Annali di Matematica Pura ed Applicata (1923 -)
https://doi.org/10.1007/s10231-024-01424-4

Fixed energy solutions to the Euler-Lagrange equations of an indefinite Lagrangian with affine Noether charge

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Received: 14 July 2023 / Accepted: 5 January 2024

Abstract

We consider an autonomous, indefinite Lagrangian L admitting an infinitesimal symmetry K whose associated Noether charge is linear in each tangent space. Our focus lies in investigating solutions to the Euler-Lagrange equations having fixed energy and that connect a given point p to a flow line $\gamma = \gamma(t)$ of K that does not cross p. By utilizing the invariance of L under the flow of K, we simplify the problem into a two-point boundary problem. Consequently, we derive an equation that involves the differential of the "arrival time" t, seen as a functional on the infinite dimensional manifold of connecting paths satisfying the semi-holonomic constraint defined by the Noether charge. When L is positively homogeneous of degree 2 in the velocities, the resulting equation establishes a variational principle that extends the Fermat's principle in a stationary spacetime. Furthermore, we also analyze the scenario where the Noether charge is affine.

Keywords Indefinite action functional · Noether charge · Fermat principle · Critical point theory

Mathematics Subject Classification 37J06 · 53C50 · 53C60 · 70H03

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Published online: 23 April 2024

1 Introduction

In the problem of finding solutions with fixed energy for an autonomous Lagrangian system with a finite number of degrees of freedom, subject to two-point or periodic boundary conditions, one viable approach is to allow for a free interval of parametrization for the involved curves. This method entails employing the action functional with fixed energy, as originally defined by Mañé (see [1] and the survey [2]). Specifically, given a pair of distinct points on a compact manifold M and a fiberwise convex and superlinear Lagrangian L, there always exists a solution connecting the two points with a fixed energy value κ, provided that κ is strictly greater than the so-called *Mañé critical value* $c(L)$ [3, Theorem X]. This approach has also demonstrated its effectiveness in addressing the challenging problem of establishing the existence of periodic solutions, as explored in works such as [4–7].

A more basic approach involves fixing the parameter interval and the initial point p, while allowing the final point to traverse along a given curve γ. This is a common framework employed in General Relativity when studying causal geodesics, which represent the paths of light rays (photons) or the worldlines of massive particles. In this scenario, the Lagrangian energy coincides with the conserved quantity of the geodesic, namely the square of the norm of its velocity vector. Consequently, the values $\kappa = 0, -1$ correspond to the energy levels of light rays and massive particles, respectively. The curve γ represents the worldline of an observer, while p signifies the event of emission or, in an alternative perspective, γ symbolizes the worldline of a source of light signals or massive particles, and p represents the event of detecting those signals (in the latter case, the geodesics originate from γ and terminate at p). In the former scenario, the parameter value of γ (more precisely, its "proper time") at the intersection point with a lightlike future-oriented curve z from p to γ is referred to as the *arrival time* of z [8]. The future-oriented lightlike geodesics are then all and only the stationary points of the arrival time with respect to any smooth variation z_ϵ made by smooth future-oriented lightlike curves between p and γ [9]. This general statement is recognized as *Fermat's principle in General Relativity*.

In fact, when the spacetime is static or stationary, Levi-Civita first introduced Fermat's principle in local coordinates in [10] and [11] and observed that the geometry of lightlike geodesics in the spacetime can be linked to a metric in a spacelike slice, called *optical metric*, which is Riemannian, when the spacetime is static, and Finslerian in the stationary case, and that allows describing and calculating various geometric and causal properties of the spacetime through it [12–16]. Subsequently, other variations of Fermat's principle have emerged, sharing fundamental elements while possessing distinct technical characteristics (see, e.g., [17–23]); moreover, it has been generalized to massive particles in [15, 24]. Light rays and massive particles via variational methods have also been studied in [25–30]. The use of Fermat's principle also emerges in the study of motion around black holes [31], as well as in the well-known phenomenon known as *gravitational lensing* [32–36], which has been specialized for particular exact solutions of the Einstein field equations, such as Schwarzschild [37] and NUT spacetimes [38]. Interestingly, the utilization of least-time principles in observational cosmology opens up the possibility of interpreting observed instances of gravitational lensing without the need to invoke the existence of dark matter [39]. We recommend referring readers to [40] for a comprehensive review on gravitational lensing in a relativistic context.

Additionally, it is worth noting that the application of variational principles related with or inspired by Fermat's principle has been extended and generalized beyond Euclidean or Lorentzian geometry as evidenced in [23, 41–50].

This work contributes to the latest type of research. We consider an indefinite Lagrangian L on a manifold M that is invariant under a one-dimensional group of local diffeomorphisms generated by a complete vector field K. The Noether charge associated with L is assumed to be linear in each tangent space T_xM. Our focus lies on solutions to the Euler-Lagrange equations of the action functional of L that connect a point p to a flow line γ of K and having fixed energy κ. Our approach is based on the variational setting in [51], which is inspired by [52]. The main result in this work, Theorem 5.1, at least when L is 2-homogeneous in the velocities, extends Fermat's principle as established in [53], that was specifically tailored to the framework explored in [52]. Leveraging this extension, we provide proof of existence (Theorem 6.9-(a)) and a multiplicity result (Theorem 6.9-(b)) for such solutions. Additionally, we delve into the analysis of the case where the Noether charge is an affine function in Appendix 1. In Appendix B we give some results that link an assumption on the manifold of curves that we consider in our variational setting, called *pseudocoercivity*, with the notion of global hyperbolicity for the cone structure associated with L in the 2-homogeneous case.

2 Notations, assumptions and a class of examples

Let M be a smooth, connected manifold of dimension $(m+1)$, where $m \geq 1$. We denote the tangent bundle of M as TM. In this paper, we consider a Riemannian metric g on M as an auxiliary metric, and we use $\|\cdot\| : TM \to \mathbb{R}$ to represent its induced norm; specifically, for any $v \in TM$, we have $\|v\|^2 = g(v, v)$. We represent an element of TM as a pair (x, v), where x belongs to M and v belongs to the tangent space T_xM.

Let $L : TM \to \mathbb{R}$ be a Lagrangian on M. For any $(x, v) \in TM$, we denote the vertical derivative of L as $\partial_v L(x, v)[\cdot]$, which is is defined as follows:

$$\partial_v L(x, v)[\xi] = \frac{\mathrm{d}}{\mathrm{d}s} L(x, v + s\xi)\bigg|_{s=0}, \quad \forall \xi \in T_xM.$$

We also need a derivative w.r.t. x, denoted by $\partial_x L(x, v)$. This is defined only locally (in a system of coordinates) as:

$$\partial_x L(x, v)[\xi] = \sum_{i=0}^{m} \frac{\partial L}{\partial x^i}(x, v)\xi^i, \quad \forall \xi \in T_xM,$$

where $(x^0, \dots, x^m)$ is a local coordinate system in a neighbourhood of x, and consequently, $(x^0, \dots, x^m, v^0, \dots, v^m)$ are the induced coordinates on TM. With this notation, the Euler-Lagrange equations for a curve $z : [0, 1] \to M$ of class C^1 are given by:

$$\frac{\mathrm{d}}{\mathrm{d}s} \partial_v L(z, \dot z) - \partial_x L(z, \dot z) = 0, \quad \forall s \in [0, 1], \tag{2.1}$$

where $\dot z$ denotes the derivative of z with respect to the parameter s. It is well-known that the *energy function* $E : TM \to \mathbb{R}$, defined as:

$$E(x, v) = \partial_v L(x, v)[v] - L(x, v),$$

is a first integral of the Lagrangian system. Therefore, if $z : [0, 1] \to M$ is a solution of the Euler-Lagrange equations, there exists a constant $\kappa \in \mathbb{R}$ such that:

$$E(z, \dot{z}) = \kappa, \quad \forall s \in [0, 1]. \tag{2.2}$$

Assumption 1 The Lagrangian $L : TM \to \mathbb{R}$ satisfies the following conditions:

- L is a C^1 function on TM;
- There exists a complete C^3 vector field K on M such that L is invariant under the one-parameter group of C^3 diffeomorphisms of M generated by K (we refer to K as an *infinitesimal symmetry of* L);
- The *Noether charge*, i.e., the map $(x, v) \in TM \mapsto \partial_v L(x, v)[K] \in \mathbb{R}$, is a C^1 one-form Q on M, namely

$$\partial_v L(x, v)[K] = Q(v); \tag{2.3}$$

- For every $x \in M$, the following equality holds:

$$Q(K) = -1 \tag{2.4}$$

Remark 2.1 In [51], the Noether charge was assumed to be an *affine* function on each tangent space. For the sake of simplicity, we present the main results under the more restrictive assumption of linearity. A discussion about the affine case is given in Appendix 1.

Remark 2.2 The C^3 regularity condition on K is needed to get that a certain map constructed by using the flow of K is a diffeomorphism (see Proposition 4.1). We don't know if the regularity of K can be lowered there to C^1.

Assumption 2 The Lagrangian $L_c : TM \to \mathbb{R}$, defined by

$$L_c(x, v) := L(x, v) + Q^2(v), \tag{2.5}$$

satisfies the following conditions:

- there exists a continuous function $C : M \to (0, +\infty)$ such that for all $(x, v) \in TM$, the following inequalities hold:

$$L_c(x, v) \le C(x)\big(\|v\|^2 + 1\big); \tag{2.6}$$

$$|\partial_x L_c(x, v)| \le C(x)\big(\|v\|^2 + 1\big); \tag{2.7}$$

$$|\partial_v L_c(x, v)| \le C(x)\big(\|v\| + 1\big); \tag{2.8}$$

- there exists a continuous function $\lambda : M \to (0, +\infty)$ such that for each $x \in M$ and for all $v_1, v_2 \in T_x M$, the following inequality holds:

$$\big(\partial_v L_c(x, v_2) - \partial_v L_c(x, v_1)\big)[v_2 - v_1] \ge \lambda(x)\|v_2 - v_1\|^2; \tag{2.9}$$

Remark 2.3 As proven in [51, Proposition 2.5], K is also an infinitesimal symmetry of L_c, and a simple computation shows that

$$\partial_v L_c(x, v)[K] = -Q(v).$$

Remark 2.4 From [51, Proposition 7.4], if L satisfies Assumptions 1 and 2, it admits a *stationary product type local structure*. This means that for each point $p \in M$, there exists a neighbourhood $U_p \subset M$, an open neighborhood S_p of $\mathbb{R}^m$, an open interval I_p of $\mathbb{R}$, and a diffeomorphism $\phi : S_p \times I_p \to U_p$ such that, denoting t as the natural coordinate of I_p,

$$\phi(\partial_t) = K|_{U_p},$$

and the function L can be expressed as follows:

$$L(x, v) = L\circ\phi\big((y, t), (v, \tau)\big) = L_0(y, v) + \omega_y(v)\tau - \frac{1}{2}\tau^2, \tag{2.10}$$

where

- $(y, t) \in S_p \times I_p$, $(v, \tau) \in \mathbb{R}^m \times \mathbb{R}$, and $(x, v) = \phi\big((y, t), (v, \tau)\big)$;
- $L_0 \in C^1(S_p)$ is a Lagrangian that satisfies the growth conditions (2.6)–(2.8) with respect to the norm $\|\cdot\|_{S_p}$ and it is fiberwise strongly convex, i.e., (2.9) holds (with L_c replaced by L_0) for some function $\lambda : S_p \to (0, +\infty)$;
- ω_y is the C^1 one-form induced by Q on S_p.

Using this notation, we have the following equalities:

$$\begin{aligned} Q(v) &= Q\circ\phi(v, \tau) = \omega_y(v) - \tau; \\ L_c(x, v) &= L_c\circ\phi_*\big((y, t), (v, \tau)\big) = L_0(y, v) + \omega_y^2(v) - \omega_y(v)\tau + \frac{1}{2}\tau^2; \end{aligned} \tag{2.11}$$

and

$$E(x, v) = E\circ\phi_*\big((y, t), (v, \tau)\big) = E_0(y, v) + \omega_y(v)\tau - \frac{1}{2}\tau^2, \tag{2.12}$$

where $E_0(y, v) = \partial_v L_0(y, v)[v] - L_0(y, v)$.

