A Rigidity Result for Some Parabolic Germs

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ABSTRACT. The goal of this article is to prove a rigidity result for unicritical polynomials with parabolic cycles. More precisely, we show that if two unicritical polynomials have conformally conjugate parabolic germs, then the polynomials are affinely conjugate.

CONTENTS

1. INTRODUCTION

The study of conformal conjugacy classes of holomorphic germs fixing the origin is a classical area of research in complex analysis. Of particular interest (and rich structure) is the set of tangent-to-identity germs

$$\mathrm{Diff}^{+1}(\mathbb{C},0) := \{f(z) = z + az^{n+1} + \cdots \mid n \geq 1,\ a \neq 0,\ f \in \mathbb{C}\{z\}\}.$$

Such a germ is also called a *parabolic germ of multiplier* 1 (this is the terminology we will use in this paper). For a parabolic germ of multiplier 1 as above, the integer $n+1$ is called the *multiplicity of the fixed point* 0. By the work of Écalle and Voronin [Eca75, Vor81], there exists an infinite-dimensional family of conformally different parabolic germs of multiplier 1, with 0 being a fixed point of multiplicity $n+1$.

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In holomorphic dynamics (in particular, in the iteration theory of polynomials), parabolic fixed points play a crucial role. A fixed point $\hat{z}$ of a polynomial $P(z)$ (in one complex variable) is called parabolic if $P'(\hat{z})$ is a q-th root of unity. Evidently, the polynomial $P^{\circ q}$ (where $P^{\circ q}$ stands for the q-th iterate of P) satisfies $(P^{\circ q})'(\hat{z}) = 1$. Conjugating $P^{\circ q}$ by a translation that sends $\hat{z}$ to 0, we obtain a polynomial parabolic germ of multiplier 1 fixing the origin. Thus, a parabolic fixed point of a polynomial determines a polynomial element of $\mathrm{Diff}^{+1}(\mathbb{C}, 0)$. Since polynomials are global objects, it is quite reasonable to expect that if the parabolic fixed points of two polynomials determine conformally conjugate elements of $\mathrm{Diff}^{+1}(\mathbb{C}, 0)$, then the global dynamics of the two polynomials are intimately related (compare [CEP15, Section 3]). The principal goal of this paper is to formalize and prove this heuristic idea for unicritical polynomials.

Any unicritical polynomial of degree $d \geq 2$ can be affinely conjugated to a map of the form $f_c(z) = z^d + c$ (an affine conjugacy respects the conformal dynamics of a polynomial). The *filled Julia set* $K(f_c)$ is defined as the set of all points which remain bounded under all iterations of f_c. The boundary of the filled Julia set is defined to be the *Julia set* $J(f_c)$. The degree d multibrot set $\mathcal{M}_d$ is the connectedness locus of degree d unicritical polynomials, that is,

$$\mathcal{M}_d := \{c \in \mathbb{C} \mid K(f_c) \text{ is connected}\}.$$

The multibrot set of degree 2 is called the *Mandelbrot set*.

A parameter $c \in \mathcal{M}_d$ is called *hyperbolic* if the forward orbit of the unique critical point 0 of f_c converges to an attracting cycle. A connected component H of the set of all hyperbolic parameters is called a *hyperbolic component* of $\mathcal{M}_d$. Every parameter on the closure of a hyperbolic component H has a unique non-repelling cycle (uniqueness is a consequence of unicriticality). The multiplier of this unique non-repelling cycle defines a $(d-1)$-fold map (which is holomorphic in the interior of H and continuous up to ∂H) from the closure of H onto the closure of the unit disk in the complex plane. For the Mandelbrot set, that is, for $d = 2$, this map yields an actual homeomorphism. This map is called the *multiplier map*, and is denoted by $\lambda_{\bar{H}}$.

A parameter c of $\mathcal{M}_d$ is called a parabolic parameter if f_c has a periodic cycle with multiplier a root of unity. Note that every parabolic cycle of a polynomial attracts the forward orbit of at least one critical point [Mil06, Theorem 10.15]. Since the polynomials f_c have a unique critical point, it follows that f_c can have at most one parabolic cycle. Every parabolic parameter of $\mathcal{M}_d$ lies on the boundary of some hyperbolic component H. A parabolic parameter c is called the *root* of a hyperbolic component H of period greater than 1 if $\lambda_{\bar{H}}(c) = 1$ and the parabolic cycle of f_c disconnects $J(f_c)$ (the period 1 hyperbolic component is exceptional in the sense that, in the dynamical plane of its root, the parabolic cycle does not disconnect the Julia set). On the other hand, a parabolic parameter c is called a *co-root* of a hyperbolic component H if $\lambda_{\bar{H}}(c) = 1$, and the parabolic cycle of f_c does not disconnect $J(f_c)$. There are exactly one root and $(d-2)$ co-roots on the boundary of every hyperbolic component H (of period greater than 1) of $\mathcal{M}_d$ (compare [EMS16, Theorem 1]). In particular, there is no co-root in the Mandelbrot set.

For $i = 1, 2$, let c_i be a root or co-root point of a hyperbolic component H_i of period n_i of $\mathcal{M}_d$, and z_i be the characteristic parabolic point of f_{c_i}, that is, the parabolic periodic point on the boundary of the critical value Fatou component (where the critical value Fatou component is the Fatou component containing the critical value). It is worthwhile to note that under the above assumptions, $(f_{c_i}^{\circ n_i})'(z_i) = 1$; that is, the restriction of $f_{c_i}^{\circ n_i}$ to a neighborhood of z_i determines an element of $\mathrm{Diff}^{+1}(\mathbb{C}, 0)$. The following theorem is the main result of this paper.

