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THE DIRICHLET AND SUBSTITUTION FORMULAS FOR RIEMANN-STIELTJES INTEGRALS IN BANACH SPACES

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An equality of the form

$$\int_a^b \left(\int_a^s d\alpha(t) \cdot h(t, s)\right) d\beta(s) = \int_a^b d\alpha(t) \left(\int_t^b h(t, s)\ d\beta(s)\right)$$

for iterated Riemann-Stieltjes integrals is called a *Dirichlet formula*.

An equality of the form

$$\int_a^b d\left(\int_a^t d\alpha(s) \cdot h(s)\right) g(t) = \int_a^b d\alpha(t) \cdot h(t)\ g(t)$$

for Riemann-Stieltjes integrals is called a *substitution formula*.

Without some restriction these formulas are not valid. In [4], p. 13 for instance, we give an example, with $\alpha(s) \equiv s$, where the formula of substitution is not true. A counterexample for the formula of Dirichlet is given in the Remark preceding Theorem 2.1.

In Secs. 2, 3, and 5 of this chapter we prove these formulas under adequate hypothesis and for different types of integrals (in Banach spaces). We need these formulas in our study of differential and integral equations with linear constraints (that is, generalized boundary conditions); see [1-4]. The different types of Riemann-Stieltjes integrals we use are necessary for the integral representation formulas of certain classes of linear operators (see [1 , Theorems 2, 3, and 4], [3, Theorem 1], and [4]); however neither the representing function α nor the function h or g is necessarily measurable (in the sense of Bochner and Lebesgue), and this fact makes their study more difficult. For instance, the identical automorphism of $C([a, b])$ has the representation

$$\varphi = \int_a^b \varphi(t)\ d\alpha(t), \quad \varphi \in C([a, b]),$$

where $\alpha(t) = \chi_{]a,t]}$, $t \in [a, b]$, and the function

$$\alpha: t \in [a, b] \longmapsto \chi_{]a,t]} \in L_\infty([a, b])$$

is not measurable (see [2, p. 59]).

In Sec. 4 we take α of bounded variation (hence measurable) and consider a more restricted class of functions h for which the analog of the Darboux criterion for integrability is valid (and which are measurable).

Sections 2 to 5 are independent of each other and may be read in any order; only Sec. 5 uses some results of Sec. 4.

1. BASIC DEFINITIONS

In order to make this discussion more self-sufficient we repeat here many of the definitions given in [4] (see also [3, 1, 2]).

DEFINITION 1.1. Given an interval $[a, b]$ of the real line, a *division* d of $[a, b]$ is a finite sequence $t_0 = a < t_1 < t_2 < \cdots < t_n = b$; we write $|d| = n$ and $\Delta d = \sup\{|t_i - t_{i-1}| \mid i = 1, 2, \ldots, |d|\}$. D or $D_{[a,b]}$ denotes the set of all divisions of $[a, b]$.

For every $\varepsilon > 0$ we define $D_\varepsilon = \{d \in D \mid \Delta d < \varepsilon\}$; the class $\{D_\varepsilon \mid \varepsilon > 0\}$ is a filter basis on D.

We say that a division d_2 is *finer* than a division d_1, and we write $d_1 \leq d_2$, if every point of d_1 is in d_2. For $d \in D$ we define $D_d = \{d' \in D \mid d \leq d'\}$; the class $\{D_d \mid d \in D\}$ is a filter basis on D finer than the preceding one and therefore we have

PROPOSITION 1.1. Given a topological space X and a function $f: D \to X$, the existence of $\lim_{\Delta d \to 0} f(d)$, i.e., according to the first filter basis, implies the existence of $\lim_{d \in D} f(d)$, i.e., according to the second filter basis, and we have the equality of both limits.

Given a Banach space X and a function $f: [a, b] \to X$, for every $d \in D$ we define the *oscillations*

$$\omega_i(f) = \sup\{\|f(t) - f(s)\| \mid t, s \in [t_{i-1}, t_i]\}$$

$$i = 1, 2, \ldots, |d|$$

and the *interior oscillations*

$$\dot{\omega}_i(f) = \sup\{\|f(t) - f(s)\| \mid t, s \in]t_{i-1}, t_i[\}$$

$$i = 1, 2, \ldots, |d|.$$

We write $\omega_d(f) = \sup\{\omega_i(f) \mid i = 1, 2, \ldots, |d|\}$ and $\dot{\omega}_d(f) = \sup\{\dot{\omega}_i(f) \mid i = 1, 2, \ldots, |d|\}$. Given $\delta > 0$ we define

$$\omega_\delta(f) = \sup\{\|f(t) - f(s)\| \mid t, s \in [a, b], |t - s| \leq \delta\}.$$

We denote by $\mathcal{D}$ the set of all pairs (d, ξ), where $d \in D$ and $\xi = (\xi_1, \ldots, \xi_{|d|})$ with $\xi_i \in [t_{i-1}, t_i]$. $\mathcal{D}^\cdot$ denotes the set of all pairs $(d, \xi^\cdot)$, where $d \in D$ and $\xi^\cdot = (\xi_1^\cdot, \ldots, \xi_{|d|}^\cdot)$ with $\xi_i^\cdot \in]t_{i-1}, t_i[$.

If t and s are real numbers, we write $t \wedge s = \inf(t, s)$ and $t \vee s = \sup(t, s)$. Given real numbers $c \leq d$, $|c, d|$ denotes any of the intervals $[c, d]$, $[c, d[$, $]c, d]$ and $]c, d[$. If A is a subset of $\mathbb{R}$, χ_A denotes its characteristic function: $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$.

DEFINITION 1.2. A *bilinear triple* (BT) is a set of three Banach spaces E, F, and G with a bilinear continuous mapping $B: E \times F \to G$ whose norm is ≤ 1; we write $x \cdot y = B(x, y)$ and denote the BT by (E, F, G).

Examples. Let W, X, and Y denote Banach spaces:

1. $E = L(X, Y)$, $F = L(W, X)$, $G = L(W, Y)$, and $B(v, u) = v \circ u$.

2. $E = L(X, Y)$, $F = X$, $G = Y$, and $B(u, x) = u(x)$; 2 is a particular case of 1: take $W = \mathbb{C}$.

3. $E = X'$, $F = X$, $G = \mathbb{C}$, and $B(x', x) = \langle x', x\rangle$; 3 is a particular case of 1: take $Y = W = \mathbb{C}$.

4. $E = G = Y$, $F = \mathbb{C}$, and $B(y, \lambda) = \lambda y$; 4 is a particular case of 1: take $X = W = \mathbb{C}$.

Given a BT (E, F, G) and functions $\alpha: [a, b] \to E$ and $f: [a, b] \to F$, for $(d, \xi) \in \mathcal{D}$ and $(d, \xi^\cdot) \in \mathcal{D}^\cdot$ we write respectively

$$\sigma_{d,\xi}(f; \alpha) = \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i)$$

and

$$\sigma_{d,\xi^\bullet}(f;\alpha) = \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i^\bullet)$$

and we define

$$\int_a^b d\alpha(t) \cdot f(t) = \lim_{\Delta d \to 0} \sigma_{d,\xi}(f;\alpha),$$

$$\oint_a^b d\alpha(t) \cdot f(t) = \lim_{d \in D} \sigma_{d,\xi}(f;\alpha),$$

$$\int_a^{\bullet b} d\alpha(t) \cdot f(t) = \lim_{d \in D} \sigma_{d,\xi^\bullet}(f;\alpha),$$

when these limits exist.

The first two integrals generalize the usual Riemann-Stieltjes integral (see [5] and also [6, 2, 4]) and the third one generalizes the Dushnik or *interior integral* (see [7, p. 96] and [3, 4]).

Given a function α: $[a, b] \to E$ we denote by $R_\alpha([a, b], F)$, $R_\alpha^-([a, b], F)$, and $R_\alpha^\bullet([a, b], F)$ the vector spaces of all functions f: $[a, b] \to F$ such that there exist, respectively, the integrals $\int_a^b d\alpha(t) \cdot f(t)$, $\oint_a^b d\alpha(t) \cdot f(t)$, and $\int_a^{\bullet b} d\alpha(t) \cdot f(t)$. For $\alpha(t) \equiv t$ we write $R([a, b], F)$ etc.

By Proposition 1.1 we have $R_\alpha([a, b], F) \subset R_\alpha^-([a, b], F) \subset R_\alpha^\bullet([a, b], F)$ with the equality of the corresponding integrals.

Given $d \in D$, *the existence of the integrals*

$$\int_{t_{i-1}}^{\bullet\, t_i} d\alpha(t) \cdot f(t), \quad i = 1, 2, \ldots, |d|,$$

implies the existence of

$$\int_a^{\bullet b} d\alpha(t) \cdot f(t) = \sum_{i=1}^{|d|} \int_{t_{i-1}}^{\bullet\, t_i} d\alpha(t) \cdot f(t);$$

for the integral $\oint_a^b$ *an analogous result is not true*; on the other hand, *for the interior integral* $\int_a^{\bullet b}$ *the integration by parts formula is not valid*, but we have

<u>THEOREM 1.2</u>. Let (E, F, G) be a BT and α: $[a, b] \to E$, f: $[a, b] \to F$ be bounded functions; if α or f is continuous, the existence of $\int_{\cdot\, a}^{b} d\alpha(t) \cdot f(t)$ implies the existence of the other two integrals and the three coincide (see [4, Theorem I.1.2]).

More generally we have:

<u>THEOREM 1.2^-</u>. Let (E, F, G) be a BT and α: $[a, b] \to E$, f: $[a, b] \to F$, bounded functions such that there exists $-\!\!\!\!\!\!\int_a^b d\alpha(t) \cdot f(t)$; if α and f have no common points of discontinuity then there exists

$$\int_a^b d\alpha(t) \cdot f(t) = -\!\!\!\!\!\!\int_a^b d\alpha(t) \cdot f(t).$$

<u>THEOREM $1.2^\bullet$</u>. Let (E, F, G) be a BT and α: $[a, b] \to E$, f: $[a, b] \to F$, bounded functions such that there exists $\int_{\cdot\, a}^{b} d\alpha(t) \cdot f(t)$; if α and f have no common points of left or right discontinuity then there exists

$$-\!\!\!\!\!\!\int_a^b d\alpha(t) \cdot f(t) = \int_{\cdot\, a}^{b} d\alpha(t) \cdot f(t).$$

In this chapter in general we give only results for the integrals $\int_a^b$ and $\int_{\cdot\, a}^{b}$; the corresponding theorems for $-\!\!\!\!\!\!\int_a^b$, and the proofs, are then obvious. Also, the proofs of the results for $\int_a^b$ in general follow from the corresponding results for $\int_{\cdot\, a}^{b}$, applying Theorem 1.2, and often they are much simpler than the direct proofs which do not use the interior integral.

Given $f \in R_\alpha([a, b], F)$, for every $(d, \xi) \in \mathcal{D}$ we define

$$f_{d,\xi} = f(\xi_1)\, \chi_{[a,t_1]} + \sum_{i=2}^{|d|} f(\xi_i)\, \chi_{]t_{i-1},t_i]};$$

we have

$$\sigma_{d,\xi}(f;\alpha) = \int_{\cdot\, a}^{b} d\alpha(t) \cdot f_{d,\xi}(t)$$

and

$$\int_a^b d\alpha(t) \cdot f(t) = \lim_{\Delta d \to 0} \int_a^{\bullet b} d\alpha(t) \cdot f_{d,\xi}(t).$$

More generally we have

PROPOSITION 1.3. If $f \in R_\alpha([a, b], F)$, for every $t \in [a, b]$ we have

$$\int_a^t d\alpha(s) \cdot f(s) = \lim_{\Delta d \to 0} \int_a^{\bullet t} d\alpha(s) \cdot f_{d,\xi}(s).$$

Given $f \in R_\alpha^\bullet([a, b], F)$, for every $(d, \xi^\bullet) \in \mathcal{D}^\bullet$ we define

$$f_{d,\xi^\bullet} = f(\xi_1^\bullet)\, \chi_{[a,t_1]} + \sum_{i=2}^{|d|} f(\xi_i^\bullet)\, \chi_{]t_{i-1},t_i]};$$

we have

$$\sigma_{d,\xi^\bullet}(f; \alpha) = \int_a^{\bullet b} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)$$

and

$$\int_a^{\bullet b} d\alpha(t) \cdot f(t) = \lim_{d \in D} \int_a^{\bullet b} d\alpha(t) \cdot f_{d,\xi^\bullet}(t).$$

In an analogous way we have

PROPOSITION 1.3$^\bullet$. If $f \in R_\alpha^\bullet([a, b], F)$, for every $t \in [a, b]$ we have

$$\int_a^{\bullet t} d\alpha(s) \cdot f(s) = \lim_{d \in D} \int_a^{\bullet t} d\alpha(s) \cdot f_{d,\xi^\bullet}(s).$$

DEFINITION 1.3. Given a parallelotope $I = \Pi_{1 \le j \le n} [a_j, b_j] \subset \mathbb{R}^n$ and a Banach space X, $E(I, X)$ denotes the vector space of all finite linear combinations of characteristic functions of parallelotopes $\Pi_{1\leq j\leq n}\,[c_j, d_j]$ contained in I (*step functions*). G(I, X) denotes the vector space of all *regulated functions*, i. e., functions that are uniform limits of step functions. Endowed with the norm $g \in G(I, X) \longmapsto \|g\| = \sup_{t\in I}\|g(t)\|$, G(I, X) is a Banach space.

It is not difficult to prove the following theorem.

THEOREM 1.4. For $f\colon [a, b] \to X$ the following properties are equivalent:

(a) $f \in G([a, b], X)$.

(b) For every $t \in [a, b[$ there exists $f(t+)$ and for every $t \in]a, b]$ there exists $f(t-)$.

(c) For every $\varepsilon > 0$ there exists $d \in D$ such that $\omega_d^{\cdot}(f) < \varepsilon$. See, for instance, [4], Theorem I.3.1.