Remark 2.5 For every $p \in M$, let $\phi_p : S_p \times I_p \to M$ be a mapping that satisfies (2.10). Since L_0 is fiberwise strongly convex, we can conclude that

$$E_0(y, v) > E_0(y, 0) = -L_0(y, 0), \qquad \forall v \neq 0, \tag{2.13}$$

Indeed, from the strict convexity of L_0 we have

$$L_0(y, 0) > L(y, v) + \partial_v L(y, v)[-v], \qquad \forall v \neq 0.$$

Assumption 3 We require:

$$\sup_{x\in M} L(x, 0) < +\infty. \tag{2.14}$$

Remark 2.6 We need the last assumption to guarantee the existence of $\kappa \in \mathbb{R}$ satisfying (4.19), which is a key condition for our main result.

2.1 Lorentz-Finsler metrics

Provided the existence of an infinitesimal symmetry, an important kind of Lagrangians that satisfy the above assumptions is given by Lorentz-Finsler metrics, introduced by J. K. Beem in [54].

Definition 2.7 Let M be a smooth, connected manifold of dimension $m+1$. A Lagrangian $L_F : TM \to \mathbb{R}$ is called *Lorentz-Finsler metric* if it satisfies the following conditions:

(a) $L_F \in C^1(TM) \cap C^2(TM \setminus 0)$, where 0 denotes the zero section of TM;

(b) $L_F(x, \lambda v) = \lambda^2 L_F(x, v)$, for all $\lambda > 0$;

(c) for any $(x, v) \in TM \setminus 0$, the vertical Hessian of L_F, i.e. the symmetric matrix

$$(g_F)_{\alpha\beta}(x, v) := \frac{\partial^2 L_F}{\partial v^\alpha \partial v^\beta}(x, v), \quad \alpha, \beta = 0, \dots, m,$$

is non-degenerate with index 1.

Remark 2.8 The regularity conditions required on L_F are sometimes too rigid and we relax them to include some interesting classes of Lagrangians (see [46, 55]). The first and the last conditions above will be replaced by:

(a') Let M be a smooth, connected, manifold of dimension $m+1$, $m \geq 1$, and $L_F \in C^1(TM) \cap C^2(\mathcal{O})$, where $\mathcal{O} \subset TM \setminus 0$ is such that $\mathcal{O}_x := \mathcal{O} \cap T_xM \neq \emptyset$ for all $x \in M$, and $\mathcal{O}_x$ is an open set in T_xM which is a linear cone (i.e. $\lambda v \in \mathcal{O}_x$, for all $\lambda > 0$, if $v \in \mathcal{O}_x$); moreover, for any $v_1, v_2 \in T_xM$ there exist two sequences of vectors v_{1k}, v_{2k} such that, for all $k \in \mathbb{N}$, the segment with extreme points v_{1k} and v_{2k} is entirely contained in $\mathcal{O}_x$ and $v_{ik} \to v_i$, $i = 1, 2$.

(c') Condition (c) is valid for each $(x, v) \in \mathcal{O}$; moreover the eigenvalues $\lambda_i(x, v)$ of $(g_F)_{\alpha,\beta}(x, v)$ are bounded away from 0 on $\mathcal{O}_x$, i.e. there exists $\lambda_+(x) > 0$ such that

$$|\lambda_i(x, v)| \geq \lambda_+(x), \tag{2.15}$$

for all $i \in \{0, \dots, m\}$ and $v \in \mathcal{O}_x$.

If L_F is a Lorentz-Finsler metric, then the couple (M, L_F) is called a *Finsler spacetime*.

The study of the notion of a Finsler spacetime has received renewed impetus from various sources. V. Perlick's work [41], which explores Fermat's principle, was particularly influential. Subsequent contributions came from [56] (also see [57]), which revived the research initiated by G. Y. Bogoslowsky [58–60], and from [61]. Additional momentum was provided by the works of V. A. Kostelecký and collaborators [62–65], as well as C. Pfeifer, N. Voicu, and their coworkers (refer to [66–71] for further details). Notable mathematical contributions include [72–74], which have influenced the field in a different manner. For a comprehensive historical overview, diverse definitions of a Finsler spacetime, and additional references, interested readers are directed to [44, 48, 75, 76].

As we will see later, the significance of Lorentz-Finsler metrics relies on the 2-homogeneity assumption. This homogeneity ensures that the solutions of the Euler-Lagrange equations, with a suitably prescribed energy value (in this case, less than or equal to 0), connecting a point to a flow line of the infinitesimal symmetry vector field, are the ones for which the time of arrival is critical. Therefore, Fermat's principle holds (see Remark 5.2).

Proposition 2.9 *Let $L_F : TM \to \mathbb{R}$ be a Lorentz-Finsler metric satisfying (a'), (b) and (c') above, and assume there exists an infinitesimal symmetry $K : M \to TM$ such that (2.3) and (2.4) hold. Then Assumptions 2 and 3 hold.*

Proof Assumption 3 is ensured by the 2–homogeneity of L_F, since $L_F(x,0) = 0$ for every $x \in M$. Let us show that Assumption 2 holds. As a first step, we notice that the Lagrangian $L_c : TM \to \mathbb{R}$, defined by

$$L_c(x,v) = L_F(x,v) + Q^2(v),$$

admits vertical Hessian at any $(x,v) \in \mathcal{O}$ that is a positive definite bilinear form on T_xM. For any $(x,v) \in \mathcal{O}$, we have

$$\partial_{vv} L_c(x,v) = \partial_{vv} L_F(x,v) + 2Q \otimes Q. \tag{2.16}$$

For each $w \in T_xM$, we have, thanks to (2.3),

$$\begin{aligned} \partial_{vv} L_F(x,v)[K,w] &= \frac{\partial^2 L_F}{\partial s \partial t}(x, v + tK + sw)\Big|_{(s,t)=(0,0)} \\ &= \frac{\partial(\partial_v L_F(x, v+sw)[K])}{\partial s}\Big|_{s=0} = \frac{\partial Q(v+sw)}{\partial s}\Big|_{s=0} = Q(w), \end{aligned} \tag{2.17}$$

hence we obtain

$$\partial_{vv} L_c(x,v)[K,K] = \partial_{vv} L_F(x,v)[K,K] + 2Q^2(K) = Q(K) + 2 = 1 > 0.$$

Now consider $w \in \ker Q$; from (2.17) we have $\partial_{vv} L_F(x,v)[w,K] = 0$, and since $\partial_{vv} L_F(x,v)$ has index 1 we obtain that $\partial_{vv} L_c(x,v)[w,w] = \partial_{vv} L_F(x,v)[w,w] > 0$, for all $w \in \ker Q$, from which we conclude that $\partial_{vv} L_c(x,v)[\cdot,\cdot]$ is positive definite.

Let

$$\lambda(x) : \inf_{v \in \mathcal{O}_x} \min_{w \in T_xM, \|w\|=1} \partial_{vv} L_c(x,v)[w,w]. \tag{2.18}$$

As any $w \in \ker Q$ is orthogonal to K_x with respect to both bilinear forms $\partial_{vv} L_c(x,v)$ and $\partial_{vv} L_F(x,v)$, by (2.16) we deduce that the determinants of $(g_F)_{\alpha\beta}$ and $(g_c)_{\alpha\beta}$ are opposite numbers and then, from (2.15) we conclude that $\lambda(x) > 0$, for all $x \in M$.

Inequality (2.9) then follows by the mean value theorem applied to the function $v \in \mathcal{O}_x \mapsto \partial_v L_c(x,v)[v_2 - v_1]$, when v_1 and v_2 both belong to $\mathcal{O}_x$ and the segment having them as extreme points is contained in $\mathcal{O}_x$ as well. Then, for each $x \in M$, (2.9) follows by continuity due to the property of approximation by segments in (a'). The inequalities (2.6), (2.7) and (2.8) are ensured by the fact that L_c is C^1 on TM and it is positive homogeneous of degree 2 w.r.t. v. □

Remark 2.10 As shown in the above proof, the vertical Hessian of L_c is positive definite on $\mathcal{O}$, the last being dense in TM. Hence, by homogeneity, L_c is a non-negative fiberwise strongly function on TM. Moreover, the vertical Hessian of $F_c := \sqrt{L_c}$ at any $(x,v) \in \mathcal{O}$ is positive semi-definite (see, e.g., [86, p. 8]). Hence, for any v_1 and v_2 belonging to $\mathcal{O}_x$ defining a segment contained in $\mathcal{O}_x$, we get by Taylor's theorem,

$$F_c(x, v_2) \geq F_c(x, v_1) + \partial_v F_c(x, v_1)[v_2 - v_1].$$

By continuity and the approximation by segments property in (a'), the above inequality holds on TM, hence F_c is fiberwise convex and therefore it is a Finsler metric on M, (i.e., $F_c(x,\cdot)$ is non-negative, positively homogeneous, and satisfies the triangle inequality on T_xM, for each $x \in M$) whose square is only of class C^1 on TM.

As a consequence of Proposition 2.9, if $L_F : TM \to M$ is a Lorentz-Finsler metric and there exists a complete vector field K such that Assumption 1 holds, then Remark 2.4 ensures that L_F can be locally expressed as follows:

$$L_F(x,v) = L_F \circ \phi((y,t),(v,\tau)) = F^2(y,v) + \omega_y(v)\tau - \frac{1}{2}\tau^2, \tag{2.19}$$

where $F : TS \to \mathbb{R}$ is a Finsler metric on S, with $F^2 \in C^1(TM)$. Whenever L_F is not twice differentiable only at the line sub-bundle of TM defined by K, F becomes a classical Finsler metric on S, (i.e. $F^2 \in C^2(TS\setminus 0)$ and, for each $y \in S$, $F(y,\cdot)$ is a Minkowski norm on T_yS, see e.g. [86, §1.2]).

Since in this case K is a *timelike Killing vector field*, namely it is an infinitesimal symmetry of L_F such that $L_F(x,K) < 0$ for every x, (M, L_F) is called *stationary* Finsler spacetime. In particular, if L_F is twice differentiable on $TM \setminus 0$, then $F^2(y,\cdot)$ in (2.19) must be the square of the norm of a positive definite inner product on T_yS. We thank the referee for this observation. In fact, a special kind of stationary Finsler spacetimes are the *stationary Lorentzian manifolds*, namely those Lorentzian manifolds (M, g_L) for which g_L is a Lorentzian metric and there exists a timelike Killing vector field for g_L. In this case, the stationary product type local structure is given by

$$g_L(v,v) = g_R(v,v) + \omega(v)\tau - \frac{1}{2}\tau^2,$$

where g_R is a Riemannian metric on an open neighbourhood S of $\mathbb{R}^m$. In this direction, the results in this paper improve previous results about stationary Lorentzian metrics (see, [15, 19, 53]), since just C^1 stationary metrics with a C^3 timelike Killing vector field are allowed and both lightlike and timelike geodesics can be considered in an unified setting.

3 Variational setting

Let us fix a point $p \in M$ and consider a flow line $\gamma : \mathbb{R} \to M$ of K that does not pass through p, i.e., $p \notin \gamma(\mathbb{R})$. We are interested in finding solutions of the Euler-Lagrange equations that connect p to points on γ with a fixed energy $\kappa \in \mathbb{R}$. Specifically, we seek to characterize curves $z \in C^1([0,1], M)$ that satisfy (2.1), with $z(0) = p$, $z(1) \in \gamma(\mathbb{R})$, and $E(z(s), \dot z(s)) = \kappa$ for all $s \in [0,1]$.

