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# A MARTINGALE ESTIMATOR FOR A STRUCTURAL IMPORTANCE MEASURE OF A COHERENT SYSTEM'S COMPONENT THROUGH ITS CRITICAL LEVEL

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# A martingale estimator for a structural importance measure of a coherent system's component through its critical level.

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**Abstract** Using martingale methods in reliability theory we define a structural importance measure through the component's critical level of a coherent system under dependence conditions. We propose a martingale estimator for this measure.

Keywords: Martingale methods in reliability theory, compensator process, structural importance measure.

## 1.Introduction

In Reliability Theory it is very useful to represent coherent systems in several alternate way, frequently in terms of minimal cut sets (Barlow and Proschan(1981)).

If the system's lifetime $T$ and the component's lifetime $T_i$, $1 \leq i \leq n$, are defined in a complete probability space $(\Omega, \Im, P)$, the system lifetime can be set as a series-parallel decomposition

$$T = \Phi(\mathrm{T}) = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i,$$

where $K_j, 1 \leq j \leq k$ are minimal cut sets, that is, a minimal set of components whose joint failure causes the system's failure. The function $\Phi$ is called structure function

In a series system the cut sets are unit sets and the lifetime is $T = \min_{1 \leq i \leq n} T_i$.

It is possible to model a coherent system as a series system considering the dynamic of the components after its critical levels, the time from which onwards a failure of the component lead to system failure. However, even if the original components are statistically independent, the components for such series system are statistically dependent and we have to use martingales methods in reliability theory to work this dependence. In this representation, the only information about the structure function is through the components critical levels and therefore they are essential to represent a coherent system as a series structure and to calculate structural importance of a component. In this paper, in Section 2 we give the necessary preliminaries in a mathematical formulation and in Section 3 we define the estimation procedure to estimate the structural importance of a component.

## 2.Mathematical Formulation

We consider a coherent system of n components with lifetime $T_i$, $1 \leq i \leq n$, defined in a complete probability space $(\Omega, \Im, P)$. The system is monitored at component's level, that is, through a family of sub $\sigma$-algebras of $\Im$, denoted $(\Im_t)_{t \geq 0}$, where

$$\Im_t = \sigma(I(T_i > s), 1 \leq i \leq n,\ s \leq t),$$

satisfies the Dellacherie's condition of right continuity and completeness. We assume that $T_i$ is totally inaccessible $\Im_t$- stopping time and that $P(T_i \neq T_j)$, for all $i, j,\ \ 1 \leq i, j \leq n$.

The system lifetime can be set as a series-parallel decomposition

$$T = \Phi(\mathbf{T}) = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i,$$

where $\mathbf{T} = (T_1, ....., T_n)$ and $K_j, 1 \leq j \leq k$ are minimal cut sets.As

$$\{T > s\} = \bigcap_{1 \leq j \leq k} \bigcup_{i \in K_j} \{T_i > s\},$$

we can say that the system lifetime $T$ is an $\Im_t$-stopping time.

In a equivalent notation we can use

$$X(t) = \Phi(\mathbf{X}(\mathbf{t})) = \min_{1 \leq j \leq k} \max_{i \in K_j} X_i(t),$$

where $\mathbf{X}(\mathbf{t}) = (X_1(t), ...., X_n(t))$, $(X(t))_{t \geq 0}$ and $(X_i(t))_{t \geq 0}$ are right continuous stochastic processes with $X(t) = 1_{\{T > t\}}$, $X_i(t) = 1_{\{T_i > t\}}$.

We denote $T_{(1)}, T_{(2)}, ..., T_{(n)}$ the ordered lifetimes $T_i$, $1 \leq i \leq n$.

For a fixed component $i \in \Lambda = \{1, ..., n\}$ the $\Im_t$- compensator $(A_i(t))_{t \geq 0}$ of the univariate counting process $(N_i(t))_{t \geq 0}$, $N_i(t) = 1_{\{T_i \leq t\}}$ is the a.s. unique right continuous increasing predictable process such that, for each $k \geq 1$, the difference process stopped at $T_{(k)}$,

$$N_i(t \wedge T_{(k)}) - A_i(t \wedge T_{(k)})),$$

is an 0 mean $(\Im_t)$-martingale. In view of the fact that $T_{(n+1)} = \infty$ we have that $N_i(t) - A_i(t)$, is an uniformly square integrable 0 mean $(\Im_t)$-martingale. $A_i(t)$ are continuous from the inaccessibility of $T_i$.