We say that functions $f, g \in G([a, b], X)$ are equivalent if for every $t \in]a, b[$ we have $f(t+) = g(t+)$ [and hence also $f(t-) = g(t-)$]; we denote by $\tilde{G}([a, b], X)$ the Banach space of these equivalence classes.

PROPOSITION 1.5. Given a BT (E, F, G), $f \in G([a, b], E)$ and $g \in G([a, b], F)$, we have $f \cdot g \in G([a, b], G)$.

DEFINITION 1.4. Given a BT (E, F, G) and $\alpha\colon [a, b] \to E$ we define the B *variation* of α (where B denotes the bilinear mapping of the BT) by $SB[\alpha] = \sup_{d\in D} SB_d[\alpha]$, where

$$SB_d[\alpha] = \sup\left\{\left\|\sum_{i=1}^{|d|}\left[\alpha(t_i) - \alpha(t_{i-1})\right]\cdot y_i\right\| \;\middle|\; y_i \in F,\ \|y_i\| \leq 1\right\}.$$

SB([a, b], E) denotes the vector space of all functions of "bounded" (i.e., finite) B variation.

Examples.

1. In the case of Examples 1 and 2 of a BT we obtain the same space (see [4, Theorem I.4.4]), which we denote by SV([a, b], L(X, Y)), and we call its elements *functions of bounded semivariation*; in this case $SV[\alpha] = SB[\alpha]$ is called the *semivariation* of α.

2. In the case of Example 3 of a BT we obtain the space BV([a, b], X') of functions of bounded variation and we have $V[\alpha] = SB[\alpha]$, where $V[\alpha] = V_{[a,b]}[\alpha]$ denotes the *variation* of α (on [a, b]): $V[\alpha] = \sup_{d \in D} V_d[\alpha]$ with $V_d[\alpha] = \Sigma_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\|$.

3. In the case of Example 4 of a BT we obtain the space BW([a, b], Y) of *functions of weak bounded variation*; in this case $W[\alpha] = SB[\alpha]$ is called the *weak variation* of α. See [2, 4, 5, or 6].

THEOREM 1.6. Let (E, F, G) be a BT and $\alpha \in SB([a, b], E)$;

(a) For every $f \in C([a, b], F)$ there exists $F_\alpha[f] = \int_a^b d\alpha(t) \cdot f(t)$; we have $F_\alpha \in L[C([a, b], F), G]$ with $\|F_\alpha\| \leq SB[\alpha]$.

(b) For $(d, \xi) \in \mathcal{D}$ we have $\|\int_a^b d\alpha(t) \cdot f(t) - \sigma_{d,\xi}(f; \alpha)\| \leq SB[\alpha]\ \omega_{\Delta d}(f)$.

(See [2, Theorem I.1.5].)

THEOREM 1.6'. Let (E, F, G) be a BT and $\alpha \in SB([a, b], E)$;

(a) For every $f \in G([a, b], F)$ there exists $F_\alpha[f] = \int_a^b d\alpha(t) \cdot f(t)$; we have $\|F_\alpha[f]\| \leq SB[\alpha]\|f\|$.

(b) $F_\alpha[f]$ depends only on the class of f in $\tilde{G}([a, b], F)$ and hence $F_\alpha \in L[\tilde{G}([a, b], X), Y]$ with $\|F_\alpha\| \leq SB[\alpha]$.

(c) For every $(d, \xi^\cdot) \in \mathcal{D}^\cdot$ we have $\|\int_a^{b\,\bullet} d\alpha(t) \cdot f(t) - \sigma_{d,\xi^\cdot}(f; \alpha)\| \leq SB[\alpha]\ \omega_d^\cdot(f)$.

See [4, Theorem I.4.12].

DEFINITION 1.5. A Banach space system with an associative system of bilinear continuous mappings, or shortly, a *bilinear associative system* (BAS) is a system of six Banach spaces E_1, E_2, E_3, E_{12}, E_{23}, and E_{123} with four bilinear continuous mappings $B^{12}: E_1 \times E_2 \to E_{12}$, $B^{23}: E_2 \times E_3 \to E_{23}$, $B^{12,3}: E_{12} \times E_3 \to E_{123}$, $B^{1,23}: E_1 \times E_{23} \to E_{123}$ such that for every $x \in E_1$, $y \in E_2$, and $z \in E_3$ we have the "associative" relation $B^{1,23}[x, B^{23}(y, z)] = B^{12,3}[B^{12}(x, y), z]$. Unless otherwise stated we suppose that the bilinear mappings have norm ≤ 1; we write $x \cdot y = B^{12}(x, y)$, $x \cdot yz = B^{1,23}[x, B^{23}(y, z)]$, etc., and we write $(E_1, \ldots, E_{123})$ to denote the BAS.

Examples.

1. W, X, Y, and Z are Banach spaces; $E_1 = L(Y, Z)$, $E_2 = L(X, Y)$, $E_3 = L(W, X)$, $E_{12} = L(X, Z)$, $E_{23} = L(W, X)$, and $E_{123} = L(W, Z)$; the bilinear mappings are the natural compositions.

2. X, Y, and Z are Banach spaces; $E_1 = L(Y, Z)$, $E_2 = L(X, Y)$, $E_3 = X$, $E_{12} = L(X, Z)$, $E_{23} = Y$, and $E_{123} = Z$; the bilinear mappings are the natural ones; 2 is a particular case of 1: take $W = \mathbb{C}$ etc.

3. All spaces are equal to a Banach algebra A [for instance, $L(X)$] the bilinear mapping being the product.

4. A BT (E, F, G) is a particular case of a BAS: we take $E_1 = E$, $E_2 = \mathbb{C}$, $E_3 = F$, $E_{12} = E$, $E_{23} = F$, and $E_{123} = G$; $B^{12}(x, \lambda) = \lambda x$, $B^{23}(\lambda, y) = \lambda y$, $B^{1,23} = B = B^{12,3}$.

5. If (E, F, G) is a BT we may define the following BAS: $E_1 = E$, $E_2 = F$, $E_3 = G'$, $E_{12} = G$, $E_{23} = E'$, $E_{123} = \mathbb{C}$ and $B^{12}(x, y) = x \cdot y$; $B^{23}(y, z')$ is the element $x' \in E'$ such that $\langle x, x'\rangle = \langle xy, z'\rangle$ for every $x \in E$ (since the linear form $x \in E \longmapsto \langle xy, z'\rangle \in \mathbb{C}$ is continuous, x' is well-defined); we write $x' = yz' = B^{23}(y, z')$; $B^{12,3}(z, z') = \langle z, z'\rangle$, $B^{1,23}(x, x') = \langle x, x'\rangle$; it is immediate that $\|yz'\| \leq \|y\|\|z'\|$, i.e., $\|B^{23}\| \leq 1$.

<u>Remark</u>. For certain theorems one needs systems with four or even five base spaces E_i. The definitions in these cases are obvious; in the case n = 4, for instance, we write the BAS $(E_1, \ldots, E_{1234})$; it is formed by ten spaces, eight bilinear mappings, and four "associative" equalities.

<u>THEOREM 1.7</u>. Let $(E_1, \ldots, E_{123})$ be a BAS that satisfies the following property:

(*) For every $u \in E_{12}$ we have $\|u\| = \sup\{\|B^{12,3}(u, z)\| \mid \|z\| \leq 1\}$.

Then if $\alpha \in SB^{1,23}([a, b], E_1)$, we have $\alpha \in SB^{12}([a, b], E_2)$ and $SB^{12}[\alpha] \leq SB^{1,23}[\alpha]$.

<u>Proof</u>. For every $d \in D$ we have

$$SB_d^{12}[\alpha]$$

$$= \sup\left\{\left\|\sum_{i=1}^{|d|}\left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot x_i^2\right\| \mid x_i^2 \in E_2, \|x_i^2\| \leq 1\right\}$$

$$= \sup\left\{\sup_{\|x^3\|\leq 1}\left\|\left[\sum_{i=1}^{|d|}\left[\alpha(t_i) - \alpha(t_{i-1})\right]x_i^2\right]x^3\right\| \mid x_i^2 \in E_2, \|x_i^2\| \leq 1\right\}$$

$$\leq \sup\left\{\left\|\sum_{i=1}^{|d|}\left[\alpha(t_i) - \alpha(t_{i-1})\right]x_i^{23}\right\| \mid x_i^{23} \in E_{23}, \|x_i^{23}\| \leq 1\right\}$$

$$= SB_d^{1,23}[\alpha].$$

The notion of BAS allows us to unify in a natural way many different situations. Let us also mention that all results of Chapter II of [4], particularly the Dirichlet and substitution formulas, extended trivially to BAS that satisfy the property (*) of Theorem 1.7.

There are also results for BT whose proof is very much simplified if we consider them as part of an adequate BAS. For instance, all results of Chapter II of [4] are still true if instead of functions of bounded semivariation (and the semivariation norm) we consider functions of bounded variation (and the variation norm). The verification of this assertion is trivial but for Lemma 1.2, whose proof does not change if we use an adequate BAS (see Theorem 1.8 that follows).

THEOREM 1.8. Let (E, F, G) be a BT, $\alpha \in BV([c, d], E)$, $h: [c, d] \times [a, b] \to F$ a function that is regulated as a function of the first variable and that is uniformly of bounded variation as a function of the second variable (i. e., for every $t \in [c,d]$ we have $h^t \in BV([a, b], F)$ and $\sup_{c\leq t\leq d} V[h^t] < \infty$); for every $s \in [a, b]$ we define $\overline{h}(s) = \int_c^d d\alpha(t) \cdot h(t, s)$. Then we have $\overline{h} \in BV([a, b], G)$ and $V[\overline{h}] \leq V[\alpha] \sup_{c\leq t\leq d} V[h^t]$.

Proof. By Theorem 1.6', $\overline{h}$ is well-defined (since $SB[\alpha] \leq V[\alpha]$). Let us take the BAS associated to the BT (E, F, G) as in Example 5. It is immediate that for $d \in D_{[a,b]}$ we have

$$V_d[\overline{h}]$$

$$= \sum_{i=1}^{|d|} \|\overline{h}(s_i) - \overline{h}(s_{i-1})\| = \sup_{\|z_i'\|\leq 1} \sum_{i=1}^{|d|} |\langle \overline{h}(s_i) - \overline{h}(s_{i-1}), z_i' \rangle|$$

$$= \sup_{\|z_i'\|\leq 1} \left| \sum_{i=1}^{|d|} \langle \overline{h}(s_i) - \overline{h}(s_{i-1}), z_i' \rangle \right| \quad \text{where } z_i' \in G'.$$

Given $z_i' \in G'$ with $\|z_i'\| \leq 1$, $i = 1, 2, \ldots, |d|$, we have

$$\left|\sum_{i=1}^{|d|} \langle \bar{h}(s_i) - \bar{h}(s_{i-1}), z_i' \rangle\right| =$$

$$= \left|\sum_{i=1}^{|d|} \left\langle \int_c^{d\bullet} d\alpha(t) \cdot \left[h(t, s_i) - h(t, s_{i-1})\right], z_i' \right\rangle\right|$$

$$= \left|\sum_{i=1}^{|d|} \int_c^d \left\langle d\alpha(t), \left[h(t, s_i) - h(t, s_{i-1})\right] z_i' \right\rangle\right|$$

$$= \left|\int_c^d \left\langle d\alpha(t), \sum_{i=1}^{|d|} \left[h(t, s_i) - h(t, s_{i-1})\right] z_i' \right\rangle\right|$$

$$\leq V[\alpha] \sup_{c \leq t \leq d} \left\| \sum_{i=1}^{|d|} \left[h(t, s_i) - h(t, s_{i-1})\right] z_i' \right\|$$

$$\leq V[\alpha] \sup_{c \leq t \leq d} \sum_{i=1}^{|d|} \|h(t, s_i) - h(t, s_{i-1})\| \leq V[\alpha] \sup_{c \leq t \leq d} V[h^t],$$

hence the result.

Remark. By the application of (c) of Theorem 1.6' one may also prove Theorem 1.8 without the consideration of BAS.

2. THE DIRICHLET FORMULA

THEOREM 2.1'. Let $(E_1, \ldots, E_{123})$ be a BAS that satisfies the property (*) of Theorem 1.7; for $\alpha \in SB^{1,23}([a, b], E_1)$, $h \in G([a, b] \times [a, b], E_2)$, and $\beta \in BV\,C([a, b], E_3) = BV([a, b], E_3) \cap C([a, b], E_3)$ we have

(D')
$$\int_a^b \left[\int_a^{s\bullet} d\alpha(t) \cdot h(t, s)\right] d\beta(s)$$
$$= \int_a^{b\bullet} d\alpha(t) \left[\int_t^b h(t, s)\, d\beta(s)\right].$$

Proof. We will show that both members of (D$^\cdot$) are well defined and that they depend continuously on h; thereafter it will be enough to prove the equality when h is a step function.

1a. We define $\bar{h}(t) = \int_t^b h(t, s)\, d\beta(s)$; by Theorem 1.6$^\cdot$ and by Theorem 1.2, for every $t \in [a, b]$ this integral is well defined and we have $\|\bar{h}(t)\| \leq \|h\|\, V[\beta]$. Therefore the mapping

$$h \in G([a, b] \times [a, b], E_2) \longmapsto \bar{h} \in B([a, b], E_{23})$$

is continuous. ($B([a, b], X)$ denotes the Banach space of all *bounded* functions $f: [a, b] \to X$). Let us prove that $\bar{h} \in G([a, b], E_{23})$; it is enough to do it when h is a step function because the general case follows by uniform convergence. Any step function is a finite linear combination of functions of the form $h = \chi_{|a,c|} \otimes \chi_{|a,d|} y$, and for this function we have

$$\bar{h}(t) = \chi_{|a,c|}(t) \int_t^b \chi_{|a,d|}(s)\, y\, d\beta(s)$$

$$= \chi_{|a,c|}(t)\, y\left[\beta(b) - \beta(t \vee d)\right],$$

which is really regulated.