Theorem 1.1 (Parabolic Germs Determine Parabolic Parameters). *Suppose there exist small neighborhoods N_1 and N_2 of z_1 and z_2 (in the dynamical planes of c_1 and c_2, respectively) such that $f_{c_1}^{\circ n_1}|_{N_1}$ and $f_{c_2}^{\circ n_2}|_{N_2}$ are conformally conjugate (i.e., $f_{c_1}^{\circ n_1}|_{N_1}$ and $f_{c_2}^{\circ n_2}|_{N_2}$ determine conformally conjugate elements of $\mathrm{Diff}^{+1}(\mathbb{C}, 0)$). Then, f_{c_1} and f_{c_2} are affinely conjugate.*

Note that f_{c_1} and f_{c_2} are affinely conjugate if and only if c_2/c_1 is a $(d-1)$-st root of unity. Therefore, the above theorem states that the conformal conjugacy class of the parabolic germ restriction of a suitable iterate of f_c determines c uniquely up to the action of a finite rotation group.

The paper is organized as follows. In Section 2, we recall some general facts about parabolic dynamics and parabolic parameters in the multibrot set $\mathcal{M}_d$. We review the theory of parabolic-like maps (a parabolic analogue of polynomial-like maps introduced by the first author in [Lom15]) in Section 3. Section 4 is devoted to the proof of Theorem 1.1. There are two essential steps in the proof of the theorem. In Lemma 4.1, we prove a rigidity theorem for parabolic-like maps. Subsequently, in Lemma 4.2, we prove a local-to-global principle to the effect that a local conformal conjugacy between certain parabolic germs can be promoted to a conformal conjugacy between suitable parabolic-like maps. The main theorem now follows by combining these two lemmas.

2. Preliminaries

Let $f(z) = z + az^{n+1} + \cdots$ be a parabolic germ of multiplier 1. The local dynamics of such a germ are well understood. Let us briefly review the situation for completeness. There are $2n$ open sectors $V_1^+, V_1^-, \ldots, V_n^+, V_n^-$ based at 0 such that they cover a deleted neighborhood of the origin. For each $i = 1, 2, \ldots, n$, we have $f(V_i^+) \subset V_i^+$, and the forward orbit (under f) of each point in V_i^+ converges to the parabolic fixed point 0 (compare Figure 2.1). These sectors are called *attracting petals*. Moreover, there exists a conformal embedding $\psi^{\mathrm{att},i}$ of V_i^+ into $\mathbb{C}$ such that the image contains a right-half plane and $\psi^{\mathrm{att},i}$ conjugates f to translation by $+1$. The maps $\psi^{\mathrm{att},i}$ are called *attracting Fatou coordinates*. On the other hand, for each $i = 1, 2, \ldots, n$, we have $f^{-1}(V_i^-) \subset V_i^-$, and the backward orbit (under f) of each point in V_i^- converges to 0. These sectors are called *repelling petals*. There exists a a conformal embedding $\psi^{\mathrm{rep},i}$ of V_i^- into $\mathbb{C}$ such that the image contains a right-half plane and $\psi^{\mathrm{rep},i}$ conjugates f^{-1} to translation by $+1$. The maps $\psi^{\mathrm{rep},i}$ are called *repelling Fatou coordinates*. These coordinates are unique up to addition of a complex constant. (For a more rigorous description of the dynamics of parabolic germs and construction of Fatou coordinates, see [Mil06, Section 10; Lor06, Section 2.3].)

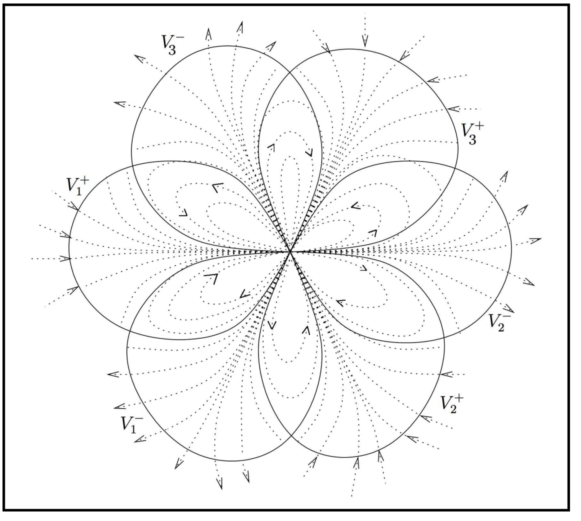

FIGURE 2.1. Dynamics of a parabolic germ with three attracting and three repelling petals.

For a general parabolic germ, these various Fatou coordinates do not agree on their common domain of definition: namely, on the intersection of two adjacent attracting and repelling petals. This leads to the collection of horn maps

$$\{\psi^{\mathrm{att},i} \circ (\psi^{\mathrm{rep},i})^{-1}, \psi^{\mathrm{att},i+1} \circ (\psi^{\mathrm{rep},i})^{-1}\}_{i\in\mathbb{Z}/n\mathbb{Z}},$$

which record the difference between two adjacent attracting and repelling Fatou coordinates. The importance of horn maps stems from the fact that they are complete conformal conjugacy invariants of parabolic germs [Eca75, Vor81]. We refer the readers to [BE02, Section 2] for a precise definition of horn maps.

