Commun Nonlinear Sci Numer Simulat 138 (2024) 108204

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Non-Newtonian incompressible fluids with nonlinear shear tensor and hereditary conditions

Heraclio Ledgar López-Lázaro [a], Pedro Marín-Rubio [b,*], Gabriela Planas [c]

[a] Universidade de São Paulo, Instituto de Ciências Matemáticas e de Computação-ICMC, 13566-590, São Carlos, SP, Brazil
[b] Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, C/ Tarfia s/n, 41012, Sevilla, Spain
[c] Departamento de Matemática, Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Rua Sergio Buarque de Holanda, 651, 13083-859, Campinas, SP, Brazil

ARTICLE INFO

MSC:
primary 35B41
76A05
secondary 35Q35
35B65
37L30

Keywords:
Ladyzhenskaya model
Pullback attractors
Regularity
Delays

ABSTRACT

We consider a mathematical model with delay for non-Newtonian incompressible fluids in a bounded domain. Existence of global weak solutions is proved under suitable regularity on the initial data and the forces. Conditions for uniqueness are also given, but in general the results are stated in a multi-valued framework. Suitable multi-valued dynamical systems are well-posed, using basically $L^2(\Omega)^n \times L^2(-h,0;L^2(\Omega)^n)$ or $C([-h,0];L^2(\Omega)^n)$ norms. Then the existence of pullback attractors is ensured acting on several universes, some of them of fixed bounded sets and others of tempered type, depending on parameters related to an integrability condition of the force and the delay term. Finally, relationships between these families of attractors are also provided, improving the characterization of attraction with respect to previous results.

1. Introduction

Ladyzhenskaya proposed several mathematical models for incompressible non-Newtonian fluids, known nowadays as Ladyzhenskaya models [1–3] or variants of the Navier–Stokes system or modified Navier–Stokes equations. This kind of equations is useful to describe the velocity and pressure of incompressible non-Newtonian fluids which are fluids that are not described by the Stokes law, e.g. see the seminal monograph [4] and the more recent [5].

For incompressible non-Newtonian fluids, the fluid velocity vector field $\mathbf{u}$ and the pressure π satisfy the following system

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \mathrm{div}\mathbb{S}(\mathbf{e}(\mathbf{u})) + \mathrm{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla\pi = \mathbf{f}(t) \\ &\mathrm{div}\mathbf{u} = 0 \end{aligned} \qquad \text{in } \Omega\times(\tau,\infty), \tag{1}$$

where $\tau \in \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ $(n \in \{2,3\})$ is a bounded domain with regular boundary $\partial\Omega$, $\mathbf{f}$ is an external force and $\mathbb{S} : \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$ is the stress tensor. In general, the stress tensor $\mathbb{S}$ is a nonlinear function that depends on the symmetric gradient of the fluid velocity, $\mathbf{e}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u}^\top + \nabla\mathbf{u}\right)$, satisfying the p-coercivity and some growth conditions.

Some examples of stress tensors are

$$\mathbb{S}^1(\mathbf{e}) = 2\nu_0|\mathbf{e}|^r\mathbf{e}, \quad \mathbb{S}^2(\mathbf{e}) = 2\nu_0(1+|\mathbf{e}|^r)\mathbf{e},$$

$$\mathbb{S}^3(\mathbf{e}) = 2\nu_0(1+|\mathbf{e}|^2)^{r/2}\mathbf{e}, \quad \text{or} \quad \mathbb{S}^{3+i}(\mathbf{e}) = 2\nu_\infty\mathbf{e} + \mathbb{S}^i, \quad i = 1,2,3,$$

* Corresponding author.
E-mail addresses: heracliolopezl@icmc.usp.br (H.L. López-Lázaro), pmr@us.es (P. Marín-Rubio), gplanas@unicamp.br (G. Planas).

https://doi.org/10.1016/j.cnsns.2024.108204
Received 17 April 2024; Received in revised form 1 July 2024; Accepted 3 July 2024
Available online 6 July 2024

where ν_0 and ν_∞ are positive constants (cf. [3,4,6]). We notice that system (1) associated to the stress tensor $\mathbb{S}^4$ with $r = 2$, is one of the models proposed by Ladyzhenskaya in [3]. For the stress tensors $\mathbb{S}^i$, $i = 1, \ldots, 6$, the fluids are known as power-law fluids.

We will assume that the stress tensor $\mathbb{S} : \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$ satisfies

$$\begin{aligned} &\mathbb{S}(\mathbf{0}) = \mathbf{0}, \\ &\big(\mathbb{S}(\mathbf{A}) - \mathbb{S}(\mathbf{B})\big) : (\mathbf{A} - \mathbf{B}) \geqslant \nu_1 \big(1 + \mu(|\mathbf{A}| + |\mathbf{B}|)\big)^{p-2} |\mathbf{A} - \mathbf{B}|^2, \\ &|\mathbb{S}(\mathbf{A}) - \mathbb{S}(\mathbf{B})| \leqslant c_1 \nu_1 \big(1 + \mu(|\mathbf{A}| + |\mathbf{B}|)\big)^{p-2} |\mathbf{A} - \mathbf{B}|, \end{aligned} \tag{2}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n^2}_{sym}$, where ν_1 and ν_2 are the viscosities (critical physical parameters of the problem), $\mu = \big(\frac{\nu_2}{\nu_1}\big)^{\frac{1}{p-2}}$ if $p \neq 2$ and $\mu = 0$ if $p = 2$. In the following, positive constants c_i will depend on the parameters of the model.

We observe that, from (2), it follows the p-coercivity and $(p-1)$-growth conditions of $\mathbb{S}$, i.e., there exist positive constants c_2, c_3 such that, for all $p \geqslant 2$ the stress tensor $\mathbb{S}$ satisfies

$$\mathbb{S}(\mathbf{D}) : \mathbf{D} \geqslant c_2(\nu_1 |\mathbf{D}|^2 + \nu_2 |\mathbf{D}|^p), \tag{3}$$

and for all $p > 1$ the stress tensor $\mathbb{S}$ satisfies

$$|\mathbb{S}(\mathbf{D})| \leqslant c_3 \nu_1 (1 + \mu |\mathbf{D}|)^{p-1},$$

for all $\mathbf{D} \in \mathbb{R}^{n^2}_{sym}$.

System (1) subject to space periodic or Dirichlet boundary conditions, with stress tensor $\mathbb{S}$ satisfying (2), has at least a weak solution when $p \geqslant 1 + 2n/(n+2)$ and weak solutions are unique in the family of weak solutions if $p \geqslant (n+2)/2$, see [4–6].

Regarding the asymptotic behavior of the solutions associated to system (1), the autonomous case is rather well understood (attractors, exponential attractors, fractal dimension, perturbed models, etc.). We can mention [5,7–10] among others.

For the non-autonomous case, i.e., the external force $\mathbf{f} = \mathbf{f}(t)$ depends on t, very recently, the authors investigated the existence of pullback attractors in [11].

In order to have more precise mathematical models, for instance after introducing measure devices or due to viscoelasticity properties and so on, we may think on constitutive models that incorporate delay effects. Picard [12], at the IV-International Congress of Mathematicians (Rome, 1908), exposed the importance of considering hereditary effects in physical systems. In the context of pseudoplastic fluids, as an application, we may cite the paper [13] and the references therein, as an example of chemical engineering real problem, where the experimental results are better approximated by the inclusion of delays in the model.

On other hand, it is worth to mention that there exist several approaches to deal with dynamical systems associated to PDE models in fluid mechanics with delay and non-autonomous effects. We have used here that of pullback attractors (e.g. cf. [14–17]). Inspired by works about the Navier–Stokes equations with delay (among many others, see [15,17,18] and the references cited therein), we formulate the following problem

$$\left.\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \mathrm{div}\mathbb{S}(\mathbf{e}(\mathbf{u})) + \mathrm{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla \pi = \mathbf{f}(t) + \mathbf{g}(t, \mathbf{u}_t) \\ &\mathrm{div}\mathbf{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega \times (\tau, \infty). \tag{4}$$

System (4) is a mathematical model for incompressible non-Newtonian fluids with delay that, unlike system (1), has an additional term in the external force, which is the delay term $\mathbf{g}(t, \mathbf{u}_t)$, that depends on time and the velocity of the fluid $\mathbf{u}_t$ defined by $\mathbf{u}_t(s) = \mathbf{u}(t+s)$ for all $s \in [-h, 0]$, where $h > 0$. The system is fulfilled with initial and Dirichlet boundary conditions.

The aim of this paper (based on HLLL's Ph.D. Thesis [19] under the supervision of the other two authors) is two-fold. Firstly we aim to establish existence results of global weak solutions to system (4) within two different settings of initial conditions and assumptions on the delay operator. Secondly, the above leads to two natural phase spaces, where we pose respective (multi-valued) dynamical systems and prove the existence of pullback attractors in each case in several universes. Moreover, relationships between these families are also established. Results are also new in the autonomous case, leading to global attractors.

Namely, the structure of the paper is the following. In Section 2 the natural operators and functional spaces involved in the problem are introduced. The assumptions on the delay operator $\mathbf{g}$ are also presented at this point, for clarity, although they will be used in two parts. Actually in Section 3 only assumptions **(I)–(III)** on $\mathbf{g}$ will be used to establish the results in the phase space $C([-h, 0]; H)$ (for short abbreviated to C_H, where H is detailed below). Firstly, existence of weak solutions, and secondly, the analysis of their long-time behavior is performed. Several aspects on the biggest admissible tempered parameter for suitable universes, necessary conditions, relations on chains of attractors and the availability of choice of such values for $p > 2$ are pointed out here. Then the analogous structure is developed in Section 4, where the phase space changes to $H \times L^2(-h, 0; H)$ (for short abbreviated to $H \times L^2_H$ or just M^2_H). This more general setting requires $\mathbf{g}$ satisfies **(I)–(IV)** for well-posedness, and a last condition **(V)** to ensure the existence of attractors. Section 5 is devoted to compare the previous families of (pullback) attractors and additional results as better attraction results in the $H \times C_H$-norm.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Given $\tau \in \mathbb{R}$, we consider the following system of partial differential equations with Dirichlet boundary condition for incompressible non-Newtonian fluids, that we will call Ladyzhenskaya Model with Delay **(LMD)**,

$$\frac{\partial \mathbf{u}}{\partial t} - \mathrm{div}\mathbb{S}(\mathbf{e}(\mathbf{u})) + \mathrm{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla \pi = \mathbf{f}(t) + \mathbf{g}(t, \mathbf{u}_t) \quad \text{in } \Omega \times (\tau, \infty),$$

$$\begin{aligned} \mathrm{div}\mathbf{u} &= 0 \text{ in } \Omega\times(\tau,\infty),\\ \mathbf{u} &= 0 \text{ on } \partial\Omega\times(\tau,\infty),\\ \mathbf{u}(\tau) &= \mathbf{u}^\tau \text{ in } \Omega,\\ \mathbf{u}(x,\tau+s) &= \phi(x,s) \text{ in } \Omega,\ s\in(-h,0), \end{aligned}$$

where $\mathbf{u} = (u_1,\dots,u_n)$ is the fluid velocity vector field, π is the pressure, $\mathbf{u}_t(s) = \mathbf{u}(t+s)$ for $s\in[-h,0]$, $h>0$, $\mathbf{f}$ is an external force, $\mathbf{g}(t,\mathbf{u}_t)$ is the delay term (whose assumptions will be specified below), ϕ is the initial condition with memory, and $\mathbb{S}:\mathbb{R}^{n^2}_{sym}\to\mathbb{R}^{n^2}_{sym}$ is the stress tensor satisfying (2).

For $s\geqslant 1$ and $p>1$, let us recall the following spaces

$$\begin{aligned} \mathcal{V} &:= \{\varphi\in C_c^\infty(\Omega)^n : \mathrm{div}\varphi = 0\},\\ H &= \text{closure of } \mathcal{V} \text{ in the } L^2(\Omega)^n\text{-norm},\\ V_p &= \text{closure of } \mathcal{V} \text{ in the } W^{1,p}(\Omega)^n\text{-norm},\\ V^s &= \text{closure of } \mathcal{V} \text{ in the } W^{s,2}(\Omega)^n\text{-norm}. \end{aligned}$$

The scalar product in H and in $L^2(\Omega)^n$ will be denoted by $(\cdot,\cdot)$ and the corresponding norm by $|\cdot|_2$. V_p^* (with norm $\|\cdot\|_*$) denotes the topological dual of V_p, and $\langle\cdot,\cdot\rangle$ stands for the duality product between these spaces. The scalar product in V^s will be denoted by $((\cdot,\cdot))_s$. Throughout the text q denotes the conjugate exponent to p.

We recall the Korn inequality. There exists a constant $c(q)>0$ such that

$$\|\nabla\mathbf{v}\|_q \leqslant c(q)\|\mathbf{e}(\mathbf{v})\|_q \quad \forall\mathbf{v}\in W_0^{1,q}(\Omega)^n,\ 1<q<\infty.$$

For short we denote $c_0 = c(2)$ and $\tilde{c}_0 = c(p)$. We also recall the Poincaré inequality

$$\lambda_1|\mathbf{v}|_2^2 \leqslant |\nabla\mathbf{v}|_2^2 \quad \forall\mathbf{v}\in V_2,$$

where λ_1 is the first eigenvalue of the Stokes operator with homogeneous Dirichlet boundary condition.

We introduce some operators, related to the matrix $\mathbf{u}\otimes\mathbf{u} = (u_iu_j)$ and the stress tensor $\mathbb{S}$.

Let $B : L^\infty(\tau,T;H)\cap L^p(\tau,T;V_p)\to L^q(\tau,T;V_p^*)$ be the operator defined by

$$\int_\tau^T \langle B(\mathbf{u})(t),\mathbf{v}(t)\rangle dt = \sum_{i,j=1}^n \int_\tau^T\int_\Omega u_j(x,t)\frac{\partial u_i(x,t)}{\partial x_j}v_i(x,t)dxdt.$$

Then, for $p\geqslant 1+2n/(n+2)$, B is a continuous operator (cf. [8, Lemma 2.8]).

Let $\mathbb{T} : L^p(\tau,T;V_p)\to L^q(\tau,T;V_p^*)$ be defined by

$$\int_\tau^T \langle\mathbb{T}(\mathbf{u})(t),\mathbf{v}(t)\rangle dt = \int_\tau^T\int_\Omega \mathbb{S}(\mathbf{e}(\mathbf{u})) : \mathbf{e}(\mathbf{v})dxdt.$$

For $p>1$ it holds that $\mathbb{T}$ is a continuous operator (cf. [8, Lemma 2.8]).

We will consider two types of initial conditions. To this end, we introduce the following spaces. Let us denote by C_H the Banach space $C([-h,0];H)$ with norm $\|\phi\|_{C_H} = \sup_{s\in[-h,0]}|\phi(s)|_2$, by L^2_H the Hilbert space $L^2(-h,0;H)$ with norm

$$\|\phi\|_{L^2_H} = \left(\int_{-h}^0 |\phi(s)|_2^2 ds\right)^{1/2},$$

and by M^2_H the Hilbert space $H\times L^2_H$ with norm

$$\|(\mathbf{v},\phi)\|_{M^2_H} = (|\mathbf{v}|_2^2 + \|\phi\|^2_{L^2_H})^{1/2} \quad \text{for } (\mathbf{v},\phi)\in M^2_H.$$

In Section 3 we first consider initial condition $\phi\in C_H$ satisfying $\phi(0) = \mathbf{u}^\tau$ and we assume that the delay term $\mathbf{g} : \mathbb{R}\times C_H\to L^2(\Omega)^n$ satisfies the following conditions.