1b. The mapping $g \in G([a, b], E_{23}) \longmapsto \int_a^{\bullet b} d\alpha(t) \cdot g(t) \in E_{123}$ is well defined and continuous, since $\alpha \in SB^{1,23}([a, b], E_1)$; hence the composed mapping

$$h \in G([a, b] \times [a, b], E_2) \longmapsto \int_a^{\bullet b} d\alpha(t) \left[\int_t^b h(t, s)\, d\beta(s)\right] \in E_{123}$$

is well defined and continuous.

Remark. 1a. and 1b remain true if we suppose only $\beta \in BV([a, b], E_3)$ and replace $\int_t^b$ by $\int_t^{\bullet b}$.

2a. We define $\tilde{h}(s) = \int_a^{\bullet s} d\alpha(t) \cdot h(t, s)$; by Theorem 1.6$^\cdot$ this integral is well defined, since $\alpha \in SB^{12}([a, b], E_1)$ by

Theorem 1.7. We have $\|\tilde{h}(s)\| \leq \|h\|\ SB^{12}[\alpha] \leq \|h\|\ SB^{1,23}[\alpha]$ and therefore the linear mapping

$$h \in G([a, b] \times [a, b], E_2) \longmapsto \tilde{h} \in B([a, b], E_{12})$$

is continuous.

2b. We will now prove that for every $h \in G([a, b] \times [a, b], E_2)$ the integral $\int_a^b \tilde{h}(s)\, d\beta(s)$ is well defined and depends continuously on h. Let $h_n \in E([a, b] \times [a, b], E_2)$ be a sequence of step functions that converges uniformly to h. Let us first prove that there exists

$$\int_a^b \tilde{h}_n(s)\, d\beta(s) = \int_a^b \left[\int_a^s \cdot\, d\alpha(t) \cdot h_n(t, s)\right] d\beta(s).$$

It is enough to show it for $h_n = \chi_{]a,c]} \otimes \chi_{]a,d]} y$; in this case we have

$$\int_a^b \chi_{]a,d]}(s) \left[\int_a^s \cdot\, d\alpha(t) \cdot y\, \chi_{]a,c]}(t)\right] d\beta(s)$$

$$= \int_a^b \chi_{]a,d]}(s) \left(\alpha(s \wedge c) - \alpha(a)\right) \cdot y\, d\beta(s)$$

$$= \int_a^d \left(\alpha(s \wedge c) - \alpha(s)\right) \cdot y\, d\beta(s),$$

and β *being continuous* we see, *using integration by parts*, that this integral exists because $\int_a^{c\wedge d} d\alpha(s) \cdot y\beta(s)$ exists.

From 2a it follows that $\tilde{h}_n$ tends uniformly to h [and that the sequence $\int_a^b \tilde{h}_n(s)\, d\beta(s)$ converges]. We will now prove that the integral $\int_a^b \tilde{h}(s)\, d\beta(s)$ exists: we have to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for $d, \bar{d} \in D$ with $\Delta d, \Delta\bar{d} < \delta$ we have

$$\|\sigma_{d,\xi}(\tilde{h}; \beta) - \sigma_{\bar{d},\bar{\xi}}(\tilde{h}; \beta)\| < \varepsilon.$$

We have

$$\|\sigma_{d,\xi}(\tilde{h};\ \beta) - \sigma_{\bar{d},\bar{\xi}}(\tilde{h};\ \beta)\|$$
$$\leq \|\sigma_{d,\xi}(\tilde{h} - \tilde{h}_n;\ \beta) - \sigma_{\bar{d},\bar{\xi}}(\tilde{h} - \tilde{h}_n;\ \beta)\|$$
$$+ \|\sigma_{d,\xi}(\tilde{h}_n;\ \beta) - \sigma_{\bar{d},\bar{\xi}}(\tilde{h}_n;\ \beta)\|.$$

The first summand is $\leq 2\|\tilde{h} - \tilde{h}_n\|\ V[\beta]$ and becomes $< \varepsilon/2$ for n sufficiently large because $\tilde{h}_n$ converges uniformly to $\tilde{h}$; if we fix such an n, there exists a $\delta > 0$ such that for Δd, $\Delta\bar{d} < \delta$ the second summand becomes smaller than $\varepsilon/2$ because we already proved that there exists $\int_a^b \tilde{h}_n(s)\ d\beta(s)$.

From $\|\int_a^b \tilde{h}(s)\ d\beta(s)\| \leq \|\tilde{h}\|\ V[\beta]$ it follows that the integral depends continuously on $\tilde{h}$; by 2a it follows, through compositions, that the linear mapping

$$h \in G([a,\ b] \times [a,\ b],\ E_2) \longmapsto \int_a^b \left[\int_a^s \cdot d\alpha(t) \cdot h(t,\ s)\right] d\beta(s) \in E_{123}$$

is continuous.

3. By our initial remarks it is now enough to prove the Dirichlet formula (D') for $h = \chi_{]a,c]} \otimes \chi_{]a,d]} y$. We have

$$\int_a^b \left[\int_a^s \cdot\ d\alpha(t) \cdot y\ \chi_{]a,c]}(t)\ \chi_{]a,d]}(s)\right]\ d\beta(s)$$
$$= \int_a^{c\wedge d} \left[\int_a^s \cdot\ d\alpha(t) \cdot y\right]\ d\beta(s)$$
$$= \int_a^{c\wedge d} \left(\alpha(s) - \alpha(a)\right)\ d\beta(s)$$

and

$$\int_a^b \cdot\ d\alpha(t)\ \left[\int_t^b \chi_{]a,c]}(t)\ \chi_{]a,d]}(s)\ y\ d\beta(s)\right]$$
$$= \int_a^c d\alpha(t) \cdot y\left(\beta(c \wedge d) - \beta(t \wedge d)\right)$$
$$= \int_a^{c\wedge d} d\alpha(t) \cdot y\left(\beta(c \wedge d) - \beta(t)\right),$$

and the equality of these two integrals follows through integration by parts.

Remark. Theorem 2.1' remains true for $\beta \in BV([a, b], E_3)$ if we suppose that α is also regulated and if we replace $\int$ by $\int^{\bullet}$ (under these hypothesis $\tilde{h}$ is regulated: it is enough to prove it for $h = \chi_{|a,c|} \otimes \chi_{|a,d|} y$, and this is immediate).

Without these restrictions the Dirichlet formula may not be valid.

Example. We take $\alpha \in BW([a, b], L_\infty([a, b]))$ defined by $\alpha(t) = \chi_{[a,t]}$ and $\beta = \chi_{[c,d]} \in BV([a, b])$, where $a < c \leq b$. We do not have $\int_a^{\bullet b} [\int_a^{\bullet s} d\alpha(t)] \, d\beta(s) = \int_a^{\bullet b} d\alpha(t) \int_t^{\bullet b} d\beta(s)$, i.e., $\int_a^{\bullet b} [\alpha(s) - \alpha(a)] \, d\beta(s) = \int_a^{\bullet b} d\alpha(t)[\beta(b) - \beta(t)]$ because $\int_a^{\bullet b} \alpha(s) \, d\beta(s)$ does not exist: it should be $\alpha(c-)$, and this limit does not exist since for $t' < t'' < c$ we have $\|\alpha(t'') - \alpha(t')\| = \|\chi_{]t',t'']}\| = 1$.

By the way, if the integration by parts formula

$$\int_a^{\bullet b} d\alpha(t) \cdot \beta(t) = \alpha(b) \cdot \beta(b) - \alpha(a) \cdot \beta(a) - \int_a^{\bullet b} \alpha(t) \cdot d\beta(t)$$

is not valid for two functions α and β, then neither is the Dirichlet formula

$$\int_a^{\bullet b} \left[\int_a^s d\alpha(t)\right] d\beta(s) = \int_a^{\bullet b} d\alpha(t) \int_t^b d\beta(s)$$

that is equivalent to it.

THEOREM 2.1. Let $(E_1, \ldots, E_{123})$ be a BAS that satisfies the property (*) of Theorem 1.7; for $\alpha \in SB^{1,23}([a, b], E_1)$, $h \in C([a, b] \times [a, b], E_2)$, and $\beta \in BV\,C([a, b], E_3)$ we have

$$\text{(D)} \qquad \int_a^b \left[\int_a^s d\alpha(t) \cdot h(t, s)\right] d\beta(s) = \int_a^b d\alpha(t) \left[\int_t^b h(t, s) \, d\beta(s)\right].$$

<u>Proof</u>. The proof follows immediately from Theorem 2.1$^{\bullet}$ by Theorem 1.2 (since under the hypothesis of Theorem 2.1 the function $t \in [a, b] \longmapsto \int_t^b h(t, s)\, d\beta(s) \in E_{23}$ is continuous, as it is easy to see).

<u>Remark</u>. (D) remains true for $\beta \in BV([a, b], E_3)$ if α is also continuous, as follows from the remark after Theorem 2.1$^{\bullet}$ (under these hypothesis $\tilde{h}$ is continuous).

<u>THEOREM 2.2$^{\bullet}$</u>. Let $(E_1, \ldots, E_{123})$ be a BAS that satisfies the property (*) of Theorem 1.7; for $\alpha \in SB^{1,23}([a, b], E_1)$, $h \in G([a, b], E_2)$ and $\beta \in BV\, C([a, b], E_3)$ we have

$$\int_a^b d_s\left[\int_a^{s} {}^{\bullet}\, d\alpha(t) \cdot h(t)\right] \beta(s) = \int_a^b {}^{\bullet}\, d\alpha(s) \cdot h(s)\, \beta(s).$$

<u>Proof</u>. By Theorem 2.1$^{\bullet}$ we have

$$\int_a^b \left[\int_a^{s} {}^{\bullet}\, d\alpha(t) \cdot h(t)\right] d\beta(s) = \int_a^b {}^{\bullet}\, d\alpha(t) \left[\int_t^b h(t)\, d\beta(s)\right].$$

Integrating by parts we get

$$\int_a^b {}^{\bullet}\, d\alpha(t) \cdot h(t)\, \beta(b) - \int_a^b d_s\left[\int_a^{s} {}^{\bullet}\, d\alpha(t) \cdot h(t)\right] \beta(s)$$

$$= \int_a^b {}^{\bullet}\, d\alpha(t) \cdot h(t) \left(\beta(b) - \beta(t)\right)$$

and hence the result.

<u>THEOREM 2.2</u>. Let $(E_1, \ldots, E_{123})$ be a BAS that satisfies the property (*) of Theorem 1.7; for $\alpha \in SB^{1,23}([a, b], E_1)$, $h \in C([a, b], E_2)$, and $\beta \in BV\, C([a, b], E_3)$ we have

$$\int_a^b d_s\left[\int_a^{s} d\alpha(t) \cdot h(t)\right] \beta(s) = \int_a^b d\alpha(s) \cdot h(s)\, \beta(s).$$

Proof. The proof follows from Theorem 2.2˙ by Theorem 1.2.

Along the same line we have the following results for iterated integrals.

THEOREM 2.3˙. Let $(E_1, \ldots, E_{123})$ be a BAS that satisfies the property (*) of Theorem 1.7; for $\alpha \in SB^{1,23}([c, d], E_1)$, $h \in G([c, d] \times [a, b], E_2)$, and $\beta \in BV([a, b], E_3)$ we have

$$\int_a^{\bullet b} \left[\int_c^{\bullet d} d\alpha(t) \cdot h(t, s)\right] d\beta(s) = \int_c^{\bullet d} d\alpha(t) \left[\int_a^{\bullet b} h(t, s)\ d\beta(s)\right].$$

The proof follows the steps of the proof of Theorem 2.1˙ with some simplifications, mainly in the part corresponding to 2b since now the function $s \in [a, b] \longmapsto \int_c^{\bullet d} d\alpha(t) \cdot h(t, s) \in E_{12}$ is regulated.

Remark. Theorem 2.1˙ cannot be deduced from Theorem 2.3˙ since the function $(t, s) \in [a, b] \times [a, b] \longmapsto Y(s - t)h(t, s) \in E_2$ is, in general, not regulated!

THEOREM 2.3. Let $(E_1, \ldots, E_{123})$ be a BAS that satisfies the property (*) of Theorem 1.7; for $\alpha \in SB^{1,23}([c, d], E_1)$, $h \in C([c, d] \times [a, b], E_2)$ and $\beta \in BV([a, b], E_3)$ and we have

$$\int_a^b \left[\int_c^d d\alpha(t) \cdot h(t, s)\right] d\beta(s) = \int_c^d d\alpha(t)\left[\int_a^b h(t, s)\ d\beta(s)\right].$$

Proof. Since the functions

$$s \in [a, b] \longmapsto \int_c^d d\alpha(t) \cdot h(t, s) \in E_{12}$$

and

$$t \in [c, d] \longmapsto \int_a^b h(t, s)\, d\beta(s) \in E_{23}$$

are continuous the result follows from Theorem 2.3˙ by Theorem 1.2.

3. THE SUBSTITUTION FORMULA (I)

The purpose of this section is to prove the following versions of the substitution formula:

<u>THEOREM 3.6˙</u>. Let $(E_1, \ldots, E_{123})$ be a BAS, $\alpha \in SB^{1,23}([a,b], E_1)$, $f \in R_\alpha^{\bullet}([a, b], E_2)$, and $g \in G([a, b], E_3)$; we have

$$(S^{\bullet}) \qquad \int_a^{\bullet b} d_t\left[\int_a^{\bullet t} d\alpha(s) \cdot f(s)\right] g(t) = \int_a^{\bullet b} d\alpha(t) \cdot f(t)\, g(t).$$

<u>THEOREM 3.6</u>. Let $(E_1, \ldots, E_{123})$ be a BAS, $\alpha \in SB^{1,23}([a,b], E_1)$, $f \in R_\alpha([a, b], E_2)$, and $g \in C([a, b], E_3)$; we have

$$(S) \qquad \int_a^b d_t\left[\int_a^t d\alpha(s) \cdot f(s)\right] g(t) = \int_a^b d\alpha(t) \cdot f(t)\, g(t).$$

In order to prove these theorems we need several preliminary results.