In what follows the concept of the critical level of a component for the system is absolutely necessary.

**Definition 2.1** The critical level of a component $i$, is the time from which onwards a failure of component $i$ leads to system failure, that is, the $\Im_t$ stopping time

$$Y_i = \inf\{t \geq o : \Phi(1_i, \mathbf{X}(\mathbf{t})) - \Phi(0_i, \mathbf{X}(\mathbf{t})) = 0\}.$$

where $(j_i, \mathbf{X}(\mathbf{t}))$ is the vector $\mathbf{X}(\mathbf{t})$ with the $i$-th coordinate equal to $j$, $\ j = 0, 1$. For a rigorous critical level definition , see Arjas (1981b).

We are going to study the $\Im_t$-compensator of the counting process $(N_\Phi(t))_{t\geq 0}$ of system failure, denoting it by $(A_\Phi(t))_{t\geq 0}$. Arjas (1981b) characterize the relationship between the component's $\Im_t$- compensator and the system's $\Im_t$-compensator processes.

**Theorem 2.2** The $\Im_t$-compensator of $N_\Phi(t) = I(T \leq t)$ is

$$A_\Phi(t) = \sum_{J\in\Lambda}[A_i(t\wedge T) - A_J(Y_i]^+ a.s.$$

where $[a]^+ = \max\{a, 0\}$.

To represent a coherent system as a series system, we consider to observe the lifetimes after its critical levels, that is, we observe:
the family of $\sigma$- algebras $(\Im_{Y_i+t})_{t\geq 0}$, where

$$\Im_{Y_i+t} = \{A \in \Im_\infty : A\bigcap\{Y_i + t \leq s\} \in \Im_s, \quad s \geq 0\};$$

the lifetimes $S_i = ((T_i - Y_i)^+|\Im_{Y_i}) \quad i \in \Lambda$, and the corresponding counting process

$$M_i(t) = 1_{\{S_i\leq t\}} = E[N_i(Y_i + t) - N_i(Y_i)|\Im_{Y_i}] = E[1_{\{Y_i<T_i\leq Y_i+t\}}|\Im_{Y_i}].$$

Under the above notation the $\Im_{Y_i+t}$ - compensator of $M_i(t)$ is $B_i(t) = E[A_i(Y_i + t) - A_i(Y_i)|\Im_{Y_i}]$.

Based in Theorem 2.1 we represent the coherent system as the series structure

$$S = \min_{\{i:Y_i<T_i\}} S_i.$$

Clearly the $\Im_{Y_i+t}$ - compensator of $M_\Phi(t) = 1_{\{S\leq t\}}$ is $B_\Phi(t) = \sum_{\{i:Y_i<T_i\}} B_i(t)$.

**Theorem 2.3** Under the above notation the system lifetime $T$ is almost surely equal to the series system lifetime $S$.
**Proof** We know that $T \in \Im_t$ and

$$\{T = s\} \Longleftrightarrow \{T \in \bigcap_{1\leq i\leq\infty} [s, s + \frac{1}{n})\} \Longleftrightarrow \{s \in \bigcap_{1\leq i\leq\infty} (T - \frac{1}{n}, T]\}$$

is a predictable set. Follows that $\{T = s\}$ is an $\Im_t$-predictable set.

Furthermore $\{T = s\} = \bigcup_{\{i:Y_i<T_i\}} \{T_i = s\}$.

Follows that $w \in \{T = s\}$ if, and only if $w \in \{T_i = s\}$ and $w \in \{Y_i < T_i\}$for some $i$. However $\{Y_i < T_i\}\bigcap\{T_i = s\}$ is equivalent to $\{Y_i < T_i = T = s\}$ and $\{Y_i < T_i\} \in \Im_{Y_i} \subset \Im_{Y_i+s}$

Therefore $\{T = s\} \in \Im_{Y_i+s}$ and we also conclude that $\{T = s\}$ is an $\Im_{Y_i+s}$- predictable set.