(I) For all $\xi\in C_H$, the mapping $t\mapsto\mathbf{g}(t,\xi)\in L^2(\Omega)^n$ is measurable.

(II) For each $t\in\mathbb{R}$, $\mathbf{g}(t,\mathbf{0}) = \mathbf{0}$.

(III) There exists $L_\mathbf{g}>0$ such that, for all $t\in\mathbb{R}$ and for any $\xi,\eta\in C_H$,

$$|\mathbf{g}(t,\xi)-\mathbf{g}(t,\eta)|_2 \leqslant L_\mathbf{g}\|\xi-\eta\|_{C_H}.$$

Observe that conditions **(I)–(III)** above imply that, given $T>\tau$, $\mathbf{u}\in C([\tau-h,T];H)$, the function $\mathbf{g}_\mathbf{u} : [\tau,T]\to L^2(\Omega)^n$ defined by $\mathbf{g}_\mathbf{u}(t) = \mathbf{g}(t,\mathbf{u}_t)$, for any $t\in[\tau,T]$, is measurable and, in fact, belongs to $L^\infty(\tau,T;L^2(\Omega)^n)$.

In Section 4 we assume less regularity on the initial data. Actually, we suppose that $(\mathbf{u}^\tau,\phi)\in M^2_H$. In order to do this, in addition to conditions **(I)–(III)**, we assume that the delay term $\mathbf{g} : \mathbb{R}\times C_H\to L^2(\Omega)^n$ also satisfies two extra hypotheses, the first one just for the existence of solutions, the second one for the long-time behavior.

(IV) There exists $C_\mathbf{g}>0$ such that, for all $\tau\leqslant t$, for any $\mathbf{u},\mathbf{v}\in C([\tau-h,t];H)$,

$$\int_\tau^t |\mathbf{g}(s,\mathbf{u}_s)-\mathbf{g}(s,\mathbf{v}_s)|_2^2 ds \leqslant C_\mathbf{g}^2\int_{\tau-h}^t |\mathbf{u}(s)-\mathbf{v}(s)|_2^2 ds.$$

Note that, thanks to **(IV)**, given $T > \tau$, the mapping

$$\mathcal{G} : \mathbf{u} \in C([\tau - h, T]; H) \to \mathbf{g}_{\mathbf{u}} \in L^2(\tau, T; L^2(\Omega)^n)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$, which is uniformly continuous from $L^2(\tau - h, T; H)$ into $L^2(\tau, T; L^2(\Omega)^n)$. From now on, we will write $\mathbf{g}(t, \mathbf{u}_t) = \widetilde{\mathcal{G}}(t)$ for each $\mathbf{u} \in L^2(\tau - h, T; H)$, and thus property **(IV)** holds for any $\mathbf{u}$, $\mathbf{v} \in L^2(\tau - h, T; H)$.

Finally the last assumption we will impose for $\mathbf{g}$ is the following.

(V) There exists a value $\eta > 0$ such that, for all $\tau \le t$, for any $\mathbf{u} \in L^2(\tau - h, t; H)$,

$$\int_\tau^t e^{\eta s} |\mathbf{g}(s, \mathbf{u}_s)|_2^2 ds \leqslant C_{\mathbf{g}}^2 \int_{\tau - h}^t e^{\eta s} |\mathbf{u}(s)|_2^2 ds.$$

Remark 1. It is straightforward to obtain examples of delay operators $\mathbf{g}$, for instance, by using $\rho : \mathbb{R} \to [0, h]$ and putting $\mathbf{g}(t, \xi) := \xi(-\rho(t))$. In this way, $\mathbf{g}$ fulfills **(I)–(III)**. Additionally suppose that $\rho \in C^1(\mathbb{R}; [0, h])$ with $\rho'(t) \le \rho^* < 1$. Then it is not difficult to check that $\mathbf{g}$ also satisfies **(IV)** and **(V)**. However, if ρ' may attach the value 1, then in general **(IV)** does not hold (as an adaptation of $\rho(s) = s$ easily shows). For several other examples on delay operators, we may refer to [15–18] and the references therein.

3. Initial conditions in C_H

In this section, we prove the existence of global weak solutions and pullback attractors to **(LMD)** assuming that

$$p \geqslant 1 + 2n/(n+2),$$
$$\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*),$$
$$\mathbf{g} : \mathbb{R} \times C_H \to L^2(\Omega)^n \text{ satisfies } (\mathbf{I}) - (\mathbf{III}),$$
$$\phi \in C_H, \ \phi(0) = \mathbf{u}^\tau.$$

3.1. Existence of weak solutions

We start by introducing the notion of weak solutions to **(LMD)**.

Definition 2. Given an initial condition $\phi \in C_H$, a weak solution to **(LMD)** is an element $\mathbf{u}$ such that for any $T > \tau$

$$\mathbf{u} \in C([\tau - h, T]; H) \cap L^p(\tau, T; V_p) \quad \text{with} \quad \frac{\partial \mathbf{u}}{\partial t} \in L^q(\tau, T; V_p^*), \tag{5}$$

which satisfies the weak formulation

$$\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \rangle + \langle \mathbb{T}(\mathbf{u}(t)), \mathbf{v} \rangle + \langle B(\mathbf{u}(t)), \mathbf{v} \rangle = \langle \mathbf{f}(t), \mathbf{v} \rangle + (\mathbf{g}(t, \mathbf{u}_t), \mathbf{v}), \tag{6}$$

for all $\mathbf{v} \in V_p$ and *a.e.* $t \in (\tau, T)$, and

$$\mathbf{u}(\tau + s) = \phi(s) \quad \forall s \in [-h, 0].$$

Remark 3. To be precise, the weak formulation (6) is not exactly equivalent to the **(LMD)**. Actually, the difference is that the original PDE system (4) contains a pair $(\mathbf{u}, \pi)$ where the pressure π should also be obtained. In order to recover the pressure one might use results by de Rham or others (e.g., cf. [20–22]) if all is settled in a distributional sense. This would be the case if, for instance, we consider $\mathbf{f} \in L^q_{loc}(\mathbb{R}; (W^{-1,q}(\Omega))^n)$ instead of $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ as we have chosen. Nevertheless, we make this abuse of notation talking about solutions for **(LMD)** for this more general setting in V_p^* since it is the most usual employed in the related literature. This observation applies not only to this section but also to the next one.

Notice that any function $\mathbf{v}$ in the class (5) can be taken as a test function in the weak formulation (6). In this way, the energy equality holds true (e.g. cf. [23, Theorem 1.8, page 33] or [5, Lemma 7.3, page 175]).

Lemma 4. *Under the assumptions of this section, any weak solution to **(LMD)** satisfies the energy equality*

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}|_2^2 + \int_\Omega \mathbb{S}(\mathbf{e}(\mathbf{u})) : \mathbf{e}(\mathbf{u}) dx = \langle \mathbf{f}, \mathbf{u} \rangle + (\mathbf{g}(t, \mathbf{u}_t), \mathbf{u}) \quad a.e.\ t > \tau. \tag{7}$$

Concerning the existence of weak solutions to **(LMD)** we have the following result.

Theorem 5. *Suppose $p \ge 1 + 2n/(n+2)$. Consider $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ and $\mathbf{g}$ fulfilling **(I)–(III)**. Then, for any $\phi \in C_H$ there exists at least one weak solution to **(LMD)**.*

Proof. Fix $T > \tau$. We employ the Faedo–Galerkin approximations and the compactness method. Consider $s > n/2+1$ and let $\{\mathbf{w}_r\}_{r=1}^{\infty}$ be a set formed by eigenfunctions to the problem

$$((\mathbf{w}_r, \mathbf{v}))_s = \lambda_r(\mathbf{w}_r, \mathbf{v}) \quad \forall \mathbf{v} \in V^s,$$

which is a Hilbert basis of H and orthogonal in V^s (cf. [4, Theorem 4.11, page 290]). Notice that, by the choice of s, for all $p > 1$ we have that $V^s \hookrightarrow V_p$. Let P^m be the orthogonal projector of H onto the linear span of the first m eigenfunctions $\mathbf{w}_j$, $j = 1, \dots, m$.

Let us define $\mathbf{u}^m(x,t) = \sum_{r=1}^{m} \gamma_r^m(t)\mathbf{w}_r$, where $\gamma_r^m(t)$ solve the Galerkin system

$$\begin{cases} \frac{d}{dt}(\mathbf{u}^m(t), \mathbf{w}_j) + \langle \mathbb{T}(\mathbf{u}^m(t)), \mathbf{w}_j\rangle + \langle B(\mathbf{u}^m(t)), \mathbf{w}_j\rangle \\ \qquad = \langle \mathbf{f}(t), \mathbf{w}_j\rangle + (\mathbf{g}(t, \mathbf{u}_t^m), \mathbf{w}_j) \qquad \text{for } 1 \leqslant j \leqslant m, \\ \mathbf{u}^m(\tau + s) = P^m\phi(s)\ \forall s \in [-h, 0]. \end{cases} \tag{8}$$

Observe that this is a system of functional ordinary differential equations in the unknown $\gamma^m(t) = (\gamma_1^m, \dots, \gamma_m^m)$, which has a maximal solution defined on an interval $[\tau - h, t_m)$ with $\tau < t_m \leqslant T$ (see for instance [24, Chapter 2]).

We multiply the jth equation of the Galerkin system (8) by $\gamma_j^m(t)$; after adding them we arrive at

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}^m(t)|_2^2 + \int_\Omega \mathbb{S}(\mathbf{e}(\mathbf{u}^m)) : \mathbf{e}(\mathbf{u}^m)dx = \langle \mathbf{f}(t), \mathbf{u}^m\rangle + (\mathbf{g}(t, \mathbf{u}_t^m), \mathbf{u}^m) \quad a.e.\ t \in (\tau, t_m),$$

because $\langle B(\mathbf{u}^m(t)), \mathbf{u}^m\rangle = 0$ due to the divergence free condition $\mathrm{div}\mathbf{u}^m = 0$.

From the coercivity of $\mathbb{S}$ (3), the assumptions on $\mathbf{g}$ and the Korn and Young inequalities, it follows that

$$\frac{d}{dt}|\mathbf{u}^m(t)|_2^2 + \frac{2c_2\nu_1}{c_0^2}|\nabla \mathbf{u}^m|_2^2 + \frac{c_2\nu_2}{\tilde{c}_0^p}\|\nabla \mathbf{u}^m\|_p^p \leqslant c_{\nu_2,p}\|\mathbf{f}\|_*^q + 2L_{\mathbf{g}}\|\mathbf{u}_t^m\|_{C_H}^2 \quad a.e.\ t \in (\tau, t_m).$$

After integration in time, using the fact that $|P^m\phi(s)|_2 \leqslant |\phi(s)|_2 \leqslant \|\phi\|_{C_H}$ for all $s \in [-h, 0]$, we deduce that

$$\begin{aligned} &\|\mathbf{u}_t^m\|_{C_H}^2 + \int_\tau^t \Big(\frac{2c_2\nu_1}{c_0^2}|\nabla \mathbf{u}^m(s)|_2^2 + \frac{c_2\nu_2}{\tilde{c}_0^p}\|\nabla \mathbf{u}^m(s)\|_p^p\Big)ds \\ \leqslant &\|\phi\|_{C_H}^2 + \int_\tau^t \big(c_{\nu_2,p}\|\mathbf{f}(s)\|_*^q + 2L_g\|\mathbf{u}_s^m\|_{C_H}^2\big)ds \quad \forall t \in [\tau, t_m). \end{aligned}$$

Hence, after the above inequalities and the Gronwall Lemma, in particular by continuation we may consider $t_m = T$ and the a priori uniform estimates give that

$$\{\mathbf{u}^m\} \text{ is bounded in } L^\infty(\tau - h, T; H) \text{ and in } L^p(\tau, T; V_p). \tag{9}$$

By using (8) (see also [4, (4.12), page 290]), there follows that $\{\frac{\partial \mathbf{u}^m}{\partial t}\}$ is bounded in $L^q(\tau, T; (V^s)^*)$. Therefore, from the Aubin–Lions Lemma (e.g. cf. [23, Theorem II.1.4, page 32]), we can extract a subsequence of $\{\mathbf{u}^m\}$ (relabeled the same) such that

$$\begin{aligned} \mathbf{u}^m &\overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(\tau, T; H), \\ \mathbf{u}^m &\rightharpoonup \mathbf{u} \text{ in } L^p(\tau, T; V_p), \\ \mathbf{u}^m &\to \mathbf{u} \text{ in } L^2(\tau, T; H), \\ \frac{\partial \mathbf{u}^m}{\partial t} &\rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \text{ in } L^q(\tau, T; (V^s)^*), \\ \mathbb{T}(\mathbf{u}^m) &\rightharpoonup \mathcal{X} \text{ in } L^q(\tau, T; V_p^*), \\ \mathbf{g}(\cdot, \mathbf{u}_\cdot^m) &\overset{*}{\rightharpoonup} \mathcal{Y} \text{ in } L^\infty(\tau, T; L^2(\Omega)^n). \end{aligned} \tag{10}$$

These convergences allow to pass to the limit in (8) obtaining

$$\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\rangle + \langle \mathcal{X}, \mathbf{v}\rangle + \langle B(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle + (\mathcal{Y}, \mathbf{v}) \quad \forall \mathbf{v} \in V_p. \tag{11}$$

It is standard to prove that

$$\mathbf{u}(\tau + \theta) = \phi(\theta) \quad \theta \in [-h, 0].$$

From (11) and the monotonicity of the operator $\mathbb{T}$ we can show that $\mathcal{X} = \mathbb{T}(\mathbf{u})$ (see for instance [6, Chapitre 2, Théorème 5.1]).

To finish the proof we show that $\mathcal{Y} = \mathbf{g}(t, \mathbf{u}_t)$, by using an energy method. First observe that $\{\mathbf{u}^m\}$ is equi-continuous in $(V^s)^*$ on $[\tau, T]$. Since $\{\mathbf{u}^m\}$ is bounded in $C([\tau, T]; H)$, by the Arzelà-Ascoli Theorem, there follows, up to a subsequence, that

$$\mathbf{u}^m \to \mathbf{u} \ \text{ in } \ C([\tau, T]; (V^s)^*).$$

This jointly with (10) give us that, if $t_m \to t$, then

$$\mathbf{u}^m(t_m) \rightharpoonup \mathbf{u}(t) \quad \text{in } H. \tag{12}$$

Our goal now is to prove that $\mathbf{u}^m \to \mathbf{u}$ in $C([\tau,T];H)$. By contradiction, if this is not true, then there exist $\varepsilon_0 > 0$, $t_0 \in (\tau,T]$ and subsequences (relabeled the same) $\{\mathbf{u}^m\}$, $\{t_m\}_{m\geq 1} \subset (\tau,T]$ such that $t_m \to t_0$ and

$$|\mathbf{u}^m(t_m) - \mathbf{u}(t_0)|_2 \geqslant \varepsilon_0. \tag{13}$$

On the one hand, by **(II)**, **(III)** and (9) there exists a constant $C > 0$ such that

$$\int_s^t |\mathcal{Y}(\theta)|_2^2 d\theta \leqslant \liminf_{m\to\infty} \int_s^t |\mathbf{g}(\theta,\mathbf{u}_\theta^m)|_2^2 d\theta \leqslant L_{\mathbf{g}}^2 C(t-s) \quad \forall s,t \in [\tau,T].$$

Therefore, by (11) and the energy equality we have that

$$\frac{1}{2}|\mathbf{z}(t)|_2^2 \leqslant \frac{1}{2}|\mathbf{z}(s)|_2^2 + \int_s^t \langle \mathbf{f}(r),\mathbf{z}(r)\rangle ds + \tilde{C}(t-s),$$

for all $\tau \leqslant s \leqslant t \leqslant T$, where $\widetilde{C} = \dfrac{L_{\mathbf{g}}^2 C c_0^2}{4c_2\nu_1\lambda_1}$, and $\mathbf{z} = \mathbf{u}^m$ or $\mathbf{z} = \mathbf{u}$.