We define the action functional $\mathcal{L} : H^1([0,1], M) \to \mathbb{R}$ as follows:

$$\mathcal{L}(z) := \int_0^1 L(z, \dot{z})\,\mathrm{d}s.$$

Similarly, we define the energy functional:

$$\mathcal{E}(z) := \int_0^1 E(z, \dot{z})\,\mathrm{d}s.$$

We note that both $\mathcal{L}$ and $\mathcal{E}$ are well-defined on $H^1([0,1], M)$ and they are respectively a C^1 and a C^0 functional due to (2.5), the growth conditions (2.6)–(2.8) and the fiberwise convexity of L_c (2.9) (see, e.g., the first part of the proof of Proposition 3.1 in [78]).

Remark 3.1 Henceforth, we will assume that $\mathcal{E}$ is a C^1 functional. This holds if L is positively homogeneous of degree 2 in the velocities, since in that case $\mathcal{E} = \mathcal{L}$; moreover it holds if L_c is a C^2, strongly convex Lagrangian on TM with second derivatives satisfying assumptions (L1') in [78, p. 605].

Recalling that we have chosen a fixed point $p \in M$, we define the set $\Omega_{p,r}(M)$ for every $r \in M$ as follows:

$$\Omega_{p,r}(M) := \big\{ z \in H^1([0,1], M) \,:\, z(0) = p, z(1) = r \big\},$$

and we denote by $\mathcal{L}_{p,r}$ the restriction of $\mathcal{L}$ to $\Omega_{p,r}(M)$.

Remark 3.2 According to [51, Proposition A.1], if z is a critical point of $\mathcal{L}_{p,r}$, then both z and the function

$$s \mapsto \partial_v L(z(s), \dot{z}(s))[\dot{z}(s)]$$

are of class C^1. As a consequence, z is a critical point of $\mathcal{L}_{p,r}$ if and only if equation (2.1) holds and there exists $\kappa \in \mathbb{R}$ such that equation (2.2) holds.

3.1 Preliminary results

Recalling that K is a complete vector field, we denote by $\psi : \mathbb{R} \times M \to M$ the flow of K, and by $\partial_u \psi$ and $\partial_x \psi$ the partial derivatives of $\psi(t, x)$ with respect to $t \in \mathbb{R}$ and $x \in M$, respectively.

Let us denote by K^c the complete lift of K to TM (see, e.g., [46]). Then, for any $(x, v) \in TM$, the flow ψ^c of K^c on TM is given by $\psi^c(t, x, v) = \big(\psi(t, x), \partial_x \psi(t, x)[v]\big)$, and we have

$$K^c(L)\big(\psi^c(t, x, v)\big) = \frac{\partial\big(L \circ \psi^c\big)}{\partial t}(t, x, v).$$

Since K is an infinitesimal symmetry of L, we have

$$\frac{\partial\big(L \circ \psi^c\big)}{\partial t}(t, x, v) = 0, \tag{3.1}$$

which implies

$$K^c(L)(x,v) = \frac{\partial L}{\partial x^h}(x,v)K^h(x) + \frac{\partial L}{\partial v^h}(x,v)\frac{\partial K^h}{\partial x^i}(x)v^i = 0. \tag{3.2}$$

Moreover, from (3.1) we also obtain

$$L(x,v) = L\big(\psi(t,x), \partial_x\psi(t,x)[v]\big), \quad \forall (x,v) \in TM,\ t \in \mathbb{R}, \tag{3.3}$$

and consequently

$$\partial_v L(x,v)[\xi] = \partial_v L\big(\psi(t,x), \partial_x\psi(t,x)[v]\big)\big[\partial_x\psi(t,x)[\xi]\big]. \tag{3.4}$$

Lemma 3.3 *If $z : [0,1] \to M$ is a weak solution of the Euler-Lagrange equation (2.1) (i.e. a critical point of $\mathcal{L}$ on $\Omega_{z(0),z(1)}(M)$), then it is a C^1 curve and its Noether charge is constant, namely there exists $c \in \mathbb{R}$ such that*

$$\partial_v L(z(s),\dot z(s))[K(z(s))] = c, \quad \forall s \in [0,1].$$

Proof By [51, Proposition A.1], both z and $\partial_v L(z,\dot z)$ are of class C^1. Therefore, it suffices to prove that, for every $s \in [0,1]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\partial_v L(z(s),\dot z(s))[K(z(s))]\Big) = 0.$$

Therefore, we can work on a local coordinate system $(x^0,\dots,x^m,v^0,\dots,v^m)$ of TM and, using (2.1) and (3.2), we obtain the following chain of equalities:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)K^i(z(s))\Big)\\
&\quad= \frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)\Big)K^i(z(s)) + \frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)\frac{\partial K^i}{\partial x^h}(z(s))\dot z^h(s)\\
&\quad= \frac{\partial L}{\partial x^i}\big(z(s),\dot z(s)\big)K^i(z(s)) + \frac{\partial L}{\partial v^i}\big(z(s),\dot z(s)\big)\frac{\partial K^i}{\partial x^h}(z(s))\dot z^h(s) = 0.
\end{aligned}$$

□

On the basis of Lemma 3.3, the curves with a constant Noether charge are the only ones that can be critical points of the action functional. The following results ensure that this subset of curves is indeed a closed manifold of class C^1, allowing for a simplification of the variational setting by considering only these curves. A detailed proof can be found in [51] and relies on the linearity assumption of the Noether charge.

Let us define the following sets:

$$\mathcal{N}_{p,r} := \{z \in \Omega_{p,r}(M) : \exists c \in \mathbb{R} \text{ such that } Q(\dot z) = c \text{ a.e. on } [0,1]\} \subset \Omega_{p,r}(M),$$

and

$$\begin{aligned}\mathcal{W}_z := \big\{\eta \in T_z\Omega_{p,r}(M) : \exists \mu \in H^1_0([0,1],\mathbb{R})\\ \text{such that } \eta(s) = \mu(s)K(z(s)), \text{ a.e. on } [0,1]\big\}.\end{aligned}$$

Since L is invariant under the one-parameter group of local C^1 diffeomorphisms generated by K, we have the following result.

Proposition 3.4 *The space $\mathcal{N}_{p,r}$ is non-empty, it is a C^1 closed submanifold of $\Omega_{p,r}(M)$ and satisfies*

$$\mathcal{N}_{p,r} = \left\{z \in \Omega_{p,r}(M) : \mathrm{d}\mathcal{L}_{p,r}(z)[\eta] = 0, \forall \eta \in \mathcal{W}_z\right\}. \tag{3.5}$$

Moreover, for every $z \in \mathcal{N}_{p,r}$, the tangent space of $\mathcal{N}_{p,r}$ at z is given by

$$T_z\mathcal{N}_{p,r} = \left\{\xi \in T_z\Omega_{p,r}(M) : \exists c \in \mathbb{R} \text{ such that } \partial_x Q(\dot z)[\xi] + Q(\dot\xi) = c \text{ a.e.}\right\}, \tag{3.6}$$

and

$$T_z\Omega_{p,r}(M) = T_z\mathcal{N}_{p,r} + \mathcal{W}_z. \tag{3.7}$$

Proof The fact that $\mathcal{N}_{p,r} \neq \emptyset$, for all $p, r \in M$, follows from [51, proposition 6.4]. Equality (3.5) is proved in [51, Proposition 4.2], and (3.6) is a particular case of [51, Proposition 4.3]. Finally, (3.7) comes from [51, Lemma 4.4],[1]. □

The above result gives the following variational principle for the critical points of $\mathcal{L}_{p,r}$, which extends a result by F. Giannoni and P. Piccione (see [52]).

Proposition 3.5 *Let $\mathcal{J}_{p,r} : \mathcal{N}_{p,r} \to \mathbb{R}$ be the restriction of $\mathcal{L}_{p,r}$ to $\mathcal{N}_{p,r}$. Then, z is a critical point for $\mathcal{L}_{p,r}$ if and only if $z \in \mathcal{N}_{p,r}$ and z is a critical point for $\mathcal{J}_{p,r}$.*

Proof See [51, Theorem 4.7]. □

4 The variational structure of the action in relation with the flow of *K*

In this section, we consider the flow of the complete vector field K and its relationship with the variational structure of the action. More precisely, let $\psi : \mathbb{R} \times M \to M$ denote the flow generated by the vector field K. Given a flow line $\gamma : \mathbb{R} \to M$ of K, there exists a point $q \in M$ such that $\gamma(t) = \psi(t, q)$.

Our goal is to prove that for each $t \in \mathbb{R}$, there is a diffeomorphism between $\mathcal{N}_{p,q}$ and $\mathcal{N}_{p,\gamma(t)}$. This enables us to define a functional (see (4.9)) on $\mathcal{N}_{p,q} \times \mathbb{R}$ and obtain an alternative equation for solutions of the Euler-Lagrange equations connecting p and γ (see (4.13)). Furthermore, recalling that we seek the solutions of Euler-Lagrange equations with a fixed energy κ, we show that for any $z \in \mathcal{N}_{p,q}$, there are two values of $t \in \mathbb{R}$ such that $\mathcal{E}(z^t) = \kappa$, where κ satisfies (4.19) and $z^t \in \mathcal{N}_{p,\gamma(t)}$ is the curve corresponding to z via the diffeomorphism. Therefore, we can simplify the problem and study a couple of functionals defined only on $\mathcal{N}_{p,q}$ (see (4.20)).

Let us define the map $F^t : \Omega_{p,q}(M) \to \Omega_{p,\gamma(t)}(M)$ as follows:

[1] We would like to draw attention to a misprint in [51, Lemma 4.4] where we note that the "direct sum" should be corrected to "sum" as it appears there.

$$\big(F^t(z)\big)(s) := \psi(ts, z(s)). \tag{4.1}$$

To simplify the notation, we write

$$z^t = F^t(z)$$

for any $z \in \Omega_{p,q}(M)$.

Proposition 4.1 *The map F^t is a diffeomorphism with its inverse being F^{-t}. Furthermore, $F^t|_{\mathcal{N}_{p,q}}$ is a diffeomorphism from $\mathcal{N}_{p,q}$ to $\mathcal{N}_{p,\gamma(t)}$. Therefore, for every $z \in \Omega_{p,q}(M)$, we have the following equivalences*:

$$\mathrm{d}F^t(z)[\xi] \in T_{z^t}\mathcal{N}_{p,\gamma(t)} \quad \text{if and only if} \quad \xi \in T_z\mathcal{N}_{p,q}, \tag{4.2}$$

and

$$\mathrm{d}F^t(z)[\eta] \in \mathcal{W}_{z^t} \quad \text{if and only if} \quad \eta \in \mathcal{W}_z. \tag{4.3}$$

Proof By utilizing a result by R. Palais [77] and considering that the flow of K is C^3, we can conclude that F^t is a diffeomorphism (cf. [53, Proposition 2.2]). Recalling that $\partial_u\psi$ is the differential of ψ with respect to the first variable, we can derive the following equalities:

$$\partial_u\psi(ts, z(s))[1] = K(\psi(ts, z(s))),$$

and

$$\partial_x\psi(ts, z(s))[K(z(s))] = K(\psi(ts, z(s))). \tag{4.4}$$

Consequently, we obtain the velocity of z^t as:

$$\frac{\mathrm{d}}{\mathrm{d}s} z^t(s) = \dot z^t(s) = \partial_u\psi(ts, z(s))[t] + \partial_x\psi(ts, z(s))[\dot z(s)]. \tag{4.5}$$

Now, considering that $Q(K) \equiv -1$, we deduce:

$$\partial_v L(z^t, \dot z^t)[K(z^t)] = Q(\dot z^t) = -t + Q\big(\partial_x\psi(ts, z(s))[\dot z(s)]\big).$$

Hence, from (3.4), we have:

$$Q(\dot z^t) = Q(\dot z) - t, \tag{4.6}$$

which implies that $z^t \in \mathcal{N}_{p,\gamma(t)}$ if and only if $z \in \mathcal{N}_{p,q}$. Therefore, this implies (4.2). Finally, (4.3) follows from $\mathrm{d}F^t(z)[v] = \partial_x\psi(ts, z(s))[v(s)]$ and (4.4). □

We introduce the functional $\mathcal{H}_{p,q} : \Omega_{p,q}(M) \times \mathbb{R} \to \mathbb{R}$ defined as follows:

$$\mathcal{H}_{p,q}(z, t) := \mathcal{L}_{p,\gamma(t)}(F^t(z)). \tag{4.7}$$

Using (4.5) and observing that $\partial_u\psi(ts, z(s))[t] = t\partial_x\psi(ts, z(s))[K(z(s)]$, we can deduce the expression:

$$\dot z^t = \partial_x\psi(ts, z(s))\big[\dot z + tK(z(s))\big], \tag{4.8}$$

so that, by applying also (3.3), we can rewrite $\mathcal{H}_{p,q}(z, t)$ as

$$\mathcal{H}_{p,q}(z,t) = \int_0^1 L\big(z, \dot z + tK(z)\big)\,\mathrm{d}s. \tag{4.9}$$

Considering that $F^t|_{\mathcal{N}_{p,q}}$ is a diffeomorphism, we obtain the following result, which allows us to focus our study on critical curves of $\mathcal{H}_{p,q}$ within $\mathcal{N}_{p,q}$.