<u>THEOREM 3.1˙</u>. Under the hypothesis of Theorem 3.6˙ we have

(a) $I_f \in SB^{12,3}([a, b], E_{12})$ where $I_f(t) = \int_a^{\bullet t} d\alpha(s) \cdot f(s)$.

(b) $SB^{12,3}[I_f] \leq \|f\|\, SB^{1,23}[\alpha]$.

(c) There exists $\int_a^{\bullet b} d_t[\int_a^{\bullet t} d\alpha(s) \cdot f(s)]\, g(t)$.

Proof. We first prove (b). Given $d \in D$ and $z_i \in E_3$ with $\|z_i\| \leq 1$, we have

$$\left\| \sum_{i=1}^{|d|} [I_f(t_i) - I_f(t_{i-1})] z_i \right\|$$

$$= \left\| \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i} d\alpha(t) \cdot f(t)\, z_i \right\|$$

$$= \left\| \int_a^b d\alpha(t) \cdot f_z(t) \right\|$$

$$\leq \|f_z\|\, SB^{1,23}[\alpha],$$

since we have obviously $f_z \in R_\alpha^{\cdot}([a, b], E_3)$, where $f_z(t) = f(t)z_i$ for $t \in]t_{i-1}, t_i[$; hence the result, since $\|f_z\| \leq \|f\|$. Part (a) follows from (b), and (c) follows from (a).

By Theorem 1.2, Theorem 3.1$^{\cdot}$ implies the following:

THEOREM 3.1. Under the hypothesis of Theorem 3.6 we have

(a) $I_f \in SB^{12,3}([a, b], E_{12})$, where $I_f(t) = \int_a^t d\alpha(s) \cdot f(s)$.

(b) $SB^{12,3}[I_f] \leq \|f\|\, SB^{1,23}[\alpha]$.

(c) There exists $\int_a^b d_t[\int_a^t d\alpha(s) \cdot f(s)]\, g(t)$.

THEOREM 3.2$^{\cdot}$. With the notations of Theorem 3.6$^{\cdot}$, if $f \in G([a, b], E_2)$ and if $g: [a, b] \to E_3$ is such that one of the integrals in $(S^{\cdot})$ exists, then so does the other and we have $(S^{\cdot})$.

Proof. Let us take approximating sums for both integrals in $(S^{\cdot})$; for $(d, \xi^{\cdot}) \in \mathcal{D}^{\cdot}$ we have

$$\left\| \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i} \!\!\!\bullet\; d\alpha(s) f(s) \cdot g(\xi_i^\bullet) - \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i^\bullet) g(\xi_i^\bullet) \right\|$$

$$= \left\| \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i} \!\!\!\bullet\; d\alpha(s) \cdot \left[f(s) - f(\xi_i^\bullet)\right] g(\xi_i^\bullet) \right\|$$

$$\leq SB^{1,23}[\alpha]\ \omega_d^\bullet(f)\ \|g\|,$$

hence the result by (c) of Theorem 1.4.

THEOREM 3.2. With the notations of Theorem 3.6, if $f \in C([a, b], E_2)$ and if $g: [a, b] \to E_3$ is such that one of the integrals in (S) exists, then so does the other and we have (S).

Proof. Let us take approximating sums for both integrals in (S); for $(d, \xi) \in \mathcal{D}$ we have

$$\left\| \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i} d\alpha(s) f(s) \cdot g(\xi_i) - \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i) g(\xi_i) \right\|$$

$$= \left\| \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i} d\alpha(s) \left[f(s) - f(\xi_i)\right] \cdot g(\xi_i) \right\|$$

$$\leq SB^{1,23}[\alpha]\ \omega_{\Delta d}(f)\ \|g\|;$$

hence the result because $\omega_{\Delta d}(f) \to 0$ if $\Delta d \to 0$, since f is uniformly continuous in $[a, b]$.

LEMMA 3.3•. Under the hypothesis of Theorem 3.6• (S•) is true if $f \in G([a, b], E_2)$.

Proof. By Proposition 1.5 the second integral in (S•) exists; by Theorem 3.1• the first one exists, hence the result by Theorem 3.2•.

LEMMA 3.3. Under the hypothesis of Theorem 3.6 (S) is true if $f \in C([a, b], E_2)$.

Proof. The proof follows from Lemma 3.3˙ by Theorem 1.2.

PROPOSITION 3.4˙. Under the hypothesis of Theorem 3.6˙ we have

$$\int_a^{b\bullet} d_t\left[\int_a^{t\bullet} d\alpha(s) \cdot f(s)\right] g(t)$$

$$= \lim_{d\in D} \int_a^{b\bullet} d_t\left[\int_a^{t\bullet} d\alpha(s) \cdot f_{d,\xi^\bullet}(s)\right] g(t).$$

Proof. $f_{d,\xi^\bullet}$ has been defined in Sec. 1. By Theorem 3.1˙ the first integral exists; we define $I(t) = \int_a^{t\bullet} d\alpha(s) \cdot f(s)$ and $I_{(d,\xi^\bullet)}(t) = \int_a^{t\bullet} d\alpha(s) \cdot f_{d,\xi^\bullet}(s)$. For $(\bar{d}, \bar{\xi}^\bullet) \in \mathcal{D}^\bullet$ we have

$$\left\|\int_a^{b\bullet} d_t\left[\int_a^{t\bullet} d\alpha(s) \cdot f(s)\right] g(t) - \int_a^{b\bullet} d_t\left[\int_a^{t\bullet} d\alpha(s) \cdot f_{d,\xi^\bullet}(s)\right] g(t)\right\|$$

$$= \left\|\int_a^{b\bullet} dI(t) \cdot g(t) - \int_a^{b\bullet} dI_{(d,\xi^\bullet)}(t) \cdot g(t)\right\|$$

$$\leq \left\|\int_a^{b\bullet} dI_{(d,\xi^\bullet)}(t) \cdot g(t) - \int_a^{b\bullet} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(t)\right\|$$

$$+ \left\|\int_a^{b\bullet} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(t) - \sum_{j=1}^{|\bar{d}|} \int_{\bar{t}_{j-1}}^{\bar{t}_j\bullet} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(\bar{\xi}_j^\bullet)\right\|$$

$$+ \left\|\sum_{j=1}^{|\bar{d}|} \int_{\bar{t}_{j-1}}^{\bar{t}_j\bullet} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(\bar{\xi}_j^\bullet)\right.$$

$$\left. - \sum_{j=1}^{|\bar{d}|} \left[I_{(d,\xi^\bullet)}(\bar{t}_j) - I_{(d,\xi^\bullet)}(\bar{t}_{j-1})\right] g(\bar{\xi}_j^\bullet)\right\|$$

$$+ \left\|\sum_{j=1}^{|\bar{d}|} \left[(I_{(d,\xi^\bullet)}(\bar{t}_j) - I(\bar{t}_j)) - (I_{(d,\xi^\bullet)}(\bar{t}_{j-1}) - I(\bar{t}_{j-1})\right] g(\bar{\xi}_j^\bullet)\right\|$$

$$+ \left\|\sum_{j=1}^{|\bar{d}|} \left[I(\bar{t}_j) - I(\bar{t}_{j-1})\right] g(\bar{\xi}_j^\bullet) - \int_a^{b\bullet} dI(t) \cdot g(t)\right\|.$$

By the definition of $I_{(d,\xi^\cdot)}$ the third summand is zero; by Lemma 3.3$^\cdot$ the first summand is zero, since $f_{d,\xi^\cdot} \in G([a, b], E_2)$. By Theorem 3.1$^\cdot$ (c) and by Theorem 1.6$^\cdot$ (c) the fifth summand is $\leq \|f\| \; SB^{1,23}[\alpha] \; \omega_{\dot{\bar{d}}}(g)$ and the same is true for the second summand, *for all* $(d, \xi^\cdot) \in \mathcal{D}^\cdot$. Hence, given $\varepsilon > 0$, there exists $d_\varepsilon \in D$ such that for $\bar{d} \geq d_\varepsilon$ the second and the fifth summands are $\leq \varepsilon/3$ for all $d \in D$; we fix such a $\bar{d} \geq d_\varepsilon$; then, by Proposition 1.3, the fourth summand becomes $\leq \varepsilon/3$ for all d sufficiently "large."

<u>PROPOSITION 3.4</u>. Under the hypothesis of Theorem 3.6 we have

$$\int_a^b d_t\left[\int_a^t d\alpha(s) \cdot f(s)\right] g(t)$$

$$= \lim_{\Delta d \to 0} \int_a^b d_t\left[\int_{\dot{a}}^t d\alpha(s) \cdot f_{d,\xi}(s)\right] g(t).$$

<u>Proof</u>. The proof follows the steps of the proof of the Proposition 3.4$^\cdot$ if we take $\mathcal{D}$ instead of $\mathcal{D}^\cdot$, $\lim_{\Delta d \to 0}$ instead of $\lim_{d \in D}$, $\omega_{\Delta \bar{d}}(g)$ instead of $\omega_{\dot{\bar{d}}}(g)$, Theorem 1.6 (b) instead of Theorem 1.6$^\cdot$ (c), etc.

<u>LEMMA 3.5$^\cdot$</u>. Under the hypothesis of Theorem 2.6$^\cdot$, for all $(d, \xi^\cdot) \in \mathcal{D}^\cdot$ there exists $\int_{\dot{a}}^b d\alpha(t) \cdot f_{d,\xi^\cdot}(t)\, g(t)$ and we have

$$\left\| \int_{\dot{a}}^b d\alpha(t) \cdot f_{d,\xi^\cdot}(t)\, g(t) - \sum_{i=1}^{|d|} \left(\alpha(t_i) - \alpha(t_{i-1})\right) \cdot f(\xi_i^\cdot)\, g(\xi_i^\cdot) \right\|$$

$$\leq SB^{1,23}[\alpha] \; \|f\| \; \omega_{\dot{d}}(g).$$

<u>Proof</u>. Since $f_{d,\xi^\cdot} \in E([a, b], E_2) \subset G([a, b], E_2)$, the existence of the integral follows from Lemma 3.3$^\cdot$. The inequality is obvious.

In the same way one proves the following:

LEMMA 3.5. Under the hypothesis of Theorem 3.6, for all $(d, \xi) \in \mathcal{D}$ there exists $\int_a^b \bullet\, d\alpha(t) \cdot f_{d,\xi}(t)\, g(t)$ and we have

$$\left\| \int_a^b \bullet\, d\alpha(t) \cdot f_{d,\xi}(t)\, g(t) - \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i)\, g(\xi_i) \right\|$$

$$\leq SB^{1,23}[\alpha]\ \|f\|\ \omega_{\Delta_d}(g).$$

Proof of Theorem 3.6'. The first integral in (S') exists by Theorem 3.1' and is, by Proposition 3.4', equal to

$$\lim_{d \in D} \int_a^b \bullet\, d_t\left[\int_a^t \bullet\, d\alpha(s) \cdot f_{d,\xi'}(s)\right] g(t),$$

which by Lemma 3.3' is equal to

$$\lim_{d \in D} \int_a^b \bullet\, d\alpha(t) \cdot f_{d,\xi'}(t)\, g(t);$$

by Lemma 3.5' this limit is equal to

$$\lim_{d \in D} \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i')\, g(\xi_i')$$

and by definition this is

$$\int_a^b \bullet\, d\alpha(t) \cdot f(t)\, g(t).$$

Proof of Theorem 3.6. The proof follows the steps of the proof of Theorem 3.6' (applying Theorem 3.1, Proposition 3.4, etc. instead of Theorem 3.1', Proposition 3.4', etc.).

4. THE DARBOUX-STIELTJES INTEGRAL

If X is a Banach space and $\alpha \in BV([a, b], X)$, we define $\hat{\alpha} \in BV([a, b], \mathbb{R})$ by $\hat{\alpha}(t) = V_{[a,t]}[\alpha]$, $t \in [a, b]$; given $d \in D$, we write $V_i[\alpha] = V_{[t_{i-1},t_i]}[\alpha] = V_{[t_{i-1},t_i]}[\hat{\alpha}] = \hat{\alpha}(t_i) - \hat{\alpha}(t_{i-1})$, $i = 1, 2, \ldots, |d|$.

Let F be a Banach space and $f: [a, b] \to F$ a bounded function. We write $f \in D_\alpha^{\cdot}([a, b], F)$, and we say that f *satisfies the interior Darboux condition* (with respect to α) if

$$\lim_{d \in D} \sum_{i=1}^{|d|} V_i[\alpha]\ \omega_i^{\cdot}(f) = 0.$$

We have $D_\alpha^{\cdot}([a, b], F) = D_{\hat{\alpha}}^{\cdot}([a, b], F)$. We write $f \in D_\alpha([a, b], F)$, and we say that f *satisfies the Darboux condition* (with respect to α) if

$$\lim_{\Delta d \to 0} \sum_{i=1}^{|d|} V_i[\alpha]\ \omega_i(f) = 0.$$

We have $D_\alpha([a, b], F) = D_{\hat{\alpha}}([a, b], F)$. Analogously we write $f \in D_\alpha^-([a, b], F)$ if we have

$$\lim_{d \in D} \sum_{i=1}^{|d|} V_i[\alpha]\ \omega_i(f) = 0;$$

we have $D_\alpha^-([a, b], F) = D_{\hat{\alpha}}^-([a, b], F)$. If $\alpha(t) \equiv t$ we write simply $D([a, b], F)$.

For the numerical Riemann integral we have

$$\begin{aligned} R([a, b], \mathbb{R}) &= D([a, b], \mathbb{R}) = R^-([a, b], \mathbb{R}) \\ &= D^-([a, b], \mathbb{R}) = R^{\cdot}([a, b], \mathbb{R}) \\ &= D^{\cdot}([a, b], \mathbb{R}); \end{aligned}$$

analogously we will prove that for $\alpha \in BV([a, b], \mathbb{R})$ we have

$$R_\alpha^{\cdot}([a, b], \mathbb{R}) = D_\alpha^{\cdot}([a, b], \mathbb{R}), \quad R_\alpha^-([a, b], \mathbb{R}) = D_\alpha^-([a, b], \mathbb{R}),$$

and

$$R_\alpha([a, b], \mathbb{R}) = D_\alpha([a, b], \mathbb{R})$$

(see Corollaries 4.10', 4.10$^-$, and 4.10) and if α is continuous these six spaces coincide (by Theorem 1.2); these facts and Theorem 4.1' (and the analogous results, "Theorem 4.1$^-$" and "Theorem 4.1" for $⨍$ and $\int$, respectively) justify the definitions given above.

