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ACTIVE REDUNDANCY FOR A COHERENT SYSTEM UNDER ITS SIGNATURE POINT PROCESS REPRESENTATION

by

Vanderlei da Costa Bueno

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Active redundancy allocation for a coherent system under its signature point process representation

Vanderlei da Costa Bueno
Instituto de Matemática e Estatística
Universidade de São Paulo
Cx. Postal 66.281 05389-970, São Paulo, Brasil
e-mail: bueno@ime.usp.br; phone: (55)(011)30916213; fax: 30916130

Abstract. We characterize active redundancy of a component with a non identically distributed spare through compensator transforms and use the cumulative hazard ordering between compensator processes to investigate the problem of where to allocate the spare in a coherent system of dependent components without simultaneous failures.

Keywords: Point processes, compensator processes, signature point processes, coherent systems, active redundancy.

1. Introduction The problem of where to allocate a redundant component in a system in order to optimize system lifetime is important in reliability theory. For instance see Boland et al. (1992), Singh and Misra (1994), Prasad et al. (1999), Kuo and Prasad (2000), Kuo et al. (2001), Bueno (2005, 2007), Belzunce et al. (2013), Zhao et al.(2012, 2013 ,2015), among others.

There are two common types of redundancy that are used, namely active redundancy, which stochastically leads to consider the maximum of random variables, and standby redundancy , which stochastically leads to consider the convolution of random variables. The problem of where to allocate a spare component, is addressed in Boland et al.(1992) for either active or standby redundancy, in a k-out-of-n system of independent components in order to stochastically increases system reliability.

For a k-out-of-n system Boland et al.(1992) consider stochastic ordering to shows that for active redundancy it is stochastically optimal always to allocate a spare to the weakest component.

Singh and Misra (1994) consider the same problem with another criterion of optimality and proves that if the lifetime of the k-out-of-n:G system resulting from an active redundancy operation of the component i is denoted by $\tau_k^i = \tau_k(T_1, ..., T_{i-1}, T_i \vee S, T_{i+1}, ..., T_n)$, then

$$P(\tau_k^i > \tau_k^j) \geq P(\tau_k^j > \tau_k^i),$$

provided the lifetime of component j is stochastically larger than that of component i, i.e., with respect to the above criterion, it is preferable to allocate the active redundant component to the stochastically weakest component for stochastically ordered component lifetimes.

Meng (1996) uses the concept of permutation equivalency to shows that if two components in a coherent system are permutation equivalent, then allocating a spare to the weaker position as an active redundant optimally improve the system lifetime. Clearly, in a k-out-of-n system, all components are permutation equivalent and the sufficient condition of Boland et al.(1992) is a consequence of this result.

Few papers had attained in the case where the components are stochastically dependent. Belzunce, et al.(2013) analyses redundancies for a k-out-of-n system of dependent components. Bueno (2005) characterizes a minimal standby redundancy through compensator transform, and proved that, under some conditions, it is optimal to perform active redundancy on the weakest component of a k-out-of-n system, with dependent components lifetimes and without simultaneous failures. Also, to work under dependence conditions Bueno and Carmo (2007), using a point process approach, characterizes an active redundancy of dependent but identically distributed component lifetimes, through compensator transform and proves that, it is optimal to perform active redundancy on the weakest component of a k-out-of-n system, with dependent components lifetimes and without simultaneous failures, if the compensators transforms $K(j,t) = 2 - e^{-A_j(t)}, \quad 1 \leq j \leq n, \quad t \in [0, \infty)$, are reversed rule of order two (RR_2).

In this paper, we intend to analyse active redundancy allocation for a coherent system of dependent components without simultaneous failures, characterize active redundancy of dependent component lifetimes and, under the signature point process representation of a coherent system, proves that it is optimal to perform active redundancy on the weakest component of a coherent system.

In Section 2 we give the mathematical details of the signature point process representation of a coherent system. In Section 3 we characterize standby redundancy through compensator transform for dependent components and in

Section 4 we investigate the best redundancy allocation in a coherent system of dependent components in order to optimize system reliability.