Proposition 4.2 *For* $(z,t) \in \Omega_{p,q}(M) \times \mathbb{R}$, *the following statements hold*:

$$\partial_z \mathcal{H}_{p,q}(z,t)[\xi] = 0, \qquad \forall \xi \in T_z\Omega_{p,q}(M), \tag{4.10}$$

if and only if $z \in \mathcal{N}_{p,q}$ *and*

$$\partial_z \mathcal{H}_{p,q}(z,t)[\xi] = 0, \qquad \forall \xi \in T_z\mathcal{N}_{p,q}. \tag{4.11}$$

Proof If (4.10) holds, we can use (4.7) and Proposition 4.1 to conclude that $z^t = F^t(z)$ is a critical point of $\mathcal{L}_{p,\gamma(t)}$, and by Proposition 3.5, z^t belongs to $\mathcal{N}_{p,\gamma(t)}$. Consequently, we have $z = F^{-t}(z^t) \in \mathcal{N}_{p,q}$, and (4.11) trivially follows from (4.10).

For the other implication, we need to show that if $z \in \mathcal{N}_{p,q}$, then

$$\partial_z \mathcal{H}_{p,q}(z,t)[\eta] = 0, \qquad \forall \eta \in \mathcal{W}_z. \tag{4.12}$$

By contradiction, let's assume that $z \in \mathcal{N}_{p,q}$ and (4.12) does not hold. According to the definition of $\mathcal{H}_{p,q}$, there exists $\eta \in \mathcal{W}_z$ such that

$$\partial_z \mathcal{H}_{p,q}(z,t)[\eta] = \mathrm{d}\mathcal{L}_{p,\gamma(t)}(F^t(z))\big[\mathrm{d}F^t(z)[\eta]\big] \neq 0.$$

Using (4.3), we know that $\mathrm{d}F^t(z)[\eta] \in \mathcal{W}_{z^t}$. Applying Proposition 3.4, we can conclude that $F^t(z) \notin \mathcal{N}_{p,\gamma(t)}$, which contradicts Proposition 4.1. □

Corollary 4.3 *If* $(z,\ t)$ *satisfies* (4.10), *then* z^t *is a critical point for* $\mathcal{L}_{p,\gamma(t)}$. *The following Euler-Lagrange equations* (*in local coordinates*) *hold*:

$$\begin{aligned}\frac{\partial L}{\partial x^i}\big(z, \dot z + tK(z)\big) - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial v^i}\big(z, \dot z + tK(z)\big)\\ + t\frac{\partial L}{\partial v^j}\big(z, \dot z + tK(z)\big)\frac{\partial K^j}{\partial x^i}(z) = 0, \quad \forall s \in [0,1],\end{aligned} \tag{4.13}$$

and there exists $\kappa \in \mathbb{R}$ *such that*

$$E\big(z, \dot z + tK(z)\big) = \kappa, \quad \forall s \in [0,1]. \tag{4.14}$$

Proof According to Proposition 4.2, if (4.10) holds, then (4.13) is an immediate consequence of (4.9) and the du Bois-Reymond lemma. By (4.7), $z^t = F^t(z)$ is a critical point of $\mathcal{L}$ on $\Omega_{p,\gamma(t)}(M)$. Hence, using Remark 3.2, we can conclude that there exists a constant κ such that $E(z^t, \dot z^t) = \kappa$. Combining (3.3), (3.4), and (4.8), we obtain (4.14). □

Proposition 4.4 *For every* $(x,v) \in TM$ *and every* $t \in \mathbb{R}$, *the following two equations hold*:

$$L\big(x, v + tK(x)\big) = L\big(x,v\big) + tQ(v) - \frac{1}{2}t^2, \tag{4.15}$$

and

$$E\big(x, v + tK(x)\big) = E\big(x, v\big) + tQ(v) - \frac{1}{2}t^2. \tag{4.16}$$

As a consequence, for every $(x, v) \in TM$, *we have*

$$L\big(x, v + tK(x)\big) - E\big(x, v + tK(x)\big) = L\big(x, v\big) - E\big(x, v\big). \tag{4.17}$$

Proof We will prove (4.15); the computations for (4.16) are analogous. Since the result has a local nature, we can use (2.10). For every $(x, v) \in TM$, we can write

$$\begin{aligned} L(x, v + tK) &= L\circ\phi_*\big((y, t), (v, \tau + t)\big) \\ &= L_0(y, v) + \omega_y(v)(\tau + t) - \frac{1}{2}(\tau + t)^2 \\ &= \Big(L_0(y, v) + \omega_y(v)\tau - \frac{1}{2}\tau^2\Big) + \big(\omega_y(v) - \tau\big)t - \frac{1}{2}t^2 \\ &= L(x, v) + tQ(v) - \frac{1}{2}t^2. \end{aligned}$$

This completes the proof. □

Using (4.15) and recalling that $Q(\dot z)$ is constant for all $z \in \mathcal{N}_{p,q}$, the functional $\mathcal{H}_{p,q}$ can be written as:

$$\mathcal{H}_{p,q}(z, t) = \int_0^1 L\big(z, \dot z\big)\mathrm{d}s + tQ(\dot z) - \frac{1}{2}t^2 = \mathcal{L}(z) + tQ(\dot z) - \frac{1}{2}t^2. \tag{4.18}$$

Proposition 4.5 *Let*

$$\kappa \le -\sup_{x\in M} L(x, 0) \tag{4.19}$$

(*recall* (2.14)). *Then the functionals* $t_+^\kappa, t_-^\kappa : \mathcal{N}_{p,q} \to \mathbb{R}$ *defined by*

$$t_\pm^\kappa(z) = Q(\dot z) \pm \sqrt{Q^2(\dot z) + 2\big(\mathcal{E}(z) - \kappa\big)}, \tag{4.20}$$

are well-defined, and they satisfy the following equation:

$$\mathcal{E}(F^{t_\pm^\kappa(z)}(z)) = \kappa. \tag{4.21}$$

Proof Since $Q(\dot z)$ is constant for every $z \in \mathcal{N}_{p,q}$, from (4.16) we have that $t_\pm^\kappa(z)$ are the only two solutions of

$$\mathcal{E}(F^t(z)) = \mathcal{E}(z^t) = \mathcal{E}(z) + tQ(\dot z) - \frac{1}{2}t^2 = \kappa.$$

Hence, it remains to prove that for every $z \in \mathcal{N}_{p,q}$, we have

$$\mathcal{E}(z) + \frac{1}{2}Q^2(\dot z) \ge \kappa,$$

provided that κ satisfies (4.19). As a consequence, it suffices to prove that

$$E(x,v) + \frac{1}{2}Q^2(v) \geq \kappa, \qquad \forall (x,v) \in TM. \tag{4.22}$$

Using the expression of L in a local chart in a neighbourhood of $x \in M$, in particular (2.11) and (2.12), and setting $(x,v) = \phi_*\big((y,t),(v,\tau)\big)$, we obtain the following equalities:

$$\begin{aligned} E(x,v) + \frac{1}{2}Q^2(v) =& E_0(y,v) + \omega_y(v)\tau - \frac{1}{2}\tau^2 + \frac{1}{2}\big(\omega_y(v) - \tau\big)^2 \\ =& E_0(y,v) + \frac{1}{2}\omega_y^2(v), \end{aligned} \tag{4.23}$$

where $E_0(y,v)$ is the energy function of the Lagrangian L_0. As a consequence, using (2.13), we obtain

$$E_0(y,v) + \frac{1}{2}\omega_y^2(v) \geq E_0(y,0) = -L_0(y,0) = -L(x,0).$$

Since κ satisfies (4.19), we infer

$$E(x,v) + \frac{1}{2}Q^2(v) \geq -L(x,0) \geq \kappa, \qquad \forall (x,v) \in TM,$$

and we are done. □

Remark 4.6 Our problem naturally leads to the condition (4.19). For a Finsler spacetime (M, L) (see Sect. 2.1), this condition means $\kappa \leq 0$. Therefore, we only consider the energy values that correspond to causal geodesics (timelike or lightlike geodesics).

Lemma 4.7 *If κ satisfies (4.19) then*

$$\mathcal{E}(z) + \frac{1}{2}Q^2(z) > \kappa, \quad \forall z \in \mathcal{N}_{p,q}.$$

Proof From (4.22), it is enough to prove that

$$\mathcal{E}(z) + \frac{1}{2}Q^2(\dot{z}) \neq \kappa.$$

By contradiction assume that $\mathcal{E}(z) + \frac{1}{2}Q^2(\dot{z}) = \kappa$. Using (4.23) and (2.13), we conclude that in any neighbourhood $U_{z(\bar{s})}$, $\bar{s} \in [0,1]$, as in Remark 2.4, and for a.e. s in a neighbourhood of $\bar{s}$, the vector $\dot{z}(s)$ corresponds through ϕ_* to a vector whose component in $TS_{z(\bar{s})}$ vanishes. This is equivalent to the existence of a function $\alpha : [0,1] \to \mathbb{R}$ such that

$$\dot{z}(s) = \alpha(s)K(z(s)), \qquad \text{for a.e. } s \in [0,1].$$

Since $Q(\dot{z})$ is constant a.e. and $Q(\alpha(s)K(z(s))) = -\alpha(s)$, we deduce that α is constant a.e. and $\dot{z}$ is equivalent to a continuous TM-valued function on $[0,1]$. Hence p and q are on the same flow line of K, which is a contradiction. □

Remark 4.8 As a consequence of Lemma 4.7, $t_\pm^\kappa$ in (4.20) are C^1 functionals on $\mathcal{N}_{p,q}$.

Corollary 4.9 *If κ satisfies (4.19), then*

$$\partial_t \mathcal{H}_{p,q}(z, t_+^\kappa(z)) \neq 0, \quad \forall z \in \mathcal{N}_{p,q},$$

and the same holds replacing $t_+^\kappa(z)$ with $t_-^\kappa(z)$.

Proof By (4.18) and (4.20), we have

$$\partial_t \mathcal{H}_{p,q}(z, t_+^\kappa(z)) = Q(\dot z) - t_+^\kappa(z) = -\sqrt{Q^2(\dot z) + 2\big(\mathcal{E}(z) - \kappa\big)}. \tag{4.24}$$

Then, the thesis follows by Lemma 4.7. □

5 Main result

We are ready to proof our main result:

Theorem 5.1 *Let $L : TM \to \mathbb{R}$ satisfy Assumptions 1, 2, and 3, and let $\kappa \in \mathbb{R}$ satisfy (4.19). A curve $\ell : [0,1] \to M$ is a solution of the Euler-Lagrange equations (2.1) joining p and γ with energy κ if and only if there exists $z \in \mathcal{N}_{p,q}$ such that $\ell = F^{t_+^\kappa(z)}(z)$ or $\ell = F^{t_-^\kappa(z)}(z)$, and the following equality holds*:

$$\mathrm{d}t_\pm^\kappa(z) = \frac{\mathrm{d}\mathcal{E}(z) - \mathrm{d}\mathcal{L}(z)}{\sqrt{Q^2(\dot z) + 2\big(\mathcal{E}(z) - \kappa\big)}}, \tag{5.1}$$

or

$$\mathrm{d}t_-^\kappa(z) = \frac{\mathrm{d}\mathcal{L}(z) - \mathrm{d}\mathcal{E}(z)}{\sqrt{Q^2(\dot z) + 2\big(\mathcal{E}(z) - \kappa\big)}}. \tag{5.2}$$

Proof Consider a critical curve $\ell \in \mathcal{N}_{p,\gamma(t)}$ with energy κ. We know that F^t is a diffeomorphism, so there exists $z \in \mathcal{N}_{p,q}$ such that $F^{-t}(\ell) = z$ and $t = t_+^\kappa(z)$ or $t = t_-^\kappa(z)$. For this proof, we will focus on the case where $t = t_+^\kappa(z)$.