Therefore if we let $N_\Phi(t) = 1_{\{T \leq t\}}$ and $M_\Phi(t) = 1_{\{S \leq t\}}$ we have

$$1 = P(T = T) = E[\int_0^\infty 1_{\{T=s\}} dN_\Phi(s)] = E[\int_0^\infty 1_{\{T=s\}} dA_\Phi(s)] =$$

$$E[\int_0^\infty 1_{\{T=s\}} dB_\Phi(s)] = E[\int_0^\infty 1_{\{T=s\}} dM_\Phi(s)] = E[1_{\{T=S\}}] = P(T = S).$$

The third equality follows from $\{T = s\}$ is $\Im_t$-predictable, the fourth follows from the conditional expectation concept and the fifty is because $\{T = s\}$ is an $\Im_{Y_i+s}$- predictable set.

For a given coherent system with structure function $T = \Phi(\mathrm{T})$, some components are more important than others in determining whether the system function or not. For example, if a component is in series with the rest of the system, then it would seem to be at least as important as any other component in the system. It is clearly of value to the designer and reliability analyst to have a quantitative measure of the structure importance of the individual components in the system.

If the coherent system is represented as a series structure

$$S = \min_{\{i: Y_i < T_i\}} S_i,$$

the only information about the system structure is through the critical levels $Y_i$. To eliminate the random effect of the component lifetimes to the structure we assume that the lifetimes $T_i$, $1 \leq i \leq n$ are uniformly distributed over the interval $(0, 1)$. We define

**Definition 3.1** If $T_i, 1 \leq i \leq n$ are independent and identically distributed with distribution Uniform on $(0, 1)$, the structural importance of component $i$ is

$$I(i) = E[T - Y_i]^+.$$

### 3. The estimation problem

We propose a martingale estimator to estimate the structural importance $E[T - Y_i]^+ = E[\int_0^T 1_{\{Y_i < s\}} ds]$. For that we are going to use the following Theorem:

**Theorem 3.2** (Karr (1986) Let, for each $i$, $1 \leq i \leq n$ the sequence $(M_i^m(t))_{m \geq 1}$ of orthogonal mean zero square integral martingale and let, for each $i$, $1 \leq i \leq n$, $(V_i(t))_{t \geq 0}$ be continuous and nondecreasing with $V_i(0) = 0$.

Suppose that:

A) For each $t$ and $i$,

$$< M_i^m >_t \rightarrow V_i(t)$$

, in distribution, as $m \rightarrow \infty$

B) There are a sequence $(c_m)_{m\geq 1}$, $c_m \to 0$ as $m \to \infty$ and

$$P(\sup|\triangle M^m(t)| \leq c_m) \to 1$$

as $m \to \infty$.

Then there exists a n-variate continuous Gaussian process $M$, with each component a martingale and with

$$< M_i, M_j >_t = 1_{\{i=j\}} V_i(t)$$

, such that $M^m \to M$ in distribution, as $m \to \infty$.

Firstly we assume that we know the $\Im_t$ intensity process $(\lambda_i(t))_{t\geq 0}$ for component $i$, such that $A_i(t) = \int_0^t \lambda_i(s)ds$ $1 \leq i \leq n$. Also, without loss of generality we assume that $\lambda_i(s) > 0, P - a.s.$.

For each $i$, $1 \leq i \leq n$, we consider the compensator transform

$$B_i(t) = \int_0^t 1_{\{Y_i < s\}} \lambda_i(s) ds.$$

An application of Girsanov Theorem gives

**Theorem 3.3**(Bremaud (1981)) If, we have $\int_0^t 1_{\{Y_i<s\}} \lambda_i(s)ds < \infty, P - a.s.$, for each $i$, $1 \leq i \leq n$, then

$$L_t = \prod_{i=1}^n [(1_{\{Y_i<S_i\}})^{N_i(t)} \exp \int_0^t 1_{\{Y_i \geq s\}} \lambda_i(s) ds]$$

is a $(P, \Im_t)$ non negative local martingale and a $(P, \Im_t)$ super martingale. Furthermore, in the case in which

$$E[L_\infty] = E[\prod_{i=1}^n [(1_{\{Y_i<S_i\}}) \exp \int_0^{Y_i} \lambda_i(s) ds]] = 1$$

, then, under the probability measure $Q$, defined through the Radon Nikodyn derivative

$$\frac{dQ}{dP} = L_\infty,$$

$$N_i(t) - \int_0^t 1_{\{Y_i<s\}} \lambda_i(s) ds$$

is an $(Q, \Im_t)$ martingale.