2. The signature point process

In our general setup, we consider the vector $(T_1, ..., T_n)$ of n component lifetimes which are finite and positive random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(T_i \neq T_j) = 1$, for all $i \neq j, i, j$ in $C = \{1, ..., n\}$, the index set of components. Therefore, the lifetimes can be dependent but simultaneous failures are ruled out. In order to simplify the notation, in this paper we assume that relations such as $\subset, =, \leq, <, \neq$ between random variables and measurable sets, always hold with probability one, which means that the term P-a.s., is suppressed.

The evolution of components in time define a marked point process given through the failure times and the corresponding marks.

We denote by $T_{(1)} < T_{(2)} < ... < T_{(n)}$ the ordered lifetimes $T_1, T_2, ..., T_n$, as they appear in time and by $X_i = \{j : T_{(i)} = T_j\}$ the corresponding marks. As a convention we set $T_{(n+1)} = T_{(n+2)} = ... = \infty$ and $X_{n+1} = X_{n+2} = ... = e$ where e is a fictitious mark not in C, the index set of the components. Therefore the sequence $(T_n, X_n)_{n \geq 1}$ defines a marked point process.

The mathematical description of our observations, the complete information level, is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t \geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)} > s\}}, X_i = j, 1 \leq i \leq n, j \in C, 0 < s \leq t\},$$

satisfies the Dellacherie conditions of right continuity and completeness.

Intuitively, at each time t the observer knows if the event $\{T_{(i)} \leq t, X_i = j\}$ have either occurred or not and if it had, he knows exactly the value $T_{(i)}$ and the mark X_i. Follows that the component and the system lifetimes T are $\Im_t$ stopping times.

Remark 2.1

An extended and positive random variable τ is an $\Im_t$-stopping time if, and only if, $\{\tau \leq t\} \in \Im_t$, for all $t \geq 0$; an $\Im_t$-stopping time τ is called predictable if an increasing sequence $(\tau_n)_{n \geq 0}$ of $\Im_t$-stopping time, $\tau_n < \tau$, exists such that $\lim_{n \to \infty} \tau_n = \tau$; an $\Im_t$-stopping time τ is totally inaccessible if

$P(\tau = \sigma < \infty) = 0$ for all predictable $\Im_t$-stopping time σ. For a mathematical basis of stochastic processes applied to reliability theory see the book of Aven and Jensen (1999), Bremaud (1981).

We consider the lifetimes $T_{(i),j}$ defined by the failure event $\{T_{(i)}, X_i = j\}$ with their sub-distribution function, suitable standardized

$$F_{(i),j}(t) = P(T_{(i),j} \leq t) = P(T_{(i)} \leq t, X_i = j).$$

Under the complete information level the behavior of the point process $P(T \leq t|\Im_t)$, as the information flows continuously in time is given by the following Theorem

Theorem 2.2 Let $T_1, T_2, ..., T_n$ be the component lifetimes of a coherent system with lifetime T. Then,

$$P(T \leq t|\Im_t) = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}}.$$

Proof From the total probability rule we have $P(T \leq t|\Im_t) =$

$$\sum_{k,j=1}^{n} P(\{T \leq t\} \cap \{T = T_{(k),j}|\Im_t) = \sum_{k,j=1}^{n} E[1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}}|\Im_t].$$

As T and $T_{(k),j}$ are $\Im_t$-stopping time and it is well known that the event $\{T = T_{(k),j}\} \in \Im_{T_{(k),j}}$ where

$$\Im_{T_{(k),j}} = \{A \in \Im_\infty : A \cap \{T_{(k),j} \leq t\} \in \Im_t, \forall t \geq 0\},$$

we conclude that $\{T = T_{(k),j}\} \cap \{T_{(k),j} \leq t\}$ is $\Im_t$-measurable.

Therefore $P(T \leq t|\Im_t) =$

$$\sum_{k,j=1}^{n} E[1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}}|\Im_t] = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}}.$$

The above decomposition allows us to define the signature process at component level.

Definition 2.3 The vector $(1_{\{T=T_{(k),j}\}}, 1 \leq k, j \leq n)$ is defined as the marked point signature process of the system ϕ.