If a parabolic germ is obtained as a restriction of a globally defined polynomial, then the attracting and (the inverse of the) repelling Fatou coordinates have natural maximal domains of definition. The extensions are obtained by iterating the dynamics. Hence, the associated horn maps extend as ramified coverings to certain natural maximal domains of definition. Definitions of these extended horn maps and their mapping properties can be found in [BE02, Section 2.5]. The main property of extended horn maps that we will use in this paper is that they have finitely many critical values and these critical values are completely determined by the conformal positions of the critical points of the polynomial [BE02, Proposition 4; Eps93]. In particular, if the polynomial is unicritical, then each of these extended horn maps has a unique critical value.

We now recall some basic facts about the parabolic parameters of the multibrot set $\mathcal{M}_d$. Every parabolic parameter c (i.e., a parameter c such that f_c has a periodic orbit of multiplier a root of unity) of $\mathcal{M}_d$ is either the root or a co-root of a unique hyperbolic component H. We will denote the characteristic parabolic point of f_c by z_c; that is, z_c is the unique parabolic periodic point of f_c that lies on the boundary of the critical value Fatou component. A parabolic parameter c is a root (respectively, a co-root) of a hyperbolic component of period greater than one if z_c is a cut point of $K(f_c)$ (respectively, z_c is not a cut point of $K(f_c)$), where z_c is called a cut point of $K(f_c)$ if $K(f_c) \setminus \{z_c\}$ is disconnected. In the parameter plane, a corresponding dichotomy holds: a root parameter of a hyperbolic component of period $n > 1$ is a cut point of $\mathcal{M}_d$, whereas a co-root parameter is not.

If c is a co-root of a hyperbolic component H of period n, then f_c has a parabolic cycle of period n and multiplier 1. Since z_c is not a cut point of $K(f_c)$, there is a unique attracting petal at z_c. Therefore, we have

$$f_c^{\circ n}(z) = z + a_c(z - z_c)^2 + O((z - z_c)^3)$$

for some $a_c \in \mathbb{C}^*$, as the Taylor series expansion of $f_c^{\circ n}$ near z_c.

The situation for roots is a bit more complicated, as they come in two different flavors. We say that a parabolic parameter c in $\mathcal{M}_d$ is a *primitive* root if z_c is a cut point of $K(f_c)$ and there is a single attracting petal at z_c. A primitive root c lies on the boundary of a unique hyperbolic component of $\mathcal{M}_d$. If the period of this unique hyperbolic component H is n, then f_c has an n-periodic parabolic cycle of multiplier 1. As in the co-root case, the Taylor series of $f_c^{\circ n}$ near z_c is given by $(z + a_c(z - z_c)^2 + O((z - z_c)^3))$ for some $a_c \in \mathbb{C}^*$. On the other hand, a parabolic parameter c is called a *satellite* root if z_c is a cut point of $K(f_c)$ and there are at least two attracting petals at z_c. A satellite root c is the (unique) common boundary point of two hyperbolic components H and H' of $\mathcal{M}_d$, one of which (say, H) has c as its root. Let the periods of the hyperbolic components H and H' be n and k, respectively. Then the unique parabolic cycle of f_c has period k and multiplier a q-th root of unity, where $q = n/k \geq 2$. Moreover, the Taylor series expansion of $f_c^{\circ n}$ near z_c is given by $(z + a_c(z - z_c)^{q+1} + O((z - z_c)^{q+2}))$ for some $a_c \in \mathbb{C}^*$. In particular, there are q attracting petals at the parabolic point z_c, and these petals are permuted transitively by $f_c^{\circ k}$. For proofs of these statements, see [EMS16, Lemma 17].

3. Parabolic-Like Maps

The theory of parabolic-like maps extends the theory of polynomial-like maps to objects with a parabolic external class. For any polynomial map P on the Riemann sphere $\hat{\mathbb{C}}$, infinity is a superattracting fixed point, and the filled Julia set K_P is the complement of the basin of attraction of infinity $\mathcal{A}(\infty)$, that is, $K_P = \hat{\mathbb{C}} \setminus \mathcal{A}(\infty)$. Thus, if U is a suitable topological disk containing K_P (e.g., if the boundary of U is a sufficiently large equipotential), then the preimage of U is a topological disk U' compactly contained in U, and $P_{|U'} : U' \to U$ is a proper holomorphic map of degree $d = \deg(P)$. The triple (P, U', U) is a (trivial) example of a polynomial-like map. Formally, a (degree d) polynomial-like map is a triple (f, U', U) where U' and U are topological disks, $U' \Subset U$, and $f : U' \to U$ is a (degree d) proper holomorphic map [DH85]. The filled Julia set K_f of a polynomial-like map is the set of points which never leave U' under iteration (for a polynomial P, this is just K_P). With any degree d polynomial-like map, one can associate a degree d covering of the unit circle $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ which encodes the dynamics of the polynomial-like map outside its filled Julia set. The map h_f is called the *external map* of the polynomial-like map f. The external map of a polynomial-like map is strictly expanding, with all periodic points repelling, and it is defined up to real-analytic diffeomorphisms of the circle. In this way, a polynomial-like map can be considered as a union of two different dynamical systems: the filled Julia set K_f and the external map h_f. By replacing the external map of a degree d polynomial-like map with the map $z \to z^d$ (which is an external map for a degree d polynomial), Douady and Hubbard proved that every degree d polynomial-like map is hybrid equivalent to a polynomial of the same degree (where a hybrid equivalence is a quasiconformal conjugacy φ with $\bar{\partial}\varphi = 0$ on K_f), and that this polynomial is unique if K_f is connected.