Since ℓ is a critical curve for $\mathcal{L}_{p,\gamma(t)}$, by the definition of $\mathcal{H}_{p,q}$ (see (4.7)), we have $\partial_z \mathcal{H}_{p,q}(z,t) = 0$. Furthermore, using (4.17) and the definition of $t_+^\kappa(z)$, we obtain the following equation:

$$\begin{aligned} \mathcal{H}_{p,q}(z, t_+^\kappa(z)) &= \int_0^1 L\big(z, \dot z + t_+^\kappa(z) K(z)\big)\mathrm{d}s \\ &= \mathcal{L}(z) - \mathcal{E}(z) + k, \qquad \forall z \in \mathcal{N}_{p,q}. \end{aligned} \tag{5.3}$$

By differentiating both sides of (5.3), we obtain:

$$\partial_z \mathcal{H}_{p,q}(z, t_+^\kappa(z)) + \partial_t \mathcal{H}_{p,q}(z, t_+^\kappa(z))\, \mathrm{d}t_+^\kappa(z) = \mathrm{d}\mathcal{L}(z) - \mathrm{d}\mathcal{E}(z). \tag{5.4}$$

Since we know that $\partial_z \mathcal{H}_{p,q}(z, t_+^\kappa(z)) = 0$, substituting this into (5.4), we get:

$$\partial_t \mathcal{H}_{p,q}(z, t_+^\kappa(z))\, \mathrm{d}t_+^\kappa(z) = \mathrm{d}\mathcal{L}(z) - \mathrm{d}\mathcal{E}(z).$$

According to Corollary 4.9, we have $\partial_t \mathcal{H}_{p,q}(z, t_+^\kappa(z)) \neq 0$ for every $z \in \mathcal{N}_{p,q}$. Using equation (4.24), we obtain ().

For the converse, if $z \in \mathcal{N}_{p,q}$ satisfies (), we can use (5.4) to conclude that $\partial_z \mathcal{H}_{p,q}(z, t_+^\kappa(z)) = 0$. By Proposition 4.2 and Corollary 4.3, we then deduce that $\ell = F^{t_+^\kappa(z)}(z)$ is a critical point of $\mathcal{L}_{p,\gamma(t)}$. Hence the thesis follows from Proposition 3.5 and Remark 3.2. □

Remark 5.2 If L is homogeneous of degree 2 in the velocities (i.e., L is a Lorentz-Finsler metric, Definition 2.7), then $\mathcal{L}(z) = \mathcal{E}(z)$ for every z and, consequently, () and () are equivalent to $\mathrm{d}t_+^\kappa(z) = 0$ and $\mathrm{d}t_-^\kappa(z) = 0$, respectively. Hence, in this case we re-obtain, for $\kappa \leq 0$, the Fermat's principle in a stationary spacetime that globally splits [19] (also known as *standard stationary spacetime*) as well as in a stationary spacetime that may not globally split [53]. Furthermore, we also obtain a Fermat's principle in a stationary Finsler spacetime that is not necessarily a stationary splitting one (compare with [46, Appendix B]), including also timelike geodesics.

6 An existence and multiplicity result

In this section we assume that L is a Lorentz-Finsler metric as in Sect. 2.1, satisfying Assumption 1. By Theorem 5.1 and Remark 5.2, the critical points z of the functionals $t_\pm^\kappa : \mathcal{N}_{p,q} \to \mathbb{R}$ give all and only the solutions ℓ of (2.1) connecting p to γ and having fixed energy $\kappa \leq 0$ (recall Remark 4.6) through the relation $\ell = F^{t_\pm^\kappa(z)}(z)$. We are going to show that $t_\pm^\kappa$ satisfy the Palais-Smale condition provided that $\mathcal{J}_{p,\gamma(t)}$ (recall Proposition 3.5) is *pseudocoercive*, for all $t \in \mathbb{R}$. Pseudocoercivity is a compactness assumption introduced in [52] and recently revived in [51]. Let us recall it:

Definition 6.1 Let $t, c \in \mathbb{R}$; the manifold $\mathcal{N}_{p,\gamma(t)}$ is said to be *c-precompact* if every sequence $(z_n)_n \subset \mathcal{J}^c_{p,\gamma(t)} := \{z \in \mathcal{N}_{p,\gamma(t)} : \mathcal{J}_{p,\gamma(t)}(z) \leq c\}$ has a uniformly convergent subsequence. We say that $\mathcal{J}_{p,\gamma(t)}$ is *pseudocoercive* if $\mathcal{N}_{p,\gamma(t)}$ is c-precompact for all $c \in \mathbb{R}$.

Remark 6.2 A sufficient condition ensuring that $\mathcal{J}_{p,r}$ is pseudocoercive, for all $p, r \in M$, is based on the existence of a C^1 function $\varphi : M \to \mathbb{R}$ such that $\mathrm{d}\varphi(K) > 0$, see [51, Proposition 8.1]. It is then natural to look at this result in the framework of causality properties of a Finsler spacetime as global hyperbolicity. We analyze this question in Appendix B.

Remark 6.3 We point out that if $\mathcal{J}_{p,r}$ is pseudocoercive then, for each $c \in \mathbb{R}$,

$$\sup_{z \in \mathcal{J}^c_{p,r}} |Q(\dot{z})| < +\infty,$$

see [51, Theorem 7.6].

Henceforth, our attention turns to the functional t_+^κ, recognizing that all the subsequent considerations can be replicated comparably for t_-^κ.

Lemma 6.4 *Let L be a Lorentz-Finsler metric, $p \in M$ and $\gamma = \gamma(t)$ be a flow line of K such that $p \notin \gamma(\mathbb{R})$. Assume that $\mathcal{J}_{p,\gamma(t)}$ is pseudocoercive for all $t \in \mathbb{R}$. Let $(z_n) \subset \mathcal{N}_{p,q}$ such that $t_+^\kappa(z_n)$ is bounded, then*

$$\sup_m |Q(\dot z_n)| < +\infty. \tag{6.1}$$

Proof Assume by contradiction that $\sup_m |Q(\dot z_n)| = +\infty$. Since $q \in \gamma(\mathbb{R})$ and $\mathcal{J}_{p,\gamma(t)}$ is pseudocoercive for all $t \in \mathbb{R}$, from Remark 6.3 necessarily $\mathcal{J}_{p,q}(z_n) \to +\infty$. Since $t_+^\kappa(z_n)$ is bounded and

$$\begin{aligned} t_+^\kappa(z_n) &= Q(\dot z_n) + \sqrt{Q^2(\dot z_n) + 2\big(\mathcal{E}(z_n) - \kappa\big)} \\ &= Q(\dot z_n) + \sqrt{Q^2(\dot z_n) + 2\big(\mathcal{L}(z_n) - \kappa\big)}, \end{aligned} \tag{6.2}$$

we get that, up to pass to a subsequence,

$$Q(\dot z_n) \to -\infty. \tag{6.3}$$

Let $C \geq 0$ such that $|t_+^\kappa(z_n)| \leq C$, for all $m \in \mathbb{N}$. From (6.2) we then get

$$2\mathcal{L}(z_n) \leq C^2 - 2C\,Q(\dot z_n) + 2\kappa. \tag{6.4}$$

Take $\bar t > C$ and consider $z_n^{\bar t} := F^{\bar t}(z_n)$. Recalling (4.7) and (4.18), we then get from (6.4):

$$\mathcal{L}(z_n^{\bar t}) = \mathcal{L}(z_n) + \bar t Q(\dot z_n) - \frac{1}{2}\bar t^2 \leq \frac{C^2}{2} + (\bar t - C)Q(\dot z_n) + \kappa \to -\infty.$$

By Remark 6.3, we deduce that $\sup_m |Q(\dot z_m^{\bar t})| < +\infty$. Then from (4.6),

$$\sup_m |Q(\dot z_n)| < +\infty,$$

in contradiction with (6.3). □

Let $z \in \mathcal{N}_{p,q}$ and $\zeta \in T_z\Omega_{p,q}(M)$; we recall that the H^1-norm of ζ is given by $\int_0^1 g_z(\zeta', \zeta')\mathrm{d}s$ where ζ' denotes the covariant derivative of ζ along z defined by the Levi-Civita connection of the auxiliary Riemannian metric g.

Lemma 6.5 *Let $(z_n)_n \subset \mathcal{N}_{p,q}$ be a bounded sequence (w.r.t. the topology induced on $\mathcal{N}_{p,q}$ by the topology of $\Omega_{p,q}(M)$) such that their images $z_n([0,1])$ are contained in a compact subset of M and, for each $n \in \mathbb{N}$, let $\zeta_n \in T_{z_n}\Omega_{p,q}(M)$. If*

$$\sup_n \int_0^1 g_{z_n}(\zeta_n', \zeta_n')\mathrm{d}s < +\infty,$$

then there exist bounded sequences $\xi_n \in T_{z_n}\mathcal{N}_{p,q}$ and $\mu_n \in H_0^1([0,1],\mathbb{R})$ such that $\zeta_n = \xi_n + \mu_n K(z_n)$.

Proof Since the images of the curves z_n are contained in a compact subset W of M, we can assume that the field K is bounded and the covariant derivatives of the fields $K(z_n)$ along z_n

are uniformly bounded in the L^2-norm. Thus, it suffices to show that there exists a bounded sequence $(\mu_n)_n \subset H_0^1([0,1],\mathbb{R})$ such that

$$\xi_n := \zeta_n - \mu_n K(z_n) \in T_{z_n}\mathcal{N}_{p,q}, \quad \forall n \in \mathbb{N}.$$

By (3.6), we need to prove that, for each $n \in \mathbb{N}$, there exists $c_n \in \mathbb{R}$ such that

$$\partial_x Q(\dot{z}_n)[\xi_n] + Q(\dot{\xi}_n) = c_n, \text{ a.e.},$$

so we need to solve, with respect to $c_n \in \mathbb{R}$ and $\mu_n \in H_0^1([0,1],\mathbb{R})$, the following ODE:

$$\partial_x Q(\dot{z}_n)[\zeta_n] + Q(\dot{\zeta}_n) - \mu_n\big(\partial_x Q(\dot{z}_n)[K(z_n)] + Q(\dot{K}_n)\big) + \mu_n' = c_n, \tag{6.5}$$

where, $\dot{K}_n$ denotes $\frac{\partial K^i}{\partial x^j}(z_n(s))\dot{z}_n^j(s)\frac{\partial}{\partial x^i}|_{z_n(s)}$. Let us re-write (6.5) as

$$\mu_n'(s) - a_n(s)\mu_n(s) = b_n(s), \tag{6.6}$$

where

$$a_n(s) = \partial_x Q(\dot{z}_n)[K(z_n)] + Q(\dot{K}_n)$$

and

$$b_n(s) = c_n - h_n(s), \qquad h_n(s) := \partial_x Q(\dot{z}_n)[\zeta_n] + Q(\dot{\zeta}_n)$$