Thus, the maps J_m, $J : [\tau,T] \to \mathbb{R}$ defined by

$$\begin{aligned} J_m(t) &= \frac{1}{2}|\mathbf{u}^m(t)|_2^2 - \int_\tau^t \langle \mathbf{f}(r),\mathbf{u}^m(r)\rangle dr - \widetilde{C}t, \\ J(t) &= \frac{1}{2}|\mathbf{u}(t)|_2^2 - \int_\tau^t \langle \mathbf{f}(r),\mathbf{u}(r)\rangle dr - \widetilde{C}t \end{aligned} \tag{14}$$

are non-increasing and continuous, and satisfy

$$J_m(t) \to J(t) \quad a.e.\ t \in (\tau,T).$$

Let us fix $\varepsilon > 0$. So, by the continuity and non-increasing character of J, there exists $\tau < \hat{t}_\varepsilon < t_0$ such that

$$\lim_{m\to\infty} J_m(\hat{t}_\varepsilon) = J(\hat{t}_\varepsilon),$$

$$0 \leqslant J(\hat{t}_\varepsilon) - J(t_0) \leqslant \varepsilon.$$

Since $t_m \to t_0$, there exists m_ε such that $\hat{t}_\varepsilon < t_m$ for all $m \geqslant m_\varepsilon$. Then, we have that

$$\begin{aligned} J_m(t_m) - J(t_0) &\leqslant J_m(\hat{t}_\varepsilon) - J(t_0) \\ &\leqslant |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)| + |J(\hat{t}_\varepsilon) - J(t_0)| \\ &\leqslant |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)| + \varepsilon \end{aligned}$$

for all $m \geqslant m_\varepsilon$. Therefore we can conclude that $\limsup_{m\to\infty} J_m(t_m) \leqslant J(t_0) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_{m\to\infty} J_m(t_m) \leqslant J(t_0). \tag{15}$$

Moreover, as $t_m \to t_0$ and

$$\int_\tau^{t_m} \langle \mathbf{f}(r),\mathbf{u}^m(r)\rangle dr \to \int_\tau^{t_0} \langle \mathbf{f}(r),\mathbf{u}(r)\rangle dr,$$

from (15) we deduce that $\limsup_{m\to\infty} |\mathbf{u}^m(t_m)|_2 \leqslant |\mathbf{u}(t_0)|_2$. This last inequality and (12) imply that $\mathbf{u}^m(t_m) \to \mathbf{u}(t_0)$ strongly in H, which is a contradiction with (13).

Notice that this also implies that $\mathbf{u}_t^m \to \mathbf{u}_t$ in C_H for all $t \geqslant \tau$. Hence, since $\mathbf{g}$ satisfies **(III)** we conclude that $\mathbf{g}(\cdot,\mathbf{u}_\cdot^m) \to \mathbf{g}(\cdot,\mathbf{u}_\cdot)$ in $L^2(\tau,T;L^2(\Omega)^n)$. By the uniqueness of the limit we identify the weak limit $\mathcal{Y} = \mathbf{g}(\cdot,\mathbf{u}_\cdot)$. Since T was arbitrary, by concatenation we may obtain a weak solution $\mathbf{u}$ to **(LMD)**. □

Let us introduce $\Phi_{C_H}(\tau,\phi)$ as the set of all weak solutions to **(LMD)** defined on $[\tau-h,\infty)$ with initial condition $\phi \in C_H$.

The following result plays an important role in Section 3.2, about existence of pullback attractors, since it implies that the multi-valued process in C_H, constructed through weak solutions, is closed.

Proposition 6. *Under the assumptions of Theorem 5, consider $\tau \in \mathbb{R}$, $\{\phi^m\} \subset C_H$ with $\phi^m \to \phi$ in C_H, and weak solutions $\mathbf{u}^m \in \Phi_{C_H}(\tau,\phi^m)$. Then, $\{\mathbf{u}^m\}$ is bounded in $C([\tau-h,T];H) \cap L^p(\tau,T;V_p)$ with $\left\{\frac{\partial \mathbf{u}^m}{\partial t}\right\}$ bounded in $L^q(\tau,T;V_p^*)$, and there exists $\mathbf{u} \in \Phi_{C_H}(\tau,\phi)$ such that (up to a subsequence) $\mathbf{u}^m \to \mathbf{u}$ in the sense described in (10).*

Proof. It is analogous to that of Theorem 5, so we omit the details. □

Next result provides uniqueness and continuity w.r.t. the initial data. This is straightforward for dimension two, but requires an extra regularity condition for dimension three.

Theorem 7. *Under the assumptions of Theorem 5, there exist positive constants K_i, $i = 1, 2, 3$ (only depending on the parameters of the model) such that for any $\tau \in \mathbb{R}$, ϕ^1, $\phi^2 \in C_H$ and $\mathbf{u} \in \Phi_{C_H}(\tau, \phi^1)$ and $\mathbf{v} \in \Phi_{C_H}(\tau, \phi^2)$, the following estimates hold,*

(i) *if $n = 2$,*

$$\|\mathbf{v}_t - \mathbf{u}_t\|^2_{C_H} \leqslant \|\phi^1 - \phi^2\|^2_{C_H} \exp\left\{ \int_\tau^t \left(K_1 + K_2 |\nabla \mathbf{u}(s)|_2^2\right) ds \right\} \quad \forall t \geq \tau, \tag{16}$$

(ii) *if $n = 3$ and $\mathbf{u} \in L^{\frac{2p}{2p-3}}(\tau, T; V_p)$ for some $T > \tau$, then*

$$\|\mathbf{v}_t - \mathbf{u}_t\|^2_{C_H} \leqslant \|\phi^1 - \phi^2\|^2_{C_H} \exp\left\{ \int_\tau^t \left(K_1 + K_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds \right\} \quad \forall t \in [\tau, T]. \tag{17}$$

Proof. Setting $\mathbf{w} = \mathbf{v} - \mathbf{u}$ and using it as a test function in the weak formulation (6), we have

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|_2^2 + \langle \mathbb{T}(\mathbf{v}) - \mathbb{T}(\mathbf{u}), \mathbf{w}\rangle + \langle B(\mathbf{v}) - B(\mathbf{u}), \mathbf{w}\rangle = (\mathbf{g}(t, \mathbf{v}_t) - \mathbf{g}(t, \mathbf{u}_t), \mathbf{w}) \quad a.e.\ t > \tau.$$

By using the coercivity of $\mathbb{S}$ (3), there follows

$$\frac{d}{dt}|\mathbf{w}|_2^2 + 2c_2\nu_1 |\mathbf{e}(\mathbf{w})|_2^2 + 2c_2\nu_2 \|\mathbf{e}(\mathbf{w})\|_p^p \leqslant 2\int_\Omega |\mathbf{w}|^2 |\nabla \mathbf{u}| dx + 2(\mathbf{g}(t, \mathbf{v}_t) - \mathbf{g}(t, \mathbf{u}_t), \mathbf{w})\ a.e.\ t > \tau.$$

Now we split the analysis into two cases.

Case $n = 2$**.** Consider $\varepsilon_1 > 0$, to be specified later. We use the Ladyzhenskaya inequality to estimate

$$\begin{aligned}\int_\Omega |\mathbf{w}|^2 |\nabla \mathbf{u}| dx &\leqslant \|\mathbf{w}\|_4^2 |\nabla \mathbf{u}|_2 \\ &\leqslant \hat{c}|\mathbf{w}|_2 |\nabla \mathbf{w}|_2 |\nabla \mathbf{u}|_2 \\ &\leqslant \frac{\varepsilon_1}{2}|\nabla \mathbf{w}|_2^2 + \frac{\hat{c}^2}{2\varepsilon_1}|\nabla \mathbf{u}|_2^2 |\mathbf{w}|_2^2.\end{aligned}$$

By using the Korn inequality and the previous estimate, we have, for any $\varepsilon_2 > 0$, that

$$\begin{aligned}&\frac{d}{dt}|\mathbf{w}|_2^2 + \left(\frac{2c_2\nu_1\lambda_1}{c_0^2} - \varepsilon_1\lambda_1 - \varepsilon_2\right)|\mathbf{w}|_2^2 + \frac{2c_2\nu_2}{\tilde{c}_0^p}\|\nabla \mathbf{w}\|_p^p \\ &\leqslant \frac{\hat{c}^2}{\varepsilon_1}|\mathbf{w}|_2^2 |\nabla \mathbf{u}|_2^2 + \frac{1}{\varepsilon_2}|\mathbf{g}(t, \mathbf{v}_t) - \mathbf{g}(t, \mathbf{u}_t)|_2^2 \quad a.e.\ t > \tau.\end{aligned}$$

We choose $\varepsilon_1 = \frac{c_2\nu_1}{c_0^2}$ and $\varepsilon_2 = \frac{c_2\nu_1\lambda_1}{c_0^2}$, integrate from τ to t, and use hypothesis **(III)** to arrive at

$$|\mathbf{w}(t)|_2^2 \leqslant |\mathbf{w}(\tau)|_2^2 + K_2 \int_\tau^t |\mathbf{w}(s)|_2^2 |\nabla \mathbf{u}(s)|_2^2 ds + K_1 \int_\tau^t \|\mathbf{w}_s\|^2_{C_H} ds \quad \forall t \geq \tau,$$

where $K_1 = \frac{L_{\mathbf{g}}^2}{\varepsilon_2}$ and $K_2 = \frac{c_0^2 \hat{c}^2}{c_2\nu_1}$. Therefore, we have that

$$\|\mathbf{w}_t\|^2_{C_H} \leqslant \|\phi^1 - \phi^2\|^2_{C_H} + K_2 \int_\tau^t \|\mathbf{w}_s\|^2_{C_H} |\nabla \mathbf{u}(s)|_2^2 ds + K_1 \int_\tau^t \|\mathbf{w}_s\|^2_{C_H} ds \quad \forall t \geq \tau.$$

The Gronwall Lemma gives (16).

Case $n = 3$**.** Applying the Hölder and interpolation inequalities, it follows that

$$\begin{aligned}\int_\Omega |\mathbf{w}|^2 |\nabla \mathbf{u}| dx &\leqslant \|\mathbf{w}\|^2_{\frac{2p}{p-1}} \|\nabla \mathbf{u}\|_p \\ &\leqslant \tilde{c}|\mathbf{w}|_2^{\frac{2p-3}{p}} |\nabla \mathbf{w}|_2^{\frac{3}{p}} \|\nabla \mathbf{u}\|_p \\ &\leqslant \frac{\nu_1}{2c_0^2}|\nabla \mathbf{w}|_2^2 + K_3 \|\nabla \mathbf{u}\|_p^{\frac{2p}{2p-3}} |\mathbf{w}|_2^2,\end{aligned}$$

where $K_3 = \frac{(2p-3)\tilde{c}^{\frac{2p}{2p-3}}}{2p\varepsilon^{\frac{2p}{2p-3}}}$ and $\varepsilon = \left(\frac{p\nu_1}{2c_0^2}\right)^{3/2p}$.

Thus we obtain

$$\|\mathbf{w}_t\|^2_{C_H} \leqslant \|\phi^1 - \phi^2\|^2_{C_H} + K_3 \int_\tau^t \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}} |\mathbf{w}(s)|_2^2 ds + K_1 \int_\tau^t \|\mathbf{w}_s\|^2_{C_H} ds \quad \forall t \in [\tau, T].$$

Since $\int_\tau^t \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}} ds < \infty$, the Gronwall Lemma again gives (17). □

Remark 8. In light of Theorem 7, not only continuity w.r.t. initial data but also uniqueness is straightforward in dimension two. Using item (ii) in the result, a sufficient condition to have both, continuity w.r.t. initial data and uniqueness in three dimensions is $p \geqslant 5/2$, since then $2p/(2p-3) \leq p$. Nevertheless, the continuity notion is not essential for the analysis of the long-time behavior of solutions, which can be carried out even without uniqueness, just using Proposition 6 and multi-valued dynamical systems as is shown in next paragraph.

3.2. Existence of pullback attractors in C_H

In this section we begin with the analysis of the asymptotic behavior of the weak solutions to **(LMD)**.

To settle properly some well-known concepts, let us recall very briefly some few basic definitions of the abstract theory concerning long-time dynamics, in particular of pullback attractors.

Definition 9. Given a metric space (X, d_X), $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : t \geq \tau\}$, and $\mathcal{P}(X)$ the class of non-emtpy subsets of X, a multi-valued process $U : \mathbb{R}^2_d \times X \to \mathcal{P}(X)$ is a family of maps such that $U(t,t) = \mathrm{Id}_X$ for any $t \in \mathbb{R}$ and $U(t,\tau)x \subset U(t,s)(U(s,\tau)x)$ for any $\tau \leq s \leq t$ and $x \in X$.

This process usually exists through the solution operator to some problem where there is lack of uniqueness, and when the system of PDE is time-dependent, so both initial and final times are required. If we cannot perform an analysis of stability of equilibria or it is incomplete, it is still possible to determine interesting bigger (but minimal somehow) objects that attract (and therefore concentrate) the essential dynamic of all the phase-space X, in the sense of proximity of the trajectories (solutions) for long time toward a desired object (with nice properties as finite-dimensionality, rate of attractions, compactness or some manifold structure, etcetera). Concerned with attractors, treated since the second half of twentieth century, but for a generically not autonomous situation, we are not focused on global attractors but in one of the several available versions of time-dependent attractors, that of pullback dynamics.

Definition 10. Given a universe $\mathcal{D} \subset \mathcal{P}(X)$ (to be specified), the minimal pullback $\mathcal{D}$-attractor for a multi-valued process (X, U) is a time-dependent family $A = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ such that (i) $A(t)$ is a non-empty compact set of X for any $t \in \mathbb{R}$; (ii) it attracts pullback the dynamics of elements in $\mathcal{D}$, i.e., for any $\widehat{D} = \{D(\tau)\}_{\tau\in\mathbb{R}} \in \mathcal{D}$ and $t \in \mathbb{R}$, the following Hausdorff semi-distance in X limit holds: $\lim_{\tau\to-\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), A(t)) = 0$; (iii) A is negatively invariant, i.e., $A(t) \subset U(t,\tau)A(\tau)$ for all $t \geq \tau$; (iv) A is minimal among all the time-dependent pullback $\mathcal{D}$-attracting families $\widehat{C} = \{C(t)\}_{t\in\mathbb{R}}$ with closed sections, i.e. $A(t) \subset C(t)$ for any $t \in \mathbb{R}$.

Depending on the structure of the universe $\mathcal{D}$ and other conditions, existence results may become simpler or not, and being furnished with additional minimality and/or invariance properties, comparison relations, dimensionality analysis and so on. Associated to the concept of pullback attractor there are two inherent concepts: absorption and asymptotic compactness.

Definition 11. A time-dependent family $B_0 \subset X$ is pullback $\mathcal{D}$-absorbing if for any $\widehat{D} = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$ and $t \in \mathbb{R}$ there exists $\tau(t,\widehat{D}) < t$ such that $U(t,\tau)D(t) \subset B_0(t)$ for any $\tau \leq \tau(t,\widehat{D})$. Given a family $\widetilde{D} = \{\tilde{D}(t)\}_{t\in\mathbb{R}}$, the process U is pullback $\widetilde{D}$-asymptotically compact if for any $t \in \mathbb{R}$, $\tau_n \leq t$ with $\tau_n \to -\infty$ and $x_n \in \tilde{D}(\tau_n)$, any subset $\{y_n\}$, with $y_n \in U(t,\tau_n)x_n$ for all n, is relatively compact in X.

