RT-MAT 96-12

A class of exceptional Bernstein algebras associated to graphs

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Junho/96

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Abstract

Given a simple graph G without loops and a field F of characteristic not 2 we construct an exceptional Bernstein algebra $F(G)$ over F and show that $F(G)$ is indecomposable if and only if G is connected. The duplicate of $F(G)$ is a Bernstein Jordan algebra whose type is determined by the number of points and lines of G. We comment briefly how to extend the theory to the case of multigraphs.

1 Basic facts on Bernstein algebras and graphs

A Bernstein algebra over a field F ($\mathrm{char} F \neq 2$) is a nonassociative commutative algebra over F, endowed with a nonzero homomorphism $\omega: A \longrightarrow F$ (called the weight function of A) such that

$$(x^2)^2 = \omega(x)^2 x^2 \tag{1}$$

for all $x \in A$. The ideal $N = ker\omega$ satisfies the identity $(x^2)^2 = 0$. Such algebras always have idempotents of weight 1 and if e is one these elements, A decomposes as $A = Fe \oplus U \oplus Z$ where

$$U = \{x \in A : \omega(x) = 0 \text{ and } 2ex = x\} \tag{2}$$

and

$$Z = \{x \in A : \omega(x) = 0 \text{ and } ex = 0\} \tag{3}$$

[1] A.M.S. (1991) subject classification number: 17D92

[2] Sponsored by CNPq of Brazil, proc. 300645/93-7 and FAPESP proc. 95/5255-7.

[3] Part of this paper was written while this author was visiting the University of Oviedo - Spain. Sponsored by FAPESP, proc. 94/2247-0.

This direct sum is the Peirce decomposition of A relative to the idempotent e. The relations

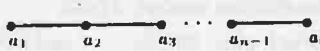
$$U^2 \subseteq Z,\ UZ \subseteq U,\ Z^2 \subseteq U \text{ and } UZ^2 = 0 \tag{4}$$

hold for every Peirce decomposition of A. The following equalities hold in every Bernstein algebra, where $u \in U$ and $z \in Z$:

$$u^3 = u(uz) = u^2(uz) = (uz)z^2 = (uz)^2 = u^2z^2 = 0 \tag{5}$$

Every idempotent of A has the form $e_0 = e + u_0 + u_0^2$ where $u_0 \in U$. The Peirce decomposition relative to e_0 is $A = Fe_0 \oplus U_0 \oplus Z_0$ where $U_0 = \{u + 2uu_0 : u \in U\}$, $Z_0 = \{z - 2(u_0 + u_0^2)z : z \in Z\}$. The mapping $u \in U \mapsto u + 2uu_0 \in U_0$ is clearly linear and bijective, the same holds for the mapping $z \in Z \mapsto z - 2(u_0 + u_0^2)z \in Z_0$. Then $\dim U = \dim U_0$ and $\dim Z = \dim Z_0$. The ordered pair of integers $(1 + \dim U, \dim Z)$ is an invariant of A, called the type of A. In particular, if $U^2 = 0$ then $U_0 = U$ so also $U_0^2 = 0$. Such Bernstein algebras are called *exceptional*. In general, if $A = Fe \oplus U \oplus Z$ then $A^2 = Fe \oplus U \oplus U^2$ so that $A = A^2$ if and only if $U^2 = Z$. These are called *nuclear*. An exceptional algebra is never nuclear (unless $Z = 0$).

The theory of Bernstein algebras is spread over several papers and can be found also in the last chapter of [15]. Bernstein algebras were introduced independently by Ph. Holgate [10] and Y. Lyubich [11] some 20 years ago, as an attempt to solve the Bernstein Problem concerning Populations Genetics, stated and partially solved by S. Bernstein in 1923, see [11]. So far, our knowledge of exceptional algebras is poor and [11] is still the best reference in the subject. We hope this paper will contribute in some way to a better understanding of the subject. On the other hand, concerning graph theory, we refer the reader to Harary's classical book [8] among many other books. Also useful is the book [2], of which we have adopted notations and terminology. We assume that our graphs don't have loops. We denote by $\delta(u)$, where u is a point (= vertex) of G, the degree of u, that is, the number of lines (= edges) incident with u. For a graph $G = (V(G), X(G))$ we denote $\delta(G) = \inf\{\delta(u) : u \in V(G)\}$ and $\Delta(G) = \sup\{\delta(u) : u \in V(G)\}$. The line graph of G, denoted by $L(G)$, has $X(G)$ as its set of points. Two points in $L(G)$ are adjacent in case they are adjacent lines in G. If the line α joins the points a and b, then $\delta(\alpha) = \delta(a) + \delta(b) - 2$. For a given point a, $\overline{a}$ denotes the set of points which are adjacent to a. We denote by P_n the graph



In Sections 2 and 3, all graphs are simple. In Section 4 we state some results for multigraphs (where two points may be linked by several lines).