THEOREM 4.1'. Given a BT (X, F, G), $\alpha \in BV([a, b], X)$, and $f \in D^{\cdot}_{\alpha}([a, b], F)$, we have:

(a) There exists $\int^{b}_{\dot a} d\alpha(t) \cdot f(t) \in G$.

(b) $\|\int^{b}_{\dot a} d\alpha(t) \cdot f(t)\| \leq V[\alpha]\, \|f\|$.

Proof. (a) Given $d, \bar d \in D$ with $d \leq \bar d$, we write $I_i = [t_{i-1}, t_i]$, $\bar I_j = [\bar t_{j-1}, \bar t_j]$. The existence of the integral follows from

$$\|\sigma_{d,\xi^{\cdot}}(f;\alpha) - \sigma_{\bar d,\bar\xi^{\cdot}}(f;\alpha)\|$$

$$= \Big\|\sum_{i=1}^{|d|} \sum_{\bar I_j \subset I_i} \Big(\alpha(\bar t_j) - \alpha(\bar t_{j-1})\Big)\Big(f(\xi^{\cdot}_i) - f(\bar\xi^{\cdot}_j)\Big)\Big\|$$

$$\leq \sum_{i=1}^{|d|} \sum_{\bar I_j \subset I_i} V_j[\alpha]\, \omega^{\cdot}_i(f) = \sum_{i=1}^{|d|} V_i[\alpha]\, \omega^{\cdot}_i(f).$$

The proof of (b) is immediate.

Example. In particular there exists $\int^{b}_{\dot a} d\hat\alpha(t)\, f(t) \in F$.

THEOREM 4.2'. Given $\alpha \in BV([a, b], X)$, a BT (E, F, G), $f \in D^{\cdot}_{\alpha}([a, b], E)$, and $g \in D^{\cdot}_{\alpha}([a, b], F)$, we have $f \cdot g \in D^{\cdot}_{\alpha}([a, b], G)$.

Proof. For $d \in D$ we have $\omega_i^{\bullet}(f \cdot g) \leq \|f\| \, \omega_i^{\bullet}(g) + \|g\| \, \omega_i^{\bullet}(f)$; hence the result follows from

$$\sum_{i=1}^{|d|} v_i[\alpha] \, \omega_i^{\bullet}(f \cdot g) \leq \|f\| \sum_{i=1}^{|d|} v_i[\alpha] \, \omega_i^{\bullet}(g) + \|g\| \sum_{i=1}^{|d|} v_i[\alpha] \, \omega_i^{\bullet}(f).$$

Remark. It is immediate that for D_α^- with $\displaystyle\fint_a^b$ and for D_α with $\displaystyle\int_a^b$ we have theorems analogous to 4.1˙ and 4.2˙.

Given a function φ: $[a, b] \to \mathbb{R}_+$ and $d \in D$, we write $M_i^{\bullet}(\varphi) = \sup\{\varphi(t) \mid t \in]t_{i-1}, t_i[\}$ and $M_i(\varphi) = \sup\{\varphi(t) \mid t \in [t_{i-1}, t_i]\}$, $i = 1, 2, \ldots, |d|$. If v: $[a, b] \to \mathbb{R}$ is nondecreasing, we consider the interior upper sums

$$S_d^{\bullet}(\varphi, v) = \sum_{i=1}^{|d|} M_i^{\bullet}(\varphi) \left[v(t_i) - v(t_{i-1})\right]$$

and the *interior upper integral*

$$\overline{\int_a^b}^{\bullet} \varphi(t) \, dv(t) = \inf_{d \in D} S_d^{\bullet}(\varphi, v);$$

analogously we define de upper sums

$$S_d(\varphi, v) = \sum_{i=1}^{|d|} M_i(\varphi) \left[v(t_i) - v(t_{i-1})\right]$$

and the *upper integral*

$$\overline{\fint_a^b} \varphi(t) \, dv(t) = \inf_{d \in D} S_d(\varphi, v).$$

If F is a Banach space, f: $[a, b] \to F$ a bounded function, and $\alpha \in BV([a, b], X)$, we define

$$\|f\|_{\hat{\alpha}}^{\bullet} = \overline{\int_a^b}^{\bullet} \|f(t)\| \, d\hat{\alpha}(t) \quad \text{and} \quad \|f\|_{\hat{\alpha}}^{-} = \overline{\fint_a^b} \|f(t)\| \, d\hat{\alpha}(t);$$

$\overline{R}_{\hat{\alpha}}^{\bullet}([a, b], F)$ $\left[\overline{R}_{\hat{\alpha}}^{-}([a, b], F)\right]$ denotes the space of all bounded functions f: $[a, b] \to F$ endowed with the seminorm $\| \ \|_{\hat{\alpha}}^{\bullet}$ $[\| \ \|_{\hat{\alpha}}^{-}]$.

PROPOSITION 4.3$^{\cdot}$ (4.3^{-}). $\overline{R}^{\cdot}_{\hat\alpha}([a, b], F)$ $[\overline{R}^{-}_{\hat\alpha}([a, b], F)]$ depends only on $\hat\alpha$ and not on α.

We write $f \in S^{\cdot}_{\alpha}([a, b], F)$ $[f \in S^{-}_{\alpha}([a, b], F)]$ if for every $\varepsilon > 0$ there exists a step function $f_\varepsilon \in E([a, b], F)$ such that $\| f - f_\varepsilon \|^{\cdot}_{\hat\alpha} < \varepsilon$ $[\| f - f_\varepsilon \|^{-}_{\hat\alpha} < \varepsilon]$. By Proposition 4.3$^{\cdot}$ (4.3^{-}) we have

PROPOSITION 4.4$^{\cdot}$ (4.4^{-}). $S^{\cdot}_{\alpha}([a, b], F)$ $[S^{-}_{\alpha}([a, b], F)]$ depends only on $\hat\alpha$.

If $\alpha \in BV([a, b], X)$, for every $t \in \,]a, b]$ there exists $\alpha(t-) = \lim_{\varepsilon \downarrow 0} \alpha(t - \varepsilon)$ and we have $\hat\alpha(t) - \hat\alpha(t-) = \|\alpha(t) - \alpha(t-)\|$; for every $t \in [a, b[$ there exists $\alpha(t+)$ and we have $\hat\alpha(t+) - \hat\alpha(t) = \|\alpha(t+) - \alpha(t)\|$ (see [2, Theorem I.2.7 and Proposition I.2.8]); since $\hat\alpha$ is nondecreasing we have

$$\sum_{a<t\leq b} \left(\hat\alpha(t) - \hat\alpha(t-)\right) + \sum_{a\leq t<b} \left(\hat\alpha(t+) - \hat\alpha(t)\right) \leq V[\hat\alpha],$$

it follows that the series $\Sigma_{a<t\leq b} \|\alpha(t) - \alpha(t-)\|$ and $\Sigma_{a\leq t<b} \|\alpha(t+) - \alpha(t)\|$ are absolutely convergent and therefore the *jump component* α_s of α is well defined:

$$\alpha_s = \sum_{a<\tau\leq b} \chi_{[\tau,b]}\left(\alpha(\tau) - \alpha(\tau-)\right) + \sum_{a\leq\tau<b} \chi_{]\tau,b]}\left(\alpha(\tau+) - \alpha(\tau)\right);$$

the *continuous component* $\alpha_c = \alpha - \alpha_s$ is a continuous function of bounded variation (α_c is continuous, since for every $t \in \,]a, b]$ ($t \in [a, b[$) we have $\alpha_s(t) - \alpha_s(t-) = \alpha(t) - \alpha(t-)$ $[\alpha_s(t+) - \alpha_s(t) = \alpha(t+) - \alpha(t)]$). It is immediate that $\hat\alpha_s = (\hat\alpha)_s$ and $\hat\alpha_c = (\hat\alpha)_c$.

If $v\colon [a, b] \to \mathbb{R}$ is nondecreasing, we denote by m_v the measure defined by v. (We recall that for $a < c < d < b$ we have $m_v([a, c[) = v(c-) - v(a)$, $m_v(]c, d[) = v(d-) - v(c+)$,

$m_v(]d, b]) = v(b) - v(d+)$, $m_v(\{c\}) = v(c+) - v(c-)$, etc.) Given a Banach space F and a function $f: [a, b] \to F$ we denote by D^f the set of all points of discontinuity of f.

THEOREM 4.5'. Let X and F be Banach spaces, $\alpha \in BV([a, b], X)$ and $f: [a, b] \to F$ a bounded function; the following properties are equivalent:

I'. $f \in D^{\cdot}_{\alpha}([a, b], F)$; $\hat{I}'$. $f \in D^{\cdot}_{\hat{\alpha}}([a, b], F)$.

II'. $f \in S_{\dot{\alpha}}([a, b], F)$; $\widehat{II}'$. $f \in S^{\cdot}_{\hat{\alpha}}([a, b], F)$.

III'. $\lim_{d \in D} \sum_{i=1}^{|d|} \omega^{\cdot}_i(f) \, \|\alpha(t_i) - \alpha(t_{i-1})\| = 0.$

IV'a. $m_v(D^f) = 0$ where $v = \hat{\alpha}_c$.

IV'b. If $\alpha(t+) \neq \alpha(t)$ then there exists $f(t+)$, and if $\alpha(t-) \neq \alpha(t)$ then there exists $f(t-)$.

Proof. I' $\iff$ $\hat{I}'$ follows from the definition of $D^{\cdot}_{\alpha}([a, b], F)$, and II' $\iff$ $\widehat{II}'$ follows from Proposition 4.4'; it is obvious that I' $\implies$ III'.

III' $\implies$ I': From the definition of $V[\alpha]$ it follows that for every $\varepsilon > 0$ there exists a $d_\varepsilon \in D$ such that for $d \geq d_\varepsilon$ we have

$$V[\alpha] - \sum_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\| < \varepsilon;$$

hence in

$$\sum_{i=1}^{|d|} V_i[\alpha] \, \omega^{\cdot}_i(f)$$

$$= \sum_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\| \, \omega^{\cdot}_i(f)$$

$$+ \left\{ \sum_{i=1}^{|d|} V_i[\alpha] \, \omega^{\cdot}_i(f) - \sum_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\| \, \omega^{\cdot}_i(f) \right\}$$

$$\leq \sum_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\| \; \omega_i^{\bullet}(f)$$

$$+ \sum_{i=1}^{|d|} \left\{V_i[\alpha] - \|\alpha(t_i) - \alpha(t_{i-1})\|\right\} \omega_d^{\bullet}(f)$$

the first summand becomes arbitrarily small by the hypothesis III$^{\bullet}$ and the second one by the reasoning we have just given.

I$^{\bullet}$ $\Longrightarrow$ II$^{\bullet}$: Given $f \in D_{\alpha}^{\bullet}([a, b], F)$, we take $d \in D$ such that $\Sigma_{i=1}^{|d|} V_i[\alpha] \; \omega_i^{\bullet}(f) < \varepsilon$. Then we have $f_{d,\xi^{\bullet}} \in E([a, b], F)$ and

$$\|f - f_{d,\xi^{\bullet}}\|_{\hat{\alpha}}^{\bullet} = \sum_{i=1}^{|d|} \overline{\int}_{\bullet t_{i-1}}^{t_i} \|f(t) - f(\xi_i^{\bullet})\| \; d\hat{\alpha}(t)$$

$$\leq \sum_{i=1}^{|d|} V_i[\alpha] \; \omega_i^{\bullet}(f) < \varepsilon.$$

II$^{\bullet}$ $\Longrightarrow$ I$^{\bullet}$: Given $f \in S_{\alpha}^{\bullet}([a, b], F)$ let $g \in E([a, b], F)$ be such that $\|f - g\|_{\hat{\alpha}}^{\bullet} < \varepsilon$, hence there exists $d \in D$ such that

$$\sum_{i=1}^{|d|} M_i^{\bullet}(\|f(\;) - g(\;)\|)\left[\hat{\alpha}(t_i) - \hat{\alpha}(t_{i-1})\right] < \varepsilon.$$

We may refine the division d and suppose that $g = \Sigma_{i=1}^{|d|} c_i \chi_{]t_{i-1},t_i[}$ and therefore we have

$$\sum_{i=1}^{|d|} M_i^{\bullet}(\|f(\;) - c_i\|) \left[\hat{\alpha}(t_i) - \hat{\alpha}(t_{i-1})\right] < \varepsilon.$$

But

$$\omega_i^{\bullet}(f) = \sup\{\|f(t) - f(s)\| \mid t, s \in]t_{i-1}, t_i[\}$$

$$\leq \sup\{\|f(t) - c_i\| + \|c_i - f(s)\| \mid t, s \in]t_{i-1}, t_i[\}$$

$$\leq 2M_i^{\bullet}(\|f(\;) - c_i\|)$$

and therefore $\Sigma_{i=1}^{|d|} V_i[\alpha] \; \omega_i^{\bullet}(f) < 2\varepsilon$.

We complete the proof of the theorem by showing that I' <===> IV'. Since $\hat{\alpha} = \hat{\alpha}_c + \hat{\alpha}_s$, it is immediate that we have I' if and only if we have I'a and I'b, where

$$\text{I'a.} \quad \lim_{d\in D} \sum_{i=1}^{|d|} \omega_i^{\cdot}(f) \left|\hat{\alpha}_c(t_i) - \hat{\alpha}_c(t_{i-1})\right| = 0,$$

$$\text{I'b.} \quad \lim_{d\in D} \sum_{i=1}^{|d|} \omega_i^{\cdot}(f) \left|\hat{\alpha}_s(t_i) - \hat{\alpha}_s(t_{i-1})\right| = 0.$$

We will prove that I'a <===> IV'a and I'b <===> IV'b. The equivalence of I'a and IV'a follows immediately from the following theorem.