A parabolic-like map is an object similar to a polynomial-like map, in the sense that it can be considered as the union of two different dynamical systems: the filled Julia set and the external map [Lom15]. However, the external map of a parabolic-like map contains a parabolic fixed point, which complicates the setting considerably.

Definition (Parabolic-like maps). A *parabolic-like map* of degree $d \geq 2$ is a 4-tuple (f, U', U, γ) where the following hold:

- U' and U are open subsets of $\mathbb{C}$, with U', U, and $U \cup U'$ isomorphic to a disc, and U' not contained in U.
- $f : U' \to U$ is a proper holomorphic map of degree $d \geq 2$ with a parabolic fixed point at $z = z_0$ of multiplier 1.
- $\gamma : [-1, 1] \to \bar{U}$ is an arc with $\gamma(0) = z_0$, forward invariant under f, C^1 on $[-1, 0]$ and on $[0, 1]$, and such that

$$f(\gamma(t)) = \gamma(dt), \quad \forall\ -\frac{1}{d} \leq t \leq \frac{1}{d},$$
$$\gamma\left(\left[\frac{1}{d}, 1\right) \cup \left(-1, -\frac{1}{d}\right]\right) \subseteq U \setminus U', \quad \gamma(\pm 1) \in \partial U.$$

  It resides in the repelling petal(s) of z_0, and it divides U' and U into Ω', Δ', and Ω, Δ, respectively, such that $\Omega' \Subset U$ (and $\Omega' \subset \Omega$), $f : \Delta' \to \Delta$ is an isomorphism (see Figure 3.1) and Δ' contains at least one attracting fixed petal of z_0. We call the arc γ a *dividing arc*.

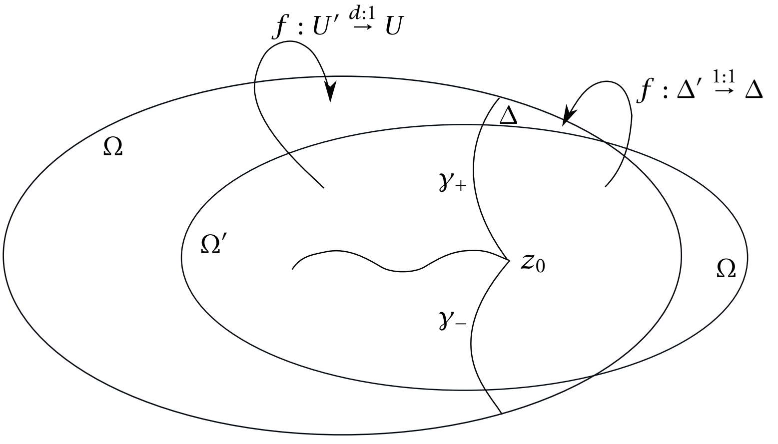

FIGURE 3.1. For a parabolic-like map (f, U', U, γ), the arc γ divides U' and U into Ω', Δ' and Ω, Δ, respectively. These sets are such that Ω' is compactly contained in U, $\Omega' \subset \Omega$, $f : \Delta' \to \Delta$ is an isomorphism, and Δ' contains at least one attracting fixed petal of the parabolic fixed point.

The filled Julia set K_f of a parabolic-like map (f, U', U, γ) is the set of points which never leave $\Omega' \cup \{z_0\}$ under iteration. The model family in degree 2 is given by the family of quadratic rational maps with a parabolic fixed point of multiplier 1 at infinity (normalized by having critical points at ± 1); this is

$$\mathrm{Per}_1(1) = \{[P_A] \mid P_A(z) = z + 1/z + A,\ A \in \mathbb{C}\}.$$

The filled Julia set for P_A with $A \neq 0$ is the complement of the parabolic basin of infinity $\mathcal{A}_A(\infty)$, so $K_A = \hat{\mathbb{C}} \setminus \mathcal{A}_A(\infty)$. (For $A = 0$, we need to make a choice, since both the left and right half planes are parabolic basins. We set $K_0 = \overline{\mathbb{H}}_\ell$.) The map

$$h_2(z) = \frac{z^2 + 1/3}{z^2/3 + 1}$$

is an external map for every P_A, $A \in \mathbb{C}$ [Lom15, Proposition 4.2]. By replacing the external map of a degree 2 parabolic-like map with the map h_2, the first author proved [Lom15] that every degree 2 parabolic-like map is hybrid equivalent to a member of the family $\mathrm{Per}_1(1)$, and that this member is unique if K_f is connected. For more detailed studies on parabolic-like maps, consult [Lom15] for a dynamical description, [Lom14a] for a parameter space (of degree 2 analytic families of parabolic-like maps) description, and [Lom14b] for an easy discussion on the results contained in the previous two articles.

4. PROOF OF THE THEOREM

In the dynamical plane of the root of a satellite hyperbolic component of the multibrot set $\mathcal{M}_d$, a suitable iterate of the polynomial admits a degree d parabolic-like restriction (see [Lom15, Section 3.1, Example 3] for details of the construction in the case $d = 2$, the case $d > 2$ being similar). The following lemma proves a rigidity statement for these parabolic-like maps.