2 The Bernstein algebra of a graph

Let $G = (V(G), X(G))$ be a finite simple graph, where $V(G)$ is the set of points and $X(G)$ is the set of lines of G. We suppose that G has p points and q lines so that $0 \le q \le \binom{p}{2}$, $p = |V(G)|$ and $q = |X(G)|$. The lines will be denoted by, say, $\alpha = \overline{ab}$, where a and b are the points linked by α, so that $\overline{ab} = \overline{ba}$. Suppose F is a field of characteristic not 2 and let $N = N(G)$ be the direct sum of U and Z, where U (resp. Z) is the F-vector space freely generated by $V(G)$ (resp. $X(G)$). We define in N the following commutative multiplication (on the basis of U and Z): for every $\alpha = \overline{ab} \in X(G)$, where $a, b \in V(G)$,

$$a\alpha = a\overline{ab} = b;\ \text{other products are zero} \tag{6}$$

Every $x \in N$ has the form $x = \sum r_a a + \sum s_\alpha \alpha$, where $a \in V(G), \alpha \in X(G), r_a$ and $s_\alpha \in F$. We have $x^2 = 2[\sum r_a a][\sum s_\alpha \alpha] = 2\sum r_a s_\alpha a\alpha \in U$ and so $(x^2)^2 \in U^2 = 0$ by (6). Hence N satisfies the identity $(x^2)^2 = 0$. In fact, N satisfies the stronger condition $(x_1x_2)(x_3x_4) = 0$, because both x_1x_2 and x_3x_4 are in U and $U^2 = 0$. So N is solvable of index 3. We can embed N in an exceptional Bernstein algebra, by taking the linear operator $\tau : N \longrightarrow N$ defined by $\tau(u) = \frac{1}{2}u$, $u \in U$, and $\tau(z) = 0$, $z \in Z$, and defining the form ω by $\omega(\alpha, a) = \alpha$ and the product in $A = F(G) = F \oplus N(G)$ by

$$(\alpha, a)(\beta, b) = (\alpha\beta, ab + \tau(\alpha b + \beta a)) \tag{7}$$

for $\alpha, \beta \in F$, $a, b \in N$, as in [4, (1)]. The Bernstein algebra $F(G)$ just defined will be referred as the Bernstein algebra associated to the graph G or the Bernstein graph algebra of G. The elements of U (resp. Z) are proper vectors of the linear operator $L_e : N(G) \longrightarrow N(G)$ defined by $x \longmapsto ex$, corresponding to the proper value $\frac{1}{2}$ (resp. 0), where $e = (1, 0)$. The equation $F(G) = Fe \oplus U \oplus Z$, where $e = (1, 0)$, is the natural Peirce decomposition of $F(G)$ and the following relations hold, as particular cases of (4):

$$U^2 = 0,\ UZ \subseteq U,\ Z^2 = 0 \tag{8}$$

as direct consequences of (6). In particular, $(F(G))^2 = Fe \oplus U$. Moreover the type of $F(G)$ is $(1 + p, q)$. The method described above produces exceptional Bernstein algebras of type $(1 + p, q)$ for any p and $q \le \binom{p}{2}$ when G runs over the set of all simple graphs with p points.

We establish some general properties of $F(G)$. The subspace U can be decomposed as $U = U_{\delta(G)} \oplus U_{\delta(G)+1} \oplus \cdots \oplus U_{\Delta(G)}$ where U_j is the subspace generated by the points of degree j, $\delta(G) \le j \le \Delta(G)$. Similarly we have

$Z = Z_{\delta(L(G))} \oplus Z_{\delta(L(G))+1} \oplus \cdots \oplus Z_{\Delta(L(G))}$, with Z_j generated by all lines of degree j. Hence we have a decomposition which refines the Peirce decomposition:

$$F(G) = Fe \oplus U_{\delta(G)} \oplus \cdots \oplus U_{\Delta(G)} \oplus Z_{\delta(L(G))} \oplus \cdots \oplus Z_{\Delta(L(G))} \quad (9)$$

which is called the δ-decomposition of $F(G)$. The following proposition is a refinement of the inclusion $UZ \subseteq U$:

Proposition 1 *Suppose we take integers i and j such that $\delta(G) \leq i \leq \Delta(G)$ and $\delta(L(G)) \leq j \leq \Delta(L(G))$. Then:*

(a) If $j - i + 2 \geq \delta(G)$ then $U_i Z_j \subseteq U_{j-i+2}$

(b) If $j - i + 2 < \delta(G)$ then $U_i Z_j = 0$

Proof: It is enough to work with points and lines, which generate U_i and Z_j respectively. In the first case, take a point $a \in U_i$ and a line $\alpha \in Z_j$. If α is not adjacent to a then necessarily $a\alpha = 0$. If α is adjacent to a then $\alpha = \overline{ab}$ for some point b and $a\alpha = a\overline{ab} = b$. But the degree of $\alpha = \overline{ab}$ is $\delta(\overline{ab}) = \delta(a) + \delta(b) - 2$ so that $\delta(b) = j - i + 2$ so $a\alpha = b \in U_{j-i+2}$. In the second case, no point $a \in U_i$ and no line $\alpha \in Z_j$ are adjacent so $a\alpha = 0$. ■

Recall that for an arbitrary commutative algebra A, the set $\mathrm{Ann}\,(A) = \{a \in A : aA = 0\}$ is an ideal of A.