For a detailed exposition on several results on the theory of autonomous and non-autonomous attractors for single-valued and multi-valued processes see, for instance, [25–29] and the references therein.

Now we split our analysis concerning problem **(LMD)** in two cases, depending on the phase-space where the initial condition for the problem is taken, according to the previous existence results. Let us start by considering the Banach space C_H.

Let us define the bi-parametric family $\mathcal{U} : \mathbb{R}^2_d \times C_H \to \mathcal{P}(C_H)$ given by

$$\mathcal{U}(t,\tau)\phi = \{\mathbf{u}_t(\cdot) : \mathbf{u} \in \Phi_{C_H}(\tau,\phi)\}.$$

Next result gives the upper-semicontinuity of the maps $\mathcal{U}(t,\tau)$.

Proposition 12. *Under the assumptions of Theorem 5, let $\{\phi^m\} \subset C_H$ and $\phi \in C_H$ be such that $\phi^m \to \phi$ in C_H, and consider $\{\mathbf{u}^m\}$, where $\mathbf{u}^m \in \Phi_{C_H}(\tau,\phi^m)$. Then, there exist a subsequence of $\{\mathbf{u}^m\}$ (relabeled the same) and $\mathbf{u} \in \Phi_{C_H}(\tau,\phi)$ such that for any $s \geq \tau$*

$$\mathbf{u}^m_s \to \mathbf{u}_s \quad \text{strongly in } C_H.$$

Proof. Consider $T > \tau$. By Proposition 6 the sequence $\{\mathbf{u}^m\}$ is bounded in $L^\infty(\tau,T;H) \cap L^p(\tau,T;V_p)$ with $\{\frac{\partial \mathbf{u}^m}{\partial t}\}$ bounded in $L^q(\tau,T;V_p^*)$ and a subsequence $\{\mathbf{u}^m\}$ converges to $\mathbf{u} \in \Phi_{C_H}(\tau,\phi)$.

Observe that $\{\mathbf{u}^m\}$ is equicontinuous in V_p^* on $[\tau,T]$. Since this sequence is bounded in $C([\tau,T];H)$, by the Arzelà-Ascoli Theorem, up to a subsequence, there follows that

$$\mathbf{u}^m \to \mathbf{u} \quad \text{strongly in } C([\tau,T];V_p^*).$$

Hence, one has

$$\mathbf{u}^m(s) \rightharpoonup \mathbf{u}(s) \quad \text{weakly in } H, \text{ for any } s \in [\tau,T].$$

As in the proof of Theorem 5, with the same arguments and the energy functions J and J_m defined in (14) we can prove that $\mathbf{u}^m_s \to \mathbf{u}_s$ in C_H for all $s \geqslant \tau$. □

As a consequence of the previous result, we have the following one.

Corollary 13. *Under the assumptions of Theorem 5, $\mathcal{U} : \mathbb{R}^2_d \times C_H \to \mathcal{P}(C_H)$ is an upper-semicontinuous and strict multi-valued process with closed values.*

Next result indicates that the non-autonomous dynamical system is dissipative. For clarity in the exposition, we perform the analysis for $p > 2$, since the case $p = 2$ may be considered more standard (this last case involves additional restriction on the range of certain parameter η, see Remark 15 below).

Lemma 14. *Suppose that the assumptions of Theorem 5 are fulfilled and consider $p > 2$. Then, there exist positive constants $\widehat{C}_1$ and K_4 such that for any $\eta > 0$ and $\phi \in C_H$, any weak solution $\mathbf{u} \in \Phi_{C_H}(\tau, \phi)$ satisfies for all $t \geqslant \tau$*

$$\|\mathbf{u}_t\|^2_{C_H} \leqslant e^{\eta h} e^{-(\eta - 2L_{\mathbf{g}} e^{\eta h})(t-\tau)} \|\phi\|^2_{C_H} + e^{\eta h} \int_\tau^t e^{-(\eta - 2L_{\mathbf{g}} e^{\eta h})(t-s)} (\widehat{C}_1 + K_4 \|\mathbf{f}(s)\|^q_*) ds \tag{18}$$

and

$$\frac{c_2 \nu_2}{\tilde{c}_0^p} \int_t^{t+1} \|\nabla \mathbf{u}(s)\|^p_p ds \leqslant |\mathbf{u}(t)|^2_2 + K_4 \int_t^{t+1} \|\mathbf{f}(s)\|^q_* ds + 2L_{\mathbf{g}} \int_t^{t+1} \|\mathbf{u}_s\|^2_{C_H} ds. \tag{19}$$

Proof. From the energy equality (7) and the coercivity of $\mathbb{S}$ (3), we have that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\mathbf{u}(t)|^2_2 + c_2\nu_1 |\mathbf{e}(\mathbf{u})|^2_2 + c_2 \nu_2 \|\mathbf{e}(\mathbf{u})\|^p_p &\leqslant \langle \mathbf{f}(t), \mathbf{u}\rangle + (\mathbf{g}(t, \mathbf{u}_t), \mathbf{u}) \\ &\leqslant \|\mathbf{f}(t)\|_* \|\nabla \mathbf{u}\|_p + |\mathbf{g}(t, \mathbf{u}_t)|_2 |\mathbf{u}|_2 \quad a.e.\ t > \tau. \end{aligned}$$

The Korn, Poincaré and Young inequalities and hypothesis **(III)** give

$$\frac{d}{dt}|\mathbf{u}(t)|^2_2 + \overline{\eta}|\mathbf{u}|^2_2 + 2\eta_2 \|\nabla \mathbf{u}\|^p_p \leqslant \frac{2^q}{\varepsilon^q q}\|\mathbf{f}(t)\|^q_* + \frac{\varepsilon^p}{p}\|\nabla \mathbf{u}\|^p_p + 2L_{\mathbf{g}}\|\mathbf{u}_t\|^2_{C_H} \quad a.e.\ t > \tau,$$

where $\overline{\eta} = 2c_2\nu_1\lambda_1 c_0^{-2}$, $\eta_2 = \frac{c_2\nu_2}{\tilde{c}_0^p}$ and $\varepsilon > 0$ is arbitrary.

By choosing $\varepsilon = (p\eta_2)^{1/p}$ there follows

$$\frac{d}{dt}|\mathbf{u}(t)|^2_2 + \overline{\eta}|\mathbf{u}|^2_2 + \eta_2 \|\nabla \mathbf{u}\|^p_p \leqslant K_4 \|\mathbf{f}(t)\|^q_* + 2L_{\mathbf{g}}\|\mathbf{u}_t\|^2_{C_H} \quad a.e.\ t > \tau, \tag{20}$$

where $K_4 = \frac{2^q}{(p\eta_2)^{q/p} q}$.

We split the analysis into two cases: $0 < \eta \leqslant \overline{\eta}$ and $\eta > \overline{\eta}$.

Case $0 < \eta \leqslant \overline{\eta}$**.** From (20), multiplying by $e^{\eta t}$ and integrating from τ to t we arrive at

$$e^{\eta t}|\mathbf{u}(t)|^2_2 \leqslant e^{\eta\tau}\|\phi\|^2_{C_H} + \int_\tau^t e^{\eta s} K_4 \|\mathbf{f}(s)\|^q_* ds + 2L_{\mathbf{g}} \int_\tau^t e^{\eta s}\|\mathbf{u}_s\|^2_{C_H} ds \quad \forall t \geq \tau.$$

With $t \geqslant \tau$, let $s \in [-h, 0]$ and suppose that $t + s \geqslant \tau$. Then

$$e^{\eta(t+s)}|\mathbf{u}(t+s)|^2_2 \leqslant e^{\eta\tau}\|\phi\|^2_{C_H} + \int_\tau^t e^{\eta\theta} K_4 \|\mathbf{f}(\theta)\|^q_* d\theta + 2L_{\mathbf{g}} \int_\tau^t e^{\eta\theta}\|\mathbf{u}_\theta\|^2_{C_H} d\theta.$$

If $s \in [-h, 0]$ is such that $t + s \leqslant \tau$, trivially

$$e^{\eta(t-h)}|\mathbf{u}(t+s)|^2_2 \leqslant e^{\eta(t+s)}|\mathbf{u}(t+s)|^2_2 \leqslant e^{\eta\tau}\|\phi\|^2_{C_H}.$$

Therefore, we deduce from above that

$$e^{\eta t}\|\mathbf{u}_t\|^2_{C_H} \leqslant e^{\eta\tau}e^{\eta h}\|\phi\|^2_{C_H} + e^{\eta h}\int_\tau^t e^{\eta s} K_4 \|\mathbf{f}(s)\|^q_* ds + 2L_{\mathbf{g}} e^{\eta h}\int_\tau^t e^{\eta s}\|\mathbf{u}_s\|^2_{C_H} ds \quad \forall t \geqslant \tau.$$

The Gronwall Lemma yields

$$\|\mathbf{u}_t\|^2_{C_H} \leqslant e^{\eta h} e^{-(\eta - 2L_{\mathbf{g}} e^{\eta h})(t-\tau)} \|\phi\|^2_{C_H} + e^{\eta h} \int_\tau^t e^{-(\eta - 2L_{\mathbf{g}} e^{\eta h})(t-s)} K_4 \|\mathbf{f}(s)\|^q_* ds \quad \forall t \geq \tau.$$

Case $\eta > \overline{\eta}$**.** Denote $\beta := \eta - \overline{\eta} > 0$. Let C_I be an embedding constant of $W^{1,p}_0(\Omega)^n \subset L^2(\Omega)^n$, i.e., $|\mathbf{u}|_2 \leq C_I \|\nabla \mathbf{u}\|_p$. By applying the Young inequality we have that

$$|\mathbf{u}|^2_2 \leq \frac{\gamma^{p/2}}{p/2}\|\nabla \mathbf{u}\|^p_p + \frac{(p-2)C_I^{2p/(p-2)}}{p\gamma^{p/(p-2)}}.$$

Choosing $\frac{\gamma^{p/2}}{p/2} = \frac{c_2\nu_2}{2\tilde{c}_0^p \beta}$ there follows

$$\beta|\mathbf{u}|^2_2 \leq \frac{c_2\nu_2}{2\tilde{c}_0^p}\|\nabla \mathbf{u}\|^p_p + \widehat{C}_1, \tag{21}$$

where $\widehat{C}_1 = \frac{(p-2)C_I^{2p/(p-2)}\beta}{p\gamma^{p/(p-2)}}$. Then (20) reduces to

$$\frac{d}{dt}|\mathbf{u}(t)|_2^2 + \eta|\mathbf{u}|_2^2 + \frac{\eta_2}{2}\|\nabla\mathbf{u}\|_p^p \leqslant K_4\|\mathbf{f}(t)\|_*^q + 2L_{\mathbf{g}}\|\mathbf{u}_t\|_{C_H}^2 + \widehat{C}_1.$$

As in the previous case, we conclude that (18) holds.

Finally, (19) follows by integrating (20). □

Remark 15. For the case $p = 2$, we can assume that $c_2 = 1$ and $\nu_2 = 0$. Similar computations to those in the previous result lead to, for some $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$,

$$\|\mathbf{u}_t\|_{C_H}^2 \leqslant e^{\eta h}e^{-(\eta-2L_{\mathbf{g}}e^{\eta h})(t-\tau)}\|\phi\|_{C_H}^2 + e^{\eta h}\frac{\lambda_1}{\beta}\int_\tau^t e^{-(\eta-2L_{\mathbf{g}}e^{\eta h})(t-s)}\|\mathbf{f}(s)\|_*^2 ds \quad \forall t \geq \tau \quad \text{and}$$

$$\frac{\nu_1}{c_0^2}\int_t^{t+1}|\nabla\mathbf{u}(s)|_2^2 ds \leqslant |\mathbf{u}(t)|_2^2 + \frac{c_0^2}{\nu_1}\int_t^{t+1}\|\mathbf{f}(s)\|_*^2 ds + 2L_{\mathbf{g}}\int_t^{t+1}\|\mathbf{u}_s\|_{C_H}^2 ds \quad \forall t \geq \tau,$$

where $\beta := 2\nu_1\lambda_1 c_0^{-2} - \eta$.

Next we introduce the tempered universes required to establish suitable asymptotic properties of the process $\mathcal{U}$. Naturally they will be related to a type of growth that dissipate the initial data, from the previous estimates.

Definition 16 (*Universe in C_H*)**.** Given $\sigma > 0$, we will denote by $\mathcal{D}_\sigma(C_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{\mathbf{v}\in D(\tau)}\|\mathbf{v}\|_{C_H}^2\right) = 0.$$

$\mathcal{D}_\sigma(C_H)$ is inclusion-closed. $\mathcal{D}_F(C_H)$ will denote the class of all families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of C_H.

To simplify notation, by estimate (18), here on we introduce the class

$$\mathcal{I}_*^{q,\sigma} = \{\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*) : \int_{-\infty}^0 e^{\sigma s}\|\mathbf{f}(s)\|_*^q ds < \infty\}$$

and denote $\sigma_\eta := \eta - 2L_{\mathbf{g}}e^{\eta h}$.

Remark 17. It is not difficult to deduce that, given $L_{\mathbf{g}}$ and h positive, the function $\eta \mapsto \sigma_\eta = \eta - 2L_{\mathbf{g}}e^{\eta h}$ attaches positive values if and only if $2L_{\mathbf{g}}h < e^{-1}$. In such a case, the maximum is attained at $\eta_* = \frac{-1}{h}\log(2L_{\mathbf{g}}h) > 0$ and its value is $\sigma_{\eta_*} = \frac{-1}{h}(1 + \log(2L_{\mathbf{g}}h))$. Therefore, $2L_{\mathbf{g}}h < e^{-1}$ is a necessary condition in the analysis of this section. Its clear meaning as a smallness condition is that either h or $L_{\mathbf{g}}$ must be small enough to compensate each other.

After the above notation, and Lemma 14, we may establish a result ensuring the existence of a pullback absorbing family.

Corollary 18. *Under the assumptions of Theorem 5, suppose also that $p > 2$ and there exists some $\sigma_\eta = \eta - 2L_{\mathbf{g}}e^{\eta h} > 0$ such that $\mathbf{f} \in \mathcal{I}_*^{q,\sigma_\eta}$. Then, the family $\widehat{B}_{\sigma_\eta,C_H} = \{B_{\sigma_\eta,C_H}(t) : t \in \mathbb{R}\}$ with $B_{\sigma_\eta,C_H}(t) = \overline{B}_{C_H}(0, \mathcal{R}_{1,\sigma_\eta}(t))$, where*

$$\mathcal{R}_{1,\sigma_\eta}^2(t) = 1 + e^{\eta h}\int_{-\infty}^t e^{-\sigma_\eta(t-s)}(\widehat{C}_1 + K_4\|\mathbf{f}(s)\|_*^q)ds, \tag{22}$$

is pullback $\mathcal{D}_{\sigma_\eta}(C_H)$-absorbing for the process $\mathcal{U}$.

Remark 19. Observe that $\widehat{B}_{\sigma_\eta,C_H} \in \mathcal{D}_{\sigma_\eta}(C_H)$.

Presented separately, just for clarity, the following result is another consequence of Lemma 14. The uniform estimates pullback starting in any family of the universe shall allow to establish a compactness property later.