**Corollary 3.4** Under the hypothesis of Theorem 3.3

$$\int_0^t \lambda_i^{-1}(s) dN_i(s) - \int_0^t 1_{\{Y_i<s\}} ds$$

is an $(Q, \Im_t)$ martingale.

**Proof** The proof follows easily since that $\lambda_i^{-1}(s)$ is $\Im_t$-predictable, and

$$\int_0^t \lambda_i^{-1}(s)dN_i(s) - \int_0^t 1_{\{Y_i<s\}}ds =$$

$$\int_0^t \lambda_i^{-1}(s)dN_i(s) - \int_0^t 1_{\{Y_i<s\}}\lambda_i^{-1}(s)\lambda_i(s)ds =$$

$$\int_0^t \lambda_i^{-1}(s)d[N_i(s) - \int_0^s 1_{\{Y_i<u\}}\lambda_i(u)du]$$

a $(Q, \Im_t)$ martingale.

Therefore, we propose the unbiased martingale estimator $\widehat{E}_i(t) = \int_0^t \lambda_i^{-1}(s)dN_i(s)$ for $E_i(t) = E_Q[\int_0^t 1_{\{Y_i<s\}}ds]$.

Note that, the mean squared error is the mean of the variation process

$$E_Q[(\widehat{E}_i(t) - E_i(t))^2] = E_Q[< \widehat{E}_i - E_i >_t].$$

where $< \widehat{E}_i - E_i >_t] = \int_0^t(\lambda_i^{-1}(s))^2 1_{\{Y_i<s\}}\lambda_i(s)ds$.

Follows that

$$E_Q[(\widehat{E}_i(t) - E_i(t))^2 = E_Q[\int_0^t \lambda_i^{-1}(s))1_{\{Y_i<s\}}ds].$$

Clearly, an estimator for

$$C_i(t) = E_Q[\int_0^t \lambda_i^{-1}(s)1_{\{Y_i<s\}}ds]$$

is

$$\widehat{C}_i(t) = \int_0^t (\lambda_i^{-1}(s))^2 dN_i(s).$$

We are going to introduce our statistical model applying Theorem 3.2 in a set of data constituting of independent and identically distributed copies of $(N_i, \lambda_i)$, $1 \leq i \leq n$.

We assume, for each $i, 1 \leq i \leq n$ a sequence of pairs $(N_i^j(t), \lambda_i^j(t)), 1 \leq j \leq m$, of $m$ copies independent and identically distributed as $(N_i(t), \lambda_i(t))$, and let, for each $i, 1 \leq i \leq n$,

$$N_i^m(t) = \sum_{j=1}^m N_i^j(t), \quad and \quad \lambda_i^m(t) = \sum_{j=1}^m \lambda_i^j(t).$$

To estimate $E_i(t) = E_Q[\int_0^t 1_{\{Y_i<s\}}ds]$, we propose the martingale estimator

$$\widehat{E}_i^m(t) = \int_0^t (\lambda_i^m(t))^{-1} dN_i^m(s)$$

. Clearly, as proved in Corollary 3.4, it is an unbiased estimator.

A natural first question to be done is :

Are the martingale estimator $\widehat{E}_i^m(t)$ consistent in the sense that $\widehat{E}_i^m(t) - E_i(t) \to 0$ under $Q$? What we can say if we consider the measure $P$?

**Theorem 3.5** Under $Q$, $\widehat{E}_i^m(t)$ is a consistent estimator for $E_i(t)$, that is

$$E_Q[\sup_{t\leq T}(\widehat{E}_i^m(t) - E_i(t))^2] \to 0$$

as $m \to \infty$

**Proof.**

$$\widehat{E}_i^m(t) - E_i(t) = \int_0^t (\lambda_i^m(s))^{-1} dN_i^m(s) - \int_0^t 1_{\{Y_i<s\}} ds =$$

$$\int_0^t (\lambda_i^m(s))^{-1} d[N_i^m(s) - \int_0^s \lambda_i^m(u) 1_{\{Y_i<u\}} du]$$

is an $(Q, \Im_t)$ martingale. Follows that the variation process

$$< \widehat{E}_i^m - E_i >_t = \int_0^t (\lambda_i^m(s))^{-2} 1_{\{Y_i<s\}} \lambda_i^m(s) ds = \int_0^t (\lambda_i^m(s))^{-1} 1_{\{Y_i<s\}} ds.$$

Therefore

$$E_Q[\sup_{s\leq t}(\widehat{E}_i^m(s) - E_i(s))^2] \leq 4E_Q[(\widehat{E}_i^m(t) - E_i(t))^2] =$$

$$4E_Q[< \widehat{E}_i^m - E_i >_t] = 4E_Q[\frac{1}{m}\int_0^t \frac{m}{\lambda_i^m(s)} 1_{\{Y_i<s\}} ds] \leq 4E_Q[\frac{1}{m}\int_0^t \frac{m}{\lambda_i^m(s)} ds] \to 0$$

for the Stronger Law of Larger Numbers.