Setting $A_n(s) = \int_0^s a_n(\tau)\mathrm{d}\tau$, and

$$c_n = \left(\int_0^1 e^{-A_n(s)}\mathrm{d}s\right)^{-1}\left(\int_0^1 e^{A_n(s)}h_n(s)\mathrm{d}s\right),$$

a solution of (6.6) which satisfies the boundary conditions $\mu_n(0) = \mu_n(1) = 0$ is given by

$$\mu_n(s) = e^{A_n(s)}\int_0^s b_n(\tau)e^{-A_n(\tau)}\mathrm{d}\tau.$$

We notice that the sequence $A_n(s) : [0,1] \to \mathbb{R}$ is uniformly bounded in L^∞ since

$$|A_n(s)| \le \int_0^1 |a_n|\mathrm{d}s \le C_1\int_0^1 \sqrt{g(\dot{z}_n,\dot{z}_n)}\mathrm{d}s,$$

where C_1 is a positive constant depending on the maxima of the absolute values of the components of Q and K and their derivatives, in each coordinate system used to cover W, and on a constant that bounds from above the Euclidean norm with the norm associated with g in each of the same of coordinate system. This implies that the sequence of functions $e^{\pm A_n(s)}$ is also uniformly bounded in L^∞ and then $\left(\int_0^1 e^{-A_n(s)}\mathrm{d}s\right)^{-1}$ is bounded as well. Analogously,

$$|h_n(s)| \le C_2\sqrt{g(\dot{z}_n,\dot{z}_n)},$$

where now $C_2 \ge 0$ is independent of K but depend on an upper bound for the L^∞-norms of the fields ζ_n. Hence c_n is bounded and b_n satisfies then

$$|b_n(s)| \leq C_3 + C_2\sqrt{g(\dot z_n,\dot z_n)},$$

for some non-negative constant C_3. Since

$$\mu_n'(s) = a_n(s)e^{A_n(s)}\int_0^s b_n(\tau)e^{-A_n(\tau)}\mathrm{d}\tau + b_n(s)$$

we get

$$|\mu_n'(s)| \leq C_4|a_n(s)| + |b_n(s)|,$$

for some non-negative constant C_4, depending also on an upper bound of the sequence $\int_0^1 \sqrt{g(\dot z_n,\dot z_n)}\mathrm{d}s$. Hence, μ_n is bounded in H_0^1-norm. □

Lemma 6.6 *Let $z \in \mathcal{N}_{p,q}$ and $\eta \in \mathcal{W}_z$, then* $\mathrm{d}t_+^\kappa(z)[\eta] = 0$.

Proof From (5.4), since $\mathcal{L} = \mathcal{E}$, we get

$$\partial_z\mathcal{H}_{p,q}(z,t_+^\kappa(z)) + \partial_t\mathcal{H}_{p,q}(z,t_+^\kappa(z))\,\mathrm{d}t_+^\kappa(z) = 0.$$

As showed in the proof of Proposition 4.2, $\partial_z\mathcal{H}_{p,q}(z,t_+^\kappa(z))[\eta] = 0$, and since, from Corollary 4.9, $\partial_t\mathcal{H}_{p,q}(z,t_+^\kappa(z)) \neq 0$, we get the thesis. □

We are now ready to prove the Palais-Smale condition for t_+^κ. We recall that a C^1 functional $f : \mathcal{M} \to \mathbb{R}$, defined on a manifold $\mathcal{M}$, satisfies the Palais-Smale condition if every sequence $z_n \subset \mathcal{M}$ such that $f(z_n)$ is bounded and $\mathrm{d}f(z_n) \to 0$, admits a converging subsequence.

Proposition 6.7 *Under the assumptions in Lemma 6.4, $t_+^\kappa : \mathcal{N}_{p,q} \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof Let $(z_n)_n \subset \mathcal{N}_{p,q}$ and $C \geq 0$ such that $|t_+^\kappa(z_n)| \leq C$ and $\mathrm{d}t_+^\kappa(z_n) \to 0$. From Lemma 6.4, we have that (6.1) holds. Hence, from (6.2) we deduce that $\mathcal{L}(z_n)$ is bounded from above. By the pseudocoercivity assumption, there exists then a subsequence, still denoted by z_n, which uniformly converge to a continuous curve $z : [0,1] \to M$ connecting p to q. Thus, the curves z_n are contained in a compact subset W of M. Hence, from Remark 2.10 there exists a positive constant α, depending on W, such that $L_c(x,v) \geq \alpha g(v,v)$, for all $x \in W$ and $v \in T_xM$. Let $\mathcal{L}_c$ denote the action functional of L_c and

$$\mathcal{S}(z) := \sqrt{Q^2(\dot z) + 2\big(\mathcal{L}(z) - \kappa\big)} = \sqrt{2\big(\mathcal{L}_c(z) - \kappa\big) - Q^2(\dot z)}.$$

Since $Q(\dot z_n)$ is bounded, $\mathcal{L}(z_n)$ is bounded from above and

$$\alpha\int_0^1 g(\dot z_n,\dot z_n) \leq \mathcal{L}_c(z_n) = \mathcal{L}(z_n) + Q^2(\dot z_n), \tag{6.7}$$

we deduce that $\mathcal{S}(z_n)$ and $\int_0^1 g(\dot z_n,\dot z_n)\mathrm{d}s$ are bounded as well. Moreover, for $z \in \mathcal{N}_{p,q}$, let us see $Q(\dot z)$ as a functional $\mathcal{Q}$ on $\mathcal{N}_{p,q}$ (recall that $Q(\dot z)$ is constant a.e. on [0, 1]).

Let then $\zeta_n \in T_{z_n}\Omega^{1,2}_{p,q}(M)$ be a bounded sequence; from Lemma 6.5 there exist two bounded sequences $\xi_n \in T_{z_n}\mathcal{N}_{p,q}$ and $\mu_n \in H^1_0([0,1],\mathbb{R})$ such that $\zeta_n = \xi_n + \mu_n K_{z_n}$. As z_n is a Palais-Smale sequence, from Lemma 6.6 we obtain

$$\mathrm{d}t^\kappa_+(z_n)[\zeta_n] = \mathrm{d}t^\kappa_+(z_n)[\xi_n] + \mathrm{d}t^\kappa_+(z_n)[\mu_n K_{z_n}] = \mathrm{d}t^\kappa_+(z_n)[\xi_n] \to 0.$$

We now apply a localization argument as in [78] (see also the proof of [51, Theorem 5.6]). Thus, we can assume that L is defined on $[0,1]\times U \times \mathbb{R}^{m+1}$, with U an open neighbourhood of 0 in $\mathbb{R}^{m+1}$. Analogously, we associate to L_c and Q, a time-dependent fiberwise strongly convex Lagrangian L_{cs} in U and a C^1 family of linear forms Q_s. Moreover, we can identify $(z_n)_n$ with a sequence in the Sobolev space $H^1([0,1],U)$. By (6.7), taking into account that the curves z_n have fixed end-points, we get that $(z_n)_n$ is bounded in $H^1([0,1],U)$ and so it admits a subsequence, still denoted by $(z_n)_n$, which weakly and uniformly converges to a curve $z \in H^1([0,1],\mathbb{R}^{m+1})$ which also satisfies the same fixed end-points boundary conditions. The differential at z_n of the localized functional obtained, that we still denote with t^κ_+, is given by

$$\mathrm{d}t^\kappa_+(z_n) = \mathrm{d}\mathcal{Q}_s(z_n) + \frac{\mathrm{d}\mathcal{L}_{cs}(z_n) - \mathcal{Q}_s(z_n)\mathrm{d}\mathcal{Q}_s(z_n)}{\mathcal{S}_s(z_n)}$$

where the index s is used to denote the localized functionals. Since $\mathcal{S}_s(z_n)$ is bounded we get

$$0 \leftarrow \mathcal{S}_s(z_n)\mathrm{d}t^\kappa_+(z_n) = \big(\mathcal{S}_s(z_n) - \mathcal{Q}_s(z_n)\big)\mathrm{d}\mathcal{Q}_s(z_n) + \mathrm{d}\mathcal{L}_{cs}(z_n).$$

In particular, since $z_n - z$ is bounded in H^1_0, we obtain

$$\big(\mathcal{S}_s(z_n) - \mathcal{Q}_s(z_n)\big)\mathrm{d}\mathcal{Q}_s(z_n)[z_n - z] + \mathrm{d}\mathcal{L}_{cs}(z_n)[z_n - z] \to 0. \tag{6.8}$$

Since $z_n \to z$ uniformly and weakly, we deduce that $\mathrm{d}\mathcal{Q}_s(z_n)[z_n - z] \to 0$. As $\mathcal{S}_s(z_n) - \mathcal{Q}_s(z_n)$ is bounded then $\big(\mathcal{S}_s(z_n) - \mathcal{Q}_s(z_n)\big)\mathrm{d}\mathcal{Q}_s(z_n)[z_n - z] \to 0$ as well. From (6.8), we then get $\mathrm{d}\mathcal{L}_{cs}(z_n)[z_n - z] \to 0$. We can then conclude that $z_n \to z$ in H^1-norm thanks to the convexity of L_{cs} as in the proof of [51, Theorem 5.6]. There exists then a subsequence z_{n_k} such that $\dot z_{n_k} \to \dot z$, a.e. on $[0,1]$. As $Q(\dot z_{n_k}) = c_k$ a.e., for some $c_k \in \mathbb{R}$, we get that also $Q(\dot z)$ is constant a.e., i.e. $z \in \mathcal{N}_{p,q}$. □

Lemma 6.8 *Under the assumptions of Lemma 6.4, the functional $t^\kappa_+ : \mathcal{N}_{p,q} \to \mathbb{R}$ is bounded from below.*

Proof By contradiction, let us assume the existence of a sequence $(z_n)_n \subset \mathcal{N}_{p,q}$ such that $\lim_{n\to\infty} t^\kappa_+(z_n) = -\infty$. From (6.2), this implies that

$$\lim_{n\to\infty} Q(\dot z_n) = -\infty,$$

hence from Remark 6.3, up to pass to a subsequence, $\mathcal{L}(z_n) = \mathcal{J}_{p,q}(z_n) \to +\infty$. Therefore, from (6.2), $t^\kappa_+(z_n) \geq 0$, for n big enough. □

We are now ready to present an existence and multiplicity results for solutions of the Euler-Lagrange equations (2.1). Previous existence results, in the case $\kappa = 0$, based on causality techniques were obtained in [46, Proposition 6.2 and Proposition B.2] for Finsler spacetimes that admit a global splitting $S \times \mathbb{R}$ endowed with a Lorentz-Finsler metric of the type (2.19) and in [47, Theorem 2.49] in the more general setting of a manifold with a proper cone structure.