Remark 2.4 As $P(T_i = T_j) = 0$ for all i, j, the collection $\{\{T = T_{(i),j}\}, 1 \leq i \leq n, \ 1 \leq j \leq n\}$ form a partition of Ω and $\sum_{k,j=1}^{n} \Sigma_{j=1}^{n} 1_{\{T=T_{(k),j}\}} = 1$. Therefore

$$P(T > t|\Im_t) = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} - \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}} =$$

$$\sum_{k,j=1}^{n} 1_{\{T=T_{(k)j}\}}[1 - 1_{\{T_{(k)j} \leq t\}}] = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} > t\}}.$$

Remark 2.5 Using Remark 2.1.4 we can calculate the system reliability as

$$P(T > t) = E[P(T > t|\Im_t)] =$$

$$E[\sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} > t\}}] = \sum_{k,j=1}^{n} P(\{T = T_{(k),j}\} \cap \{T_{(k),j} > t\}).$$

If the component lifetimes are totally inaccessible, independent and identically distributed we have,

$$P(T > t) = \sum_{k=1}^{n} P(T = T_{(k)}) P(T_{(k)} > t)$$

recovering the classical result as in Samaniego (1985).

Remark 2.6 The marked point $N_t((i), j) = 1_{\{T_{(i)} \leq t, X_i = j\}}$ is an $\Im_t$-submartingale, that is, $T_{(i),j}$ is $\Im_t$-measurable and $E[N_t((i), j)|\Im_s] \geq N_s((i), j)$ for all $0 \leq s \leq t$.

From Doob-Meyer decomposition, there exists an unique $\Im_t$-predictable process, $(A_t((i), j)_{t \geq 0}$, $A_0((i), j) = 0$, called the $\Im_t$-compensator of $N_t((i), j)$, such that $M_t((i), j) = N_t((i), j) - A_t((i), j)$ is a zero mean uniformly integrable $\Im_t$-martingale. We assume that $T_i, 1 \leq i \leq n$ are totally inaccessible $\Im_t$-stopping time and, under this assumption, $A_t((i), j)$ is continuous. In certain sense, an absolutely continuous lifetime is totally inaccessible.

The compensator process $(A_t((i),j))_{t\geq 0}$ generalizes the classical hazard function notion, on the basis of all observations available up to, but not including, the present.

As $N_t((i),j)$ can only count on the time interval $(T_{(i-1)},T_{(i)}]$, the corresponding compensator differential $dA_t((i),j)$ must vanish outside this interval.

Note that, to count the $i-th$ failure we let $N_t((i)) = \Sigma_{j\geq 1} N_t((i),j)$ with $\Im_t$-compensator process $A_t((i)) = \Sigma_{j\geq 1} A_t((i),j)$. $N_t(j) = \Sigma_{i\geq 1} N_t((i),j)$, counts the component failure and it has $\Im_t$-compensator process $A_t(j) = \Sigma_{i\geq 1} A_t((i),j)$.

The $\Im_t$-compensator of $1_{\{T\leq t\}}$, where T is the system lifetime is set in the following Theorem:

Theorem 2.7 Let $T_1, T_2, ..., T_n$, be the components lifetimes of a coherent system with lifetime T. Then, the $\Im_t$-submartingale $P(T\leq t|\Im_t)$, has the $\Im_t$-compensator

$$\sum_{k,j=1}^{n}\int_0^t 1_{\{T=T_{(k),j}\}} dA_s((k),j).$$

Proof

We consider the process

$$1_{\{T=T_{(k),j}\}}(w,s) = 1_{\{S=T_{(k),j}\}}(w).$$

It is left continuous and $\Im_t$-predictable. Therefore

$$\int_0^t 1_{\{T=T_{(k),j}\}}(s) dM_s((k),j)$$

is an $\Im_t$-martingale.

