W. \tag{4.11}$$

Next, if we consider the C^1-curve γ connecting (p_0, q_0) and $(p_-, 0)$ on the level curve $E(u,v) = E(p_0, q_0) = E_0(p_0)$, $E_0(p_0) \in (E_*, 0)$, then the line integral of dW along of γ is given by

$$\int_\gamma \mathrm{d}W = -\int_{-\gamma} \mathrm{d}W = -\int_{(p_-,0)}^{(p_0,q_0)} \mathrm{d}W = -[W(p_0,q_0) - W(p_-,0)] = -W(p_0,q_0). \tag{4.12}$$

We recall that the value of $\mathcal{T}(p_0)$ in (4.9) is obtained from the level curve $E(u,v) = E(p_0, q_0) = E_0(p_0)$, $q_0 = -\frac{1}{2}\sqrt{A(p_0)}$, thus from (4.7), (4.11) and (4.12), for $p_0 \in (0,1)$ we obtain the relation

$$\begin{aligned}
[E_0(p_0) + A(p_*)]\mathcal{T}(p_0) &= -\int_{p_-}^{p_0} \frac{E(u,v) + A(p_*)\,\mathrm{d}u}{v} = -\int_{p_-}^{p_0} \left[v - \frac{A(u) - A(p_*)}{v}\right] \mathrm{d}u \\
&= -\int_{p_-}^{p_0} \left[3 - \frac{2(A(u) - A(p_*))A''(u)}{[A'(u)]^2}\right] v\mathrm{d}u \\
&\quad + \frac{2(A(p_0) - A(p_*))}{A'(p_0)} q_0.
\end{aligned} \tag{4.13}$$

Therefore, it follows immediately that the mapping $p_0 \in (0,1) \to \mathcal{T}(p_0) \in (0, +\infty)$ is C^1.

In the following we prove that $\mathcal{T}'(p_0) > 0$ for every $p_0 \in (0,1)$. Indeed, by differentiating the expression in (4.13), using $E_0'(p_0) = -\frac{3}{4}A'(p_0)$, $\frac{\partial v}{\partial p_0} = \frac{1}{2v}E_0'(p_0)$ and $\frac{\mathrm{d}q_0}{\mathrm{d}p_0} = -\frac{1}{4}\frac{A'(p_0)}{\sqrt{A(p_0)}}$, it follows

$$\begin{aligned}
[E_0(p_0) + A(p_*)]\mathcal{T}'(p_0) &= \frac{A(p_*)}{2\sqrt{A(p_0)}} + \frac{3}{8}A'(p_0)\int_{p_-}^{p_0} \left[1 - \frac{2(A(u) - A(p_*))A''(u)}{[A'(u)]^2}\right] \frac{1}{v}\mathrm{d}u \\
&= \frac{A(p_*)}{2\sqrt{A(p_0)}} + \frac{3}{32}A'(p_0)\int_{p_-}^{p_0} \frac{1 - 2u^2}{u^2 v}\mathrm{d}u.
\end{aligned} \tag{4.14}$$

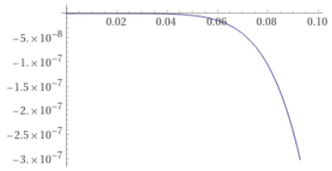

Figure 6. Graph of $F(p_0)$ for $p_0 \in (0, 0.1)$.

Next, by using integration by parts we obtain

$$\int_{p_-}^{p_0} \frac{1-2u^2}{u^2 v}\mathrm{d}u = -\int_{p_-}^{p_0} \frac{A'(u)}{2u^3\sqrt{E_0(p_0)+A(u)}}\mathrm{d}u = -\int_{p_-}^{p_0} \frac{1}{u^3}\frac{\mathrm{d}}{\mathrm{d}u}\sqrt{E_0(p_0)+A(u)}\mathrm{d}u$$
$$= \frac{q_0}{p_0^3} + 3\int_{p_-}^{p_0} \frac{v}{u^4}\mathrm{d}u. \tag{4.15}$$

Substituting this into (4.14) yields

$$[E_0(p_0)+A(p_*)]\mathcal{T}'(p_0) = \frac{A(p_*)}{2\sqrt{A(p_0)}} + \frac{3}{32}A'(p_0)\frac{q_0}{p_0^3} + \frac{9}{32}A'(p_0)\int_{p_-}^{p_0}\frac{v}{u^4}\mathrm{d}u. \tag{4.16}$$

Since $A'(p_0) \leqq 0$ for $p_0 \in [p_*, 1)$, and $q_0, v < 0$, we get immediately from (4.16) that $\mathcal{T}'(p_0) > 0$ for every $p_0 \in [p_*, 1)$.

The case $p_0 \in (0, p_*)$ is more delicate. Initially, we have the following estimative by using that $E_0(p_0) = -A(p_-)$,

$$\int_{p_-}^{p_0} \frac{\sqrt{E_0(p_0)+A(u)}}{u^4}\mathrm{d}u = \int_{p_-}^{p_0}\frac{1}{u^4}\sqrt{u^2-p_-^2}\sqrt{1-(u^2+p_-^2)}\mathrm{d}u \leqq \frac{1}{3p_0^3p_-^2}\left(p_0^2-p_-^2\right)^{3/2}. \tag{4.17}$$

Thus, from (4.16), (4.17) and from the facts that $q_0 = -\frac{1}{2}\sqrt{A(p_0)}$, $A'(p_0) > 0$, and $A(p_*) = \frac{1}{4}$, we have that $\mathcal{T}'(p_0) > 0$ for every $p_0 \in (0, p_*)$ as long as we have

$$F(p_0) \equiv p_-^2 A'(p_0)A(p_0) + 2\sqrt{A(p_0)}A'(p_0)\left(p_0^2-p_-^2\right)^{3/2} - \frac{8}{3}p_0^3p_-^2 < 0$$

for $p_0 \in (0, p_*)$. Indeed, first from relation $A(p_-) = \frac{3}{4}A(p_0)$ we get that

$$p_-^2 = \frac{1-\sqrt{1-3A(p_0)}}{2}.$$

Then, it follows immediately that $\lim_{p_0 \to 0^+} F(p_0) = 0$. Moreover, $F'(p_0) < 0$ for $p_0 \in (0, p_*)$ (see figure 6). Therefore, $F(p_0) < 0$ for $p_0 \in (0, p_*)$.