As $\widehat{E}_i^m(t) - E_i(t)$ is an uniformly integrable martingale and $T$ is an $\Im_t$-stopping time, we can apply the Optimal Sampling Theorem to get

$$E_Q[\sup_{t\leq T}(\widehat{E}_i^m(t) - E_i(t))^2] \to 0$$

as $m \to \infty$

**Corollary 3.6** Under $P$, $\widehat{E}_i^m(t)$ is a consistent estimator for $E_i(t)$.
**Proof.** As

$$E[L_\infty] = E[\prod_{i=1}^n [(1_{\{Y_i<S_i\}}) \exp(A_i(Y_i))] > 0$$

we also have

$$\frac{dP}{dQ} = \frac{1}{L_\infty}$$

and

$$E_P[\sup_{t\le T}(\widehat{E}_i^m(t) - E_i(t))^2] =$$

$$E_Q[\frac{1}{\prod_{i=1}^{n}[(1_{\{Y_i<S_i\}})\exp(A_i(Y_i))]}\sup_{t\le T}(\widehat{E}_i^m(t) - E_i(t))^2] \le$$

$$E_Q[\sup_{t\le T}(\widehat{E}_i^m(t) - E_i(t))^2] \to 0$$

as $m \to \infty$.

A second important question is: Are the estimator error process $\widehat{E}_i^m(t) - E_i(t)$, suitable scaled, asymptotically normal , in the sense that , under $Q$, they satisfy a martingale central limit theorem?

**Theorem 3.7** Assume that, for each $i, 1 \le i \le n$

$$\int_0^t E_Q[\frac{1_{\{Y_i<s\}}}{\lambda_i(s)}]ds < \infty$$

Then the processes $m^{\frac{1}{2}}[\widehat{E}^m - E]$ converge in distribution, under $Q$, to a continuous $n$ variate Gaussian martingale $M$ satisfying

$$< M_i, M_j >_t = 1_{\{i=j\}} \int_0^t E_Q \, \frac{1_{\{Y_i<s\}}}{\lambda_i(s)} ds$$

**Proof.** We verify the hypothesis of Theorem 3.2.

Jumps in the martingale $m^{\frac{1}{2}}[\widehat{E}^m - E]$ arise only from $\widehat{E}_i^m$, whose jumps are of size $\lambda_i^m(s)]^{-1} \sim n^{-1}$. Therefore, part B) of Theorem is true if we take $c_n = n^{-\frac{1}{4}}$.

To check part A) we note that

$$m\int_0^t \lambda_i^m(s)]^{-2} 1_{\{Y_i<s\}} \lambda_i^m(s) ds = m\int_0^t \lambda_i^m(s)^{-1} 1_{\{Y_i<s\}} ds \to$$

$$\int_0^t E_Q[\frac{1_{\{Y_i<s\}}}{\lambda_i(s)}]ds.$$

**Remark 1** To apply Theorem 3.7 it is necessary to estimate the variation process

$$\int_0^t \frac{m}{\lambda_i^m(s)} 1_{\{Y_i<s\}} ds$$

which converges to the process $\int_0^t E_Q[\frac{1_{\{Y_i<s\}}}{\lambda_i(s)}]ds$. Its martingale estimator is

$$\int_0^t (\lambda_i^m(t))^{-2} dN_i^m(s).$$

**Remark 2** We also can estimate the mean critical level $E[Y_i] = E[\int_0^t 1_{\{Y_i \geq s\}} ds$ with the same above procedure considering the probability measure $Q$ defined by

$$\prod_{i=1}^{n} [(1_{\{S_i \leq Y_i\}}) \exp \int_{Y_i}^{S_i} \lambda_i(s) ds].$$

We can say that a component $i$ is more critical than a component $j$ if and only if $E[Y_i] \leq E[Y_j]$.

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