Proposition 2 *For every graph G,* $\mathrm{Ann}\,(F(G)) = 0$.

Proof: It is well known that, given a Peirce decomposition $A = Fe \oplus U \oplus Z$ of a general Bernstein algebra then $\mathrm{Ann}\,(A) \subseteq Z$. In our case, if G has no lines then $F(G)$ has type $(1+p, 0)$ so necessarily $\mathrm{Ann}\,(F(G)) = 0$. We may assume $q \geq 1$. Suppose $z \in \mathrm{Ann}\,(F(G))$, say $z = \sum r_\alpha \alpha$ where $r_\alpha \in F$ and $\alpha \in X(G)$. Fix one index α, where $\alpha = \overline{ab}$. Then

$$za = (r_{\overline{ab}}\overline{ab} + \sum_{\alpha \neq \overline{ab}} r_\alpha \alpha)a = r_{\overline{ab}} b + \left[\sum_{c \in \overline{a}} r_{\overline{ac}}\overline{ac} + \sum_{c,d \neq a} r_{\overline{cd}}\overline{cd}\right] a = r_{\overline{ab}} b + \sum_{c \in \overline{a}} r_{ac} c = 0,$$

implies $r_{\overline{ab}} = 0$, so $z = 0$. ■

Proposition 3 *With the above notations, the following conditions on $F(G)$ are equivalent:*

(a) $UZ = U$;

(b) G has no isolated points.

Proof: Suppose b is an isolated point of G. If $UZ = U$, then, from (6) $b = \sum_{j=1}^{n} r_j b_j$, with $r_j \in F$, where all b_j must be distinct of b, as b is an isolated point. This equality is a contradiction to the linear independence of the points as elements of U. Conversely, if no point is isolated, given $b \in V(G)$, there is an $a \in V(G)$ such that $a\overline{ab} = b$ so $b \in UZ$. As $V(G) \subseteq UZ$ we have $U \subseteq UZ$ and so $U = UZ$. ∎

We recall that in any Bernstein algebra A with Peirce decomposition $A = Fe \oplus U \oplus Z$, the ideal generated by all elements of the form $a^2 - \omega(a)a$, $a \in A$, can be expressed as $(UZ + Z^2) \oplus Z$ so it is contained in $N = U \oplus Z$. This ideal is called the Etherington's ideal of A. In our case, we have the following corollary:

Corollary *For every graph G, the Etherington's ideal of $F(G)$ equals $N = U \oplus Z$ if and only if G has no isolated points.*

The graph G with p points and no lines yields a Bernstein algebra of type $(1+p, 0)$, that is, the gametic algebra for simple Mendelian inheritance, which is the only commutative baric algebra of dimension $1 + p$ satisfying $x^2 - \omega(x)x = 0$. Due to this, we shall always assume that G has at least 2 points and at least one line.

One of the central questions in the theory of Bernstein algebras is to study the nilpotency of $N = ker\omega$. For recent results, see [1] and [9] and references therein. In our case, N is solvable of index 2 but never nilpotent (unless $Z = 0$) as $a\,\overline{ab} = b$, $(a\,\overline{ab})\overline{ab} = a$ and so on, where $a, b \in V(G)$ are linked by $\overline{ab}$. So every graph yields an exceptional Bernstein algebra such that $N = ker\omega$ is solvable but not nilpotent. Another question in the theory of Bernstein algebras is to state necessary and sufficient conditions for such algebras being train, that is, to satisfy some equation of the form $x^n + \gamma_1\omega(x)x^{n-1} + \cdots + \gamma_{n-1}\omega(x)^{n-1}x = 0$, where $\gamma_i \in F$ (in particular if $\omega(x) = 0$ then $x^n = 0$). Recently it has been proved that the only possible train equations satisfied by Bernstein algebras have the fatorized form $x^2(x-1)(x-\frac{1}{2})^{n-3}$ for elements for weight 1. In our case, $F(G)$ is never train. In fact if we take adjacent points a, b then $a + \overline{ab} \in N$ and its powers are never zero, as $(a + \overline{ab})^2 = b$, $(a + \overline{ab})^3 = a$ and so on, oscillating between a and b.