Next, from relation $E_0(p_0) + A(u) = A(u) - A(p_-) \leqq A(u)$, we have

$$\mathcal{T}(p_0) \geqq \int_{p_-}^{p_0}\frac{\mathrm{d}u}{u\sqrt{1-u^2}},$$

and so, since $p_-(p_0)\to 0$ as $p_0\to 1$ and $\int_0^1 \frac{du}{u\sqrt{1-u^2}}du=+\infty$, we get $\mathcal{T}(p_0)\to+\infty$ as $p_0\to 1$. It remains to prove that $\mathcal{T}(p_0)\to\alpha_0$ as $p_0\to 0$, for $\alpha_0>0$. In fact, for $p_0>0$ small enough, we obtain from the mean value theorem and from $A'(u)$ being a strictly increasing positive function on $(0,p_0)$, that

$$\mathcal{T}(p_0)\geqq\int_{p_-}^{p_0}\frac{1}{\sqrt{uA'(u)}}du\geqq\frac{1}{\sqrt{2}\sqrt{1-2p_-^2}}\int_{p_-}^{p_0}\frac{1}{u}du\geqq\frac{1}{\sqrt{2}\sqrt{1-2p_-^2}}\left(1-\frac{p_-}{p_0}\right).$$

Then, since $\lim_{p_0\to 0}\frac{p_-}{p_0}=\frac{\sqrt{3}}{2}$, we arrive that

$$\alpha_0\equiv\lim_{p_0\to 0}\mathcal{T}(p_0)\geqq\frac{(2-\sqrt{3})\sqrt{2}}{4}.$$

Thus, for $\pi\epsilon>\alpha_0$ there exists exactly one root p_0 of (4.9) and such that $p_0=sech(a(\epsilon))\in(0,1)$ with $a(\epsilon)\in(-\infty,0)$. Now, by using that $\omega=-\epsilon^2<-\alpha_0^2/\pi^2$, the scaling transformation (4.1), and the solitons formulas (1.7) and (4.3), we obtain a unique two-lobe state solution $(\Phi_\omega,\Psi_\omega)\in D_0$ satisfying the stationary NLS equation (1.5) under the conditions $p=1$, $L=\pi$ and $Z=0$. Moreover, by the analysis above, we get the C^1-mapping $\omega\to(-\infty,-\frac{\alpha_0^2}{\pi^2})\to p_0\in(0,1)$, with $\mathcal{T}(p_0)=\pi\sqrt{-\omega}$, and so a C^1-diffeomorphism shift mapping $\omega\to(-\infty,-\frac{\alpha_0^2}{\pi^2})\to a(\omega)\in(-\infty,0)$ with $a(\omega)$ such that $sech(a(\omega))=p_0$. Lastly, we obtain a C^1-mapping $\omega\in(-\infty,-\frac{\alpha_0^2}{\pi^2})\to\Theta(\omega)=(\Phi_\omega,\Psi_\omega)\in D_0$ of positive two-lobe states for the cubic-NLS on a tadpole graph. The proof of the lemma is complete. □

Lemma 4.3. *Consider the C^1-mapping $\omega\in(-\infty,-\frac{\alpha_0^2}{\pi^2})\to\Theta(\omega)=(\Phi_\omega,\Psi_\omega)\in D_0$ of two-lobe state constructed in lemma 4.1 for the cubic-NLS on a tadpole graph. Then, the mass $\Omega(\omega)\equiv Q(\Phi_\omega,\Psi_\omega)$ satisfies*

$$\frac{d}{d\omega}\Omega(\omega)<0.$$

Proof. Let's consider the two-lobe state $(\Phi_\omega,\Psi_\omega)$ given by lemma (4.1). Then, via the change of variable $\psi(x)\equiv\Psi(x+\pi)$, $x>0$, the scaling transformation (4.1), the soliton formula (4.3), the fact that

$$-\varphi^2dy=\frac{\varphi^2}{\sqrt{E_0+A(\varphi)}}d\varphi,$$

with $y\in[-\epsilon\pi,0]$, and from the even property of φ, the mass Ω can be rewritten as

$$\Omega(\omega)=2\epsilon\int_{-\epsilon\pi}^{0}\varphi^2(y)dy+\epsilon[1-\tanh(a)]=2\epsilon\int_{p_-}^{p_0}\frac{u^2}{\sqrt{E_0+A(u)}}du+\epsilon\left[1+\sqrt{1-p_0^2}\right],\tag{4.18}$$

where $\epsilon^2=-\omega$ and $p_0=p_0(\omega)\in(0,1)$, such that $\mathcal{T}(p_0)=\pi\epsilon$, where $E_0\equiv E_0(p_0)=q_0^2-A(p_0)=-\frac{3}{4}A(p_0)$ is the energy level associated to $E(u,v)=v^2-A(u)$, with $A(u)=u^2-u^4$.

Thus, formula in (4.18) can be reformulated in the following p_0-dependence formula,

$$\mathcal{N}(p_0)\equiv\pi\Omega(\omega)=\mathcal{T}(p_0)\left[2\int_{p_-}^{p_0}\frac{u^2}{\sqrt{E_0+A(u)}}du+\left[1+\sqrt{1-p_0^2}\right]\right].\tag{4.19}$$

Now, we prove that the mapping $p_0 \in (0,1) \to \mathcal{H}(p_0) \equiv \int_{p_-}^{p_0} \frac{u^2}{\sqrt{E_0 + A(u)}} \mathrm{d}u \in (0,+\infty)$ is C^1. Indeed, from the differential formula

$$d\left[\frac{2\left(A\left(u\right)-A\left(p_*\right)\right)u^2 v}{A'\left(u\right)}\right] = 2\left[1+\frac{2\left(1+2u^2\right)\left(A\left(u\right)-A\left(p_*\right)\right)}{\left[A'\left(u\right)\right]^2}\right]u^2 v\mathrm{d}u + \frac{2\left(A\left(u\right)-A\left(p_*\right)\right)}{A'\left(u\right)}u^2\mathrm{d}v, \tag{4.20}$$

and for (u,v) such that $E(u,v) = E_0(p_0)$ (and so $2vdv = A'(u)du$, with $v<0$), we get