Theorem 6.9 *Let M be a smooth, connected finite dimensional manifold, $L : TM \to \mathbb{R}$ be a Lorentz-Finsler metric on M satisfying Assumption 1, $p \in M$ and $\gamma : \mathbb{R} \to M$ be a flow line of K such that $p \notin \gamma(\mathbb{R})$. Let us assume that $\mathcal{J}_{p,\gamma(t)}$ is pseudocoercive for all $t \in \mathbb{R}$. Let $\kappa \leq 0$. Then,*

(a) *There exists a curve $z : [0,1] \to M$ that is a solution of Euler-Lagrange equations (2.1) with energy κ, joining p and $\gamma(\mathbb{R})$ and minimizes t_+^κ;*
(b) *If M is a non-contractible manifold, then there exists a sequence of curves $z_n : [0,1] \to M$ that are solutions of Euler-Lagrange equations (2.1) with energy κ joining p and $\gamma(\mathbb{R})$ and such that $\lim_{n\to\infty} t_+^\kappa(w_n) = +\infty$.*

Proof Since t_+^κ is a bounded from below, C^1 functional defined on a C^1 manifold and it satisfies the Palais-Smale condition, both part (a) and (b) follows from [79, Theorem (3.6)], Theorem 5.1 and Remark 5.2, taking into account, for part (b), that if M is non-contractible then the Lusternik-Schnirelmann category of $\mathcal{N}_{p,q}$ is $+\infty$ as follows from [51, Proposition 6.4] and [80, Proposition 3.2]. □

Remark 6.10 Assumption (2.4) could be considered quite restrictive; however, for solutions with energy $\kappa = 0$, that is not the case for the following reasons:

(1) Since L is a Lorentz-Finsler metric, solutions $z : [0,1] \to M$ of the Euler-Lagrange equations (2.1) with $\kappa = 0$ satisfy $L\big(z(s), \dot z(s)\big) = 0$ for all $s \in [0,1]$ and therefore they are lightlike geodesics (see, e.g., [41, 81, 82]).
(2) According to [83, Proposition 4.4] (see also [82, Proposition 3.4] and [81, Proposition 12]) for any smooth function $\varphi : M \to (0, +\infty)$ and for any lightlike geodesic $z : [0,1] \to M$ of L, there exists a reparametrization of z (on some interval $[0, a_z]$) which is a lightlike geodesic of the Lorentz-Finsler metric φL.
(3) Let $\tilde L$ be a Lorentz-Finsler metric on M which satisfies Assumption 1 with (2.4) replaced by $\tilde Q(K) < 0$ (where $\tilde Q$ is the Noether charge of $\tilde L$). Hence, $L := -\tilde L/\tilde Q(K)$ satisfies (2.4).
(4) The infinitesimal symmetry K of $\tilde L$ remains an infinitesimal symmetry for L. This is a consequence of the fact that the flow ψ of K preserves $\tilde Q(K)$ (see the proof of [51, Proposition 2.5-(iv)]), and then

$$\begin{aligned}\frac{\partial\big(L\circ\psi^c\big)}{\partial t}(t,x,v) =& K^c(L)\big(\psi^c(t,x,v)\big) = K^c\big(-\tilde L/\tilde Q(K)\big)\big(\psi^c(t,x,v)\big)\\ =& -\frac{\partial\big(\tilde L\circ\psi^c/(\tilde Q(K)\circ\psi)\big)}{\partial t}(t,x,v)\\ =& \Big(\big(\tilde Q(K)\circ\psi\big)^{-2}\frac{\partial\big(\tilde Q(K)\circ\psi\big)}{\partial t}\tilde L\circ\psi^c\\ &-\big(\tilde Q(K)\circ\psi\big)^{-1}\frac{\partial\big(\tilde L\circ\psi^c\big)}{\partial t}\Big)(t,x,v) = 0,\end{aligned}$$

(recall the beginning of Sect. 3.1).

Summing up, Theorem 6.9 also holds (replacing [0, 1] with unknown interval of parametrizations $[0, a_z]$) for a Lorentz-Finsler metric $\tilde{L}$ on M which satisfies Assumption 1 with (2.4) replaced by $\tilde{Q}(K) < 0$.

Appendix A Affine Noether charge

In this section, we briefly show that Theorem 5.1 holds even if the Noether charge is an affine function with respect to v. Specifically, there exists a C^1 one-form Q on M and a C^1 function $d : M \to \mathbb{R}$ such that (2.3) is replaced by

$$N(x, v) := \partial_v L(x, v)[K] = Q(v) + d(x); \tag{A1}$$

and d is invariant under the one-parameter group of C^3 diffeomorphisms generated by K. In such a case, the stationary type local structure is given by

$$L(x, v) = L \circ \phi_* \big((y, t), (v, \tau)\big) = L_0(y, v) + \big(\omega_y(v) + d(y)\big)\tau - \frac{1}{2}\tau^2, \tag{A2}$$

so we have

$$E(x, v) = E \circ \phi_* \big((y, t), (v, \tau)\big) = E_0(y, v) + \omega_y(v)\tau - \frac{1}{2}\tau^2. \tag{A3}$$

Moreover, the set $\mathcal{N}_{p,r}$ is given by

$$\mathcal{N}_{p,r} := \{z \in \Omega_{p,r}(M) : \exists c \in \mathbb{R} \text{ such that } N(z, \dot{z}) = c, \text{ a.e. on } [0, 1]\} \subset \Omega_{p,r}(M),$$

and Proposition 3.5 still holds. Moreover, defining $F^t : \Omega_{p,q}(M) \to \Omega_{p,\gamma(t)}(M)$ as (4.1) and $\mathcal{H}_{p,q} : \Omega_{p,q}(M) \times \mathbb{R} \to \mathbb{R}$ as in (4.7), it is possible to prove both Proposition 4.2 and Corollary 4.3. The main difference with the linear case is that Proposition 4.4 doesn't hold and it is replaced by the following result, whose proof is a based on a computation in local charts which employs (A2).

Proposition A.1 *For every $(x, v) \in TM$ and for every $t \in \mathbb{R}$, the following two equations holds*:

$$L\big(x, v + tK(x)\big) = L\big(x, v\big) + tN(x, v) - \frac{1}{2}t^2,$$

and

$$E\big(x, v + tK(x)\big) = E\big(x, v\big) + tQ(v) - \frac{1}{2}t^2.$$

As a consequence, for every $(x, v) \in TM$ we have

$$L\big(x, v + tK(x)\big) - E\big(x, v + tK(x)\big) = L\big(x, v\big) - E\big(x, v\big) + td(x). \tag{A4}$$

Since on any curve $z \in \mathcal{N}_{p,q}$ the quantities $Q(\dot{z})$ and $d(z)$ are not necessarily constant, let us introduce the functionals $\mathcal{Q} : \mathcal{N}_{p,q} \to \mathbb{R}$ and $\mathcal{D} : \mathcal{N}_{p,q} \to \mathbb{R}$ as follows:

$$\mathcal{Q}(z) := \int_0^1 Q(\dot{z})\mathrm{d}s, \quad \text{and} \quad \mathcal{D}(z) := \int_0^1 d(z)\mathrm{d}s.$$

Using this notation, for every $z \in \mathcal{N}_{p,q}$ the two quantities $t_\pm^\kappa(z)$ that satisfy (4.21) are given by

$$t_\pm^\kappa(z) = \mathcal{Q}(z) \pm \sqrt{\mathcal{Q}^2(z) + 2\big(\mathcal{E}(z) - \kappa\big)},$$

and they are still well defined if κ satisfies (4.19). Moreover, since the function d doesn't appear in the expression of $E(x, v)$ in local coordinates (see (A3)), Corollary 4.9 still holds. Because of the difference between (A4) and (4.17), we have that in the affine case the equation analogous to (5.3) is

$$\begin{aligned}\mathcal{H}_{p,q}(z, t_+^\kappa(z)) &= \mathcal{L}(z) - \mathcal{E}(z) + \kappa + t_+^\kappa(z)\big(N(z,\dot{z}) - \mathcal{Q}(z)\big)\\ &= \mathcal{L}(z) - \mathcal{E}(z) + \kappa + t_+^\kappa(z)\mathcal{D}(z).\end{aligned}$$

As a consequence, using a similar proof of the one of Theorem 5.1, we obtain the following result.

Theorem A.2 *Let $L : TM \to \mathbb{R}$ satisfy assumptions 1, 2, and 3, with (A1) instead of (2.3), and let $\kappa \in \mathbb{R}$ satisfy (4.19). A curve ℓ is a solution of the Euler-Lagrange equations (2.1) joining p and γ with energy κ if and only if there exists $z \in \mathcal{N}_{p,q}$ such that $\ell = F^{t_+^\kappa(z)}(z)$ or $\ell = F^{t_-^\kappa(z)}(z)$, and the following equality holds:*

$$\mathrm{d}t_+^\kappa(z) = \frac{\mathrm{d}\mathcal{E}(z) - \mathrm{d}\mathcal{L}(z) - t_+^\kappa(z)\mathrm{d}\mathcal{D}(z)}{\sqrt{\mathcal{Q}^2(\dot{z}) + 2\big(\mathcal{E}(z) - \kappa\big)}},$$

or

$$\mathrm{d}t_-^\kappa(z) = \frac{\mathrm{d}\mathcal{L}(z) - \mathrm{d}\mathcal{E}(z) + t_-^\kappa(z)\mathrm{d}\mathcal{D}(z)}{\sqrt{\mathcal{Q}^2(\dot{z}) + 2\big(\mathcal{E}(z) - \kappa\big)}}.$$

Corollary A.3 *Let $L : TM \to \mathbb{R}$ satisfy assumptions 1, 2, and 3, with (A1) instead of (2.3), and let $\kappa \in \mathbb{R}$ satisfy (4.19). Moreover, assume that $d : M \to \mathbb{R}$ is a constant function. Then, a curve ℓ is a solution of the Euler-Lagrange equations (2.1) joining p and γ with energy κ if and only if there exists $z \in \mathcal{N}_{p,q}$ such that $\ell = F^{t_+^\kappa(z)}(z)$ or $\ell = F^{t_-^\kappa(z)}(z)$, and () or () holds.*

Acknowledgements D. Corona is partially supported by the FAPESP (São Paulo, Brazil) grant n. 2022/13010-3. E. Caponio is partially supported by European Union - Next Generation EU - PRIN 2022 PNRR "P2022YFAJH Linear and Nonlinear PDE's: New directions and Applications". E. Caponio, D. Corona, R. Giambòthank the partial support of GNAMPA INdAM - Italian National Institute of High Mathematics, project CUP-E55F22000270001. P. Piccione is partially sponsored by FAPESP (São Paulo, Brazil), grant n. 2022/16097-2, and by CNPq, Brazil. We would like to thank a referee for providing us with valuable comments.

Funding Open access funding provided by Politecnico di Bari within the CRUI-CARE Agreement.

Appendix B Pseudocoercivity and global hyperbolicity

In this appendix we show that pseudocoercivity and global hyperbolicity of a Finsler spacetime (M, L) as defined in Subsection 2.1, are connected notions. We refer to [47, 73] for the needed notions of causality, and in particular of global hyperbolicity and of a Cauchy hypersurface, in Finsler spacetimes and in the more general framework of proper cone structures (see [47, Definition 2.4]). We notice indeed that M is endowed with a continuous cone structure $\mathcal{C}:=\{(x,v)\in TM : L(x,v)\leq 0, Q(v)<0\}$. In fact, from the local expression of L (2.19), we deduce that $(v,\tau)\in \mathcal{C}_{(y,t)}:=\mathcal{C}\cap T_{(y,t)}M$ if and only if

$$\tau \geq \omega(v)+\sqrt{\omega^2(v)+2F^2(y,v)}, \tag{B1}$$

and since $F^2(y,\cdot)$ is strongly convex, we deduce that $\mathcal{C}_{(y,t)}\cup\{0\}$ is a closed, convex, sharp cone with non-empty interior.

Our first aim would be to extend [84, Theorem 5.1], which states that if a stationary Lorentzian manifold is globally hyperbolic with a complete Cauchy hypersurface then it is pseudocoercive. We obtain a result in that direction, namely Proposition B.2, that ensures pseudocoerciveness from the global hyperbolicity in our setting requiring some other technical assumptions that are trivially satisfied in the Lorentzian setting.

Lemma B.1 *Let (M, L) be a Finsler spacetime (i.e. $L_F : TM\to\mathbb{R}$ satisfies (a'), (b) and (c') in Definition 2.7) such that Assumption 1 holds. If (M, L) is globally hyperbolic (i.e. the cone structure $\mathcal{C}$ associated to L is globally hyperbolic) then M globally splits as $S\times\mathbb{R}$ and L is given on $S\times\mathbb{R}$ by an expression of the type (2.10), with $L_0:=L|_{TS}$ and ω the one-form induced by Q on S.*

Proof From [85, Theorem 1.3], we have that there exists a smooth Cauchy time function $T : M\to\mathbb{R}$. Let then $S:=T^{-1}(0)$. Being $K_x\in\mathcal{C}_x$, for all $x\in M$, we have that $\mathrm{d}T(K)>0$ by definition of a smooth time function, and then K is transversal to S. Thus, for any vector $(x,w)\in TM$ with $x\in S$, we can write $w=w_S+\tau_w K_x$ where $w_S\in T_xS$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}L(x,w_S+s\tau_wK)=\tau_w\partial_vL(x,w_S+s\tau_wK)[K]=\tau_wQ(w_S)-s\tau_w^2,$$

by integrating w.r.t. s between 0 and 1, we get

$$L(x,w)=L(x,w_S+\tau_wK)=L(x,w_S)+\tau_wQ(w_S)-\frac{1}{2}\tau_w^2,$$

which gives the required expression for L restricted to vectors $(x,w)\in TM$ with $x\in S$. Let ϕ be the restriction to $S\times\mathbb{R}$ of the flow of K. Since T is a Cauchy time function, it is strictly increasing on the flow lines γ of K and it satisfies $\lim_{s\to\pm\infty} T(\gamma(s)) = \pm\infty$. Therefore, $\phi : S \times \mathbb{R} \to M$ is a diffeomorphism. Using that L is invariant by the flow of K^c we obtain

$$L \circ \phi_*\big((x,t),(v,\tau)\big) = L_0(x,v) + \omega(v)\tau - \frac{1}{2}\tau^2, \tag{B2}$$

where $L_0 = L|_{TS}$ and ω is the one-form induced by Q on S. □

Let us denote by g_S the C^1 Riemannian metric on S induced by g. We assume that the one-form ω has sublinear growth w.r.t. the distance d_S induced by g_S, i.e. there exist $\alpha \in [0,1)$ and two non-negative constants k_0 and k_1 such that

$$\|\omega\| \leq k_0 + k_1\big(d_S(x,x_0)\big)^\alpha, \tag{B3}$$

for some $x_0 \in S$ and all $x \in S$. From [51, Proposition 8.1], we immediately obtain the following result.