Lemma 4.1 (Rigidity of Parabolic-like Mappings). *Let c_1 and c_2 be the root points of two satellite hyperbolic components H_1 and H_2 (of period n_1 and n_2, respectively) of the multibrot set $\mathcal{M}_d$. If the parabolic-like mappings defined by the restrictions of $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ (around their critical value Fatou components) are conformally conjugate, then $c_1 = c_2$ up to affine conjugacy.*

Proof. Since c_i is the root of a satellite component H_i of period n_i, the polynomial f_{c_i} has a unique parabolic cycle of period k_i $(< n_i)$ with multiplier a q_i-th root of unity, where $q_i = n_i/k_i$. The Taylor series expansion of $f_{c_i}^{\circ n_i}$ near z_i is given by $(z + a_i(z - z_i)^{q_i+1} + O((z - z_i)^{q_i+2}))$ for some $a_i \in \mathbb{C}^*$. In particular, there are q_i attracting petals at the parabolic point z_i, and these petals are permuted transitively by $f_{c_i}^{\circ k_i}$. Let us start by noticing that if the parabolic-like mappings defined by the restrictions of $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ are conformally conjugate, then the parabolic germs of $f_{c_i}^{\circ n_i}$ at z_i (for $i = 1, 2$) are conformally conjugate. As the number of attracting petals of a parabolic germ is preserved by a topological conjugacy, it follows that $q_1 = q_2 = q$, for example.

We label the q Fatou components of f_{c_i} touching at the characteristic parabolic point z_i counter-clockwise such that U_i^1 is the Fatou component of f_{c_i} containing the critical value c_i. Since c_i is the root of a satellite component with a k_i-periodic parabolic cycle, the polynomial $f_{c_i}^{\circ k_i}$ has a polynomial-like restriction (h_i, V_i', V_i) that is hybrid equivalent to some (degree d) p_i/q-rabbit (basilica if $q = 2$) parameter on the boundary of the principal hyperbolic component of $\mathcal{M}_d$ (more precisely, $f_{c_i}^{\circ k_i}$ has a polynomial-like restriction h_i that is hybrid equivalent to some polynomial $f_{c_i'}$ with a fixed point of multiplier $e^{2\pi i p_i/q}$).

Let η be a conformal conjugacy between the parabolic-like restrictions of $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ in neighborhoods of $\overline{U_1^1}$ and $\overline{U_2^1}$ (respectively). *A priori*, η is defined only in a neighborhood W of $\overline{U_1^1}$. We can assume, possibly after shrinking W (but ensuring that it still contains $\overline{U_1^1}$), that $f_{c_1}^{\circ n_1}$ has a unique critical point in W. We will now use the dynamics $f_{c_1}^{\circ n_1}$ to extend η to a conformal conjugacy between $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ from a neighborhood of $K(h_1)$ to a neighborhood of $K(h_2)$.

Let us define the "characteristic ear" $\mathcal{E}_1$ of $K(h_1)$ to be the closure of the connected component of $K(h_1) \setminus \{z_1\}$ containing the critical value c_1 (see Figure 4.1, Left). Note that $f_{c_1}^{\circ n_1}(\mathcal{E}_1) = K(h_1)$. More precisely, there is a unique (strictly) pre-periodic point z_1' on ∂U_1^1 such that $f_{c_1}^{\circ n_1}(z_1') = z_1$, and the closures $\mathcal{E}_2, \dots, \mathcal{E}_q$ of the connected components of $\mathcal{E}_1 \setminus \{z_1'\}$ *not* containing c_1 are

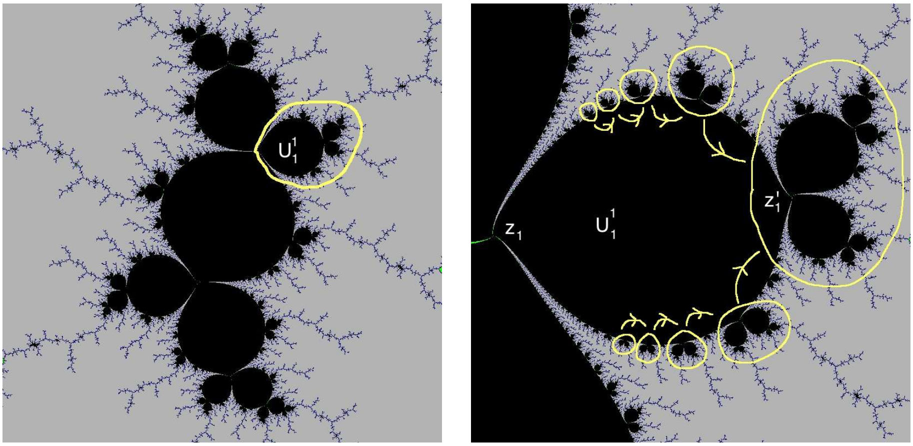

FIGURE 4.1. Left: The characteristic ear $\mathcal{E}_1$ of $K(h_1)$ is enclosed by the yellow curve. Right: Under $f_{c_1}^{\circ n_1}$, each region enclosed by a yellow curve univalently maps to the "next" one. The initial conformal conjugacy between the two parabolic-like maps can be extended to a neighborhood of $\mathcal{E}_1$ by using the univalent restrictions of $f_{c_1}^{\circ n_1}$ indicated in the figure.

univalently mapped by $f_{c_1}^{\circ n_1}$ onto the closures of the connected components of $K(h_1) \setminus \{z_1\}$ *not* containing c_1 (compare Figure 4.2). Therefore, it suffices to first extend the conjugacy η to a neighborhood of $\mathcal{E}_1$, and then use the above-mentioned univalent restrictions of $f_{c_1}^{\circ n_1}$ (on neighborhoods of $\mathcal{E}_2, \ldots, \mathcal{E}_q$) to spread it to a neighborhood of $K(h_1)$ (via the formula $f_{c_2}^{\circ n_2} \circ \eta \circ (f_{c_1}^{\circ n_1})^{-1}$).