$$\frac{A\left(u\right)-A\left(p_*\right)}{v}u^2\mathrm{d}u = d\left[\frac{2\left(A\left(u\right)-A\left(p_*\right)\right)u^2 v}{A'\left(u\right)}\right] - 2\left[1+\frac{2\left(1+2u^2\right)\left(A\left(u\right)-A\left(p_*\right)\right)}{\left[A'\left(u\right)\right]^2}\right]u^2 v\mathrm{d}u, \tag{4.21}$$

where the quotients are not singular for every $u>0$ (included the critical point $u = p_*$). Then, by following a similar analysis as in the proof of lemma 4.1 (see formulas (4.12) and (4.13)) we obtain

$$\begin{aligned}
\left[E_0\left(p_0\right)+A\left(p_*\right)\right]\mathcal{H}\left(p_0\right) &= -\int_{p_-}^{p_0}\frac{\left(E\left(u,v\right)+A\left(p_*\right)\right)u^2}{v}\mathrm{d}u\\
&= -\int_{p_-}^{p_0}\left[u^2 v - \frac{\left(A\left(u\right)-A\left(p_*\right)\right)u^2}{v}\right]\mathrm{d}u\\
&= -\int_{p_-}^{p_0}\left[3+\frac{4\left(1+2u^2\right)\left(A\left(u\right)-A\left(p_*\right)\right)}{\left[A'\left(u\right)\right]^2}\right]u^2 v\mathrm{d}u\\
&\quad + \frac{2\left(A\left(p_0\right)-A\left(p_*\right)\right)}{A'\left(p_0\right)}p_0^2 q_0.
\end{aligned} \tag{4.22}$$

Because the integrands are free of singularities and $E_0(p_0) + A(p_*) > 0$, the mapping $p_0 \in (0,1) \to \mathcal{H}(p_0) \in (0,+\infty)$ is C^1. Thus, the mapping $p_0 \in (0,1) \to \mathcal{N}(p_0) \in (0,+\infty)$ is C^1.

In the following we will see that $\mathcal{N}'(p_0) > 0$ for every $p_0 \in (0,1)$. We begin by differentiating (4.19) with respect to p_0, to give

$$\mathcal{N}'\left(p_0\right) = \mathcal{T}'\left(p_0\right)\left[2\mathcal{H}\left(p_0\right)+\left[1+\sqrt{1-p_0^2}\right]\right] + 2\mathcal{T}\left(p_0\right)\mathcal{H}'\left(p_0\right) + \mathcal{T}\left(p_0\right)\frac{-p_0}{\sqrt{1-p_0^2}}, \tag{4.23}$$

and we consider two cases for p_0:

(a) let $p_0 \in (0,p_*]$ and we take

$$J \equiv 2\mathcal{T}\left(p_0\right)\mathcal{H}'\left(p_0\right) + \mathcal{T}\left(p_0\right)\frac{-p_0}{\sqrt{1-p_0^2}}.$$

Then, via a long differentiation of (4.22), we get

$$[E_0(p_0)+A(p_*)]\mathcal{H}'(p_0) = -\frac{p_0^2}{16q_0} - \frac{3}{32}A'(p_0)\int_{p_-}^{p_0}\frac{1-2u^2}{v}\mathrm{d}u.$$

Therefore, from the former formula and from the facts that $E_0(p_0) = -\frac{3}{4}A(p_0)$, $A(p_*) = \frac{1}{4}$ and $2q_0 = -\sqrt{A(p_0)}$, we arrive

$$I \equiv [E_0(p_0)+A(p_*)]J = \frac{3}{4}\mathcal{T}(p_0)\left[\frac{p_0 A(p_0)}{\sqrt{1-p_0^2}} - \frac{A'(p_0)}{4}\int_{p_-}^{p_0}\frac{1-2u^2}{v}\mathrm{d}u\right].$$

For $p_0 \in (0,p_*]$, we have $A'(p_0) \geqq 0$ and $1-2u^2>0$, therefore $I>0$ and so $J>0$. From lemma 4.1 we have $\mathcal{T}'(p_0)>0$, hence from (4.23) it follows $\mathcal{N}(p_0)>0$ for $p_0 \in (0,p_*]$.

(b) let $p_0 \in (p_*,1)$ and we consider

$$\begin{aligned}[E_0(p_0)+A(p_*)]\mathcal{N}'(p_0) &= 2[E_0(p_0)+A(p_*)]\mathcal{T}'(p_0)\mathcal{H}(p_0)\\ &\quad - \frac{3}{16}\mathcal{T}(p_0)A'(p_0)\int_{p_-}^{p_0}\frac{1-2u^2}{v}\mathrm{d}u + P(p_0)\\ &\equiv II + P(p_0),\end{aligned} \tag{4.24}$$

where $P(p_0)>0$ for every $p_0\in(0,1)$, since $\mathcal{T}'(p_0)>0$. In the following we show that $II>0$ for $p_0\in(p_*,1)$. Indeed, from (4.14) we get

$$II = -\frac{3}{16}A'(p_0)\left[\mathcal{T}(p_0)\int_{p_-}^{p_0}\frac{1-2u^2}{v}\mathrm{d}u - \mathcal{H}(p_0)\int_{p_-}^{p_0}\frac{1-2u^2}{u^2v}\mathrm{d}u\right] + \mathcal{H}(p_0)\frac{1}{4\sqrt{A(p_0)}}. \tag{4.25}$$

Now, by using the expressions to $\mathcal{T}(p_0)$ and $\mathcal{H}(p_0)$ we get

$$\mathcal{T}(p_0)\int_{p_-}^{p_0}\frac{1-2u^2}{v}\mathrm{d}u - \mathcal{H}(p_0)\int_{p_-}^{p_0}\frac{1-2u^2}{u^2v}\mathrm{d}u = \left(\int_{p_-}^{p_0}\frac{u^2}{v}\mathrm{d}u\right)\left(\int_{p_-}^{p_0}\frac{1}{u^2v}\mathrm{d}u\right) - \left(\int_{p_-}^{p_0}\frac{1}{v}\mathrm{d}u\right)^2 \geqq 0,$$

in which we have used the Cauchy–Schwarz inequality. Hence, since $A'(p_0)<0$ for $p_0\in(p_*,1)$, it follows from (4.25) that $II>0$ for $p_0\in(p_*,1)$. Therefore, from (4.24) we get $\mathcal{N}'(p_0)>0$ for $p_0\in(p_*,1)$.