<u>THEOREM 4.6</u>. If $v: [a, b] \to \mathbb{R}$ is continuous nondecreasing and $f: [a, b] \to F$ a bounded function, the following properties are equivalent:

(a) $\lim_{d\in D} \sum_{i=1}^{|d|} \omega_i^{\cdot}(f) \left|v(t_i) - v(t_{i-1})\right| = 0$, i.e.,

$f \in D_v^{\cdot}([a, b], F)$.

(b) $\lim_{\Delta d\to 0} \sum_{i=1}^{|d|} \omega_i(f) \left|v(t_i) - v(t_{i-1})\right| = 0$, i.e.,

$f \in D_v([a, b], F)$.

(c) $m_v(D^f) = 0$.

<u>Proof</u>. (a) ===> (c): the set D_δ^f of points where the oscilation of f is $\geq \delta$ is closed and $D^f = \bigcup_{n\in N} D_{1/n}^f$; hence if $m_v(D^f) > 0$, there exist an $n \in N$ and $m > 0$ such that $m_v(D_{1/n}^f) = m$. Hence for any $d \in D$ we have

$$\sum_{i=1}^{|d|} \omega_i^{\cdot}(f) \left|v(t_i) - v(t_{i-1})\right|$$
$$\geq \sum \{\omega_i^{\cdot}(f) \left|v(t_i) - v(t_{i-1})\right| \mid]t_{i-1}, t_i[\cap D_{1/n}^f \neq \phi\} \geq \frac{m}{n}.$$

(c) $\Longrightarrow$ (a): if $m_v(D^f) = 0$ then we have $m_v(D^f_{1/n}) < \varepsilon$ for all $\varepsilon > 0$ and $n \in N$. Since $D^f_{1/n}$ is compact it may be covered by a finite number of intervals $I_1, \ldots, I_p$ such that $\Sigma_{j=1}^p m_v(I_j) < \varepsilon$. At every point of $F = \complement(I_1 \cup \cdots \cup I_p)$ the oscilation of f is $< 1/n$ and since F is the union of a finite number of closed intervals, by subdivisions we may suppose that the oscilation of f in every one of these intervals is $< \varepsilon$. Hence, if d denotes the division of $[a, b]$ formed by the extremities of all these intervals we have

$$\sum_{i=1}^{|d|} \omega_i(f) \left|v(t_i) - v(t_{i-1})\right|$$

$$= \sum \{\omega_i(f) \left|v(t_i) - v(t_{i-1})\right| \mid D^f_{1/n} \cap [t_{i-1}, t_i] \neq \phi\}$$

$$+ \sum \{\omega_i(f) \left|v(t_i) - v(t_{i-1})\right| \mid D^f_{1/n} \cap [t_{i-1}, t_i] = \phi\}$$

$$\leq 2\|f\|\varepsilon + \frac{1}{n}[v(b) - v(a)];$$

hence the result, since ε and n are arbitrary.

In order to prove (a) $\Longrightarrow$ (b) we need the following lemma.

<u>LEMMA 4.7</u>. If $d, \tilde{d} \in D$ are such that $\Delta d < \inf\{\tilde{t}_j - \tilde{t}_{j-1} \mid j = 1, \ldots, |\tilde{d}|\}$ then we have

$$\sum_{i=1}^{|d|} \omega_i(f) \left|v(t_i) - v(t_{i-1})\right|$$

$$\leq \sum_{j=1}^{|\tilde{d}|} \tilde{\omega}_j^\bullet(f) \left|v(\tilde{t}_j) - v(\tilde{t}_{j-1})\right| + (5|\tilde{d}| - 6)\|f\| \, \omega_{\Delta d}(v).$$

<u>Proof</u>. We write $I_d = \Sigma_{i=1}^{|d|} \omega_i(f) \, [v(t_i) - v(t_{i-1})]$; $I_{d \vee \tilde{d}}$, $I^\bullet_{\tilde{d}}$, etc., have analogous meanings. We may suppose that for $j \neq 0, |\tilde{d}|$ we have $\tilde{t}_j \neq t_i$ for $i = 1, 2, \ldots, |d| - 1$ since v is continuous and f bounded, hence we may move slightly the points $\tilde{t}_j$, $0 < j < |\tilde{d}|$, changing $I^\bullet_{\tilde{d}}$ by an arbitrarily small amount.

We denote by $\overline{d}$ the division obtained from $\hat{d} = d \vee \tilde{d}$ by omitting the points $\hat{t}_j$ which are such that $\hat{t}_{j-1}, \hat{t}_{j+1} \notin \tilde{d}$. We denote by $I^*_{\overline{d}}$ the sum obtained by replacing $\omega_j(f)$ in $I_{\overline{d}}$ with $\omega^{\bullet}_j(f)$ if $\overline{t}_{j-1} \notin \tilde{d}$ or $\overline{t}_j \notin \tilde{d}$; then we have $I^*_{\overline{d}} \leq I^{\bullet}_{\tilde{d}}$.

We have $I_d - I_{d\vee\tilde{d}} \leq (|\tilde{d}| - 2)\|f\|\omega_{\Delta d}(v)$ because if $\hat{t}_j$ is one of the $|\tilde{d}| - 2$ points of $\tilde{d}$ that we add to d in order to obtain $\hat{d} = d \vee \tilde{d}$ then in I_d we have to replace $\omega_{[\hat{t}_{j-1},\hat{t}_{j+1}]}(f) \times [v(\hat{t}_{j+1}) - v(\hat{t}_{j-1})]$ by

$$\omega_{[\hat{t}_{j-1},\hat{t}_j]}(f)\left(v(\hat{t}_j) - v(\hat{t}_{j-1})\right) + \omega_{[\hat{t}_j,\hat{t}_{j+1}]}(f)\left(v(\hat{t}_{j+1}) - v(\hat{t}_j)\right);$$

hence we increase I_d by

$$c_j \leq 2\|f\| \tfrac{1}{2}[v(\hat{t}_{j+1}) - v(\hat{t}_{j-1})] \leq \|f\| \, \omega_{\Delta d}(v).$$

Hence

$$\begin{aligned} I_d &\leq I_{d\vee\overline{d}} + (|\tilde{d}| - 2)\|f\| \, \omega_{\Delta d}(v) \\ &= I_{\overline{d}} + (|\tilde{d}| - 2)\|f\| \, \omega_{\Delta d}(v) \\ &= [I_{\overline{d}} - I^*_{\overline{d}}] + I^*_{\overline{d}} + (|\tilde{d}| - 2)\|f\| \, \omega_{\Delta d}(v) \\ &\leq 2(|\tilde{d}| - 1)2\|f\|\omega_{\Delta d}(v) + I^*_{\overline{d}} + (|\tilde{d}| - 2)\|f\| \, \omega_{\Delta d}(v) \\ &\leq I^{\bullet}_{\tilde{d}} + (5|\tilde{d}| - 6)\|f\| \, \omega_{\Delta d}(v) \qquad \text{Q.E.D.} \end{aligned}$$

<u>Proof of (a) ⟹ (b) (of Theorem 4.6)</u>. Given $\varepsilon > 0$ we take $\tilde{d} \in D$ such that $I^{\bullet}_{\tilde{d}} \leq \varepsilon/2$, since v is uniformly continuous there exists $\delta > 0$ such that for $\Delta d \leq \delta$ we have

$$\omega_{\Delta d}(v) \leq \frac{\varepsilon}{2(5|\tilde{d}| - 6)\|f\|}$$

hence by Lemma 4.7 for $\Delta d \leq \delta$ we have

$$\sum_{i=1}^{|d|} \omega_i(f) \left(v(t_i) - v(t_{i-1})\right) \leq \varepsilon.$$

It is obvious that (b) ⟹ (a).

Proof of I'b $\Longleftrightarrow$ IV'b (of Theorem 4.5'). Let us write $v = \hat{\alpha}_s$.

I'b $\Longrightarrow$ IV'b: For every $t \in [a, b[$ we write $\omega^{\bullet}_{t+}(f) = \lim_{\varepsilon \downarrow 0} \omega^{\bullet}_{]t, t+\varepsilon[}(f)$; hence we have $\omega^{\bullet}_{t+}(f) = 0$ if and only if there exists $f(t+)$. If we have $v(\bar{t}+) \neq v(\bar{t})$, then for any $d \in D$ that contains $\bar{t} = t_j$ we have

$$\sum_{i=1}^{|d|} \omega^{\bullet}_i(f) \left| v(t_i) - v(t_{i-1}) \right| \geq \omega_{]t_j, t_{j+1}[}(f) \left| v(t_{j+1}) - v(t_j) \right|$$
$$\geq \omega_{\bar{t}+}(f) \left| v(\bar{t}+) - v(\bar{t}) \right|$$

and analogously for the discontinuities at the left.

IV'b $\Longrightarrow$ I'b: By the definition of $v = \hat{\alpha}_s$ for every $\varepsilon > 0$ there exist $a = s_0 < s_1 < \cdots < s_n = b$ such that

$$\bar{V} = \sum_{j=1}^{n} \left| v(s_j) - v(s_j -) \right| + \sum_{j=0}^{n-1} \left| v(s_j +) - v(s_j) \right| \tag{3}$$
$$> V[v] - \frac{\varepsilon}{6\|f\|}.$$

We take a division

$$d:\ a = t_0 < t_1 = a + \delta_0 < t_2 = s_1 - \delta_1 < t_3 = s_1 < t_4 = s_1 + \delta_1$$
$$< t_5 = s_2 - \delta_2 < \cdots < t_{3n} = s_n.$$

We have

$$\sum_{i=1}^{|d|} \omega^{\bullet}_i(f) \left| v(t_i) - v(t_{i-1}) \right|$$

$$= \sum_{j=1}^{n} \omega^{\bullet}_{3j}(f) \left| v(t_{3j}) - v(t_{3j-1}) \right|$$
$$+ \sum_{j=0}^{n-1} \omega^{\bullet}_{3j+1}(f) \left| v(t_{3j+1}) - v(t_{3j}) \right| + \sum_{j=1}^{n} \omega^{\bullet}_{3j-1}(f) \left| v(t_{3j-1}) - v(t_{3j-2}) \right|.$$

We take $\delta_0, \delta_1, \ldots, \delta_n$ so small that we have (a), (b), (c), and (d) where

(a) $\omega^{\bullet}_{3j}(f) \leq \varepsilon/3nV[v]$ if $v(t_{3j}) - v(t_{3j}-) > 0$.

(b) $v(t_{3j}) - v(t_{3j-1}) \leq \varepsilon/6n\|f\|$ if $v(t_{3j}-) = v(t_{3j})$.

(c) $\omega^{\bullet}_{3j+1}(f) \leq \varepsilon/3nV[v]$ if $v(t_{3j}+) - v(t_{3j}) > 0$.

(d) $v(t_{3j+1}) - v(t_{3j}) \leq \varepsilon/6n\|f\|$ if $v(t_{3j}+) = v(t_{3j})$.

Then we have

$$\sum_{j=1}^{n} \omega^{\bullet}_{3j}(f) \left[v(t_{3j}) - v(t_{3j-1})\right] \leq \frac{\varepsilon}{3} \quad \text{[by (a) and (b)]},$$

$$\sum_{j=0}^{n-1} \omega^{\bullet}_{3j+1}(f) \left[v(t_{3j+1}) - v(t_{3j})\right] \leq \frac{\varepsilon}{3} \quad \text{[by (c) and (d)]},$$

$$\sum_{j=1}^{n} \omega^{\bullet}_{3j-1}(f) \left[v(t_{3j-1}) - v(t_{3j-2})\right] \leq 2\|f\| \left[V[v] - \overline{V}\right] < \frac{\varepsilon}{3} \quad \text{[by (3)]}. \quad \text{Q.E.D.}$$

In an analogous way as Theorem 4.5$^{\bullet}$ we prove

<u>THEOREM 4.5^{-}</u>. Let X and F be Banach spaces, $\alpha \in BV([a, b], X)$ and $f: [a, b] \to F$ a bounded function; the following properties are equivalent:

I^{-}. $f \in D^{-}_{\alpha}([a, b], F)$ $\quad$ Î$^{-}$. $f \in D^{-}_{\hat{\alpha}}([a, b], F)$

II^{-}. $f \in S^{-}_{\alpha}([a, b], F)$ $\quad$ ÎI^{-}. $f \in S^{-}_{\hat{\alpha}}([a, b], F)$

III^{-}. $\lim\limits_{d \in D} \sum_{i=1}^{|d|} \omega_i(f) \|\alpha(t_i) - \alpha(t_{i-1})\| = 0$

IV^{-}a. $m_v(D^f) = 0$ where $v = \hat{\alpha}_c$

b. α and f have no common left or right discontinuity, i.e., if $\alpha(t+) \neq \alpha(t)$ then there exists $f(t+) = f(t)$ and if $\alpha(t-) \neq \alpha(t)$ then there exists $f(t-) = f(t)$.

Analogously to Theorem 4.5' and 4.5$^-$ we have

THEOREM 4.5. Let X and F be Banach spaces, $\alpha \in BV([a, b], X)$ and $f: [a, b] \to F$ a bounded function; the following properties are equivalent:

I. $f \in D_\alpha([a, b], F)$ $\quad$ $\hat{I}$. $f \in D_{\hat{\alpha}}([a, b], F)$

III. $\lim_{\Delta d \to 0} \sum_{i=1}^{|d|} \omega_i(f) \|\alpha(t_i) - \alpha(t_{i-1})\| = 0$

IVa. $m_v(D^f) = 0$ where $v = \hat{\alpha}_c$

b. α and f have no common points of discontinuity

V. $m_{\hat{\alpha}}(D^f) = 0$.

Proof. I $\Longleftrightarrow$ $\hat{I}$ follows from the definition of $D_\alpha([a, b], F)$; I $\Longrightarrow$ III is obvious and the proof of III $\Longrightarrow$ I is analogous to the proof of III' $\Longrightarrow$ I' in Theorem 4.5'. The equivalence of $\hat{I}$, IV, and V follows immediately from Theorem 4.6 and from

LEMMA 4.8. Under the hypothesis of Theorem 4.5 the following properties are equivalent:

$\hat{I}$b. $\lim_{\Delta d \to 0} \sum_{i=1}^{|d|} \omega_i(f) \left|\hat{\alpha}_s(t_i) - \hat{\alpha}_s(t_{i-1})\right| = 0.$

IVb. α and f have no common points of discontinuity.