As a finite sum of $\Im_t$-martingales is an $\Im_t$-martingale, we have

$$\sum_{k,j=1}^{n}\int_0^t 1_{\{T=T_{(k),j}\}} dM_s((k),j) =$$

$$\sum_{k,j=1}^{n}\int_0^t 1_{\{T=T_{(k),j}\}} d1_{\{T_{(k),j}\leq s\}} - \sum_{k,j=1}^{n}\int_0^t 1_{\{T=T_{(k),j}\}} dA_s((k),j).$$

is an $\Im_t$-martingale. As the compensator is unique we finish the proof.

Example 2.8 Consider a complex system under a maintenance program, so that when the system fails it is repaired and continued in operation. The successive failure times may follows a Weibull process. In practical we consider the ordered lifetimes with a conditional survival function given by

$$\overline{F}(t_i|t_1,t_2,...,t_{i-1}) = \exp[-(\frac{t_i-\eta_i}{\theta})^\beta + (\frac{t_i-\eta_{i-1}}{\theta})^\beta]$$

for $\eta_i \vee t_{i-1} < t_i$ where t_i are the ordered observations. The density function is

$$f(t_1,t_2,...,t_n) = \pi_{i=1}^n f(t_i|t_1,t_2,...,t_{i-1}) =$$

$$(\frac{\beta}{\theta})(\frac{t_1-\eta_1}{\theta})^{\beta-1}\exp[(\frac{t_1-\eta_1}{\theta})^\beta]\pi_{i=2}^n(\frac{\beta}{\theta})(\frac{t_i-\eta_i}{\theta})^{\beta-1}$$

$$\exp[-(\frac{t_i-\eta_i}{\theta})^\beta + (\frac{t_i-\eta_{i-1}}{\theta})^\beta].$$

Follows that

$$dA_t(i|t_1,t_2,...,t_{i-1} = \frac{f(t_1,t_2,...,t_n)}{\overline{F}(t_i|t_1,t_2,...,t_{i-1})} = (\frac{\beta}{\theta})(\frac{t-\eta_i}{\theta})^{\beta-1}, \quad t_{i-1} \le t < t_i,$$

$t_0 = 0$.

Therefore

$$A_t(i) = \frac{\beta}{\theta^\beta}\int_{\eta_i}^t (s-\eta_i)^{\beta-1}ds = (\frac{t-\eta_i}{\theta})^\beta, \quad t > \eta_i, \quad t_{i-1} \le t < t_i, \quad t_0 = 0.$$

3.Active redundancy through compensator transforms

Parallel operations are very important in reliability theory: the performance of a parallel system are always better than the performance of any coherent system with the same components and, also, it is used in replacement models to optimize system reliability through active redundancy.

Without loss of generality, we analyse a parallel system of two components. We consider to observe two component lifetimes T and S, which are positive random variables defined in a complete probability space $(\Omega, \Im, P)$ through the family of sub σ-algebras $(\Im_t)_{t\geq 0}$ of $\Im$, where

$$\Im_t = \sigma\{1_{\{S>s\}}, 1_{\{T>s\}}, 0 \leq s \leq t\}, \quad \Im_0 = \{\Omega, \emptyset\}$$

satisfies Dellacherie's conditions.

In what follows we assume that S and T are totally inaccessible $\Im_t$-stopping time and that $P(S = T) = 0$, that is, the lifetimes can be dependent but simultaneous failures are ruled out.

The parallel operation of S and T is defined by the maximum between S and T and denoted by

$$S \vee T = \max\{S, T\}.$$

If we denote the survival functions of S and T as $\overline{G}(t) = P(S > t|\Im_t)$ and $\overline{F}(t) = P(T > t|\Im_t)$ respectively, it follows from Arjas and Yashin (1988), that, under some conditions, the $\Im_t$-compensator processes of $N_t^B = 1_{\{S\leq t\}}$ and $N_t^A = 1_{\{T\leq t\}}$ are given by $B_t = -\ln P(S > t|\Im_t) = -\ln(\overline{G}(t \wedge S))$ and $A_t = -\ln P(T > t|\Im_t) = -\ln(\overline{F}(t \wedge T))$, respectively. We assume such conditions and as S and T are totally inaccessible $\Im_t$-stopping time the compensator processes are continuous.