Since $K(h_1)$ is locally connected, the iterated pre-images of $\mathcal{E}_2 \cup \cdots \cup \mathcal{E}_q$ under $f_{c_1}^{\circ n_1}$ (choosing suitable inverse branches of $f_{c_1}^{\circ n_1}$ such that the pre-images are attached to ∂U_1^1) shrink to the point z_1. Therefore, W contains infinitely many such inverse images of $\mathcal{E}_2 \cup \cdots \cup \mathcal{E}_q$. We can now iteratively extend η to a neighborhood of $\mathcal{E}_1$ using the formula $f_{c_2}^{\circ n_2} \circ \eta \circ (f_{c_1}^{\circ n_1})^{-1}$ (cf. Figure 4.1, Right).

To summarize, we have defined a (finite) sequence of maps $\{\eta_s\}_{s=0}^N$ such that η_0 is the original conjugacy η, and $\eta_s := f_{c_2}^{\circ n_2} \circ \eta_{s-1} \circ (f_{c_1}^{\circ n_1})^{-1}$ for $s = 1, 2, \ldots, N$ (choosing suitable inverse branches). Moreover, $\mathrm{Dom}(\eta_s) \cap U_1^1 \neq \emptyset$, for $s = 1, 2, \ldots, N$, and $\bigcup_{s=0}^N \mathrm{Dom}(\eta_s) \supset K(h_1)$. Since $f_{c_1}^{\circ n_1}$ fixes U_1^1 and η_0 is a conjugacy between $f_{c_1}^{\circ n_1}|_{U_1^1}$ and $f_{c_2}^{\circ n_2}|_{U_2^1}$, it follows from the construction that each η_s extends the conformal map η_0 defined on U_1^1. Hence, by uniqueness of analytic continuation, all these extensions match up to yield a conformal map η defined on a neighborhood of $K(h_1)$ such that it conjugates $f_{c_1}^{\circ n_1}$ to $f_{c_2}^{\circ n_2}$.

Therefore, η is a conformal conjugacy between the polynomial-like maps $h_1^{\circ q}$ $(= f_{c_1}^{\circ n_1})$ and $h_2^{\circ q}$ $(= f_{c_2}^{\circ n_2})$. By [IM16, Corollary 10.2], we conclude that f_{c_1} and f_{c_2} are affinely conjugate. □

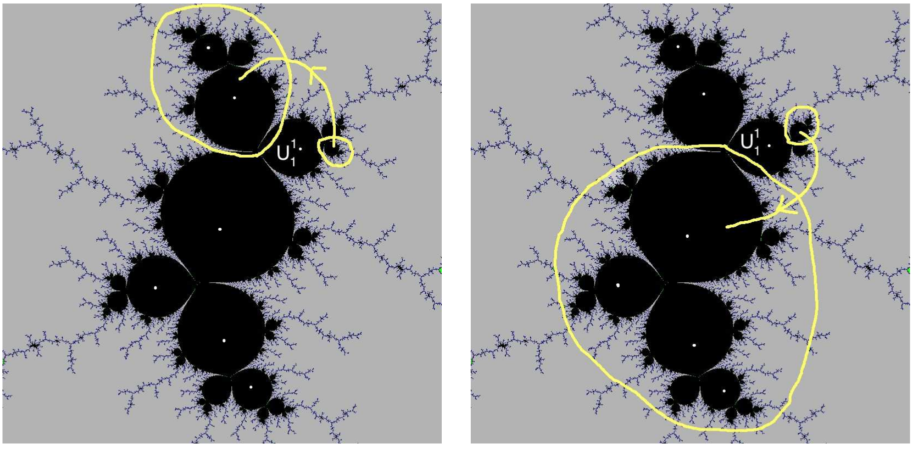

FIGURE 4.2. Both figures show the small filled Julia set $K(h_1)$ in the dynamical plane of f_{c_1}, where c_1 is a satellite root of the Mandelbrot set with $q = 3$. The critical points of $f_{c_1}^{\circ n_1}$ in $K(h_1)$ are marked. A small neighborhood of the characteristic ear $\mathcal{E}_1$ contains exactly one of these critical points. We obtain the required extension of η by first extending it univalently to a neighborhood of the characteristic ear $\mathcal{E}_1$ and then using the univalent restrictions of $f_{c_1}^{\circ n_1}$ indicated in the figures to spread η to a neighborhood of $K(h_1)$.

We continue to work with parameters c_1 and c_2 that are root points of satellite hyperbolic components H_1 and H_2 (of period n_1 and n_2, respectively) of the multibrot set $\mathcal{M}_d$. Our next lemma shows that a local conjugacy between the parabolic germs of $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ can be promoted to a conformal conjugacy between two suitable degree d parabolic-like mappings. As in the proof of the previous lemma, we label the Fatou components of f_{c_i} touching at the characteristic parabolic point z_i counter-clockwise such that U_i^1 is the Fatou component of f_{c_i} containing the critical value c_i. To investigate the consequences of a conformal conjugacy between two polynomial parabolic germs, we will need to use the concept of extended horn maps (see Section 2).