Lemma 20. *Suppose that the assumptions of Corollary 18 are satisfied. Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_{\sigma_\eta}(C_H)$ there exists $\tau(\widehat{D}, t, h) < t - 2h - 1$ such that for any $\tau \leq \tau(\widehat{D}, t, h)$ and $\mathbf{u} \in \Phi_{C_H}(\tau, D(\tau))$ it holds*

$$|\mathbf{u}(r)|_2 \leqslant \varrho_1(t) \quad \forall r \in [t-2h-1, t],$$

$$\int_{r-1}^r \|\nabla\mathbf{u}(s)\|_p^p ds \leqslant \varrho_2(t) \quad \forall r \in [t-h, t],$$

where

$$\varrho_1^2(t) = 1 + e^{\eta h}e^{\sigma_\eta(h+1)}\int_{-\infty}^t e^{-\sigma_\eta(t-s)}(\widehat{C}_1 + K_4\|\mathbf{f}(s)\|_*^q)ds,$$

$$\varrho_2(t) = \frac{\tilde{c}_0^p}{c_2\nu_2}\Big(\big(1 + 2L_{\mathbf{g}}\big)\varrho_1^2(t) + K_4\int_{t-h-2}^t \|\mathbf{f}(s)\|_*^q ds\Big).$$

Now we are ready to prove the asymptotic compactness of $\mathcal{U}$ in C_H.

Proposition 21. *Under the assumptions of Corollary 18, the process $\mathcal{U}$ is pullback $\mathcal{D}_{\sigma_\eta}(C_H)$-asymptotically compact.*

Proof. By Corollary 18 and Remark 19 it is equivalent to consider just the family $\widehat{B}_{\sigma_\eta,C_H}$. Fix $t_0 \in \mathbb{R}$, and let $\{\tau_m\}$ be a sequence with $\tau_m \to -\infty$. Without loss of generality, according to Lemma 20, consider that $\tau_m \leq \tau(\widehat{B}_{\sigma_\eta,C_H}, t_0, h)$ for all m. Then, using the estimates of Lemma 20, for any sequence of weak solutions $\mathbf{u}^m \in \Phi_{C_H}(\tau_m, B_{\sigma_\eta,C_H}(\tau_m))$, we will check that the sequence $\{\mathbf{u}^m_{t_0}\}$ is relatively compact in C_H.

From Lemma 20 there exist a subsequence (relabeled the same) and an element $\mathbf{u}$ such that

$$\begin{aligned}
\mathbf{u}^m &\overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(t_0-h-1,t_0;H),\\
\mathbf{u}^m &\rightharpoonup \mathbf{u} \text{ in } L^p(t_0-h-1,t_0;V_p),\\
\frac{\partial \mathbf{u}^m}{\partial t} &\rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \text{ in } L^q(t_0-h-1,t_0;V_p^*),\\
\mathbf{u}^m &\to \mathbf{u} \text{ in } L^2(t_0-h-1,t_0;H),\\
\mathbf{u}^m(t) &\to \mathbf{u}(t) \text{ in } \ H \ \textit{a.e.}\ t \in (t_0-h-1,t_0).
\end{aligned}$$

Observe also that $\mathbf{u} \in C([t_0-h-1,t_0];H)$, and the Arzelà-Ascoli Theorem ensures for a subsequence (relabeled the same) that $\mathbf{u}^m \to \mathbf{u}$ in $C([t_0-h-1,t_0];V_p^*)$. So, for any sequence $\{t_m\} \in [t_0-h-1,t_0]$ with $t_m \to t^*$, one has

$$\mathbf{u}^m(t_m) \to \mathbf{u}(t^*) \quad \text{in } V_p^*.$$

Moreover, by **(III)** and Lemma 20, it holds

$$\int_{t_0-h-1}^{t_0} |\mathbf{g}(s,\mathbf{u}^m_s)|_2^2 ds \leqslant L_{\mathbf{g}}^2 (h+1)\varrho_1^2(t_0).$$

Therefore, there exists $\xi \in L^2(t_0-h-1,t_0;L^2(\Omega)^n)$ such that, up to a subsequence,

$$\mathbf{g}(\cdot,\mathbf{u}^m_\cdot) \rightharpoonup \xi \text{ weakly in } L^2(t_0-h-1,t_0;L^2(\Omega)^n).$$

Observe that, again by **(III)** and Lemma 20, there follows

$$\begin{aligned}
\int_s^t |\mathbf{g}(r,\mathbf{u}^m_r)|_2^2 dr &\leqslant C(t-s),\\
\int_s^t |\xi(r)|_2^2 dr \leqslant \liminf_{m\to\infty} \int_s^t |\mathbf{g}(r,\mathbf{u}^m_r)|_2^2 dr &\leqslant C(t-s),
\end{aligned} \tag{23}$$

for all $t_0-h-1 \leqslant s \leqslant t \leqslant t_0$, where $C = L_{\mathbf{g}}^2\varrho_1^2(t_0)$. Then, in a standard way, one can prove that $\mathbf{u}$ is a weak solution to

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} - \operatorname{div}(\mathbb{S}(\mathbf{e}(\mathbf{v}))) + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla\pi = \mathbf{f}(t) + \xi(t) \quad \text{in } \Omega\times(t_0-h-1,t_0),\\
&\operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega\times(t_0-h-1,t_0),\\
&\mathbf{v} = 0 \quad \text{on } \partial\Omega\times(t_0-h-1,t_0),\\
&\mathbf{v}(x,t_0-h-1) = \mathbf{u}(x,t_0-h-1), \quad x\in\Omega.
\end{aligned}$$

By the energy equality (7) and (23), we obtain

$$\frac{1}{2}|\mathbf{z}(t)|_2^2 \leqslant \frac{1}{2}|\mathbf{z}(s)|_2^2 + \int_s^t \langle \mathbf{f}(r),\mathbf{z}(r)\rangle ds + \tilde{C}(t-s),$$

for all $t_0-h-1 \leqslant s \leqslant t \leqslant t_0$, where $\tilde{C} = \frac{Cc_0^2}{4c_2\nu_1\lambda_1}$, and $\mathbf{z} = \mathbf{u}^m$ or $\mathbf{z} = \mathbf{u}$.

Then, the maps J_m, $J : [t_0-h-1,t_0] \to \mathbb{R}$ defined by

$$\begin{aligned}
J_m(t) &= \frac{1}{2}|\mathbf{u}^m(t)|_2^2 - \int_{t_0-h-1}^t \langle \mathbf{f}(r),\mathbf{u}^m(r)\rangle dr - \tilde{C}t\\
J(t) &= \frac{1}{2}|\mathbf{u}(t)|_2^2 - \int_{t_0-h-1}^t \langle \mathbf{f}(r),\mathbf{u}(r)\rangle dr - \tilde{C}t,
\end{aligned}$$

are non-increasing and continuous, and satisfy

$$J_m(t) \to J(t) \quad \textit{a.e.}\ t \in (t_0-h-1,t_0).$$

As in the proof of Theorem 5, with the help of the functions J_m and J we conclude that $\mathbf{u}^m \to \mathbf{u}$ in $C([t_0-h,t_0];H)$. This completes the proof. □

We are in position to state the result about the existence of minimal pullback attractors for the process $\mathcal{U}$ on C_H.

Theorem 22. *Suppose that $p > 2$, there exists some $\sigma_\eta = \eta - 2L_{\mathbf{g}}e^{\eta h} > 0$ such that $\mathbf{f} \in \mathcal{I}_*^{q,\sigma_\eta}$ and $\mathbf{g}$ satisfies **(I)–(III)**. Then, there exist the minimal pullback $\mathcal{D}_F(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_F(C_H)} = \{\mathcal{A}_{\mathcal{D}_F(C_H)}(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_{\sigma_\eta}(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)} = \{\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) : t \in \mathbb{R}\}$ for the multi-valued process $\mathcal{U} : \mathbb{R}_d^2 \times C_H \to \mathcal{P}(C_H)$. The minimal pullback $\mathcal{D}_{\sigma_\eta}(C_H)$-attractor belongs to $\mathcal{D}_{\sigma_\eta}(C_H)$, it is invariant and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) \subset \overline{B}_{C_H}(0, \mathcal{R}_{1,\sigma_\eta}(t)) \quad \forall t \in \mathbb{R}, \tag{24}$$

where $\mathcal{R}_{1,\sigma_\eta}$ is given in (22). Furthermore, if $\mathbf{f}$ satisfies that

$$\sup_{s\leq 0} \int_{s-1}^{s} \|\mathbf{f}(\theta)\|_*^q d\theta < \infty, \tag{25}$$

then

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) \quad \forall t \in \mathbb{R}. \tag{26}$$

Proof. The existence of pullback attractors for the multi-valued process $\mathcal{U}$ in the universes $\mathcal{D}_{\sigma_\eta}(C_H)$ and $\mathcal{D}_F(C_H)$ follows from [27, Theorem 3], and the invariance of $\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}$ as well. The inclusions (24) are given by [27, Theorem 4] (see also [14, Theorem 3.15]).

Finally, under the additional condition (25), which is equivalent to

$$\sup_{s\leqslant 0}\left(e^{-\sigma_\eta s} \int_{-\infty}^{s} e^{\sigma_\eta \theta} \|\mathbf{f}(\theta)\|_*^q d\theta \right) < \infty,$$

we see by estimate (22) that for all $T \in \mathbb{R}$, $\bigcup_{t\leqslant T} \overline{B}_{C_H}(0, \mathcal{R}_{1,\sigma_\eta}(t))$ is a bounded subset of C_H. Hence, (26) follows from [27, Corollary 1]. □

Remark 23. If σ_η in the above statement is strictly less than σ_{η_*}, the maximum of $\eta \mapsto \sigma_\eta$ given in Remark 17, then it is possible to obtain more families of pullback attractors. Indeed, since $\sigma_\eta \leq \sigma_\mu$ implies $I_*^{q,\sigma_\eta} \subset I_*^{q,\sigma_\mu}$, one may apply directly the same result for $\sigma_\mu \in [\sigma_\eta, \sigma_{\eta_*}]$. Besides that, $\mathcal{D}_{\sigma_\eta}(C_H) \subset \mathcal{D}_{\sigma_\mu}(C_H)$ and there exists the corresponding minimal pullback $\mathcal{D}_{\sigma_\mu}(C_H)$-attractor, $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$. Moreover, from [27, Theorem 4], there follows that, for any $t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t)$.

Finally, if $\mathbf{f}$ satisfies (25), then, as above, by (26) we get

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\eta}(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \text{ for all } t \in \mathbb{R}.$$

Remark 24. (i) If $p = 2$ the results are still valid if there exists $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$ such that $\eta > 2L_{\mathbf{g}}e^{\eta h}$ and $\mathbf{f} \in \mathcal{I}_*^{q,\sigma_\eta}$. The restrictions on the parameter η are in agreement with the analysis of the two-dimensional Navier–Stokes equations with delays and initial condition in C_H (cf. [15]).

(ii) A remarkable feature of the case $p > 2$ in the non-autonomous setting is the following. The well-posedness of the attractor firstly requires a right universe where the dissipation happens. The upper limitation for η in the case $p = 2$ is removed when $p > 2$. This means that, although the coupling link among $\mathbf{f}$ and σ_η must exist, the case $p > 2$ allows that if $\mathbf{f}$ belongs to I_*^{q,σ_η} for some $\eta > 0$, this value of η can be chosen, no matter how large it be. Recall from Remark 17 that $\eta_* \to \infty$ if $h \to 0$.

4. Initial conditions in M_H^2

In this section we study the existence of global weak solutions and pullback attractors to **(LMD)** with less regular initial data and simpler integrability condition for $\mathbf{f}$. Due to these relaxations, we will impose two additional hypotheses on the delay term $\mathbf{g}$ to make up for this lack of regularity. More precisely, here on we assume that

$$\begin{aligned}
&p \geqslant 1 + 2n/(n+2),\\
&\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*),\\
&\mathbf{g} : \mathbb{R} \times C_H \to L^2(\Omega)^n \text{ satisfies } (\mathbf{I}) - (\mathbf{V}),\\
&(\mathbf{u}^\tau, \phi) \in M_H^2.
\end{aligned}$$

To be precise, the existence and upper-semicontinuity results only require assumptions **(I)–(IV)**, while for the asymptotic behavior condition **(V)** will be also imposed.

4.1. Existence of weak solutions

In this paragraph we summarize the results about existence and uniqueness of weak solutions in this new setting, which can be proved in a similar way when the initial data is more regular (cf. Section 3.1).

Definition 25. Given an initial condition $(\mathbf{u}^\tau, \phi) \in M_H^2$, a weak solution to **(LMD)** is an element $\mathbf{u}$ such that for any $T > \tau$

$$\mathbf{u} \in L^2(\tau - h, T; H) \cap L^p(\tau, T; V_p) \cap L^\infty(\tau, T; H) \quad \text{with} \quad \frac{\partial \mathbf{u}}{\partial t} \in L^q(\tau, T; V_p^*),$$

which satisfies the weak formulation (6) for all $\mathbf{v} \in V_p$ and *a.e.* $t \in (\tau, T)$, and

$$\mathbf{u}(\tau) = \mathbf{u}^\tau \quad \text{and} \quad \mathbf{u}(\tau + s) = \phi(s) \quad a.e.\ s \in (-h, 0). \tag{27}$$

Remark 26. A weak solution has a representative in the class $\mathbf{u} \in C([\tau, T]; H)$, whereby first part of (27) is meaningful. Further observe that, if $\phi \in C_H$ and $\phi(0) = \mathbf{u}^\tau$ then $\mathbf{u} \in C([\tau - h, T]; H)$.

Theorem 27. *Assume that $p \geq 1 + 2n/(n+2)$, $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ and $\mathbf{g}$ fulfills **(I)–(IV)**. Consider $(\mathbf{u}^\tau, \phi) \in M_H^2$. Then there exists at least one weak solution to **(LMD)**.*

Proof. We employ the Faedo–Galerkin approximations and follow analogously as in Theorem 5. Just observe that the proof is even simpler in this case since it is not necessary the use of the energy method to identify that $\mathbf{g}(\cdot, \mathbf{u}^m_\cdot)$ converges to $\mathbf{g}(\cdot, \mathbf{u}_\cdot)$. Indeed, by using the Aubin–Lions Lemma and thanks to hypothesis **(IV)**, this is straightforward. □

Analogously to Section 3.2, let us define $\Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$ as the set of all weak solutions to **(LMD)** defined (*a.e.*) on $(\tau - h, \infty)$ with initial condition $(\mathbf{u}^\tau, \phi) \in M_H^2$.

Remark 28. Observe that the energy equality (7) is also true for weak solutions in this setting.

We also have the following result that will allow to show that the multi-valued processes, defined from weak solutions, are closed.

Proposition 29. *Under the hypotheses of Theorem 27, consider $\tau \in \mathbb{R}$, $\{(\mathbf{u}^{\tau,m}, \phi^m)\} \subset M_H^2$ such that $(\mathbf{u}^{\tau,m}, \phi^m) \to (\mathbf{u}^\tau, \phi)$ in M_H^2 and $\mathbf{u}^m \in \Phi_{M_H^2}(\tau, (\mathbf{u}^{\tau,m}, \phi^m))$ for all m. Then, for any $T > \tau$, $\{\mathbf{u}^m\}$ is bounded in $L^2(\tau - h, T; H) \cap L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p)$ with $\left\{\frac{\partial \mathbf{u}^m}{\partial t}\right\}$ bounded in $L^q(\tau, T; V_p^*)$, and there exists $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$ such that, up to a subsequence, the following convergences hold*

$$\begin{aligned}
\mathbf{u}^m &\to \mathbf{u} \text{ in } L^2(\tau - h, T; H),\\
\mathbf{u}^m &\overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(\tau, T; H),\\
\mathbf{u}^m &\rightharpoonup \mathbf{u} \text{ in } L^p(\tau, T; V_p),\\
\frac{\partial \mathbf{u}^m}{\partial t} &\rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \text{ in } L^q(\tau, T; V_p^*).
\end{aligned} \tag{28}$$

Similarly as in Theorem 7 we have the following result.