Lastly, the mapping $p_0\in(0,1)\to\mathcal{N}(p_0)$ is C^1 and strictly increasing, where $\mathcal{N}(p_0) = \pi\Omega(\omega)$. In addition, by lemma 4.1 and (4.9), the mapping $\epsilon\in(\alpha_0/\pi,+\infty)\to p_0(\epsilon)\in(0,1)$ is C^1 and strictly increasing. Therefore, from $\epsilon=\sqrt{-\omega}$ and from the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\Omega(\omega) = \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}p_0}\frac{\mathrm{d}p_0}{\mathrm{d}\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{d}\omega}<0,$$

and so the proof of lemma 4.3 is complete. □

Remark 4.4. From the proof of lemmas 4.1 and 4.3, we get that the shift-mapping $\omega\in(-\infty,\omega_0)\to a(\omega)\in(-\infty,0)$ has the following properties: $a'(\omega)<0$, $a(\omega)\to-\infty$ as $\omega\to\omega_0$ and $a(\omega)\to 0$ as $\omega\to-\infty$.

Proof (theorem 1.7). By lemma 4.1 we obtain item (1). Next, for $Z=0$, from theorem 1.3-item (2)–(a) we get that the Morse index for $\mathcal{L}_{+,0}$ in (1.13), $n(\mathcal{L}_{+,0})$, satisfies $n(\mathcal{L}_{+,0})=1$ for $a=a(\omega)\in[a^*,0)$ since $\gamma=\gamma(a)\leqq 0$ in (1.15). Moreover, from theorems 1.4 and 1.6 follow that for any shift-parameter value of a, $Ker(\mathcal{L}_{+,0})=\{0\}$, $Ker(\mathcal{L}_{-,0})=span\{(\Phi_\omega,\Psi_\omega)\}$ and $\mathcal{L}_{-,0}\geqq 0$. Thus, by lemma 4.3 and theorem 2.1, we obtain that $e^{-i\omega t}(\boldsymbol{\Phi}_\omega,\boldsymbol{\Psi}_\omega)$ is orbitally stable in $\mathcal{E}(\mathcal{G})$ for $\omega\leqq\omega^*<0$ such that $a(\omega)\in[a^*,0)$.

Next, we show that $n(\mathcal{L}_{+,0})=1$ for ω such that $a(\omega)<a^*$. Indeed, by a similar analysis as in the proof of lemma 4.1, the mapping $\omega\to(\Phi_\omega,\Psi_\omega)$ is a real-analytic mapping depending of the shift-parameter a, with $a\in(-\infty,0)\to\Theta(\omega(a))=(\Phi_{\omega(a)},\Psi_{\omega(a)})$. Then, we can see the family of operators $\mathcal{L}_{+,0}$ in (1.13) depending of the parameter a, and so we will use the notation $\mathcal{L}_a$. Moreover, $\Theta(\omega(a))\to\Theta(\omega^*)$ as $a\to a^*$ in the sense that $\|\Theta(\omega(a))-\Theta(\omega^*)\|_{\mathcal{E}(\mathcal{G})}\to 0$ as $a\to a^*$. Thus, we obtain that $\mathcal{L}_a$ converges to $\mathcal{L}_{a^*}$ as $a\to a^*$ in the generalized sense. Indeed, denoting $W_a=diag(-\omega(a)-6\Phi_{\omega(a)},-\omega(a)-6\Psi_{\omega(a)})$ we obtain

$$\widehat{\delta}(\mathcal{L}_a,\mathcal{L}_{a^*})=\widehat{\delta}(\mathcal{L}_{a^*}+(W_a-W_{a^*}),\mathcal{L}_{a^*})\leqq\|W_a-W_{a^*}\|_{L^2(\mathcal{G})}\to 0,\qquad \text{as }\ a\to a^*,$$

where $\widehat{\delta}$ is the gap metric (see [41, chapter IV]).

Now, it denotes by $N=n(\mathcal{L}_a)$. Thus, from $Ker(\mathcal{L}_a)=\{0\}$ for all $a<0$, $n(\mathcal{L}_{a^*})=1$, and the generalized convergence of $\mathcal{L}_a$ to $\mathcal{L}_{a^*}$ as $a\to a^*$, we get $n(\mathcal{L}_a)=N=1$ for $a\in[a^*-\delta_1,a^*+\delta_1]$, with $\delta_1>0$ small enough. Lastly, by using a continuation argument based on the Riesz-projection, we obtain that $n(\mathcal{L}_a)=1$ for any $a<a^*$ (see proof of lemmas 3.6 and 3.7). This finishes the proof. □

Remark 4.5. From the theory of Jacobi elliptic functions (see [25]), we can give an explicit formula for the profile Φ_ω and the shift-value $a(\omega)$ obtained in lemma 4.1. Indeed, we consider $\omega<0$, $p=1$ in (1.5), $L=\pi$ and Ψ_ω in (1.7) given by

$$\Psi_\omega(x)=\sqrt{-\omega}sech\left(\sqrt{-\omega}(x-\pi)+a\right),\quad x\geqq\pi, \tag{4.26}$$

with a to be chosen by the condition $\Phi_\omega(\pi)=\Phi_\omega(-\pi)=\sqrt{-\omega}sech(a)$, in (1.5). The profile Φ_ω is given by a profile of dnoidal type, dn (see Angulo [9, 10] and Cacciapuoti *et al* [26]), namely,

$$\Phi_\omega(x)=\sqrt{\frac{|\omega|}{2-k^2}}dn\left(\sqrt{\frac{|\omega|}{2-k^2}}\left(x-\frac{T_{dn}(k)}{2}\right);k\right), \tag{4.27}$$

with

$$T_{dn}(k)=2\sqrt{\frac{2-k^2}{|\omega|}}K(k) \tag{4.28}$$

where $k\in(0,1)$ is a free parameter, so-called the modulus of the Jacobian elliptic function dn, and K is the Legendre's complete elliptic integral of the first type [25]

$$K(k)=\int_0^1\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt. \tag{4.29}$$

Thus, since Φ_ω satisfies the first equation in (1.5), $k=k(\omega)$. Moreover, as $dn(\cdot;k)$ is an even periodic function with fundamental period being $2K(k)$ ($dn(x+2K(k);k)=dn(x;k)$), it follows $\Phi_\omega(-\pi)=\Phi_\omega(\pi)$. Next, since $dn\in[\sqrt{1-k^2},1]$, we have that Φ_ω is bounded away from

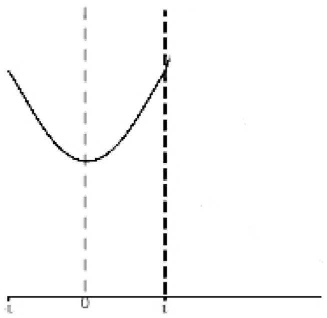

Figure 7. Profile of Φ_ω in (4.27) on $[-L,L]$ with $L=\pi$.