We look more closely to the Peirce decompositions of $F(G)$, corresponding to other idempotents of the form $e_0 = e + u_0$, where $e = (1,0)$ and $u_0 \in U$. In general, for exceptional algebras we have $U_0 = U$ and $Z_0 = \{z - 2u_0 z : z \in Z\}$. In our case $U^2 = U_0^2 = Z^2 = 0$ but in general Z_0^2 will not be 0. In fact, the product of two typical elements of Z_0 is $(z_1 - 2u_0z_1)(z_2 - 2u_0z_2) = z_1z_2 - 2z_1(u_0z_2) - 2z_2(u_0z_1) + 4(u_0z_1)(u_0z_2)$. The first summand is 0 as well as the fourth, because $(u_0z_1)(u_0z_2) \in U^2 = 0$. But, in general, $(u_0z_1)z_2 + (u_0z_2)z_1 \neq 0$. For instance, if $b \in \overline{a}$, then $(a\,\overline{ab})\overline{ab} = a \neq 0$.

Lemma 1 *Let G be any graph with no isolated points and let $F(G) = Fe \oplus U \oplus Z$ be the natural Peirce decomposition of $F(G)$. If $u \in U$ and $\alpha(\alpha u) = 0$ for every $\alpha \in X(G)$, then $u = 0$.*

Proof: Denote by $a_1, \ldots, a_n$ the points of G so that $u = \sum_{i=1}^{n} \lambda_i a_i$. Choose any $k \in \{1, \ldots, n\}$. As G has no isolated points, there is a line $\overline{a_k a_j} \in X(G)$ (j depending on k, $j \neq k$). Then $\overline{a_k a_j}\, u = \lambda_k a_j + \lambda_j a_k$ and $0 = \overline{a_k a_j}(\overline{a_k a_j}\, u) = \lambda_k a_k + \lambda_j a_j$, so $\lambda_j = \lambda_k = 0$ and $u = 0$. ■

Proposition 4 *Suppose G has no isolated points. If $e_0 = e + u_0$ is an idempotent of $F(G)$ such that $Z_0^2 = 0$ then $u_0 = 0$ (and so $e_0 = e$).*

Proof: We have seen above that $Z_0 = \{z - 2u_0 z : z \in Z\}$ so that $Z_0^2 = 0$ implies $(u_0z_1)z_2 + (u_0z_2)z_1 = 0$ for all $z_1, z_2 \in Z$. Then, in particular, $(u_0\alpha)\alpha = 0$ for all $\alpha \in X(G)$ and by Lemma 1, $u_0 = 0$. ■

Lemma 2 *Let G be a connected graph with at least 3 points. If an ideal I of $F(G)$ intersects nontrivially the subspace U then $U \subseteq I$.*

Proof: Choose some nonzero element $x \in U \cap I$ and express x as $x = \lambda_1 a_1 + \ldots + \lambda_n a_n$ where $a_1, \ldots, a_n$ are the points of G and $\lambda_1, \ldots, \lambda_n \in F$, with say, $\lambda_1 \neq 0$. We may suppose also that a_2 is adjacent to a_1. Then $(x\overline{a_1a_2})\overline{a_1a_2} = \lambda_1 a_1 + \lambda_2 a_2 \in I$. Suppose the degree of a_1 is at least 2. Then, say, a_3 is adjacent to a_1 so that $(\lambda_1 a_1 + \lambda_2 a_2)\overline{a_1a_3} = \lambda_1 a_3 \in I$ and $a_3 \in I$. Let $a_k \in V(G)$ with $k \neq 3$. As G is connected there is a chain of lines $\overline{a_3 a_{i_1}}, \overline{a_{i_1} a_{i_2}}, \ldots, \overline{a_{i_p} a_k}$ in $X(G)$ such that $(\ldots((a_3\overline{a_3 a_{i_1}})\overline{a_{i_1} a_{i_2}})\ldots)\overline{a_{i_p} a_k} = a_k \in I$, so $V(G) \subseteq I$ and $U \subseteq I$, as desired. Suppose now the degree of a_1 is 1. Then the degree of a_2 must be at least 2 because otherwise P_2 would be a connected component of G. As G has at least 3 points, then G would be disconnected, a contradiction. Then a_2 has degree at least 2. Suppose a_l is adjacent to a_2, with $l \neq 1$. Then $x\overline{a_1a_2} = \lambda_1 a_2 + \lambda_2 a_1 \in I$ and so $(x\overline{a_1a_2})\overline{a_2a_l} = \lambda_1 a_l \in I$. Then $a_l \in I$ and we can repeat the chain argument used above to conclude that any point $a_k \in I$. ■

In any exceptional algebra $A = Fe \oplus U \oplus Z$, U is an ideal of A. We have a precise description of the baric ideals of $F(G)$, when G is connected. Recall that a baric ideal of A is any ideal of A contained in $N = U \oplus Z$.

Theorem 1 *Let G be a simple connected graph. Then:*

(a) If G has at least 3 points, the nonzero baric ideals of $F(G)$ are exactly the subspaces $U \oplus Z'$ where Z' is an arbitrary subspace of Z.