Theorem 30. *Under the hypotheses of Theorem 27, there exist positive constants $\widetilde{K}_i$, $i = 1, 2, 3$, (only depending on the parameters of the model) and $\widetilde{C}_{\mathbf{g}} = \max\{1, \widetilde{K}_1\}$ such that for any $(\mathbf{v}^\tau, \phi^1)$, $(\mathbf{u}^\tau, \phi^2) \in M_H^2$, any weak solutions $\mathbf{v} \in \Phi_{M_H^2}(\tau, (\mathbf{v}^\tau, \phi^1))$ and $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi^2))$ satisfy the following estimates,*

(i) *if $n = 2$,*

$$|\mathbf{v}(t) - \mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi^1) - (\mathbf{u}^\tau, \phi^2)\|^2_{M_H^2} \exp\left\{\int_\tau^t \left(\widetilde{K}_1 + \widetilde{K}_2 |\nabla \mathbf{u}(s)|_2^2\right) ds\right\} \quad \forall t \geq \tau,$$

(ii) *if $n = 3$ and $\mathbf{u} \in L^{\frac{2p}{2p-3}}(\tau, T; V_p)$ for some $T > \tau$, then*

$$\begin{aligned}
|\mathbf{v}(t) - \mathbf{u}(t)|_2^2 \leqslant & \widetilde{C}_{\mathbf{g}} \|(\mathbf{v}^\tau, \phi^1) - (\mathbf{u}^\tau, \phi^2)\|^2_{M_H^2}\\
& \times \exp\left\{\int_\tau^t \left(\widetilde{K}_1 + \widetilde{K}_3 \|\nabla \mathbf{u}(s)\|_p^{\frac{2p}{2p-3}}\right) ds\right\} \quad \forall t \in [\tau, T].
\end{aligned}$$

Again, as pointed out in Remark 8, the above result provides uniqueness of solution for $n = 2$, and also for $n = 3$ subject to extra regularity, and continuity with respect to initial data. Nevertheless we do not require strictly continuity. Indeed in the multi-valued framework it suffices to have upper-semicontinuity, which holds thanks to Proposition 29.

4.2. Existence of pullback attractors in M_H^2

In this section we analyze the asymptotic behavior of the weak solutions to **(LMD)** in the Hilbert space M_H^2.
Now, we introduce the bi-parametric family of mappings $\mathcal{S} : \mathbb{R}_d^2 \times M_H^2 \to \mathcal{P}(M_H^2)$, given by

$$\mathcal{S}(t, \tau)(\mathbf{u}^\tau, \phi) = \left\{\left(\mathbf{u}(t), \mathbf{u}_t(\cdot)\right) : \mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))\right\}.$$

Next result ensures that the process $\mathcal{S}$ has closed values.

Proposition 31. *Under the hypotheses of Theorem 27, let* $\{(\mathbf{u}^{\tau,m}, \phi^m)\} \subset M_H^2$ *and* $(\mathbf{u}^\tau, \phi) \in M_H^2$ *be such that* $(\mathbf{u}^{\tau,m}, \phi^m) \to (\mathbf{u}^\tau, \phi)$ *in* M_H^2*. Then, for any sequence* $\{\mathbf{u}^m\}$ *with* $\mathbf{u}^m \in \Phi_{M_H^2}(\tau, (\mathbf{u}^{\tau,m}, \phi^m))$*, there exist a subsequence (relabeled the same) and* $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$ *such that for any* $t \geqslant \tau$

$$\mathbf{u}^m(t) \to \mathbf{u}(t) \quad \textit{strongly in } H, \tag{29}$$

$$\mathbf{u}^m_t \to \mathbf{u}_t \quad \textit{strongly in } L^2_H. \tag{30}$$

Proof. Consider $T > \tau$. Observe that by Proposition 29 the sequence $\{\mathbf{u}^m\}$ is bounded in $L^2(\tau - h, T; H) \cap L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p)$ and $\{\frac{\partial \mathbf{u}^m}{\partial t}\}$ is bounded in $L^q(\tau, T; V_p^*)$. Therefore, there exist a subsequence of $\{\mathbf{u}^m\}$ (relabeled the same) and $\mathbf{u} \in L^2(\tau - h, T; H) \cap L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p)$ with $\frac{\partial \mathbf{u}}{\partial t} \in L^q(\tau, T; V_p^*)$ such that convergences (28) hold true.

Again by Proposition 29, we have that $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$. In particular, by one of the convergences in (28), claim (30) follows straightforward for $t \in [\tau, T]$.

So it remains to prove (29). By the Arzelà-Ascoli Theorem there follows, up to a subsequence, that $\mathbf{u}^m \to \mathbf{u}$ strongly in $C([\tau, T]; V_p^*)$. Since $\{\mathbf{u}^m\}$ is bounded in $C([\tau, T]; H)$ we have that

$$\mathbf{u}^m(s) \rightharpoonup \mathbf{u}(s) \quad \text{weakly in } H \text{ for any } s \in [\tau, T].$$

From the energy equality (7) it holds that

$$\frac{1}{2}|\mathbf{z}(t)|_2^2 \leqslant \frac{1}{2}|\mathbf{z}(s)|_2^2 + \int_s^t \langle \mathbf{f}(r), \mathbf{z}(r)\rangle ds + \frac{1}{2\overline{\eta}}\int_s^t |\mathbf{g}(\theta, \mathbf{z}_\theta)|_2^2 d\theta \quad \forall \tau \leqslant s \leqslant t \leqslant T,$$

where $\mathbf{z} = \mathbf{u}^m$ or $\mathbf{z} = \mathbf{u}$, being $\overline{\eta} = 2c_2\nu_1\lambda_1 c_0^{-2}$.

Thus, let us consider the functions $J, J_m : [\tau, T] \to \mathbb{R}$ given by

$$J_m(t) = |\mathbf{u}^m(t)|^2 - 2\int_\tau^t \langle \mathbf{f}(\theta), \mathbf{u}^m(\theta)\rangle d\theta - \frac{1}{\overline{\eta}}\int_\tau^t |\mathbf{g}(\theta, \mathbf{u}^m_\theta)|_2^2 d\theta,$$

$$J(t) = |\mathbf{u}(t)|^2 - 2\int_\tau^t \langle \mathbf{f}(\theta), \mathbf{u}(\theta)\rangle d\theta - \frac{1}{\overline{\eta}}\int_\tau^t |\mathbf{g}(\theta, \mathbf{u}_\theta)|_2^2 d\theta.$$

From above we have that J and J_m are non-increasing and continuous, and

$$J_m(t) \to J(t) \quad \textit{a.e. } t \in (\tau, T),$$

by **(IV)** and the convergences in (28).

By the properties of the functions J_m and J, we can deduce, again as in Theorem 5, that (29) holds for all $t \in [\tau, T]$. Observe that $T > \tau$ is arbitrary, so the proof is complete. □

From above and Proposition 29 we obtain the following result.

Corollary 32. *Under the hypotheses of Theorem 27, the family of mappings* $S : \mathbb{R}_d^2 \times M_H^2 \to \mathcal{P}(M_H^2)$ *is an upper-semicontinuous and strict multi-valued process with closed values.*

In order to go further on the analysis of the solutions to **(LMD)** we require additional estimates and include the final assumption **(V)** on $\mathbf{g}$. We perform the main exposition for $p > 2$. The case $p = 2$ is similar and will be commented at the end of the paragraph.

Lemma 33. *Assume that* $p > 2$, $\mathbf{f} \in L^q_{loc}(\mathbb{R}; V_p^*)$ *and* $\mathbf{g}$ *fulfills* ***(I)–(V)****. Then there exist positive constants* C_η *and* $\widetilde{K}_4$ *such that any weak solution* $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$ *satisfies*

$$|\mathbf{u}(t)|_2^2 \leqslant \widetilde{C}_{\mathbf{g}} e^{-\eta(t-\tau)}\|(\mathbf{u}^\tau, \phi)\|^2_{M_H^2} + \int_\tau^t e^{-\eta(t-s)}\left(C_\eta + \widetilde{K}_4\|\mathbf{f}(s)\|_*^q\right)ds \quad \forall t \geqslant \tau, \tag{31}$$

where $\widetilde{C}_{\mathbf{g}} = \max\{1, \frac{C_{\mathbf{g}}^2 c_0^2}{c_2\nu_1\lambda_1}\}$.

Proof. By the energy equality (7), the coercivity of $\mathbb{S}$ (3), and the Korn, Young and Poincaré inequalities, we have that

$$\frac{d}{dt}|\mathbf{u}(t)|_2^2 + \tilde{\eta}|\mathbf{u}|_2^2 + \eta_2\|\nabla\mathbf{u}\|_p^p \leqslant \widetilde{K}_4\|\mathbf{f}(t)\|_*^q + \frac{1}{\tilde{\eta}}|\mathbf{g}(t, \mathbf{u}_t)|_2^2, \tag{32}$$

where $\tilde{\eta} = c_2\nu_1\lambda_1 c_0^{-2}$, $\eta_2 = \frac{c_2\nu_2}{\tilde{c}_0^p}$ and $\widetilde{K}_4 = \frac{2^q}{\eta_2^{q/p}p^q q}$.

We split the analysis into two cases: $\eta \in (0, \tilde{\eta}]$ and $\eta > \tilde{\eta}$.

Case $\eta \in (0, \tilde{\eta}]$**.** From (32), multiplying by $e^{\eta t}$, integrating from τ to t, and using hypothesis **(V)**, we have for all $t \geq \tau$

$$e^{\eta t}|\mathbf{u}(t)|_2^2 + \eta_2\int_\tau^t e^{\eta s}\|\nabla\mathbf{u}\|_p^p ds \leqslant \widetilde{C}_{\mathbf{g}} e^{\eta\tau}\|(\mathbf{u}^\tau, \phi)\|^2_{M_H^2} + \int_\tau^t e^{\eta s}\widetilde{K}_4\|\mathbf{f}(s)\|_*^q ds + \frac{C_{\mathbf{g}}^2}{\tilde{\eta}}\int_\tau^t e^{\eta s}|\mathbf{u}(s)|_2^2 ds,$$

where $\widetilde{C}_{\mathbf{g}} = \max\{1, \frac{C_{\mathbf{g}}^2}{\tilde{\eta}}\}$.

Since $|\mathbf{u}|_2 \leqslant C_I \|\nabla \mathbf{u}\|_p$ (recall that C_I denotes an embedding constant introduced in Section 3) there exists $\widetilde{C} = \widetilde{C}(p, C_{\mathbf{g}}, C_I, \eta_2) > 0$ such that $\frac{C_{\mathbf{g}}^2}{\tilde{\eta}}|\mathbf{u}|_2^2 \leqslant \frac{\eta_2}{2}\|\nabla \mathbf{u}\|_p^p + \widetilde{C}$. Therefore, from above we obtain (31).

Case $\eta > \tilde{\eta}$**.** Denote $0 < \beta := \eta - \tilde{\eta}$. Let $\widehat{C}_1$ be as in (21). Then (32) reduces to

$$\frac{d}{dt}|\mathbf{u}(t)|_2^2 + \eta|\mathbf{u}|_2^2 + \frac{\eta_2}{2}\|\nabla \mathbf{u}\|_p^p \leqslant \widehat{C}_1 + \widetilde{K}_4\|\mathbf{f}(t)\|_*^q + \frac{1}{\tilde{\eta}}|\mathbf{g}(t, \mathbf{u}_t)|_2^2.$$

In the same way as the previous case, we recover (31), concluding the proof. □

Remark 34. For the case $p = 2$, we have that $c_2 = 1$ and $\nu_2 = 0$. Assume that $\nu_1\lambda_1 c_0^{-2} > C_{\mathbf{g}}$ and that there exists $\eta \in (0, 2[\nu_1\lambda_1 c_0^{-2} - C_{\mathbf{g}}])$ satisfying hypothesis **(V)**. A similar argument as in Lemma 33 shows that for any $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$

$$|\mathbf{u}(t)|_2^2 \leqslant \max\{1, C_{\mathbf{g}}^2\}e^{-\eta(t-\tau)}\|(\mathbf{u}^\tau, \phi)\|_{M_H^2}^2 + \beta^{-1}\int_\tau^t e^{-\eta(t-s)}\|\mathbf{f}(s)\|_*^2 ds,$$

where $\beta := 2\nu_1 c_0^{-2} - (\eta + 2C_{\mathbf{g}})\lambda_1^{-1}$.

The previous lemma allows to derive a bound for the solutions in L_H^2.

Lemma 35. *Under the assumptions of Lemma 33, given $\tau \in \mathbb{R}$ and $(\mathbf{u}^\tau, \phi) \in M_H^2$, any weak solution $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$ satisfies*

$$\|\mathbf{u}_t\|_{L_H^2}^2 \leqslant \widetilde{C}_{h,\mathbf{g}} e^{-\eta(t-\tau)}\|(\mathbf{u}^\tau, \phi)\|_{M_H^2}^2 + he^{\eta h}\int_\tau^t e^{-\eta(t-s)}(C_\eta + \widetilde{K}_4\|\mathbf{f}(s)\|_*^q)ds \quad \forall t \geq \tau,$$

where $\widetilde{C}_{h,\mathbf{g}} = (1 + h\widetilde{C}_{\mathbf{g}})e^{\eta h}$.

Proof. From (31) for $\tau + h \leqslant t$, there follows

$$|\mathbf{u}(t+\theta)|_2^2 \leqslant \widetilde{C}_{\mathbf{g}} e^{-\eta(t+\theta-\tau)}\|(\mathbf{u}^\tau, \phi)\|_{M_H^2}^2 + \int_\tau^{t+\theta} e^{-\eta(t+\theta-s)}\left[C_\eta + \widetilde{K}_4\|\mathbf{f}(s)\|_*^q\right]ds \quad \forall \theta \in [-h, 0].$$

Thus, integrating in θ between $-h$ and 0, we have that

$$\|\mathbf{u}_t\|_{L_H^2}^2 \leqslant h\widetilde{C}_{\mathbf{g}} e^{\eta h}e^{-\eta(t-\tau)}\|(\mathbf{u}^\tau, \phi)\|_{M_H^2}^2 + he^{\eta h}\int_\tau^t e^{-\eta(t-s)}(C_\eta + \widetilde{K}_4\|\mathbf{f}(s)\|_*^q)ds \quad \forall t \geqslant \tau + h.$$

Now, suppose that $\tau \leqslant t \leqslant \tau + h$. Then, splitting

$$\|\mathbf{u}_t\|_{L_H^2}^2 = \int_{t-h}^\tau |\mathbf{u}(\theta)|_2^2 d\theta + \int_\tau^t |\mathbf{u}(\theta)|_2^2 d\theta,$$

first addend is controlled by $\|\phi\|_{L_H^2}^2$, and second addend is estimated by using (31). Arranging constants, we obtain the inequality in the statement, which concludes the proof. □

Now we introduce a tempered universe in M_H^2, not for a general parameter but for the value η appearing in assumption **(V)**.

Definition 36 (*Universe in M_H^2*)**.** Given $\eta > 0$, we will denote by $\mathcal{D}_\eta(M_H^2)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M_H^2)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\eta\tau}\sup_{(\mathbf{v},\phi)\in D(\tau)}\|(\mathbf{v}, \phi)\|_{M_H^2}^2\right) = 0.$$

Observe that $\mathcal{D}_\eta(M_H^2)$ is inclusion-closed. Moreover, we will denote by $\mathcal{D}_F(M_H^2)$ the class of all families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of M_H^2.