0, $x=0$ is the unique minimal point for Φ_ω on $[-\pi,\pi]$ and that $\Phi_\omega(0)=\sqrt{\frac{|\omega|}{2-k^2}}\sqrt{1-k^2}$ (see figure 7).

From relation $dn(x+K(k);k)=\sqrt{1-k^2}\frac{1}{dn(x;k)}$, the profile Φ_ω takes the form

$$\Phi_\omega(x)=\sqrt{\frac{|\omega|}{2-k^2}}\frac{\sqrt{1-k^2}}{dn\left(\sqrt{\frac{|\omega|}{2-k^2}}x;k\right)}. \tag{4.30}$$

Thus, we get the following formula for the shift-value $a=a(\omega)$ in function of k:

$$\frac{\sqrt{1-k^2}}{\sqrt{2-k^2}}\frac{1}{dn(\alpha(k);k)}=\operatorname{sech}(a),\quad \alpha(k)\equiv\sqrt{\frac{|\omega|}{2-k^2}}\pi. \tag{4.31}$$

Next, from relation $2\Phi'_\omega(\pi)=\Psi'_\omega(\pi)$ we can give an estimative of constant ω_0 in lemma 4.1. Indeed, from $sn(x+K(k);k)=\frac{cn(x;k)}{dn(x;k)}$, $cn(x+K(k);k)=-\sqrt{1-k^2}\frac{sn(x;k)}{dn(x;k)}$ (see formula (112.03) in [25]), we get from the former derivative condition that

$$3k^4sn^2(\alpha(k);k)\,cn^2(\alpha(k);k)=dn^4(\alpha(k);k), \tag{4.32}$$

where sn,cn are the Jacobian elliptic functions snoidal and cnoidal, respectively (see [25]). Thus, for obtaining a positive two-single lobe profile solution on $[-\pi,\pi]$ we need to find k_0 such that for

$$H_{\pi,\omega}(k)\equiv 3k^4sn^2(\alpha(k);k)\,cn^2(\alpha(k);k)-dn^4(\alpha(k);k),\qquad k\in(0,1) \tag{4.33}$$

we have $H_{\pi,\omega}(k_0)=0$. In general, equation $H_{\pi,\omega}(k)=0$ not always admits one solution. In fact, from the following limits

$$\lim_{k\to 0}H_{\pi,\omega}(k)=-1\ \text{ and }\ \lim_{k\to 1^-}H_{\pi,\omega}(k)=\operatorname{sech}^2\left(\sqrt{|\omega|}\pi\right)\left(3-4\operatorname{sech}^2\left(\sqrt{|\omega|}\pi\right)\right)\equiv\ell, \tag{4.34}$$

we get the following: there is an unique $\omega_0<0$ such that $\operatorname{sech}^2(\sqrt{|\omega_0|}\pi)=\frac{3}{4}$ (so $\sqrt{|\omega_0|}\pi\approx 0.5493$). Therefore, since $H_{\pi,\omega}$ is a strictly increasing mapping for $k\in(0,1)$ and for $\omega\in(\omega_0,0)$, we have $\ell<0$, then $H_{\pi,\omega}(k)=0$ is not possible. For $\omega<\omega_0$ we obtain $\ell>0$ and so $H_{\pi,\omega}(k)=0$ has at least one solution. Thus, $\omega_0=-\alpha_0^2/\pi^2$, with α_0 being the positive root of $sech^2(\alpha_0)=3/4$.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix

A.1. Extension theory and Morse index estimative

In this subsection we formulate some tools and applications of the extension theory of symmetric operators of Krein & von Neumann suitable for our needs (see [46, 52] for further information). In particular, we establish for $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$ in (3.9)–(3.10), the Morse index estimative $n(\widetilde{\mathcal{L}}) \leqq 3$. Moreover, for $(\mathcal{L}_1, S_\gamma)_{\gamma\in\mathbb{R}}$ in (3.19), we also obtain that $n(\mathcal{L}_1) \leqq 2$. These two specific estimates for the Morse index were used in section 3 above.

The following two results from the extension theory of symmetric operators are classical and can be found in [52].

Theorem A.1 (von-Neumann decomposition). *Let A be a closed, symmetric operator, then*

$$D(A^*) = D(A) \oplus \mathcal{N}_{-i} \oplus \mathcal{N}_{+i}. \tag{A.1}$$

with $\mathcal{N}_{\pm i} = \ker(A^* \mp iI)$. *Therefore, for* $u \in D(A^*)$ *and* $u = x + y + z \in D(A) \oplus \mathcal{N}_{-i} \oplus \mathcal{N}_{+i}$,

$$A^* u = Ax + (-i)\,y + iz. \tag{A.2}$$

Remark A.2. The direct sum in (A.1) is not necessarily orthogonal.

The following proposition provides a strategy for estimating the Morse index of the self-adjoint extensions of a minimal symmetric operator (see [46], [52]-chapter X). We recall the notation $n_\pm(A) = dim \ker(A^* \mp iI)$.