Vb. $m_u(D^f) = 0$ where $u = \hat{\alpha}_s$.

Proof. $\hat{I}$b $\Longrightarrow$ Vb: We have $m_u(D^f) = \Sigma_{t \in D^f} [u(t+) - u(t-)]$ [where we take $u(b+) = u(b)$ and $u(a-) = u(a)$]. Hence if we have $m_u(D^f) > 0$ there exists $\bar{t} \in D^f$ such that $u(\bar{t}+) - u(\bar{t}-) > 0$. Let us suppose that $\bar{t} \in]a, b[$ (if $\bar{t} = a$ or $\bar{t} = b$, the reasoning is analogous); for any $d \in D$ and $]t_{j-1}, t_j[\ni \bar{t}$ we have

$$\sum_{i=1}^{|d|} \omega_i(f) \left|u(t_i) - u(t_{i-1})\right| \geq \omega_j(f) \left|u(t_j) - u(t_{j-1})\right|$$
$$\geq \omega_{\bar{t}}(f) \left|u(\bar{t}+) - u(\bar{t}-)\right|$$

i.e., we do not have $\hat{I}$b. Vb $\Longrightarrow$ IVb is obvious since for every $t \in D^f$ we have

$$m_u(D^f) \geq m_u(\{t\}) = u(t+) - u(t-) = \|\alpha(t+) - \alpha(t-)\|.$$

IVb $\Longrightarrow$ $\hat{I}$b: If α and f have no common point of discontinuity then given $\varepsilon > 0$, for every $s \in [a, b]$ there exists $\delta_s > 0$ such that if $s \in]a, b[$ we have

$$\omega_{[s-\delta_s, s+\delta_s]}(f) \leq \frac{\varepsilon}{2V[u]} \quad \text{or} \quad u(s + \delta_s) - u(s - \delta_s) \leq \frac{\varepsilon}{2\|f\|};$$

if $s = a$ we have

$$\omega_{[a,a+\delta_a]}(f) \leq \frac{\varepsilon}{2V[u]} \quad \text{or} \quad u(a+) - u(a) \leq \frac{\varepsilon}{2\|f\|}$$

and analogously for $s = b$. Let $[a, a + \delta_a[$, $]s_j - \delta_j, s_j + \delta_j[$, $]b - \delta_b, b]$, $j = 1, \ldots, m$ (where $\delta_j = \delta_{s_j}$) be a finite subcovering of the open covering $[a, a + \delta_a[$, $]s - \delta_s, s + \delta_s[$ $(s \in]a, b[)$, $]b - \delta_b, b]$ of $[a, b]$. If $\delta > 0$ is such that any interval $[c, d] \subset [a, b]$ with $d - c \leq \delta$ is contained in one of the intervals of the finite subcovering then for $d \in D$ with $\Delta d \leq \delta$ we have

$$\sum_{i=1}^{|d|} \omega_i(f) \left|u(t_i) - u(t_{i-1})\right| \leq \varepsilon.$$

Theorem 4.6 implies

<u>THEOREM 4.5[c]</u>. Let X and F be Banach spaces; if $\alpha \in BV([a, b], X)$ is a continuous function we have $D^{\bullet}_{\alpha}([a, b], F) = D_{\alpha}([a, b], F)$

and the following properties are equivalent: I$^{\bullet}$, $\hat{I}^{\bullet}$, II$^{\bullet}$, $\hat{II}^{\bullet}$, III$^{\bullet}$, I^{-}, $\hat{I}^{-}$, II^{-}, $\hat{II}^{-}$, III^{-}, I, $\hat{I}$, III, and V (of Theorems 4.5$^{\bullet}$, 4.5^{-}, and 4.5).

COROLLARY 4.9. Given a BT (E, F, G), $\alpha \in AC([a, b], E)$ (i.e., $\alpha: [a, b] \to E$ is absolutely continuous) and $f \in D([a, b], F)$ then there exists $\int_a^b d\alpha(t) \cdot f(t)$ and $f \in D_\alpha([a, b], F)$.

Proof. By Theorem 4.5 (and by Theorem 4.1) it is enough to prove that $f \in D_\alpha([a, b], F)$ or that $m_{\hat{\alpha}}(D^f) = 0$. Since $f \in D([a, b], F)$, D^f has (Lebesgue) measure zero and since α is absolutely continuous so is $\hat{\alpha}$, and therefore $m(D^f) = 0$ implies $m_{\hat{\alpha}}(D^f) = 0$.

COROLLARY 4.10$^{\bullet}$. If $v \in BV([a, b], \mathbb{R})$ we have $R_v^{\bullet}([a, b], \mathbb{R}) = D_v^{\bullet}([a, b], \mathbb{R}) = D_{\hat{v}}^{\bullet}([a, b], \mathbb{R}) = R_{\hat{v}}^{\bullet}([a, b], \mathbb{R})$.

Proof. By Theorem 4.5$^{\bullet}$ we have $D_v^{\bullet}([a, b], \mathbb{R}) \subset R_v^{\bullet}([a, b], \mathbb{R})$; hence it remains to prove that $R_v^{\bullet}([a, b], \mathbb{R}) \subset D_v^{\bullet}([a, b], \mathbb{R})$. If $f \in R_v^{\bullet}([a, b], \mathbb{R})$ then given $\varepsilon > 0$ there exists a $d_\varepsilon \in D$ such that for every $d \in D$ with $d \geq d_\varepsilon$ we have

$$|\sigma_{d,\xi^{\bullet}}(f; v) - \sigma_{d,\eta^{\bullet}}(f; v)| \leq \frac{\varepsilon}{2};$$

for every $i = 1, 2, \ldots, |d|$ we take $\xi_i^{\bullet}, \eta_i^{\bullet} \in]t_{i-1}, t_i[$ such that

$$f(\xi_i^{\bullet}) - f(\eta_i^{\bullet}) \geq \omega_i^{\bullet}(f) - \frac{\varepsilon}{2V[v]} \quad \text{if} \quad v(t_i) - v(t_{i-1}) \geq 0$$

and such that

$$f(\eta_i^{\bullet}) - f(\xi_i^{\bullet}) \geq \omega_i^{\bullet}(f) - \frac{\varepsilon}{2V[v]} \quad \text{if} \quad v(t_i) - v(t_{i-1}) < 0;$$

then we have

$$\sum_{i=1}^{|d|} \omega_i^{\cdot}(f) \left[v(t_i) - v(t_{i-1})\right]$$

$$\leq \sum_{i=1}^{|d|} \left[f(\xi_i^{\cdot}) - f(\eta_i^{\cdot}) + \frac{\varepsilon}{2V[v]}\right]\left[v(t_i) - v(t_{i-1})\right]$$

$$\leq |\sigma_{d,\xi^{\cdot}}(f; v) - \sigma_{d,\eta^{\cdot}}(f; v)| + \frac{\varepsilon}{2V[v]} \sum_{i=1}^{|d|} |v(t_i) - v(t_{i-1})|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;$$

hence by the equivalence of III' and I' of Theorem 4.5' we have $f \in D_v^{\cdot}([a, b], \mathbb{R})$.

In an analogous way one proves the

COROLLARY 4.10 (4.10^-). If $v \in BV([a, b], \mathbb{R})$ we have

$$R_v([a, b], \mathbb{R}) = D_v([a, b], \mathbb{R}) = D_{\hat{v}}([a, b], \mathbb{R}) = R_{\hat{v}}([a, b], \mathbb{R})$$

$$\left(R_v^-([a, b], \mathbb{R}) = D_v^-([a, b], \mathbb{R}) = D_{\hat{v}}^-([a, b], \mathbb{R}) = R_{\hat{v}}^-([a, b], \mathbb{R})\right)$$

Remark. In Corollaries 4.10', 4.10, and 4.10^- we may replace $\mathbb{R}$ by a finite-dimensional Banach space F ($\simeq \mathbb{R}^n$) considering the components of the function f.

COROLLARY 4.11'. If $v \in BV([a, b], \mathbb{R})$ we have

$$f \in R_v^{\cdot}([a, b], \mathbb{R}) \Longrightarrow |f| \in R_v^{\cdot}([a, b], \mathbb{R}).$$

Proof. By Corollary 4.10' we have $f \in D_{\hat{v}}^{\cdot}([a, b], \mathbb{R})$, hence $|f| \in D_{\hat{v}}^{\cdot}([a, b], \mathbb{R})$ [since $\omega_i^{\cdot}(|f|) \leq \omega_i^{\cdot}(f)$], and therefore we have $|f| \in R_v^{\cdot}([a, b], \mathbb{R})$, again by Corollary 4.10'.

Remark. It is not difficult to prove that if $v: [a, b] \to R$ is such that $v \notin BV([a, b], \mathbb{R})$ then there exists $f \in R_v^{\cdot}([a, b], \mathbb{R})$ such that $|f| \notin R_v^{\cdot}([a, b], \mathbb{R})$.

COROLLARY 4.11 (4.11$^-$). If $v \in BV([a, b], \mathbb{R})$ we have

$$f \in R_v([a, b], \mathbb{R}) \Longrightarrow |f| \in R_v([a, b], \mathbb{R})$$

$$\left(f \in R_v^-([a, b], \mathbb{R}) \Longrightarrow |f| \in R_v^-([a, b], \mathbb{R})\right)$$

If X and Y are Banach spaces and $\alpha \in BV([a, b], X)$, $\beta \in BV([a, b], Y)$ we write $\alpha \leq \beta$ if there exists $\lambda > 0$ such that $\lambda\beta - \hat{\alpha}$ is nondecreasing.

PROPOSITION 4.12$^{\cdot}$ (4.12, 4.12$^-$). Given $\alpha \leq \beta$ for any Banach space F we have

$$D_\alpha^{\cdot}([a, b], F) \supset D_\beta^{\cdot}([a, b], F),$$

$$\left(D_\alpha([a, b], F) \supset D_\beta([a, b], F),\ D_\alpha^-([a, b], F) \supset D_\beta^-([a, b], F)\right).$$

Proof. The proof follows immediately from

$$v_i[\alpha] = v_i[\hat{\alpha}] \leq \lambda v_i[\hat{\beta}] = \lambda v_i[\beta].$$

Given $v \in BV([a, b], \mathbb{R})$ we denote by $v_\uparrow$ and $v_\downarrow$, respectively, its nondecreasing and nonincreasing components ($v_\uparrow = \frac{1}{2}(\hat{v} + v)$ and $v_\downarrow = \frac{1}{2}(\hat{v} - v)$).

THEOREM 4.13$^{\cdot}$. Given a Banach space F and $v \in BV([a, b], \mathbb{R})$ we have

$$D_v^{\bullet}([a, b], F) = D_{v_\uparrow}^{\bullet}([a, b], F) \cap D_{v_\downarrow}^{\bullet}([a, b], F)$$
$$= D_{\hat{v}}^{\bullet}([a, b], F).$$

Proof. Since $v_\uparrow, v_\downarrow \leq v \leq \hat{v} = v_\uparrow + v_\downarrow$ we have, by Proposition 3.12$^\bullet$, that

$$D_{v_\uparrow}^{\bullet}([a, b], F) \cap D_{v_\downarrow}^{\bullet}([a, b], F) \supset D_v^{\bullet}([a, b], F) = D_{\hat{v}}^{\bullet}([a, b], F);$$

from $V_i[v] = V_i[v_\uparrow] + V_i[v_\downarrow]$ follows the other inclusion.

Analogously we have

THEOREM 4.13 (4.13^-). Given $v \in BV([a, b], \mathbb{R})$ and a Banach space F we have

$$D_v([a, b], F) = D_{v_\uparrow}([a, b], F) \cap D_{v_\downarrow}([a, b], F) = D_{\hat{v}}([a, b], F),$$

$$\left(D_v^-([a, b], F) = D_{v_\uparrow}^-([a, b], F) \cap D_{v_\downarrow}^-([a, b], F) = D_{\hat{v}}^-([a, b], F)\right)$$

Let $v: [a, b] \to \mathbb{R}$ be a nondecreasing function; given a Banach space F and a bounded function $f: [a, b] \to F$ we write

$$f_{d,\xi} \to f \quad v\text{-a.e.} \quad (v\text{-almost everywhere})$$

if for any sequence $(d_n, \xi^{(n)}) \in \mathcal{D}$ with $\Delta d_n \to 0$ we have

$$f_{d_n,\xi^{(n)}}(t) \to f(t)$$

for all points $t \in [a, b]$ outside of a set of v-measure zero.

<u>THEOREM 4.14</u>. Let X and F be Banach spaces and $\alpha \in BV([a, b], X)$; we have $f \in D_\alpha([a, b], F)$ if and only if $f_{d,\xi} \to f$ $\hat{\alpha}$-a.e.