(b) If $G = P_2$, with points a and b, the nonzero baric ideals of $F(G)$ are $F(a+b)$, $F(a-b)$, U and N.

Proof: It is easily seen that $U \oplus Z'$ is an ideal of $F(G)$ in cases (a) or (b) and $F(a+b)$ and $F(a-b)$ are also ideals in case (b). Now to the converse. Take initially any nonzero baric ideal I so that $I = (I \cap U) \oplus (I \cap Z)$. Suppose $I \cap Z \neq 0$ and choose an element $z \in I \cap Z$ with $z \neq 0$. The element z is a linear combination of all lines of G so there is at least one line of the form $\overline{ab_1}$ in this linear combination with nonzero coefficient λ_1. Let $b_1, b_2, \ldots, b_n$ $(n \geq 1)$ be the (distinct) adjacent points to a and $\overline{c_1d_1}, \ldots, \overline{c_md_m}$ $(m \geq 0)$ the lines which are not incident with a. Then $z = \lambda_1\overline{ab_1} + \ldots + \lambda_n\overline{ab_n} + \gamma_1\overline{c_1d_1} + \ldots + \gamma_m\overline{c_md_m}$, $\lambda_1 \neq 0$, with λ_i, $\gamma_j \in F$. Now $za = \lambda_1 b_1 + \ldots + \lambda_n b_n \in I \cap U$. By Lemma 2, $U \subseteq I$ and so $I = (I \cap U) \oplus (I \cap Z) = U \oplus (I \cap Z) = U \oplus Z'$. Suppose now $I \cap Z = 0$; then $I \subseteq U$ and again by Lemma 2, $I = U$. This finishes the proof of case a).

In order to complete the case (b), we suppose G has two points a and b so that $F(G) = Fe \oplus Fa \oplus Fb \oplus F\overline{ab}$. Let I be any one dimensional ideal contained in $Fa \oplus Fb$. Take $x = \lambda_1 a + \lambda_2 b \in I$, $\lambda_1 \neq 0$. Then $\overline{ab}x = \lambda_1 b + \lambda_2 a \in I$ and so $\lambda_1 x - \lambda_2 \overline{ab}x = (\lambda_1^2 - \lambda_2^2)a \in I$. If $\lambda_1^2 \neq \lambda_2^2$ then $a \in I$ so $b \in I$ and $I = U$, contradiction. Then $\lambda_1^2 = \lambda_2^2$, so $\lambda_1 = \pm\lambda_2$ and $x = \lambda_1(a \pm b)$. As x runs over I, we must always have the same sign ($+$ or $-$) because otherwise $\dim I = 2$. Thus finally $I = F(a+b)$ or $I = F(a-b)$. ■

Corollary 1 *For a connected graph G the principal ideal generated by any point a is U and the principal ideal generated by any line $\overline{ab}$ is $U \oplus F\overline{ab}$.*

As a rule the product of two baric ideals in $F(G)$ is not an ideal, according to the following corollary

Corollary 2 *For a connected graph G we have equivalence between:*

(a) The product of two arbitrary baric ideals in $F(G)$ is an ideal;

(b) G is the graph P_2.

Proof: *a)* $\Longrightarrow$ *b)* : If, by contradiction, G has at least 3 points a, b, c and lines $\overline{ab}$ and $\overline{ac}$, then $U\left(U \oplus F\overline{ab}\right) = Fa \oplus Fb$, which is not an ideal as $(a+b)\overline{ac} = c \notin Fa \oplus Fb$.

b) $\Longrightarrow$ *a)* : The set of all nonzero baric ideals is $F(a+b)$, $F(a-b)$, U and N and we see directly that the condition (a) holds. ■

Corollary 3 *If G is a simple connected graph with $p \geq 3$ points and q lines then all homomorphic images of $F(G)$ are $F(G)$ itself or (trivial) Bernstein algebras of type $(1, s)$, for some s, $0 \leq s \leq q$.*

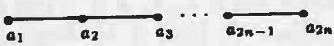
Proof: Follows from Theorem 1. ■

Let G be a connected graph with at least 3 points. By Theorem 1, the baric ideals of $F(G)$ have the form $U \oplus Z'$ with Z' any subspace of Z. In general, $(U \oplus Z')(U \oplus Z'') \subseteq U$ and so it will be a nonzero baric ideal if and only if $(U \oplus Z')(U \oplus Z'') = U$.

Corollary 4 *If $Z' + Z'' = Z$ then $(U \oplus Z')(U \oplus Z'') = U$.*

Proof: Suppose $a \in V(G)$ and consider any line $\overline{ab} \in X(G)$. Then $\overline{ab} = z' + z''$ with $z' \in Z'$ and $z'' \in Z''$. Now $a = \overline{ab}b = bz' + bz'' \in UZ' + UZ'' = (U \oplus Z')(U \oplus Z'')$ so that $V(G) \subseteq (U \oplus Z')(U \oplus Z'')$ and this implies $U \subseteq (U \oplus Z')(U \oplus Z'')$ and the equality $(U \oplus Z')(U \oplus Z'') = U$. ■

The converse of this corollary is not true. Consider the graph P_{2n}:



and $\alpha = \overline{a_1a_2} + \overline{a_2a_3} + \ldots + \overline{a_{2n-1}a_{2n}} \in Z$. We have for $Z' = F\alpha$, $(U \oplus Z')^2 = Z'U = U$, as it follows from an elementary calculation.