Proposition A.3. *Let A be a densely defined lower semi-bounded symmetric operator (that is,* $A \geqslant mI$*) with finite deficiency indices,* $n_\pm(A) = k < \infty$*, in the Hilbert space H, and let* $\widehat{A}$ *be a self-adjoint extension of A. Then the spectrum of* $\widehat{A}$ *in* $(-\infty, m)$ *is discrete and consists of, at most, k eigenvalues counting multiplicities.*

Proposition A.4. *We consider the family of self-adjoint operators* $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$ *where*

$$\widetilde{\mathcal{L}} = -\partial_x^2 - \omega - (p+1)(2p+1)\,\psi_{\text{even}}^{2p},$$

with $\psi_{\text{even}} \in H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R})$ *being the even-extension of the bump-profile* $\psi_{0,a}(x) = \psi_a(x+L)$, $x > 0$ *in (1.14), on whole the line, and* $D_{\delta,\gamma}$ *begin the following* δ*-interaction type domain*

$$D_{\delta,\gamma} = \left\{f \in H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R}) : f'(0+) - f'(0-) = \gamma f(0)\right\}.$$

Then $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$ *is a branch of the family of self-adjoint extensions of the following symmetric operator* $\mathcal{M}$

$$\mathcal{M} = -\partial_x^2 - \omega - (p+1)(2p+1)\psi_{\text{even}}^{2p}, \quad D(\mathcal{M}) = \left\{v \in H^2(\mathbb{R}) : v(0) = 0, v(\pm\mu) = 0\right\},$$

where μ *denotes the unique positive zero of* ψ_a. *Moreover, we have the Morse index estimative* $n(\widetilde{\mathcal{L}}) \leqq 3$.

Proof. From the Krein & von Neumann's extension theory (see chapter II.2 in [6]) we have that the symmetric operator $\mathcal{N}$,

$$\mathcal{N} = -\frac{d^2}{dx^2}, \quad D(\mathcal{N}) = \left\{v \in H^2(\mathbb{R}) : v(0) = 0, v(\pm\mu) = 0\right\},$$

has deficiency numbers $n_\pm(\mathcal{N}) = 3$. Thus, since $\psi_{\text{even}} \in L^\infty(\mathbb{R})$, we have $n_\pm(\mathcal{M}) = 3$, and hence all self-adjoint extensions of $\mathcal{N}$ are given by a nine-parameter family of self-adjoint operators. Then, when we restrict to the case of separated boundary conditions at each point $0, \pm\mu$, we obtain a three-parameters family of self-adjoint operators $(L_{\gamma,\alpha_1,\alpha_2}, D(L_{\gamma,\alpha_1,\alpha_2}))$, depending on $\gamma, \alpha_i \in \mathbb{R}$, and it given (via theorem A.1) by

$$\begin{cases} L_{\gamma,\alpha_1,\alpha_2} = -\partial_x^2 - \omega - (p+1)(2p+1)\psi_{even}^{2p} \\ D(L_{\gamma,\alpha_1,\alpha_2}) = \left\{ \begin{array}{c} v \in H^2(\mathbb{R} - \{0, \pm\mu\}) \cap H^1(\mathbb{R}) : v(0+) - v(0-) = \gamma v'(0), \\ v'(-\mu+) - v'(-\mu-) = \alpha_1 v(-\mu),\ v'(\mu+) - v'(\mu-) = \alpha_2 v(\mu) \end{array} \right\}. \end{cases}$$

It is easily seen that we arrive at $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$ for $\alpha_i = 0$, $i = 1, 2$.

Next, we estimative $n(\widetilde{\mathcal{L}})$. By proposition A.3 it is enough to prove that $\mathcal{M}$ is a non-negative operator. In fact, it is easy to verify that for $v \in D(\mathcal{M})$ the following identity holds for $\psi_e \equiv \psi_{\text{even}}$

$$\mathcal{M}v = \frac{-1}{\psi_e'} \frac{d}{dx}\left[(\psi_e')^2 \frac{d}{dx}\left(\frac{v}{\psi_e'}\right)\right], \quad x \neq 0, \pm\mu. \tag{A.3}$$

Now, integration by parts yields

$$\begin{aligned} \langle \mathcal{M}v, v\rangle = &\int_{-\infty}^{-\mu-} (\psi_e')^2 \left(\frac{d}{dx}\left(\frac{v}{\psi_e'}\right)\right)^2 dx + \int_{-\mu+}^{0-} (\psi_e')^2 \left(\frac{d}{dx}\left(\frac{v}{\psi_e'}\right)\right)^2 dx \\ &+ \int_{0+}^{\mu-} (\psi_e')^2 \left(\frac{d}{dx}\left(\frac{v}{\psi_e'}\right)\right)^2 dx + \int_{\mu+}^{+\infty} (\psi_e')^2 \left(\frac{d}{dx}\left(\frac{v}{\psi_e'}\right)\right)^2 dx \\ &- \left[v'v - v^2\frac{\psi_e''}{\psi_e'}\right]_{-\infty}^{-\mu-} - \left[v'v - v^2\frac{\psi_e''}{\psi_e'}\right]_{-\mu+}^{0-} - \left[v'v - v^2\frac{\psi_e''}{\psi_e'}\right]_{0+}^{\mu-} \\ &- \left[v'v - v^2\frac{\psi_e''}{\psi_e'}\right]_{\mu+}^{+\infty}. \end{aligned} \tag{A.4}$$

Noting $v(\pm\mu) = 0$ and that $\pm\mu$ are the first-order zeroes for ψ_e' (namely, $\psi_e''(\pm\mu) \neq 0$), we have, for instance, for the eighth term in (A.4) that

$$-\left[v'v - v^2\frac{\psi_e''}{\psi_e'}\right]_{\mu+}^{\infty} = -\psi_e''(\mu)\lim_{x\to\mu+}\frac{v^2(x)}{\psi_e'(x)} = -2\psi_e''(\mu)\lim_{x\to\mu+}\frac{v(x)v'(x)}{\psi_e'(x)} = 0. \tag{A.5}$$

Analogously the fifth term in (A.4) is zero. Next, since $\psi_e'(0\pm) \neq 0$ and from the limit in (A.5), the sixth and seventh terms in (A.4) are also zero.