Now we calculate

$$P(S \vee T > t|\Im_t) = P(S > t|\Im_t) + P(T > t|\Im_t) - P(S > t, T > t|\Im_t) =$$

$$e^{-[A_t+B_t]}\{e^{A_t} + e^{B_t} - 1\}$$

and therefore, in the set $\{t < S \vee T\}$ the $\Im_t$- compensator of $S \vee T$ is

$$-\ln[P(S \vee T > t|\Im_t)] = A_t + B_t - \ln[e^{A_t} + e^{B_t} - 1].$$

We intend to define a probability measure under what the above expression will be the $\Im_t$-compensator transformation of $1_{\{S\vee T\le t\}}$.

Firstly, we consider the compensator transform

$$A^*(t) = \int_0^t \frac{e^{B_s}-1}{e^{A_s}+e^{B_s}-1}dA_s = \int_0^t 1 - [\frac{e^{A_s}}{e^{A_s}+e^{B_s}-1}]dA_s =$$

$$A(t) - \int_0^t \frac{e^{A_s}}{e^{A_s}+e^{B_s}-1}dA_s.$$

To prove the main Theorem of this section we are going to use the following Lemma:

Lemma 3.1 Under the above hypothesis the following process

$$L_t^A) = (\frac{e^{B_T}-1}{e^{A_T}+e^{B_T}-1})^{1_{\{T\le t\}}} e^{\int_0^t \frac{e^{A_s}}{e^{A_s}+e^{B_s}-1}dA_s}$$

is a nonnegative local $\Im_t$-martingale with $E[L_t^A]=1$.

Proof We consider the localization sequence, the $\Im_t$-stopping time defined by

$$V_n = \inf\{t\ge 0 : A_t \ge n \ \ or \ \ B_t \ge n\}.$$

It is sufficient to prove that the process

$$L_t^{A^n} = (\frac{e^{B_T}-1}{e^{A_T}+e^{B_T}-1})^{1_{\{T\le t\wedge V_n\}}} e^{\int_0^{t\wedge V_n} \frac{e^{A_s}}{e^{A_s}+e^{B_s}-1}dA_s}$$

is a bounded $\Im_t$-martingale.

For any $\Im_t$-stopping time $V\le V_n$ we can write

$$L_V^{A^n} = 1 - \int_0^V e^{\int_0^s \frac{e^{A_u}}{e^{A_u}+e^{B_u}-1}dA_u} \frac{e^{A_s}}{e^{A_s}+e^{B_s}-1}d(N_s - A_s)$$

where $N_t = 1_{\{T\le t\}}$. The procedure is easy:

On the set $\{V<T\}$ we have

$$1 + \int_0^V e^{\int_0^s \frac{e^{A_u}}{e^{A_u}+e^{B_u}-1}dA_u} \frac{e^{A_s}}{e^{A_s}+e^{B_s}-1}dA_s =$$

$$e^{\int_0^V \frac{e^{A_u}}{e^{A_u}+e^{B_u}-1}dA_u} = L_V^{A^n}.$$

Otherwise, on the set $\{V \geq T\}$

$$1 - \int_0^V e^{\int_0^s \frac{e^{A_u}}{e^{A_u}+e^{B_u}-1}dA_u} \frac{e^{A_s}}{e^{A_s}+e^{B_s}-1} d(N_s - A_s) =$$

$$e^{\int_0^T \frac{e^{A_u}}{e^{A_u}+e^{B_u}-1}dA_u}[1 - \frac{e^{A_T}}{e^{A_T}+e^{B_T}-1}] = L_V^{A^n}.$$

As the integrand $e^{\int_0^s \frac{e^{A_u}}{e^{A_u}+e^{B_u}-1}dA_u} \frac{e^{A_s}}{e^{A_s}+e^{B_s}-1}$ is an $\Im_t$-predictable process and $N_s - A_s$ is an $\Im_t$-martingale, $L_t^{A^n}$ is an $\Im_t$-martingale with $E[L_t^{A^n}] = 1$.