Lemmas 33 and 35 furnish the existence of pullback absorbing families for the process $\mathcal{S}$.

Corollary 37. *Under the assumptions of Lemma 33, if $\mathbf{f} \in I_*^{q,\eta}$, then the family $\widehat{B}_{\eta,M_H^2} = \{B_{\eta,M_H^2}(t) : t \in \mathbb{R}\}$ with $B_{\eta,M_H^2}(t) = \overline{B}_{M_H^2}(0, \mathcal{R}_2(t))$ is pullback $\mathcal{D}_\eta(M_H^2)$-absorbing for the process $\mathcal{S}$, where*

$$\mathcal{R}_2^2(t) = 1 + (1 + he^{\eta h})\int_{-\infty}^t e^{-\eta(t-s)}(C_\eta + \widetilde{K}_4\|\mathbf{f}(s)\|_*^q)ds.$$

Remark 38. Observe that $\widehat{B}_{\eta,M_H^2} \in \mathcal{D}_\eta(M_H^2)$.

Lemma 39. *Assume that $p > 2$, $\mathbf{g}$ fulfills **(I)–(V)** and $\mathbf{f} \in I_*^{q,\eta}$. Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$ there exists $\tau_1(\widehat{D}, t, h) < t - h - 2$ such that for all $\tau \leqslant \tau_1(\widehat{D}, t, h)$ and any $(\mathbf{u}^\tau, \phi^\tau) \in D(\tau)$ and $\mathbf{u} \in \Phi_{M_H^2}(\tau, (\mathbf{u}^\tau, \phi))$ it holds*

$$|\mathbf{u}(r)|_2 \leqslant \rho_3(t) \quad \forall r \in [t - h - 2, t],$$

$$\int_{r-1}^{r} \|\nabla \mathbf{u}(\theta)\|_p^p d\theta \leqslant \varrho_4(t) \quad \forall r \in [t-h-1, t],$$

where

$$\varrho_3^2(t) = 1 + e^{-\eta(t-h-2)} \int_{-\infty}^{t} e^{\eta s}(C_\eta + \widetilde{K}_4 \|\mathbf{f}(s)\|_*^q) ds \quad \forall t \in \mathbb{R},$$

$$\varrho_4(t) = \frac{\tilde{c}_0^p}{c_2 \nu_2} \Big((1 + \frac{L_{\mathbf{g}}^2}{\tilde{\eta}}) \varrho_3^2(t) + \widetilde{K}_4 \int_{t-h-2}^{t} \|\mathbf{f}(s)\|_*^q ds \Big) \quad \forall t \in \mathbb{R}.$$

Proof. From Lemma 33, choose $\tau_1(\widehat{D}, t) < t - h - 2$ such that

$$\widetilde{C}_{\mathbf{g}} e^{\eta(t-\tau)} \|(\mathbf{u}^\tau, \phi^\tau)\|_{M_H^2}^2 < 1,$$

for all $(\mathbf{u}^\tau, \phi^\tau) \in D(\tau)$, with $\tau \leqslant \tau_1(\widehat{D}, t)$. Thus, we obtain the first estimate.

Now, observe that $\mathbf{u} \in C([t-h-1, t]; H)$. Integrating (32) for $r \in [t-h-1, t]$ and using **(III)** we have

$$\eta_2 \int_{r-1}^{r} \|\nabla \mathbf{u}(s)\|_p^p ds \leqslant |\mathbf{u}(r-1)|_2^2 + \widetilde{K}_4 \int_{r-1}^{r} \|\mathbf{f}(s)\|_*^q ds + \frac{L_{\mathbf{g}}^2}{\tilde{\eta}} \int_{r-1}^{r} \|\mathbf{u}_s\|_{C_H}^2 ds.$$

Therefore, by using the first bound we deduce the second one. □

Proposition 40. *Assume that $p > 2$, $\mathbf{g}$ fulfills **(I)–(V)** and $\mathbf{f} \in I_*^{q,\eta}$. Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\eta(M_H^2)$, $\{\tau_m\}$ with $\tau_m \to -\infty$, and $\mathbf{u}^m \in \Phi_{M_H^2}(\tau_m, D(\tau_m))$, the sequence $\{\mathbf{u}_t^m\}$ is relatively compact in C_H. In particular, $\mathcal{S}$ is pullback $\mathcal{D}_\eta(M_H^2)$-asymptotically compact.*

Proof. It suffices to consider the absorbing family $\widehat{B}_{\eta, M_H^2} \in \mathcal{D}_\eta(M_H^2)$. Fix $t \in \mathbb{R}$ and $\{\tau_m\}$ and $\{\mathbf{u}^m\}$ as in the statement. We will prove that the sequence $\{\mathbf{u}_t^m\}$ is relatively compact in C_H.

It follows from Lemma 39 that there exists $m(t,h)$ such that for $m \geqslant m(t,h)$

$$|\mathbf{u}^m(r)|_2 \leqslant \varrho_3(t) \quad \forall r \in [t-h-2, t],$$

$$\int_{r-1}^{r} \|\nabla \mathbf{u}^m(s)\|_p^p ds \leqslant \varrho_4(t) \quad \forall r \in [t-h-1, t].$$

Now, the proof is analogous as the one in Proposition 21, replacing ϱ_1 by ϱ_3. Thus, we conclude that $\mathbf{u}_t^m \to \mathbf{u}_t$ in C_H. □

Theorem 41. *Assume that $p > 2$, $\mathbf{g}$ fulfills **(I)–(V)** and $\mathbf{f} \in I_*^{q,\eta}$. Then there exist the minimal pullback $\mathcal{D}_F(M_H^2)$-attractor $\mathcal{A}_{\mathcal{D}_F(M_H^2)} = \{\mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_\eta(M_H^2)$-attractor $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)} = \{\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) : t \in \mathbb{R}\}$ for the multi-valued process $\mathcal{S} : \mathbb{R}_d^2 \times M_H^2 \to \mathcal{P}(M_H^2)$. The minimal pullback $\mathcal{D}_\eta(M_H^2)$-attractor belongs to $\mathcal{D}_\eta(M_H^2)$, it is invariant and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t) \subset \overline{B}_{M_H^2}(0, \mathcal{R}_2(t)) \quad \forall t \in \mathbb{R}. \tag{33}$$

If $\mathbf{f}$ satisfies (25), then $\mathcal{A}_{\mathcal{D}_F(M_H^2)}(t) = \mathcal{A}_{\mathcal{D}_\eta(M_H^2)}(t)$ for all $t \in \mathbb{R}$.

Proof. The existence of pullback attractors for $\mathcal{S}$ in the universes $\mathcal{D}_\eta(M_H^2)$ and $\mathcal{D}_F(M_H^2)$ and the invariance of $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}$ follow from [27, Theorem 3]. The inclusions (33) are consequence of [27, Theorem 4]. The last statement, of equality of attractors in the universes $\mathcal{D}_F(M_H^2)$ and $\mathcal{D}_\eta(M_H^2)$, is again consequence of [27, Corollary 1] since for each $T \in \mathbb{R}$, $\sup_{t \le T} \mathcal{R}_2(t) < \infty$ when $\mathbf{f}$ satisfies (25). □

Remark 42. If $p = 2$ the above results are still valid assuming that $\nu_1 \lambda_1 c_0^{-2} > C_{\mathbf{g}}$ and that the value η from hypothesis **(V)** satisfies $\eta \in (0, 2[\nu_1 \lambda_1 c_0^{-2} - C_{\mathbf{g}}])$. These restrictions on the coefficients and parameter η are in agreement with the analysis of the two-dimensional Navier–Stokes equations with delays and initial condition in M_H^2 (cf. [16]).

Remark 43. The chosen presentation of the above results is according to the standard theory with a multi-valued process well-established in M_H^2. This has been done just for clarity. The counterpart is that the theses are unpleasantly incomplete, in the sense that the asymptotic compactness has been really obtained in C_H, not only in M_H^2. It is possible to reformulate the result, improving the characteristics of these families, where they are compact and in which stronger sense they are attracting. To make a clear exposition we postpone it to next section.

5. Improvements and comparison of attractors

In this section we present some improvements on the results obtained previously. Namely, these are two types. On the one hand, relationships of the constructed attractors in M_H^2 and C_H universes, and on the other hand, with respect to the attractor norm, since at last it will not only be the M_H^2-norm but with respect to the C_H-norm. As consequence, some extra compactness properties arise.

We continue assuming that all the assumptions **(I)–(V)** for $\mathbf{g}$ are satisfied and $\mathbf{f} \in I_*^{q,\eta}$.

Existence of attractors in C_H is consequence of previous results.

Corollary 44. *Assume that $p \geq 2$, $\mathbf{g}$ fulfills **(I)–(V)** and $\mathbf{f} \in I_*^{q,\eta}$, and additionally $\eta \in (0, 2[\nu_1\lambda_1 c_0^{-2} - C_{\mathbf{g}}])$ if $p = 2$. Then there exist pullback attractors for the multi-valued process $\mathcal{U}$ in the universes $\mathcal{D}_\eta(C_H)$ and $\mathcal{D}_F(C_H)$ and the following relationship holds*

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t) \quad \forall t \in \mathbb{R}. \tag{34}$$

If $\mathbf{f}$ satisfies (25), then $\mathcal{A}_{\mathcal{D}_F(C_H)}(t) = \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)$ for all $t \in \mathbb{R}$.

Proof. Suppose $p > 2$ (the case $p = 2$ is treated similarly with the usual restriction, we omit it for short). From Lemma 33 we have that the family $\widehat{B}_{\eta,C_H} = \{B_{\eta,C_H}(t) : t \in \mathbb{R}\}$ with $B_{\eta,C_H}(t) = \overline{B}_{C_H}(0, \widetilde{\mathcal{R}}_2(t))$, where

$$\widetilde{\mathcal{R}}_2^2(t) = 1 + e^{\eta h} \int_{-\infty}^{t} e^{-\eta(t-s)}(C_\eta + \widetilde{K}_4\|\mathbf{f}(s)\|_*^q)ds,$$

is pullback $\mathcal{D}_\eta(C_H)$-absorbing for the process $\mathcal{U}$. On the other hand, as in Proposition 40, $\mathcal{U}$ is pullback $\widehat{B}_{\eta,C_H}$-asymptotically compact in C_H. Then, (34) follows by [27, Theorems 3 and 4].

The last statement, of equality of attractors in the universes $\mathcal{D}_F(C_H)$ and $\mathcal{D}_\eta(C_H)$, is again consequence of [27, Corollary 1] since for each $T \in \mathbb{R}$, $\sup_{t\leq T} \widetilde{\mathcal{R}}_2(t) < \infty$ when $\mathbf{f}$ satisfies (25). □

Remark 45. (i) Oppositely to the assumption about $\mathbf{f} \in I_*^{q,\sigma_\eta}$ with $\sigma_\eta > 0$ provided in Theorem 22, the above result does not require this stronger condition on $\mathbf{f}$. However, this relaxation has the counterpart of $\mathbf{g}$ assuming the extra conditions **(IV)–(V)**, and coupled with $\mathbf{f} \in I_*^{q,\eta}$.

(ii) Since η is fixed in this context, related to **(V)**, we cannot consider a chain of attractors in the universes $\mathcal{D}_\mu(C_H) \supset \mathcal{D}_\eta(C_H)$ for $\mu \geq \eta$, because $\mathbf{g}$ does not need to satisfy **(V)** with such new values μ.

For convenience, let us introduce the canonical injection $j : C_H \to M_H^2$ given by $j(\phi) = (\phi(0), \phi)$. Observe that $j \in \mathcal{L}(C_H, M_H^2)$ and $\|j\|^2_{\mathcal{L}(C_H,M_H^2)} \leq 1 + h$.

Remark 46. Given any element $\widehat{D} \in \mathcal{D}_\eta(C_H)$, then $j(\widehat{D}) \in \mathcal{D}_\eta(M_H^2)$.

The injection j allows to establish a natural relation between the basic bricks, i.e., the omega-limit families $\Lambda_{M_H^2}$ and Λ_{C_H} (that is the accumulation points of pullback converging sequences in their respective topologies).

Corollary 47. *Under the assumptions of Corollary 44, it holds*

$$\Lambda_{M_H^2}(j(\widehat{D}), t) = j(\Lambda_{C_H}(\widehat{D}, t)) \quad \forall \widehat{D} \in \mathcal{D}_\eta(C_H), \ \forall t \in \mathbb{R}.$$

Proof. It is a straightforward consequence of Proposition 40, concerning weak solutions to **(LMD)** regardless of using the processes $\mathcal{S}$ or $\mathcal{U}$, and the injection $j : C_H \to M_H^2$. □

A sufficient condition, useful to establish comparison results, is to find a smoothing transformation, throughout solutions to **(LMD)**, of any family of one universe into an element of a second universe after an elapsed time.

Proposition 48. *Assume the hypotheses of Corollary 44. Then, for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\eta(M_H^2)$, it holds that*

(i) $\widehat{D}^h = \{D^h(t) : t \in \mathbb{R}\} \in \mathcal{D}_\eta(M_H^2)$, where $D^h(\tau + h) = \mathcal{S}(\tau + h, \tau)D(\tau)$,

(ii) $\widetilde{D}^h = \{\widetilde{D}^h(t) : t \in \mathbb{R}\} \in \mathcal{D}_\eta(C_H)$, where $\widetilde{D}^h(\tau + h) = \pi_{L_H^2} \mathcal{S}(\tau + h, \tau)D(\tau)$, being $\pi_{L_H^2} : M_H^2 \to L_H^2$ the projector $\pi_{L_H^2}(\mathbf{u}, \phi) = \phi$ for $(\mathbf{u}, \phi) \in M_H^2$.

Proof. Since $\mathbf{f} \in I_*^{q,\eta}$, then $\int_\tau^{\tau+h} e^{\eta s}\|\mathbf{f}(s)\|_*^q ds \to 0$ as $\tau \to -\infty$. Therefore, the result is consequence of Lemmas 33 and 35, with the extra restriction given in Remark 42 when $p = 2$. □

Theorem 49. *Assume that $p \geq 2$, $\mathbf{g}$ fulfills **(I)–(V)** and $\mathbf{f} \in I_*^{q,\eta}$, and additionally $\eta \in (0, 2[\nu_1\lambda_1 c_0^{-2} - C_{\mathbf{g}}])$ if $p = 2$. Then, the attractors $\mathcal{A}_{\mathcal{D}_F(M_H^2)}$ and $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)}$, ensured by Theorem 41 and Remark 42, attract in $H \times C_H$-norm and their sections are compact in $H \times C_H$.*

Proof. Consider the space $j(C_H) \subset H \times C_H$ endowed with norm $(|\mathbf{u}^\tau|_2^2 + \|\phi\|_{C_H}^2)^{1/2}$ for any pair $(\mathbf{u}^\tau, \phi) \in H \times C_H$, the (restricted) process (we do not change the name, since no confusion arises)

$$\mathcal{S} : \mathbb{R}_d^2 \times j(C_H) \to \mathcal{P}(H \times C_H)$$

and the auxiliary universe

$$\mathcal{D}_\eta(M_H^2) \cap j(C_H) = \{\widehat{D} \in \mathcal{D}_\eta(M_H^2) : D(t) \subset j(C_H)\}.$$

For such process, the existence of minimal pullback attractor $\mathcal{A}_{\mathcal{D}_\eta(M_H^2)\cap j(C_H)}$, with compact sections in $H \times C_H$ and attracting w.r.t. $H \times C_H$-norm follows by the previous results.