Proposition B.2 *Under the assumptions of Lemma B.1, assume also that g is complete, (B3) holds, L_0 is non-negative and satisfies*

$$\big(\partial_v L_0(x,v_2) - \partial_0 L_0(x,v_1)\big)[v_2 - v_1] \geq \lambda_0(x)\|v_2 - v_1\|^2, \tag{B4}$$

for each $x \in S$, and all $v_1, v_2 \in T_xS$. If $\inf_{x\in S} \lambda_0(x) > 0$, then $\mathcal{J}_{p,r}$ is pseudocoercive for all $p, r \in M$.

Remark B.3 We notice that the condition (B4) is always satisfied in the Lorentzian setting, since S can be taken to be a smooth spacelike Cauchy hypersurface; moreover if S is complete then a possible auxiliary Riemannian metric g on $S \times \mathbb{R}$ is the natural product metric which is then also complete. Therefore, $\lambda_0(x) = 1$, for each $x \in S$, in the Lorentzian setting. We point out that in [84, Theorem 5.1] the completeness of S is a required assumption. The more technical assumption in Proposition B.2 is (B3). It is needed to get the boundedness of the constants $Q(\dot z)$, for all z in a fixed sublevel $\mathcal{J}^c_{p,r}$, a property called *c-boundedness* in [51], that implies pseudocoerciveness if satisfied for each $c \in \mathbb{R}$ (see [51, Proposition 7.2]). Actually, when L is a 2-positive homogeneous Lagrangian and $L_0 \in C^1(TS)$ is the square of a Finsler metric on S, a close inspection of the proof of [84, Theorem 5.1] makes clear that (B3) can be removed, and an analogous proof can be repeated by using the action functional of L_0 instead of the energy functional of the Riemannian metric on S. In fact, using the global splitting $S \times \mathbb{R}$ and (B2), the arrival time functional of a lightlike curve $z(s) = \big(x(s), t(s)\big)$, (i.e., a causal curve $z : [0,1] \to M$ such that $L\big(z(s), \dot z(s)\big) = 0$, a.e. on $[0,1]$) between $p = (x_0, 0) \in S \times \{0\}$ and a flow line of K, $\gamma(t) = (x_1, t)$, is given by

$$J : \Omega_{x_0,x_1}(S) \to \mathbb{R}, \qquad J(x) = \int_0^1 \left(\omega(\dot x) + \sqrt{\omega^2(\dot x) + 2L_0(x,\dot x)}\right)\mathrm{d}s,$$

and this is a key point in the proof of [84, Theorem 5.1] (refer to [84, Lemma 5.4]). Moreover, the completeness of the Riemannian metric on S can be replaced by the forward or backward completeness of $\sqrt{L_0}$. Another fundamental point is the compactness of $S \cap J^-(q)$, for any $q \in M$, (see (B6) for the definition of $J^-(q)$), used in the proof of [84, Lemma 5.5]. In our setting, this is an immediate consequence of [47, Theorem 2.44]. Summing up, the following result extending [84, Theorem 5.1] holds:

Theorem B.4 *Under the assumptions of Lemma B.1, assume also that $L_0 \in C^1(TS)$ is the square of a forward or backward complete Finsler metric on S. Then $\mathcal{J}_{p,r}$ is pseudocoercive for all $p, r \in M$.*

Remark B.5 In light of Theorem B.4, it becomes important to give conditions ensuring that L_0 is the square of a Finsler metric on S. A first observation is that L_0 is non-negative and (B4) holds if, for each $x \in S$

$$\lambda(x) - \max_{v \in T_xS, \|v\|=1} 2Q_x^2(v) > 0,$$

where $\lambda(x)$ is defined in (2.9) (see [51, Remark 2.14]).

We also notice that, if $\mathcal{O}_0 := \mathcal{O} \cap TS$, satisfies, relatively to TS, the same properties satisfied by $\mathcal{O}$ in Remark 2.8-(a'), then (B4) holds if

$$\inf_{v \in \mathcal{O}_x} \left(\min_{v \in T_xS, \|v\|=1} \left(\partial_{vv} L_c(x,v)[v,v] - 2Q_x^2(v) \right) \right) > 0. \tag{B5}$$

Moreover, in this case, $\sqrt{L_0}$ in (B2) is a Finsler metric on S such that L_0 is of class C^1. Indeed, from (2.16) and (B5) we immediately get that $\partial_{vv} L(x,v)|_{T_xS \times T_xS}$ is a positive definite bilinear form, for every $v \in T_xS \cap \mathcal{O}_0$. Therefore, recalling that $L_0 = L|_{TS}$ and it is fiberwise positively homogeneous, we have that $L_0(v) \geq 0$ for all $v \in \mathcal{O}_0$ and then on TS by density of $\mathcal{O}_0$ in TS. Arguing as in Remark 2.10, we then conclude that $\sqrt{L_0}$ is a Finsler metric.

Actually, in this last setting, (B5) is also a necessary condition for L_0 being the square of a Finsler metric. In fact, let $\{e_1, \dots, e_m\} \subset T_xS$ be an orthonormal basis of T_xS with respect to the auxiliary Riemannian metric g. Using this basis, we can write the one-form $\omega : T_xS \to \mathbb{R}$ given by $Q|_{T_xS}$ as $(\omega_1, \dots, \omega_m)$. Let us denote by $g_0(v)_{ij}$ the vertical Hessian matrix of L_0 in $v \in T_xS \cap \mathcal{O}_x$ with respect to this basis. Similarly, we denote by $g_c(v)_{ij}$ the vertical Hessian matrix of L_c restricted to T_xS. With this notation, first we notice that, $g_0(v)_{ij}$ has $m-1$ positive eigenvalues, since it coincides with $g_c(v)_{ij}$ on $\ker(\omega)$. By [86, Proposition 11.2.1], applied to the vector $i\sqrt{2}(w_1, \dots, w_m) \in \mathbb{C}^m$, we have

$$\det(g_0(v)_{ij}) = \det\left(g_c(v)_{ij} - 2\omega_i\omega_j\right) = \left(1 - 2g_c(v)^{ih}\omega_h\omega_i\right)\det(g_c(v)_{ij}),$$

where $g_c(v)^{ij}$ denotes the inverse matrix of $g_c(v)_{ij}$. Since $g_c(v)_{ij}$ is positive definite, then $g_0(v)_{ij}$ is positive definite if and only if $1 - 2g_c(v)^{ih}\omega_h\omega_i > 0$, namely if and only if the norm of ω with respect to $g_c(v)$ is strictly less than 1/2 for every $v \in T_xS \cap \mathcal{O}_x$.

Let us now analyze the converse situation, i.e. we assume now that $\mathcal{J}_{p,r}$ is pseudocoercive for all $p, r \in M$ and we prove that global hyperbolicity holds. We recall (see, e.g., [47, §2.1]) that an absolutely continuous curve $\gamma : [a,b] \to M$ is *causal* if $\dot\gamma(t) \in \mathcal{C}_{\gamma(t)}$, for a.e. $t \in [a,b]$. For any $p \in M$, we set

$$J^+(p) := \left\{ \begin{array}{c} r \in M \,:\, r = p \text{ or there exists a causal curve } \gamma \,:\, [0,1] \to M \\ \text{such that } \gamma(0) = p \text{ and } \gamma(1) = r \end{array} \right\},$$

and, analogously, we define

$$J^-(p) := \left\{ \begin{array}{c} r \in M \,:\, r = p \text{ or there exists a causal curve } \gamma \,:\, [0,1] \to M \\ \text{such that } \gamma(0) = r \text{ and } \gamma(1) = p \end{array} \right\}. \tag{B6}$$

We call *causal diamond* a set given by $J^+(p) \cap J^-(r)$, for some $p, r \in M$.

According to [47, Corollary 2.4], global hyperbolicity on a proper cone structure $\mathcal{C}$ is equivalent to the non-existence of absolutely continuous closed causal curves plus compactness of every causal diamond. We use this characterization to prove the next result that extends to Lorentz-Finsler stationary spacetimes [52, Proposition B1].

Theorem B.6 *Let* (M, L) *be a Finsler spacetime such that Assumption 1 holds. If* $\mathcal{J}_{p,r}$ *is pseudocoercive for all* $p, r \in M$, *then* (M, L) *is globally hyperbolic.*

Before proving the above result we need the following lemma.

Lemma B.7 *Any absolutely continuous causal curve* $\gamma \,:\, [a, b] \to M$ *admits a reparametrization on* $[0, 1]$ *as an* H^1 *curve with* $Q(\dot\gamma(s)) = \text{const.}$.

Proof By the local splitting and homogeneity in (B1), we can use the locally defined functions t to parametrize locally γ as $\gamma(t) = (x(t), t)$, so that $t \mapsto \|\dot x(t)\|$ is locally bounded. As the support of γ is compact, we can patch together the locally defined reparametrization to get an H^1 curve defined on an interval $[0, c]$, and a further reparametrization gives the thesis. □

Proof of Proposition B.6 From Lemma B.7, there is no loss of generality in considering just H^1 curves parametrized on $[0, 1]$ with $Q(\dot\gamma(s)) = \text{const.}$. Assume that there exists a closed causal curve $\gamma \,:\, [0, 1] \to M$. We take the sequence γ_n, $n \geq 1$, defined by concatenating the n curves $\gamma_j(s) := \gamma(n(s - j/n))$ for $s \in [j/n, (j + 1)/n]$, $j = 0, \dots, n - 1$. The sequence satisfies $\mathcal{J}(\gamma_n) \leq 0$ but it does not admit any uniformly converging subsequence in contradiction with pseucoercivity of $\mathcal{J}_{\gamma(0),\gamma(0)}$, hence (M, L) must be causal. Let us now assume by contradiction that $J^+(p) \cap J^-(r)$ is not compact. Then there exists a sequence of points $(q_n)_{n\in\mathbb{N}} \subset J^+(p) \cap J^-(r)$ that does not admit any subsequence converging to a point in $J^+(p) \cap J^-(r)$. We take then a sequence of causal curves $(\gamma_n)_{n\in\mathbb{N}} \subset J^+(p) \cap J^-(r)$ such that $q_n \in \gamma_n([0, 1])$, for each $n \in \mathbb{N}$. Moreover, by Lemma B.7 we can assume that the sequence $(\gamma_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{J}_{p,r}$. Since $\mathcal{J}(\gamma_n) \leq 0$ for every $n \in \mathbb{N}$, by pseudocoercivity $(\gamma_n)_{n\in\mathbb{N}}$ admits a uniformly converging subsequence $(\gamma_{n_k})_k$. The uniform limit is then a causal curve $\gamma \,:\, [0, 1] \to M$ connecting p to r, by theorem [47, Theorem 2.12]. This implies that $(q_{n_k})_k$ must admit a converging subsequence to a point in $J^+(p) \cap J^-(r)$, which is a contradiction. □

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