Secondly, we consider the compensator transform

$$B_t^* = \int_0^t \frac{e^{A_s}-1}{e^{A_s}+e^{B_s}-1} dB_s = B_t - \int_0^t \frac{e^{B_s}}{e^{A_s}+e^{B_s}-1} dB_s$$

and with the same argument to prove Lemma 2.1 we can prove Lemma 3.2:

Lemma 3.2 Under the above hypothesis the following process

$$L_t^B = \left(\frac{e^{A_S}-1}{e^{A_S}+e^{B_S}-1}\right)^{1_{\{S\leq t\}}} e^{\int_0^t \frac{e^{B_s}}{e^{A_s}+e^{B_s}-1}dB_s}$$

is a nonnegative local $\Im_t$-martingale with $E[L_t^B] = 1$.

Observe that the same expression for the $\Im_t$-compensator of $1_{\{T \vee S \leq t\}}$ e is obtained through the transformation:

$$A_t^* + B_t^* = A_t + B_t - \int_0^t \frac{e^{A_s}dA_s + e^{B_s}dB_s}{e^{A_s}+e^{B_s}-1} =$$

$$A_t + B_t - \ln[e^{A_t} + e^{B_t} - 1].$$

Then, we propose the compensators transforms:

$$A_t^* = \int_0^t \alpha(s) dA_s, \quad \alpha(s) = \frac{e^{B_s}-1}{e^{A_s}+e^{B_s}-1},$$

and

$$B_t^* = \int_0^t \beta(s)dB_s, \quad \beta(s) = \frac{e^{A_s}-1}{e^{A_s}+e^{B_s}-1}.$$

to prove the main Theorem:

Theorem 3.3 Under the above hypothesis the following process

$$L(t) = L_t^A L_t^B = (\alpha(T))^{1_{\{T\le t\}}}(\beta(S))^{1_{\{S\le t\}}}[e^{A_t}+e^{B_t}-1]$$

is a nonnegative local $\Im_t$-martingale and $E[L(t)] = 1$.
Proof. Using Lemma 3.1, Lemma 3.2 and the Stieltjes differentiation rule we have

$$L_t^A L_t^B - 1 = \int_0^t L_{s-}^A dL_s^B + \int_0^t L_{s-}^B dL_s^A +$$

$$\sum_{s\le t} \Delta L_s^A \Delta L_s^B.$$

As by assumption, A_t and B_t are continuous and $P(S = T) = 0$ we have $\sum_{s\le t} \Delta L_s^A \Delta L_s^B = 0$ and therefore $L_t^A L_t^B$ in an local $\Im_t$-martingale with $E[L_t^A L_t^B] = 1$ and the theorem is proved.

Now, we are looking for a probability measure Q, such that, under Q, $C_t^* = A_t^* + B_t^*$ becomes the $\Im_t$-compensator of $1_{\{S\vee T\le t\}}$ with respect to this modified probability measure.

Under certain conditions, it is possible to find Q. Indeed, assume that the process L_t is uniformly integrable. Then it follows from well known results on point process martingales (Girsanov Theorem, Bremaud (1981)) that the desired measure Q is given by the Radon Nikodyn derivative

$$\frac{dQ}{dP} = L(\infty).$$

Remark 3.4

In the case where T and S are identically distributed, we have $A_t = B_t$ and the compensator transform is given by

$$A_t^* + B_t^* = 2\int_0^t \frac{e^{A_s} - 1}{2e^{A_s} - 1} dA_s = \int_0^t \frac{2 - 2e^{-A_s}}{2 - e^{-A_s}} dA_s.$$

which is used in Bueno and Carmo (2007), to define active redundancy operation when the component and the spare are dependent but identically distributed.

4. Active redundancy in a coherent system of dependent components

We are concerned with the problem of where to allocate a spare component using active redundancy in a coherent system in order to optimize system reliability improvement. We let $T = \phi(\mathbf{T})$ the lifetime of a coherent system with component lifetimes $\mathbf{T} = (T_1, ..., T_n)$, $P(T_i = T_j) = 0$, for all $i \neq j, 1 \leq i, j \leq n$ under the hypothesis and notation of Section 2. Furthermore, let $T^i = \phi(T_1, ..., T_{i-1}, T_i \vee S, T_{i+1}, ..., T_n)$ the system lifetime resulting from an active redundancy operation of component i through a spare with lifetime S, not identically distributed as T_i. In particular we count this system failure through $N_t^i = 1_{\{T^i \leq t\}}$, a counting process with $\Im_t$-compensator $A_t^i \quad 1 \leq i \leq n$.