Therefore, we get $\mathcal{M} \geqq 0$ on $D(\mathcal{M})$, and consequently by proposition A.3, the Morse index of $\widetilde{\mathcal{L}}$ acting on $D_{\delta,\gamma}$ satisfies $n(\widetilde{\mathcal{L}}) \leqq 3$. This finishes the proof. □

Proposition A.5. *We consider the family of self-adjoint operators* $(\mathcal{L}_1, S_\gamma)_{\gamma\in\mathbb{R}}$, *where*

$$\mathcal{L}_1 = -\partial_x^2 - \omega - (p+1)(2p+1)\psi_a^{2p}, \tag{A.6}$$

with $\psi_a(x) = \psi_0(x+a)$, $a < 0$, $x \in (0,+\infty)$, *being the bump-profile determined by the soliton* ψ_0 *in (1.8), and with* S_γ *given by*

$$S_\gamma = \left\{f \in H^2(0,+\infty) : f'(0+) = \gamma f(0)\right\}. \tag{A.7}$$

Then the following Morse index estimative, $n(\mathcal{L}_1) \leqq 2$, *is satisfied.*

Proof. Let $\mu > 0$ be the unique positive zero of ψ_a and it considers for $\mathcal{N} = -\frac{d^2}{dx^2}$ the minimal-symmetric operator $(\mathcal{N}, C_0^\infty[(0,+\infty) - \{\mu\}])$. Then, the closure of $\mathcal{N}$, denoted by $\dot{\mathcal{N}}$, has deficiency number $n_\pm(\dot{\mathcal{N}}) = 2$. Thus, since $\psi_a \in L^\infty(0,+\infty)$, we have $n_\pm(\mathcal{L}_1) = 2$ with $D(\mathcal{L}_1) = D(\dot{\mathcal{N}})$, and hence all the self-adjoint extensions of $\dot{\mathcal{N}}$ are given by a four-parameter family of self-adjoint operators. Thus, when we restrict to the case of separated boundary conditions at each point $0, \mu$, we get a two-parameter family of self-adjoint operators $(L_{\gamma,\alpha}, D(L_{\gamma,\alpha}))$, depending on $\gamma, \alpha \in \mathbb{R}$, and it given by (via theorem A.1)

$$\begin{cases} L_{\gamma,\alpha} = -\partial_x^2 - \omega - (p+1)(2p+1)\psi_a^{2p} \\ D(L_{\gamma,\alpha}) = \left\{\begin{array}{c} v \in H^2((0,+\infty) - \{\mu\}) \cap H^1(0,+\infty) : v'(0+) = \gamma v(0), \\ v'(\mu+) - v'(\mu-) = \alpha v(\mu) \end{array}\right\}. \end{cases}$$

Then, it is easily seen that $(\mathcal{L}_1, S_\gamma)_{\gamma\in\mathbb{R}}$ is a branch of the family $(L_{\gamma,\alpha}, D(L_{\gamma,\alpha}))$ for $\alpha = 0$. Thus, by using a similar analysis as in proposition A.4 and via proposition A.3, we get $n(\mathcal{L}_1) \leqq 2$. This finishes the proof. □

A.2. Extension theory for the Laplacian operator on a tadpole graph

In this subsection, by completeness in the exposition, we establish an extension theory for the Laplacian operator $-\Delta$,

$$-\Delta = \left(-\frac{d^2}{dx^2}\right)_{\mathbf{e}\in\mathbf{E}} \tag{A.8}$$

on a tadpole graph $\mathcal{G}$ with structure represented by the set $\mathbf{E} = (-L, L) \cup (L, +\infty)$ and a single vertex at $\nu = L$, such that it induces the one-parameter family of self-adjoint operators $(-\Delta, D_Z)_{Z\in\mathbb{R}}$ in (1.3). With these notations we have the following result.

Theorem A.6. *The Schrödinger type operator* $-\Delta$ *in (A.8) on* $L^2(\mathcal{G})$*, with domain*

$$D(-\Delta) = \left\{(\phi,\psi) \in H^2(\mathcal{G}) : \phi(-L) = \phi(L) = \psi(L) = 0,\ \phi'(L) - \phi'(-L) = \psi'(L)\right\}, \tag{A.9}$$

is a densely defined symmetric operator with deficiency indices $n_\pm(-\Delta) = 1$*. Therefore,* $(-\Delta, D(-\Delta))$ *has a one-parameter family of self-adjoint extensions defined by* $(-\Delta, D_Z)_{Z\in\mathbb{R}}$ *with*

$$D_Z = \left\{(f,g) \in H^2(\mathcal{G}) : f(L) = f(-L) = g(L),\ and,\ f'(L) - f'(-L) = g'(L) + Zg(L)\right\}. \tag{A.10}$$

Proof. The symmetric property of $(-\Delta, D(-\Delta))$ is immediate. Since, $C_c^\infty(-L,L) \bigoplus C_c^\infty(L,+\infty) \subset D(-\Delta)$ we obtain the density property of $D(-\Delta)$ in $L^2(\mathcal{G})$. Now, it is not difficult to see that the adjoint operator $(-\Delta^*, D(-\Delta^*))$ of $(-\Delta, D(-\Delta))$ is given by

$$-\Delta^* = -\Delta, \quad D(-\Delta^*) = \left\{(u,v) \in H^2(\mathcal{G}) : u(-L) = u(L) = v(L)\right\}. \tag{A.11}$$

Next, the deficiency subspaces $\mathcal{D}_\pm = ker(-\Delta^* \mp i)$ have dimension one. Indeed, by using the definition of $D(-\Delta^*)$ in (A.11) we can see that $\mathcal{D}_\pm = span\{(f_{\pm i}, g_{\pm i})\}$ with

$$\begin{cases} f_{+i}(x) = e^{\sqrt{-i}(x+L)} + e^{-\sqrt{-i}(x-L)}, & x \in (-L,L),\ Im\left(\sqrt{-i}\right) > 0 \\ g_{+i}(x) = e^{\sqrt{-i}(x-L)} + e^{\sqrt{-i}(x+L)}, & x > L, \end{cases} \tag{A.12}$$

and,

$$\begin{cases} f_{-i}(x) = e^{\sqrt{i}(x+L)} + e^{-\sqrt{i}(x-L)}, & x \in (-L,L),\ Im\left(\sqrt{i}\right) < 0 \\ g_{-i}(x) = e^{\sqrt{i}(x-L)} + e^{\sqrt{i}(x+L)}, & x > L. \end{cases} \tag{A.13}$$

Thus, from theorem A.1 in [6] (appendix A), from the formulas in (A.12)–(A.13) and the von-Neumann decomposition in theorem A.1, we can deduce that $(-\Delta, D(-\Delta))$ has a one-parameter family of self-adjoint extensions $(-\Delta, D_Z)_{Z\in\mathbb{R}}$ with D_Z defined in (A.10). This finishes the proof. □

A.3. Perron–Frobenius property

In the following we establish the Perron–Frobenius property for the family of self-adjoint operators $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ in (3.9)–(3.10). We note that there are several results in the literature related to the Perron-Frobenius property for Schrödinger operator of the type $-\Delta + V(x)$ with an external potential. In the case of metric graphs, some results have been obtained depending of the graph's topology (see [27, 31], and reference therein). Here, by convenience of the reader, we give an unified proof of this property for $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ with any value of γ. For that, we start with the following two remarks: by Weyl's essential spectrum theorem [53], we have that the essential spectrum $\sigma_{\text{ess}}(\widetilde{\mathcal{L}})$ of $\widetilde{\mathcal{L}}$, it satisfies $\sigma_{\text{ess}}(\widetilde{\mathcal{L}}) = [-\omega, +\infty)$. Furthermore, from $\langle \widetilde{\mathcal{L}}\psi_{\text{even}}, \psi_{\text{even}}\rangle < 0$ and the extension theory, it follows that $1 \leqq n(\widetilde{\mathcal{L}}) \leqq 3$ for any γ.