We recall (see [3], [4] and [5]) that a Bernstein algebra $A = Fe \oplus N$ is decomposable when there exist Bernstein algebras A_1 and A_2 of dimension at least 2 such that $A = A_1 \vee A_2$. This is equivalent to say that there exist nonzero baric ideals I_1 and I_2 of A such that $N = I_1 \oplus I_2$. Otherwise, it is indecomposable. The following lemma is very natural.

Lemma 3 *Let G be a simple graph with connected components $G_1, \ldots, G_k$. Then $F(G) \cong F(G_1) \vee \ldots \vee F(G_k)$.*

Proof: The proof can be done simply by constructing the multiplication table of $F(G)$ and assuming, by induction, that $k = 2$. We omit the details. ■

Theorem 2 *The following conditions are equivalent for the simple graph G:*

(a) G is connected

(b) $F(G)$ is indecomposable

Proof: $(a) \Longrightarrow (b)$: Suppose G has at least 3 points. The baric ideals of $F(G)$, according to Theorem 1, have the form $U \oplus Z'$ so that it is impossible to find two nonzero baric ideals with zero intersection. In the case $G = P_2$, the nonzero baric ideals of $F(G)$ are $F(a+b)$, $F(a-b)$, U and N and again it is impossible to obtain such a decomposition of N.

$(b) \Longrightarrow (a)$: Call $G_1, \ldots, G_n$ the connected components of G. Then by Lemma 3, $F(G) = F(G_1) \vee \ldots \vee F(G_n)$. As $F(G)$ is indecomposable then $n = 1$ and $G = G_1$ is connected. ∎

Corollary 5 *Let $G = (V(G), X(G))$ be a connected graph with p points. Then $F(G)$ is an indecomposable exceptional Bernstein algebra of type $(1+p, q)$ where $p-1 \leq q \leq \binom{p}{2}$. In particular, $F(P_p)$ has type $(1+p, p-1)$.*

3 The duplicate of F(G)

The purpose of this section is to investigate properties of the commutative duplicate of $F(G)$, for G a simple graph. It is well known that every commutative baric algebra (A, ω) has a commutative duplicate (A^D, ω_D) where A^D is the symmetric product of the vector space A by itself, endowed with the multiplication $(x * y)(x' * y') = xy * x'y'$ for every $x, x', y, y' \in A$. The weight function $\omega_D \colon A^D \longrightarrow F$ is defined by $\omega_D(x * y) = \omega(xy)$. If e is an idempotent of weight 1 in A then $e * e$ is an idempotent of weight 1 in A^D so that $A^D = F(e * e) \oplus ker\omega_D$. The following result goes back essentially to Etherington [6]. For a proof, see [12].

Theorem 3 *The algebras A^D and $A^2 \rtimes \mathrm{Ann}(A^D)$ are isomorphic, where $\rtimes$ means semidirect product.*

In [13] it was proved effectively that if A^2 is a Bernstein algebra of order n then A^D is a Bernstein algebra of order $n+1$. We recall that a Bernstein algebra of order n $(n \geq 0)$ is a commutative baric algebra (A, ω) such that the identity $x^{[n+2]} = \omega(x)^{2^n} x^{[n+1]}$ holds, where the plenary powers $x^{[k]}$ are defined by $x^{[1]} = x$ and $x^{[k]} = (x^{[k-1]})^2$ for $k \geq 2$. In our case, $F(G)$ is exceptional so that $(F(G))^2$ has type $(1+p, 0)$ and so is Bernstein of order 0. Its commutative duplicate must be Bernstein of order 1, according to [13].

We prove, in this Section, that the duplicate of $F(G)$ is, in fact, a Bernstein Jordan algebra. Moreover, we determine the type of $F(G)^D$ and its absolute zero divisors, which, in the majority of cases, form a high dimensional ideal. We work initially in the more general setting of algebras satisfying $U^2 = Z^2 = 0$ for some idempotent.