To compare the system lifetimes resulting from redundancy operations we are going to compare compensators of the components point processes through the cumulative hazard order as in Shaked and Shanthikumar (1993):

Definition 4.1 Consider two point processes, N_t corresponding to the component lifetime vector $\mathbf{T}$ defined in a complete probability space $(\Omega, \Im, P_{\mathbf{T}})$, and N_s , corresponding to the component lifetime vector $\mathbf{S}$, possibly defined on a different probability spaces, with corresponding continuous compensator processes

$$A_t(n) = A_{t|t_0,t_1,...,t_{n-1}} \quad on \quad [T_{(n-1)}, T_{(n)});$$

$$B_t(m) = B_{t|s_0,s_1,...,s_{m-1}} \quad on \quad [S_{(m-1)}, S_{(m)});$$

which are, $P_{\mathbf{T}}$ almost surely, continuous in t. If for all $t \geq \max\{t_{n-1}, s_{m-1}\}$ and $n \leq m$, $A_t(n) \leq B_t(m)$, for all $0 = s_0 < s_1 < ... < s_{n-1}$, $0 = t_0 < t_1 < ... < t_{n-1}$ and $s_i \leq t_i$, $0 \leq i \leq n-1$, we say that $\mathbf{S}$ is smaller than $\mathbf{T}$ in cumulative hazard order, denoted as $\mathbf{S} \leq^{ch} \mathbf{T}$.

Also, we are going to use the following result from Kwieciński and Szekli (1991)

Theorem 4.2 Kwieciński and Szekli (1991).Consider two point processes N_t corresponding to the component lifetime vector $\mathbf{T}$, defined in a complete probability space $(\Omega, \Im, P_{\mathbf{T}})$, and M_s , corresponding to the component lifetime vector $\mathbf{S}$, possibly defined on a different probability space.

If $\mathbf{S}$ is smaller than $\mathbf{T}$ in cumulative hazard order, ($\mathbf{S} \leq^{ch} \mathbf{T}$), then

$$E_{P_{\mathbf{T}}}[\Psi(N_t)] \leq E_{P_{\mathbf{T}}}[\Psi(M_t)]$$

for all decreasing real and right continuous function with left hand limits Ψ, that is, equivalent to $N_t \leq^{st} M_t$, $t \geq 0$.

In what follows we consider an unique spare with lifetime S, as in Section 3, with $1_{\{S \leq t\}}$ $\Im_t$-compensator B_t, to be allocated between the components, in order to optimize the system lifetime:

Theorem 4.3 Let $T = \phi(\mathbf{T})$ the lifetime of a coherent system with component lifetimes $\mathbf{T} = (T_1, ..., T_n)$, $P(T_i = T_j) = 0$, for all $i \neq j, 1 \leq i, j \leq n$ under the hypothesis and notation of Section 2. If $A_t(i) \geq A_t(j)$, $t \geq 0$, $1 \leq i < j \leq n$, then $N_t^i \leq^{st} N_t^j$, $t \geq 0$, $1 \leq i < j \leq n$.

Proof

Follows from Theorem that the active redundancy through compensator transform of the component i by a spare with compensator B_t is

$$A_t(i) + B_t - \int_0^t \frac{e^{A_s(i)} dA_s + e^{B_s} dB_s}{e^{A_s(i)} + e^{B_s} - 1} =$$

$$A_t(i) + B_t - \ln[e^{A_t(i)} + e^{B_t} - 1].$$

From Theorem 2.7 we have to compare system's compensators expectation values on the form

$$A_t^i = \sum_{k=1}^{n} \sum_{j=1, j\neq i}^{n} 1_{\{T=T_{(k),j}\}} A_t((k),j) +$$

$$\sum_{k=1}^{n} 1_{\{T=T_{(k),i}\}} A_t((k),i) + \sum_{k=1}^{n} \sum_{j=i+1, j\neq i}^{n} 1_{\{T=T_{(k),j}\}} A_t((k),j), \quad 1 \leq i \leq n,$$

where the notation $A_t((k),j)$ means the restriction of $A_t(j)$ to the interval $(T_{(k-1)}, T_{(k)}]$.