Lemma 4.2. *Let c_1 and c_2 be the root points of two satellite hyperbolic components H_1 and H_2 (of period n_1 and n_2, respectively) of the multibrot set $\mathcal{M}_d$, and z_1 and z_2 be the characteristic parabolic points of f_{c_1} and f_{c_2} (respectively). Then, the following are equivalent:*

- *The degree d parabolic-like mappings defined by the restrictions of $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ (with filled Julia set $\overline{U}_1^1$ and $\overline{U}_2^1$, respectively) are conformally conjugate.*
- *The (tangent-to-identity) parabolic germs given by the restrictions of $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ (around z_1 and z_2, respectively) are conformally conjugate.*

Proof. Conformal conjugacy of the parabolic-like maps clearly implies conformal conjugacy of the corresponding germs. Thus, we only need to show that when $g_1 := f_{c_1}^{\circ n_1}|_{N_1}$ and $g_2 := f_{c_2}^{\circ n_2}|_{N_2}$ are conformally conjugate by some local biholomorphism $\varphi_1 : N_1 \to N_2$ (where N_i is a small neighborhood of z_i), the degree d parabolic-like maps $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ (with filled Julia set $\overline{U}_1^1$ and $\overline{U}_2^1$, respectively) are also conformally conjugate. We now proceed to prove this.

For $i = 1, 2$, let us suppose that the period of the characteristic parabolic point z_i of f_{c_i} be k_i. Then, f_{c_i} has $q_i = n_i/k_i$ attracting petals at z_i which are permuted transitively by $f_{c_i}^{\circ k_i}$. Since two conformally conjugate germs have the same number of attracting petals, we have that, say, $q_1 = q_2 = q$. Note that φ_1 must map an attracting petal $\mathcal{P}_{c_1}^{\text{att},1} \subset N_1 \cap U_1^1$ to an attracting petal $\mathcal{P}_{c_2}^{\text{att},k} \subset N_2 \cap U_2^k$ for some $k \in \{1, 2, \ldots, q\}$. Thus, $\varphi := f_{c_2}^{\circ k_2(1-k)} \circ \varphi_1$ is a conformal conjugacy between g_1 and g_2 such that it maps $\mathcal{P}_{c_1}^{\text{att},1}$ to a petal $\mathcal{P}_{c_2}^{\text{att},1} \subset N_2 \cap U_2^1$.

For $k \in \mathbb{Z}/q\mathbb{Z}$, if $\psi_{c_2}^{\text{att},k}$ is an extended attracting Fatou coordinate for $f_{c_2}^{\circ n_2}$ in U_2^k, then there exists an extended attracting Fatou coordinate $\psi_{c_1}^{\text{att},k}$ for $f_{c_1}^{\circ n_1}$ in U_1^k such that $\psi_{c_1}^{\text{att},k} = \psi_{c_2}^{\text{att},k} \circ \varphi$ in their common domain of definition.

Similarly for $k \in \mathbb{Z}/q\mathbb{Z}$, if $\psi_{c_2}^{\text{rep},k}$ is a repelling Fatou coordinate for $f_{c_2}^{\circ n_2}$ at z_2, then $\psi_{c_1}^{\text{rep},k} := \psi_{c_2}^{\text{rep},k} \circ \varphi$ is a repelling Fatou coordinate of $f_{c_1}^{\circ n_1}$ at z_1. The inverses of these repelling Fatou coordinates admit global extensions, and we denote them by $\Psi_{c_i}^{\text{rep},k}$ (compare [BE02, Section 2.5]).

Using the extended attracting and repelling Fatou coordinates described above, we can define extended horn maps $h_{c_i,k}^{+} := \psi_{c_i}^{\text{att},k} \circ \Psi_{c_i}^{\text{rep},k}$ and $h_{c_i,k}^{-} := \psi_{c_i}^{\text{att},k+1} \circ \Psi_{c_i}^{\text{rep},k}$ of $f_{c_i}^{n_i}$ at z_i. Since these extended attracting and repelling Fatou coordinates are related by φ in neighborhoods of z_i, it follows that $h_{c_1,k}^{\pm} = h_{c_2,k}^{\pm}$ for $k = 1, 2, \ldots, k$. By [BE02, Proposition 4], each $h_{c_i,1}^{+}$ is a ramified covering with a unique critical value $\Pi(\psi_{c_i}^{\text{att},1}(c_i))$, where $\Pi(Z) = e^{2\pi i Z}$. It follows that $\Pi(\psi_{c_1}^{\text{att},1}(c_1)) = \Pi(\psi_{c_2}^{\text{att},1}(c_2))$. Therefore, $\psi_{c_1}^{\text{att},1}(c_1) - \psi_{c_2}^{\text{att},1}(c_2) = r \in \mathbb{Z}$. Thus, we can normalize the attracting Fatou coordinates such that $\psi_{c_1}^{\text{att},1}(c_1) = 0$ and $\psi_{c_2}^{\text{att},1}(c_2) = -r$.

Let us define the map $\tilde{\psi}_{c_2}^{\text{att},1} := \psi_{c_2}^{\text{att},1} \circ g_2^{\circ r}$ on $\mathcal{P}_{c_2}^{\text{att},1}$. Clearly, $\tilde{\psi}_{c_2}^{\text{att},1}$ is an attracting Fatou coordinate for $f_{c_2}^{\circ n_2}$ at z_2 such that, for all $x \in \mathcal{P}_{c_2}^{\text{att},1}$, we have $\tilde{\psi}_{c_2}^{\text{att},1}(x) - \psi_{c_2}^{\text{att},1}(x) = r$. By analytic continuation, we have an extended attracting Fatou coordinate $\tilde{\psi}_{c_2}^{\text{att},1} : U_2^1 \to \mathbb{C}$ for $f_{c_2}^{\circ n_2}$ such that

$$\tilde{\psi}_{c_2}^{\text{att},1}(c_2) = \psi_{c_2}^{\text{att},1}(c_2) + r = 0 = \psi_{c_1}^{\text{att},1}(c_1).$$