On the other hand, by comparison (cf. Proposition 48 (i) and [27, Theorem 4]) we have that

$$\mathcal{A}_{\mathcal{D}_\eta(M^2_H)\cap j(C_H)} = \mathcal{A}_{\mathcal{D}_\eta(M^2_H)}.$$

This proves the result concerning $\mathcal{A}_{\mathcal{D}_\eta(M^2_H)}$.

Now we verify the same for $\mathcal{A}_{\mathcal{D}_F(M^2_H)}$. Indeed, since each section $\mathcal{A}_{\mathcal{D}_F(M^2_H)}(t)$ is contained in the section $\mathcal{A}_{\mathcal{D}_\eta(M^2_H)}(t)$, which is compact in $H\times C_H$, it suffices to check that $\mathcal{A}_{\mathcal{D}_F(M^2_H)}(t)$ is closed in $H\times C_H$ (and not only in M^2_H). This can be proved as follows: consider $\{x_n\}\subset \mathcal{A}_{\mathcal{D}_F(M^2_H)}(t)$ with $x_n\to x$ in $H\times C_H$. By the continuous injection $H\times C_H\subset M^2_H$, $x_n\to x$ in M^2_H. Since $\mathcal{A}_{\mathcal{D}_F(M^2_H)}(t)$ is closed in M^2_H, then $x\in \mathcal{A}_{\mathcal{D}_F(M^2_H)}(t)$.

Finally, that $\mathcal{A}_{\mathcal{D}_F(M^2_H)}$ attracts in $H\times C_H$-norm is consequence of Proposition 40. □

Remark 50. (i) Although initially $\mathcal{S}:\mathbb{R}^2_d\times M^2_H\to \mathcal{P}(M^2_H)$ cannot be settled with the metric $H\times C_H$, the regularity of the solutions after an elapsed time h allows, at last, to consider $\Lambda_{H\times C_H}(\widehat{D},t)$ for any $\widehat{D}\in\mathcal{D}_\eta(M^2_H)$ (in the spirit of (X,Y)-attraction). The last claim proved above means that it holds

$$\overline{\bigcup_{\widehat{D}\in\mathcal{D}_F(M^2_H)}\Lambda_{H\times C_H}(\widehat{D},t)}^{H\times C_H} = \mathcal{A}_{\mathcal{D}_F(M^2_H)}(t)\quad \forall t\in\mathbb{R},$$

(observe this is not only an inclusion but an equality, again by the injection $H\times C_H\subset M^2_H$).

(ii) The attraction w.r.t. $H\times C_H$-norm of the pullback attractors for the universes in M^2_H and the property of having compact sections in $H\times C_H$ also hold for similar previous results for Navier–Stokes models with delay (cf. [16,17]).

In the same way as in [17], let us compare the pullback attractors in C_H and M^2_H via the introduced canonical injection j.

Theorem 51. *Assume the hypotheses of Theorem 49. Then, the following relationships hold*

$$j(\mathcal{A}_{\mathcal{D}_F(C_H)}(t))\subset \mathcal{A}_{\mathcal{D}_F(M^2_H)}(t)\quad \forall t\in\mathbb{R}, \tag{35}$$

$$j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)) = \mathcal{A}_{\mathcal{D}_\eta(M^2_H)}(t)\quad \forall t\in\mathbb{R}. \tag{36}$$

If **f** *fulfills (25), then (35) becomes an equality for all $t\in\mathbb{R}$.*

Proof. Let us start proving (35). Firstly, observe that trivially $j(\mathcal{D}_F(C_H))\subset\mathcal{D}_F(M^2_H)$ and for any $B\subset C_H$, the canonical injection j satisfies $j(\overline{B}^{C_H})\subset \overline{j(B)}^{M^2_H}$. Then,

$$\begin{aligned}
j(\mathcal{A}_{\mathcal{D}_F(C_H)}(t)) &= j\left(\overline{\cup_{\widehat{D}\in\mathcal{D}_\eta(C_H)}\Lambda_{C_H}(\widehat{D},t)}^{C_H}\right)\\
&\subset \overline{j(\cup_{\widehat{D}\in\mathcal{D}_\eta(C_H)}\Lambda_{C_H}(\widehat{D},t))}^{M^2_H} = \overline{\cup_{\widehat{D}\in\mathcal{D}_\eta(C_H)} j(\Lambda_{C_H}(\widehat{D},t))}^{M^2_H}\\
&= \overline{\cup_{\widehat{D}\in\mathcal{D}_\eta(C_H)}\Lambda_{M^2_H}(j(\widehat{D}),t)}^{M^2_H}\subset \mathcal{A}_{\mathcal{D}_F(M^2_H)}(t),
\end{aligned}$$

where we have used Corollary 47 for the last equality. Thus (35) is proved.

Analogously, it follows the inclusion

$$j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))\subset \mathcal{A}_{\mathcal{D}_\eta(M^2_H)}(t)\quad \forall t\in\mathbb{R}.$$

To obtain the opposite inclusion and conclude (36), consider an arbitrary family $\widehat{D}=\{D(t):t\in\mathbb{R}\}\in\mathcal{D}_\eta(M^2_H)$. Recalling that $j\in\mathcal{L}(C_H,M^2_H)$ satisfies $\|j\|_{\mathcal{L}(C_H,M^2_H)}\leqslant(1+h)^{1/2}$, we deduce that for $\tau\leqslant t-h$

$$\begin{aligned}
&dist_{M^2_H}\left(\mathcal{S}(t,\tau)D(\tau), j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))\right)\\
=&dist_{M^2_H}\left(\mathcal{S}(t,\tau+h)\big(\mathcal{S}(\tau+h,\tau)D(\tau)\big), j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))\right)\\
=&dist_{M^2_H}\left(\mathcal{S}(t,\tau+h)\big(j(\pi_{L^2_H}\mathcal{S}(\tau+h,\tau)D(\tau))\big), j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))\right)\\
=&dist_{M^2_H}\left(\mathcal{S}(t,\tau+h)j(\widetilde{D}^h(\tau+h)), j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))\right)\\
=&dist_{M^2_H}\left(j\big(\mathcal{U}(t,\tau+h)\widetilde{D}^h(\tau+h)\big), j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t))\right)\\
\leqslant&(1+h)^{1/2}dist_{C_H}\left(\mathcal{U}(t,\tau+h)\widetilde{D}^h(\tau+h),\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)\right),
\end{aligned}$$

where we have denoted $\widetilde{D}^h(s+h)=\pi_{L^2_H}(\mathcal{S}(s+h,s)D(s))$ for all $s\in\mathbb{R}$. Observe that, by Proposition 48 (ii), $\widetilde{D}^h\in\mathcal{D}_\eta(C_H)$. Since $\mathcal{A}_{\mathcal{D}_\eta(C_H)}$ is pullback $\mathcal{D}_\eta(C_H)$-attracting, from previous inequality we obtain that $j(\mathcal{A}_{\mathcal{D}_\eta(C_H)})$ is pullback $\mathcal{D}_\eta(M^2_H)$-attracting in $\mathcal{D}_\eta(M^2_H)$. Thus, being $\mathcal{A}_{\mathcal{D}_\eta(M^2_H)}(t)$ the minimal closed family that pullback attracts any family $\widehat{D}\in\mathcal{D}_\eta(M^2_H)$, we conclude that $\mathcal{A}_{\mathcal{D}_\eta(M^2_H)}(t)\subset j\big(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)\big)$ for all $t\in\mathbb{R}$.

Finally, the fact that (35) becomes an equality, if $\mathbf{f}$ satisfies (25), follows from [27, Corollary 1] since for each $T \in \mathbb{R}$, $\sup_{t\leq T} \mathcal{R}_2(t) < \infty$. □

Remark 52. When proving (35), in fact the first inclusion is an equality, namely

$$j\left(\overline{\cup_{\hat{D}\in\mathcal{D}_\eta(C_H)} \Lambda_{C_H}(\hat{D},t)}^{C_H}\right) = \overline{j(\cup_{\hat{D}\in\mathcal{D}_\eta(C_H)} \Lambda_{C_H}(\hat{D},t))}^{M_H^2},$$

thanks to the relatively compact character in C_H of the set $\cup_{\hat{D}\in\mathcal{D}_\eta(C_H)} \Lambda_{C_H}(\hat{D},t)$, which gives the required opposite inclusion. Therefore, if we coin a new universe $j(\mathcal{D}_F(C_H))$, we may put $j(\mathcal{A}_{\mathcal{D}_F(C_H)}) = \mathcal{A}_{j(\mathcal{D}_F(C_H))}$, another minimal pullback attractor in M_H^2.

6. Conclusions

In this research a version of the Ladyzhenskaya model including delay effects in the force has been studied. Actually two different frameworks are handled, depending on whether assumptions (**I**)-(**III**) are taken for the delay operator or also including (**IV**)-(**V**), namely $C_H := C([-h,0];H)$ or $M_H^2 := H \times L^2(-h,0;H)$ respectively. Existence of (global) weak solutions in both situations is established. It is not strictly necessary to manage under uniqueness conditions. After that, a dynamical analysis is performed, obtaining attractors. Our study focuses on non-autonomous terms, and we use pullback attractors for multi-valued processes. Of course the results also apply to the autonomous case, which would lead to global attractors. Some of main highlights of the work are the following:

(i) In the context of fluid dynamics flows, this paper extends previous studies on attractors for Navier–Stokes models with delays, having a unified presentation of different situations, types of results, assumptions and relations among them.
(ii) As a proper non-autonomous approach, in the C_H framework, it is remarked the maximal expected tempered value σ_{η_*} that can be considered (see Remarks 17, 23 and 24). We emphasize that the case $p > 2$ plays a crucial role for using any value of the parameter η, as large as required, in this non-autonomous framework.
(iii) Finally in Sections 4 and 5 the M_H^2 analysis is performed, obtaining attractors for both phase spaces; some relations among attractors are established, improving the characterization of attraction with respect to similar previous results.

CRediT authorship contribution statement

Heraclio Ledgar López-Lázaro: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Pedro Marín-Rubio:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Gabriela Planas:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

HLLL was partially supported by CAPES-Brazil, Finance Code 001, CNPq-Brazil, Unicamp/Edital DERI 031/2018 Programa Santander, FAPESP-Brazil grants #2021/01931-4 and #2022/13001-4, and IMUS.

PMR was partially supported by MCIU-AEI-Spain/FEDER under the projects PGC2018-096540-B-I00 and PID2021-122991NB-C21, US-1254251, and Junta de Andalucía/FEDER/US, project P18-FR-4509.

GP was partially supported by CNPq-Brazil grant 310274/2021-4 and FAPESP-Brazil grant 2019/02512-5.

References

[1] Ladyzhenskaya OA. New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems (Russian). Tr Mat Inst Steklova 1967;102:85–104, English translation in *Boundary Value Problems of Mathematical Physics V.*, AMS, Providence, Rhode Island, 1970.
[2] Ladyzhenskaya OA. Modifications of the Navier–Stokes equations for large gradients of the velocities (Russian). Zap Naučn Sem Leningrad Otdel Mat Inst Steklov (LOMI) 1968;7:126–54, English translation in *Boundary Value Problems of Mathematical Physics and Related Aspects of Function Theory, Part II*, 57–69, Consultants Bureau, New York, 1970.
[3] Ladyzhenskaya OA. The mathematical theory of viscous incompressible flow. New York: Gordon and Breach; 1969.
[4] Málek J, Nečas J, Rokyta M, Ružička M. Weak and measure-valued solutions to evolutionary PDEs. London: Chapman & Hall; 1996.

[5] Feireisl E, Pražák D. Asymptotic behavior of dynamical systems in fluid mechanics. Springfield, MO: American Institute of Mathematical Sciences (AIMS); 2010.
[6] Lions JL. Quelques méthodes de résolution des problèmes aux limites non lineaires. Paris: Dunod; 1969.
[7] Bulíček M, Ettwein F, Kaplický P, Pražák D. The dimension of the attractor for the 3D flow of a non-Newtonian fluid. Commun Pure Appl Anal 2009;8:1503–20.
[8] Málek J, Nečas J. A finite-dimensional attractor for three-dimensional flow of incompressible fluids. J Differential Equations 1996;127:498–518.
[9] Málek J, Pražák D. Finite fractal dimension of the global attractor for a class of non-Newtonian fluids. Appl Math Lett 2000;13:105–10.
[10] Pražák D, Žabenský J. On the dimension of the attractor for a perturbed $3d$ Ladyzhenskaya model. Cent Eur J Math 2013;11:1264–82.
[11] López-Lázaro HL, Marín-Rubio P, Planas G. Pullback attractors for non-Newtonian fluids with shear dependent viscosity. J Math Fluid Mech 2021;23:20.
[12] Picard ME. La mathématique dans ses rapports avec la physique. Nuovo Cim 1908;16:165–83.
[13] Patel VR, Ein-Mozaffari F, Upreti SR. Effect of time delays in characterizing the continuous mixing of non-Newtonian fluids in stirred-tank reactors. Chem Eng Res Des 2011;89:1919–28.
[14] García-Luengo J, Marín-Rubio P, Real J. Pullback attractors in V for non-autonomous 2D-Navier–Stokes equations and their tempered behaviour. J Differential Equations 2012;252:4333–56.
[15] García-Luengo J, Marín-Rubio P, Real J. Pullback attractors for 2D Navier–Stokes equations with delays and their regularity. Adv Nonlinear Stud 2013;13:331–57.
[16] García-Luengo J, Marín-Rubio P, Real J. Some new regularity results of pullback attractors for 2D Navier–Stokes equations with delays. Commun Pure Appl Anal 2015;14:1603–21.
[17] Marín-Rubio P, Real J. Pullback attractors for 2D-Navier–Stokes equations with delays in continuous and sub-linear operators. Discrete Contin Dyn Syst 2010;26:989–1006.
[18] Caraballo T, Real J. Navier–Stokes equations with delays. Proc R Soc Lond Ser A Math Phys Eng Sci 2001;457:2441–53.
[19] López-Lázaro HL. Asymptotic analysis to non-autonomous systems of incompressible non-Newtonian fluids (Ph.D. thesis), Campinas-Brazil: Universidade Estadual de Campinas; 2020.
[20] Simon J. On the existence of the pressure for solutions of the variational Navier–Stokes equations. J Math Fluid Mech 1999;1:225–34.
[21] Simon J. Équations de Navier–Stokes, cours de DEA 2002-2003. Clermont-Ferrand: Université Blaise Pascal; 2003.
[22] Tartar L. Topics in nonlinear analysis. Orsay: Publ. Math. Orsay; 1978.
[23] Chepyzhov VV, Vishik MI. Attractors for equations of mathematical physics. Providence, RI: American Mathematical Society; 2002.
[24] Hale JK, Verduyn Lunel SM. Introduction to functional differential equations. New York: Springer Verlag; 1993.
[25] Marín-Rubio P, Real J. On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems. Nonlinear Anal 2009;71:3956–63.
[26] Caraballo T, Herrera-Cobos M, Marín-Rubio P. Robustness of nonautonomous attractors for a family of nonlocal reaction–diffusion equations without uniqueness. Nonlinear Dynam 2016;84:35–50.
[27] Caraballo T, Herrera-Cobos M, Marín-Rubio P. Robustness of time-dependent attractors in H^1-norm for nonlocal problems. Discrete Contin Dyn Syst Ser B 2018;23:1011–36.
[28] Caraballo T, Kloeden PE. Non-autonomous attractors for integro-differential evolution equations. Discrete Contin Dyn Syst Ser S 2009;2:17–36.
[29] Melnik VS, Valero J. On attractors of multi-valued semi-flows and differential inclusions. Set-Valued Anal 1998;6:83–111.