Clearly, it is sufficient to prove for $i = 1$ and $j = 2$

$$A_t^1 = \sum_{k=1}^{n} 1_{\{T=T_{(k),1}\}} [A_t((k),1) + B_t - \ln[e^{A_t((k),1)+e^{B_t}-1]]+$$

$$\sum_{k=1}^{n} \sum_{j=2, j\neq 1}^{n} 1_{\{T=T_{(k),j}\}} A_t((k),j) \leq$$

$$\sum_{k=1}^{n} 1_{\{T=T_{(k),1}\}} A_t((k),1) + \leq \sum_{k=1}^{n} 1_{\{T=T_{(k),2}\}} [A_t((k),2)$$

$$+B_t - \ln[e^{A_t((k),2)} + e^{B_t} - 1]] + \sum_{k=1}^{n} \sum_{j=3}^{n} 1_{\{T=T_{(k),j}\}} A_t((k),j), \quad 1 \leq i \leq n = A_t^2$$

$$\leftrightarrow -\ln(e^{A_t((k),1)} + e^{B_t} - 1) \leq -\ln(e^{A_t((k),2)} + e^{B_t} - 1)$$

$$\leftrightarrow A_t((k),1) \geq A_t((k),2).$$

The final result follows from Theorem 4.2.

As by hypothesis, $A_t(i) \geq A_t(j), \quad 1 \leq i < j \leq n$ we are considering component i weaker than component j in the sense that the hazard process for failure of component i is larger than the hazard process for failure of component j (it is, also, implies that T_i is stochastically less than T_j).Follows that, under Theorem 4.3, we understand that it is optimal to perform active redundancy allocation on the weakest component of a coherent system of continuous dependent components with no simultaneous failures.

We can, also consider two spares with lifetimes S_1, S_2, $1_{\{S_1 \leq t\}}$ with $\Im_t$-compensator $B_t(1)$ and $1_{\{S_2 \leq t\}}$ with $\Im_t$-compensator $B_t(2)$, to be allocated between the components, in order to optimize the system lifetime. The following Corollary can be easily proved using the same argument of Theorem 4.3.

Corollary 4.4 Let $T = \phi(\mathbf{T})$ the lifetime of a coherent system with component lifetimes $\mathbf{T} = (T_1, ..., T_n)$, $P(T_i = T_j) = 0$, for all $i \neq j, 1 \leq i, j \leq n$ under the hypothesis and notation of Section 2. If $A_t(i) \geq A_t(j)$, $1 \leq i < j \leq n$ and $B_t(1) \geq B_t(2)$, then $N_t^i \leq^{st} N_t^j$ $1 \leq i < j \leq n$, where $T^i = \phi(T_1, ..., T_{i-1}, T_i \vee S_1, T_{i+1}, ..., T_n)$ and $T^j = \phi(T_1, ..., T_{i-1}, T_j \vee S_2, T_{i+1}, ..., T_n)$.

Remark 4.5

In the case where S_i, $i = 1, 2$ are independent and identically distributed as $T_i, i = 1, 2$, respectively we have $A_t(i) = B_t(i)$, $i = 1, 2$ and the compensator transform is given by

$$2\int_0^t \frac{e^{A_s} - 1}{2e^{A_s} - 1} dA_s = \int_0^t \frac{2 - 2e^{-A_s}}{2 - e^{-A_s}} dA_s.$$

which is used in Bueno and Carmo (2007), to define active redundancy operation when the components are dependent, without simultaneous failures in the case where the spare and the component are identically distributed, to get same results for a k-out-of-n system.

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Departamento de Estatística
IME-USP
Caixa Postal 66.281
05314-970 - São Paulo, Brasil