Define $\eta := (\tilde{\psi}_{c_2}^{\text{att},1})^{-1} \circ \psi_{c_1}^{\text{att},1} : N_1 \cap U_1^1 \to N_2 \cap U_2^1$; then, η is a conformal conjugacy between $f_{c_1}^{\circ n_1}|_{N_1}$ and $f_{c_2}^{\circ n_2}|_{N_2}$ which extends by iterated lifting to a conformal conjugacy $\eta : U_1^1 \to U_2^1$ between $f_{c_1}^{\circ n_1}|_{U_1^1}$ and $f_{c_2}^{\circ n_2}|_{U_2^1}$.[1] Abusing notation, we will denote this extended conjugacy by η. Since the basin boundaries are locally connected, by Caratheodory's theorem the conformal conjugacy η extends as a homeomorphism from ∂U_1^1 onto ∂U_2^1. Note also that by definition, $\eta = g_2^{\circ(-r)} \circ \varphi$ in their common domain of definition. Therefore, η admits an analytic continuation to a neighborhood V of the point z_1, and continues to be a conjugacy between the germs g_1 and g_2.

We will now extend this conformal conjugacy to a conformal conjugacy η between a neighborhood of $\overline{U_1^1}$ and a neighborhood of $\overline{U_2^1}$. By Montel's theorem, we have $\bigcup_{s\in\mathbb{N}} f_{c_1}^{\circ sn_1}(V \cap \partial U_1^1) = \partial U_1^1$. Since none of the $f_{c_1}^{\circ sn_1}$ has a critical point on ∂U_1^1, we can extend η to a neighborhood of each point of ∂U_1^1 by using the functional equation $\eta \circ f_{c_1}^{\circ sn_1} = f_{c_2}^{\circ sn_2} \circ \eta$. Since all of these extensions at various points of ∂U_1^1 extend the already defined (and conformal) common map η, uniqueness of analytic continuation yields an analytic extension of η to a neighborhood of $\overline{U_1^1}$. By construction, this extension is clearly a proper holomorphic map, and assumes every point in U_2^1 precisely once. Therefore, the extended η from a neighborhood of $\overline{U_1^1}$ onto a neighborhood of $\overline{U_2^1}$ has degree one. Thus, η is a conformal conjugacy between $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$. This shows that the parabolic-like mappings defined by $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ in neighborhoods of $\overline{U_1^1}$ and $\overline{U_2^1}$ (respectively) are conformally conjugate. □

We are now ready to prove the main result of this paper.

Proof of Theorem 1.1. By hypothesis, c_1 (respectively, c_2) is the root or a co-root point of a hyperbolic component H_1 (respectively, H_2) of period n_1 (respectively, n_2) of $\mathcal{M}_d$. Also, z_1 (respectively, z_2) is the characteristic parabolic point of f_{c_1} (respectively, f_{c_2}). For $i = 1, 2$, the Taylor series expansion of $f_{c_i}^{\circ n_i}$ around z_i is given by $(z + a_i(z - z_i)^{q_i+1} + O((z - z_i)^{q_i+2}))$ for some $q_i \geq 1$ and $a_i \in \mathbb{C}^*$. Note that q_i is the number of attracting petals of f_{c_i} at z_i.

We assume that there exist neighborhoods N_1 and N_2 of z_1 and z_2 such that the parabolic germs $f_{c_1}^{\circ n_1}|_{N_1}$ and $f_{c_2}^{\circ n_2}|_{N_2}$ are conformally conjugate. Evidently, such a conjugacy implies that $q_1 = q_2$. We will now consider two cases.

Case 1: $q_1 = q_2 = 1$. In this case, each c_i is a co-root or a primitive root of H_i (compare Section 2). Hence, the unique parabolic cycle of f_{c_i} has period n_i; that is, the period of z_i is n_i. Since the germs $f_{c_1}^{\circ n_1}|_{N_1}$ and $f_{c_2}^{\circ n_2}|_{N_2}$ are conformally conjugate, it follows by [IM16, Theorem 1.4] that f_{c_1} and f_{c_2} are affinely conjugate.

[1] Here is an alternative route to extend η to the entire Fatou component. We can choose Riemann maps $\varphi_{c_i} : U_i^1 \to \mathbb{D}$ with $\varphi_{c_i}(c_i) = 0$ such that φ_{c_i} conjugates $f_{c_i}^{\circ n_i}|_{U_i^1}$ to the Blaschke product $B(z) = (3z^2 + 1)/(3 + z^2)$. An easy computation in Fatou coordinates now shows that $\varphi_{c_2}^{-1} \circ \varphi_{c_1}$ extends the local conjugacy η to the entire immediate basin U_1^1 such that it conjugates $f_{c_1}^{\circ n_1}$ on U_1^1 to $f_{c_2}^{\circ n_2}$ on U_2^1.

Case 2: $q_1 = q_2 > 1$. In this case, each c_i is a satellite root of H_i. Since the germs $f_{c_1}^{\circ n_1}|_{N_1}$ and $f_{c_2}^{\circ n_2}|_{N_2}$ are conformally conjugate, it follows by Lemma 4.2 that the degree d parabolic-like mappings defined by the restrictions of $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ (around the critical value Fatou components U_1^1 and U_2^1, respectively) are conformally conjugate. By Lemma 4.1, we can now conclude that f_{c_1} and f_{c_2} are affinely conjugate. ◻

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