Theorem A.7 (Perron–Frobenius property). *We consider the family of self-adjoint operators* $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$ *defined in (3.9)–(3.10). For* γ *fixed, it assumes that* $\beta = \inf\sigma(\widetilde{\mathcal{L}}) < -\omega$ *is the smallest eigenvalue. Then,* β *is simple and its corresponding eigenfunction* ζ_β *is positive (after replacing* ζ_β *by* $-\zeta_\beta$ *if necessary) and even.*

Proof. The result follows by a slight twist of standard abstract Perron–Frobenius arguments (see proposition 2 in Albert *et al* [7]). The basic point in the analysis is to show that the Laplacian operator $-\Delta_\gamma \equiv -\frac{d^2}{dx^2}$ on the domain $D_{\delta,\gamma}$ has its resolvent $R_\mu = (-\Delta_\gamma + \mu)^{-1}$ represented by a positive kernel for some $\mu > 0$ sufficiently large. Namely, for $f \in L^2(\mathbb{R})$

$$R_\mu f(x) = \int_{-\infty}^{+\infty} K(x,y) f(y)\, dy$$

with $K(x,y) > 0$ for all $x, y \in \mathbb{R}$. By convenience of the reader we show this main point, the remainder of the proof follows the same strategy as in [7]. In fact, for γ fixed, let $\mu > 0$ be sufficiently large (with $-2\sqrt{\mu} < \gamma$ in the case $\gamma < 0$), then from the Krein formula (see theorem 3.1.2 in [6]) we obtain

$$K(x,y) = \frac{1}{2\sqrt{\mu}} \left[e^{-\sqrt{\mu}|x-y|} - \frac{\gamma}{\gamma + 2\sqrt{\mu}} e^{-\sqrt{\mu}(|x|+|y|)} \right].$$

Moreover, for every x fixed, $K(x,\cdot) \in L^2(\mathbb{R})$. Thus, the existence of the integral above is guaranteed by Holder's inequality. Now, since $K(x,y) = K(y,x)$, it is sufficient to show that $K(x,y) > 0$ in the following cases.

(1) Let $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$: for $\gamma \geqq 0$, we obtain from $\frac{\gamma}{\gamma + 2\sqrt{\mu}} < 1$ and $|x - y| \leqq |x| + |y|$, that $K(x,y) > 0$. For $\gamma < 0$ and $-2\sqrt{\mu} < \gamma$, it follows immediately $K(x,y) > 0$.
(2) Let $x > 0$ and $y < 0$: in this case,

$$K(x,y) = \frac{1}{\gamma + 2\sqrt{\mu}} e^{-\sqrt{\mu}(x-y)} > 0$$

for any value of γ (where again $-2\sqrt{\mu} < \gamma$ in the case $\gamma < 0$).

The proof of the theorem is complete. □

A.4. Orbital stability criterion

By convenience of the reader, in this subsection we adapted the abstract stability results from Grillakis *et al* in [34, 35] for the case of a tadpole graph and standing waves being a positive two-lobe state. This criterion was used in the proof of theorem 1.7.

Theorem A.8. *Suppose that there is a C^1-mapping $\omega \to (\Phi_\omega, \Psi_\omega)$ of positive two-lobe states for the NLS model (1.1) on a tadpole graph $\mathcal{G}$. We consider the assertions in theorems 1.3, 1.4 and 1.6, associated to the Morse index and the nullity index for the operators $\mathcal{L}_{+,Z}$ and $\mathcal{L}_{-,Z}$. Then, for $Ker(\mathcal{L}_{+,Z}) = \{0\}$ we have*

(1) *if $n(\mathcal{L}_{+,Z}) = 1$ and $\partial_\omega ||(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)||^2_{L^2(\mathcal{G})} < 0$, then $e^{-i\omega t}(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)$ is orbitally stable in $\mathcal{E}(\mathcal{G})$,*
(2) *if $n(\mathcal{L}_{+,Z}) = 1$ and $\partial_\omega ||(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)||^2_{L^2(\mathcal{G})} > 0$, then $e^{-i\omega t}(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)$ is orbitally unstable in $\mathcal{E}(\mathcal{G})$,*
(3) *if $n(\mathcal{L}_{+,Z}) = 2$ and $\partial_\omega ||(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)||^2_{L^2(\mathcal{G})} < 0$, then $e^{-i\omega t}(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)$ is linearly unstable in $\mathcal{E}(\mathcal{G})$.*

By completeness in the exposition, we have the following comment about the instability part, item (3), of the above theorem: it is known from [35] that when $n(\mathcal{L}_{+,Z}) = 2$ and $\partial_\omega ||(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)||^2_{L^2(\mathcal{G})} < 0$, we obtain only spectral instability of the standing wave $e^{-i\omega t}(\mathbf{\Phi}_\omega, \mathbf{\Psi}_\omega)$. Generally, to obtain the orbital instability from the spectral one is not an easy task. The approach in [36] can be a good tool for obtaining this connection (see [14–18] where the results in [36] has been applied successfully for models on metric graphs). Indeed, the key point of the approach is to use the fact that if the mapping data-solution associated to (1.1) is of class C^2, then we obtain the orbital instability from a result of spectral instability (for instance, see theorem 3.12, Corollary 3.13 and the proof of theorem 3.5 in [18], where the link between the C^2-property and the orbital instability is explained more precisely). Therefore, by using theorem 2.1 in the case $2p > 1$, we can change linearly unstable statement in item (3) by orbitally unstable.

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