Lemma 4 *Let $A = Fe \oplus U_e \oplus Z_e$ be the Peirce decomposition of a Bernstein algebra of type $(1+n, m)$ relative to an idempotent e such that $U_e^2 = Z_e^2 = 0$. Consider the commutative duplicate $A^D = F(e*e) \oplus ker\omega_D$ and the linear operator L_{e*e}: $ker\omega_D \longrightarrow ker\omega_D$. Then the proper values of L_{e*e} are $\frac{1}{2}$ and 0 and the corresponding proper spaces are respectively $Fe*U_e$ and $H = U_e*U_e \oplus Z_e*Z_e \oplus Fe*Z_e \oplus H'$, where H' is the subspace generated by the elements of the form $u*z - 2e*uz$ where $u \in U_e$ and $z \in Z_e$.*

Proof: According to general properties of duplicates we have $A^D = A*A = F(e*e) \oplus Fe*U_e \oplus Fe*Z_e \oplus U_e*U_e \oplus U_e*Z_e \oplus Z_e*Z_e$. Consider now the following equalities:

$$L_{e*e}(e*u) = \tfrac{1}{2}e*u,\ L_{e*e}(e*z) = 0,\ L_{e*e}(u*u') = 0,\ L_{e*e}(z*z') = 0,$$
$$L_{e*e}(u*z - 2e*uz) = (e*e)(u*z) - 2(e*e)(e*uz) = e*uz - e*uz = 0.$$

¿From this, we see that $Fe*U_e$ is the proper subspace of the proper value $\frac{1}{2}$ and $H = U_e*U_e \oplus Z_e*Z_e \oplus Fe*Z_e \oplus H'$ is contained in the proper subspace of L_{e*e} relative to the proper value 0 (that is, is contained in the kernel of L_{e*e}). But if we count dimensions, we get

$$\dim H = \dim(U_e*U_e \oplus Z_e*Z_e \oplus Fe*Z_e \oplus H') =$$
$$\tfrac{1}{2}n(n+1) + \tfrac{1}{2}m(m+1) + m + nm = \tfrac{1}{2}\left((m+n)^2 + n + 3m\right),$$

where $\dim U_e = n$ and $\dim Z_e = m$. As the dimension of A^D is equal to $\frac{1}{2}(1+m+n)(2+m+n)$, we are obliged to conclude that $ker\omega_D = Fe*U_e \oplus (U_e*U_e \oplus Z_e*Z_e \oplus Fe*Z_e \oplus H') = Fe*U_e \oplus H$. ■

We can say something more about these proper subspaces $Fe*U_e$ and H of the operator L_{e*e}.

Lemma 5 *Under the same conditions of the Lemma 4, H is the annihilator of A^D and $U_e*U_e = (Fe*U_e)^2$. In particular, $(A^D)^2 = Fe*e \oplus Fe*U_e \oplus U_e*U_e$.*

Proof: From the definition of the product in A^D and $U_e^2 = Z_e^2 = 0$, we have immediately $U_e*U_e \oplus Z_e*Z_e \oplus Fe*Z_e \subseteq \mathrm{Ann}(A^D)$. Moreover if $a, a' \in A$, $(a*a')(u*z - 2e*uz) = (a*a')(u*z) - 2(a*a')(e*uz) = aa'*uz - aa'*uz = 0$, so that $H' \subseteq \mathrm{Ann}(A^D)$, hence $H \subseteq \mathrm{Ann}(A^D)$. Conversely, if $x \in \mathrm{Ann}(A^D)$ with $x = \lambda e*e + \overline{u} + \overline{z}$ where $\lambda \in F$, $\overline{u} \in Fe*U_e$, $\overline{z} \in H$, then $0 = (e*e)x = \lambda(e*e) + \frac{1}{2}\overline{u}$ so that $\lambda = 0$, $\overline{u} = 0$ and $x = \overline{z} \in H$. Therefore $\mathrm{Ann}(A^D) = H$. ■

Lemma 6 *Under the same conditions of the previous lemmas, the kernel of ω_D is nilpotent of index 2 or 3 according to $U_e = 0$ or $U_e \neq 0$.*

Proof: We have $(ker\omega_D)^2 = (Fe{*}U_e \oplus H)^2 = (Fe{*}U_e)^2 = U_e{*}U_e$, according to Lemma 5. If $U_e = 0$ we are done. If $U_e \neq 0$ then $(ker\omega_D)^3 = (ker\omega_D)(U_e{*}U_e) = 0$ as $U_e * U_e$ is contained in Ann (A^D). ■

Proposition 5 *Under the same conditions of the previous lemmas, A^D satisfies the train equation $x^3 - \omega_D(x)x^2 = 0$ and so it is a Bernstein-Jordan algebra. Moreover A^D is decomposable.*

Proof: For any $\overline{x} = \lambda e * e + \overline{u} + \overline{z}$, for $\lambda \in F$, $\overline{u} \in Fe * U_e$ and $\overline{z} \in H$, we have $\overline{x}^2 = \lambda^2 e * e + \overline{u}^2 + \lambda\overline{u}$ (using the previous lemmas) and so $\overline{x}^3 = \lambda^3 e * e + \lambda^2\overline{u} + \lambda\overline{u}^2 = \lambda x^2$. The second statement is a well known result, see [14]. The third statement is a particular case of [4, Prop.5]. ■

As a particular case of this proposition, we have

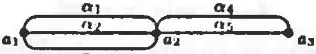
Corollary 6 *Let G be any simple graph with p points and q lines. Then $F(G)^D$ is a decomposable train algebra satisfying $x^3 - \omega_D(x)x^2 = 0$ and of type $\left(1 + p, \frac{1}{2}\left((q+p)^2 + p + 3q\right)\right)$. In particular, $F(G)^D$ is a Bernstein Jordan algebra.*

The process of duplication can be iterated, constructing the second duplicate of A, denoted $\left(A^D\right)^D$, and so on. Using [13, Théorème 2.1], we know that the second duplicate of $F(G)$ satisfies the train equation $x^4 - \omega(x)x^3 = 0$. In general, we have

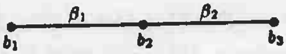
Corollary 7 *For every simple graph G, the k-th iterated duplicate of $F(G)$ satisfies the train equation $x^{k+2} - \omega(x)x^{k+1} = 0$.*

4 Complements

We have assumed in the beginning of the paper that G is a simple graph. But the same definition of multiplication in N, as given in (6), applies when G is a multigraph. In this case, the number of lines joining two points of G is arbitrary. The Bernstein algebra so obtained has type $(1 + p, q)$ where now q is arbitrary. The previous Theorem 1 is still valid but Theorem 2 is no longer valid. The following example illustrates what nappens in this case. Let G be the connected multigraph

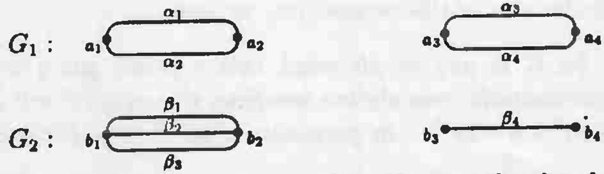


and consider the associated simple graph P_3, which is obtained by identification of all lines joining two given points:



Then $F(G) \cong F(P_3) \oplus F(\alpha_1 - \alpha_2) \oplus F(\alpha_1 - \alpha_3) \oplus F(\alpha_4 - \alpha_5) \cong F(P_3) \vee (Fe \oplus F(\alpha_1 - \alpha_2)) \vee (Fe \oplus F(\alpha_1 - \alpha_3)) \vee (Fe \oplus F(\alpha_4 - \alpha_5))$, where the four baric algebras on the right are indecomposable, the first one by Theorem 2 and the remaining 3 as two dimensional algebras. In fact, each of these two dimensional algebras is isomorphic to the algebra with basis $\{e, v\}$ and multiplication table $e^2 = e$, $ev = v^2 = 0$. The reader can easily obtain the decomposition of any multigraph, proceeding as above.

Moreover, it is easily seen that two nonisomorphic multigraphs may have the same associated Bernstein algebra. Here is an example (due to C. Martínez) using two multigraphs each with two connected components:

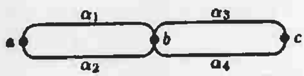


Clearly G_1 and G_2 are not isomorphic. On the other hand, according to our comments and denoting by $\langle X \rangle$ the vector space generated by X, we have

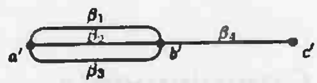
$$F(G_1) = Fe_1 \oplus \langle a_1, \ldots, a_4 \rangle \oplus \langle \alpha_1, \alpha_3 \rangle \oplus \langle \alpha_1 - \alpha_2 \rangle \oplus \langle \alpha_3 - \alpha_4 \rangle$$
$$F(G_2) = Fe_2 \oplus \langle b_1, \ldots, b_4 \rangle \oplus \langle \beta_1, \beta_4 \rangle \oplus \langle \beta_1 - \beta_2 \rangle \oplus \langle \beta_1 - \beta_3 \rangle$$

and the mapping $e_1 \mapsto e_2$, $a_i \mapsto b_i$ $(i = 1, \ldots, 4)$, $\alpha_1 \mapsto \beta_1$, $\alpha_3 \mapsto \beta_4$, $\alpha_1 - \alpha_2 \mapsto \beta_1 - \beta_2$, $\alpha_3 - \alpha_4 \mapsto \beta_1 - \beta_3$ is an isomorphism of baric algebras.

Similarly, the non isomorphic connected multigraphs



and



have the same associated Bernstein algebras. The following theorem is valid.

Theorem 4 *If G_1 and G_2 are multigraphs such that the associated simple graphs are isomorphic and $|X(G_1)| = |X(G_2)|$, then $F(G_1) \cong F(G_2)$.*

The proof of this theorem is clear from the decomposition of $F(G)$, G a multigraph, as the join of $F(G')$, G' the associated simple graph of G, and a certain number of trivial Bernstein algebras of dimension 2, as described above.

Acknowledgment

We thank the University of Oviedo for the kind hospitality during our stay. Special thanks to Profs. S. González and C. Martínez of the Faculty of Sciences for their continuous encouragement.

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