<u>Proof</u>. We write $v = \hat{\alpha}$; if there exists a sequence $(d_n, \xi^{(n)}) \in \mathcal{D}$ with $\Delta d_n \to 0$ and such that $f_{d_n,\xi^{(n)}}(t) \not\to f(t)$ at all points of a set $M \subset [a, b]$ with $m_v(M) > \delta > 0$, then for every $p \in N$ let M_p be the set of all points $t \in M$ such that

$$\| f_{d_n,\xi^{(n)}}(t) - f(t) \| > \frac{1}{p}$$

for an infinity of elements $n \in N$. Since we have $M_p \subset M_{p+1}$, $\bigcup_{p \in N} M_p = M$, and $m_v(M) > \delta$ there exists a $q \in N$ such that $m_v(M_q) > \delta$, i.e., such that at every point $t \in M_q$ we have

$$\| f_{d_n,\xi^{(n)}}(t) - f(t) \| > \frac{1}{q}$$

for an infinity of elements $n \in N$. If $t \in M_q$ we have

$$\omega_{[t-\varepsilon,t]}(f) + \omega_{[t,t+\varepsilon]}(f) > \frac{1}{q}$$

for all $\varepsilon > 0$, hence for any $d \in D$ the union of the intervals $[t_{i-1}, t_i]$ that contain points of M_q has v-measure $> \delta$, and therefore we have

$$2 \sum_{i=1}^{|d|} \omega_i(f) \left(v(t_i) - v(t_{i-1})\right) > \frac{1}{q}\delta,$$

i.e., $f \notin D_v([a, b], F)$.

Reciprocally we will show that if $f: [a, b] \to F$ is a bounded function such that $f \notin D_v([a, b], F)$ then there exists a sequence $(d_n, \xi^{(n)}) \in \mathcal{D}$ with $\Delta d_n \to 0$ such that we do not have $f_{d_n,\xi^{(n)}} \to f$ v-a.e.

Indeed, since f is bounded and $f \notin D_v([a, b], F)$, by Theorem 4.5 we have $m_v(D^f) > 0$, hence there exists $\delta > 0$ such that $m_v(D_\delta^f) = m > 0$. Therefore if we take any division $d = d_n \in D$ we have

$$\sum_{i=0}^{|d|} m_v(\{t_i\} \cap D_\delta^f) + \sum_{i=1}^{|d|} m_v(]t_{i-1}, t_i[\cap D_\delta^f) \geq m;$$

if $m_v(]t_{i-1}, t_i[\cap D_\delta^f) > 0$ we have diam $f(]t_{i-1}, t_i[) \geq \delta$ and if $m_v(\{t_i\} \cap D_\delta^f) > 0$ we have diam $f(]t_{i-1}, t_i]) \geq \delta/2$ or diam $f([t_i, t_{i+1}[) \geq \delta/2$. Hence we have

$$\text{(a)} \quad \sum \{m_v(]t_{i-1}, t_i]) \mid \text{diam } f(]t_{i-1}, t_i]) \geq \tfrac{\delta}{2}\} \geq \tfrac{m}{2}$$

or

$$\text{(b)} \quad \sum \{m_v([t_{i-1}, t_i[) \mid \text{diam } f([t_{i-1}, t_i[) \geq \tfrac{\delta}{2}\} \geq \tfrac{m}{2}.$$

Let us consider case (a) (the reasoning is analogous in case (b)); if $m_v(]t_{i-1}, t_i] \cap D_\delta^f) = m_i > 0$ and diam $f(]t_{i-1}, t_i]) \geq \delta/2$ there exist points $\bar{s}_i, \bar{t}_i \in]t_{i-1}, t_i]$ such that $\|f(\bar{s}_i) - f(\bar{t}_i)\| > \delta/3$, and by the theorem of Hahn-Banach there exist $\varphi \in F'$ and $c \in \mathbb{R}$ such that

$$H = \varphi^{-1}(]-\infty, c]) \supset \{x \in F \mid \|x - f(\bar{s}_i)\| \leq \tfrac{\delta}{6}\}$$

and

$$\complement H = \varphi^{-1}(]c, \infty[) \supset \{x \in F \mid \|x - f(\bar{t}_i)\| \leq \tfrac{\delta}{6}\}.$$

We have

$$m_v(]t_{i-1}, t_i] \cap D_\delta^f \cap f^{-1}(H)) \geq \tfrac{1}{2} m_i$$

or

$$m_v(]t_{i-1}, t_i] \cap D_\delta^f \cap f^{-1}(\complement H)) \geq \tfrac{1}{2} m_i.$$

In the first case we take $\xi_i^{(n)} \in]t_{i-1}, t_i] \cap D_\delta^f$ such that $f(\xi_i^{(n)}) = f(\bar{t}_i)$, then we have $\|f(\xi_i^{(n)}) - f(t)\| \geq \delta/6$ for every $t \in]t_{i-1}, t_i] \cap D_\delta^f \cap f^{-1}(H)$, i.e., on a set with v-measure $\geq \frac{1}{2} m_i$; analogously we take $f(\xi_i^{(n)}) = f(\bar{s}_i)$ in the second case and we choose $\xi_i^{(n)} \in \,]t_{i-1}, t_i]$ arbitrarily if we do not have $m_v(]t_{i-1}, t_i] \cap D_\delta^f) > 0$ with diam $f(]t_{i-1}, t_i]) \geq \delta/2$. Thus we have

$$\|f_{d_n,\xi^{(n)}}(t) - f(t)\| \geq \frac{\delta}{6}$$

on a set of points $t \in [a, b]$ that [by (a)] has v-measure greater then

$$\frac{1}{2} \sum \{m_v(]t_{i-1}, t_i] \cap D_\delta^f) \mid \text{diam } f(]t_{i-1}, t_i]) \geq \frac{\delta}{2}\} \geq \frac{1}{4} m.$$

Hence by the theorem of Egoroff we cannot have $f_{d_n,\xi^{(n)}} \to f$ v-a.e.

COROLLARY 4.15. The elements of $D_\alpha([a, b], F)$ are Bochner-Lebesgue integrable (see Sec. 5) with respect to the measure defined by $\hat{\alpha}$.

Proof. With the notations of Theorem 4.14, if $f \in D_\alpha([a, b], F)$ we have $f_{d_n,\xi^{(n)}} \to f$ $\hat{\alpha}$-a.e.; hence the result follows by the theorem of dominated convergence.

5. THE SUBSTITUTION FORMULA (II)

The purpose of this section is to prove the following versions of the substitution formula:

THEOREM 5.8˙. Let $(E_1, \ldots, E_{123})$ be a BAS, $\alpha \in BV([a, b], E_1)$, $f \in R_\alpha^{\cdot}([a, b], E_2)$, and $g \in D_\alpha^{\cdot}([a, b], E_3)$; then we have $f \cdot g \in R_\alpha^{\cdot}([a, b], E_{23})$ and

$$(S^{\bullet})\qquad \int_a^{\overset{\bullet}{b}} d_t\left[\int_a^{\overset{\bullet}{t}} d\alpha(s)\cdot f(s)\right] g(t) = \int_a^{\overset{\bullet}{b}} d\alpha(t)\cdot f(t)\, g(t).$$

<u>THEOREM 5.8</u>. Let $(E_1, \ldots, E_{123})$ be a BAS, $\alpha \in BV([a, b], E_1)$, $f \in R_\alpha([a, b], E_2)$, and $g \in D_\alpha([a, b], E_3)$; then we have $f \cdot g \in R_\alpha([a, b], E_{23})$ and

$$(S)\qquad \int_a^b d_t\left[\int_a^t d\alpha(s)\cdot f(s)\right] g(t) = \int_a^b d\alpha(t)\cdot f(t)\, g(t).$$

In order to prove these theorems we need several preliminary results.

<u>THEOREM 5.1$^{\bullet}$</u>. Let (E, F, G) be a BT, $\alpha \in BV([a, b], E)$, and $f \in R^{\bullet}_\alpha([a, b], F)$; we define $I_{f;\alpha}(t) = \int_a^{\overset{\bullet}{t}} d\alpha(s)\cdot f(s)$; we have

(a) $I_{f;\alpha} \in BV([a, b], G)$.

(b) $I_{f;\alpha} \leq \alpha$.

(c) $V[I_{f;\alpha}] \leq \|f\|\, V[\alpha]$.

<u>Proof</u>. (a) and (c) follow from

$$\sum_{i=1}^{|d|} \|I_{f;\alpha}(t_i) - I_{f;\alpha}(t_{i-1})\| = \sum_{i=1}^{|d|} \left\|\int_{t_{i-1}}^{\overset{\bullet}{t_i}} d\alpha(s)\cdot f(s)\right\|$$
$$\leq \sum_{i=1}^{|d|} V_i[\alpha]\, \|f\|$$
$$= V[\alpha]\, \|f\|.$$

(b) follows from the fact that for $a \leq s < t \leq b$ we have

$$\|I_{f;\alpha}(t) - I_{f;\alpha}(s)\| = \left\|\int_s^{\overset{\bullet}{t}} d\alpha(\sigma)\cdot f(\sigma)\right\|$$

$$\leq V_{[s,t]}[\alpha]\ \|f\|$$

$$= \|f\|[\hat{\alpha}(t) - \hat{\alpha}(s)].$$

In the same way we have the following:

THEOREM 5.1. Let (E, F, G) be a BT, $\alpha \in BV([a, b], E)$, and $f \in R_\alpha([a, b], F)$; we define $I_{f;\alpha}(t) = \int_a^t d\alpha(s) \cdot f(s)$; we have

(a) $I_{f;\alpha} \in BV([a, b], G)$.

(b) $I_{f;\alpha} \leq \alpha$.

(c) $V[I_{f;\alpha}] \leq \|f\|\ V[\alpha]$.

THEOREM 5.2'. Under the hypothesis of Theorem 5.8' we have

(a) $g \in D^{\bullet}_{I_{f;\alpha}}([a, b], E_3)$.

(b) $$\left\| \int_a^{b\,\bullet} d_t\left(\int_a^{t\,\bullet} d\alpha(s) \cdot f(s)\right) g(t) - \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i\,\bullet} d\alpha(s)\ f(s) \cdot g(\xi_i^{\bullet}) \right\| \leq \sum_{i=1}^{|d|} V_i[I_{f;\alpha}]\ \omega_i^{\bullet}(g).$$

Proof. By Theorem 5.1' and Proposition 4.12' we have (a); hence the first integral in (b) exists; (b) is then immediate.

Analogously we have

THEOREM 5.2. Under the hypothesis of Theorem 5.8 we have

(a) $g \in D_{I_{f;\alpha}}([a, b], E_3)$.

$$\text{(b)} \quad \left\| \int_a^b d_t \left(\int_a^t d\alpha(s) \cdot f(s) \right) g(t) - \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i} d\alpha(s)\, f(s) \cdot g(\xi_i) \right\| \le \sum_{i=1}^{|d|} V_i[I_{f;\alpha}]\, \omega_i(g).$$

THEOREM 5.3$^\bullet$. Let $(E_1, \ldots, E_{123})$ be a BAS, $\alpha \in BV([a, b], E_1)$, $f \in D_\alpha^\bullet([a, b], E_2)$, and $g: [a, b] \to E_3$; if one of the integrals in $(S^\bullet)$ exists, so does the other and both are equal.

Proof. Given $(d, \xi^\bullet) \in \mathcal{D}^\bullet$ we consider the approximating sums to both integrals; we have

$$\left\| \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i}{}^{\bullet} d\alpha(s)\, f(s) \cdot g(\xi_i^\bullet) - \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i^\bullet)\, g(\xi_i^\bullet) \right\|$$

$$= \left\| \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i}{}^{\bullet} d\alpha(s) \cdot \left[f(s) - f(\xi_i^\bullet)\right] g(\xi_i^\bullet) \right\|$$

$$\le \|g\| \sum_{i=1}^{|d|} V_i[\alpha]\, \omega_i^\bullet(f),$$

and hence the result because $f \in D_\alpha^\bullet([a, b], E_2)$.

In the same way one proves

THEOREM 5.3. Let $(E_1, \ldots, E_{123})$ be a BAS, $\alpha \in BV([a, b], E_1)$, $f \in D_\alpha([a, b], E_2)$, and $g: [a, b] \to E_3$; if one of the integrals in (S) exists, so does the other and both are equal.

LEMMA 5.4$^\bullet$. With the notations of Theorem 5.3$^\bullet$, if $g \in R^\bullet_{I_{f;\alpha}}([a, b], E_3)$, then we have $f \cdot g \in R_\alpha^\bullet([a, b], E_{23})$ and $(S^\bullet)$.

Proof. By the hypothesis, the first integral of $(S^\cdot)$ exists; the result follows from Theorem 5.3$^\cdot$.

LEMMA 5.4. With the notations of Theorem 5.3, if $g \in R_{I_{f;\alpha}}([a, b], E_3)$, then we have $f \cdot g \in R_\alpha([a, b], E_{23})$ and (S).

LEMMA 5.5$^\cdot$. With the notations of Theorem 5.3$^\cdot$, if $g \in D^\cdot_\alpha([a, b], E_3)$, then $g \in D^\cdot_{I_{f;\alpha}}([a, b], E_3)$ and we have $(S^\cdot)$.

Proof. By Theorem 4.2$^\cdot$ we have $f \cdot g \in D^\cdot_\alpha([a, b], E_{23})$ and in Theorem 5.1$^\cdot$ we saw that $I_{f;\alpha} \leq \alpha$; hence by Proposition 4.12$^\cdot$ we have $g \in D^\cdot_{I_{f;\alpha}}([a, b], E_3)$ and so by Lemma 5.4$^\cdot$ the second integral exists too and we have $(S^\cdot)$.

LEMMA 5.5. With the notations of Theorem 5.3, if $g \in D_\alpha([a, b], E_3)$, then we have $g \in D_{I_{f;\alpha}}([a, b], E_3)$ and (S).

PROPOSITION 5.6$^\cdot$. Under the hypothesis of Theorem 5.8$^\cdot$ we have

$$\int_a^{b\,\cdot} d_t\left[\int_a^{t\,\cdot} d\alpha(s) \cdot f(s)\right] g(t) = \lim_{d\in D} \int_a^{b\,\cdot} d_t\left[\int_a^{t\,\cdot} d\alpha(s) \cdot f_{d,\xi^\cdot}(s)\right] g(t).$$

Proof. By Theorem 5.2$^\cdot$ the first integral exists. We write

$$I(t) = \int_a^{t\,\cdot} d\alpha(s) \cdot f(s) \quad \text{and} \quad I_{(d,\xi^\cdot)}(t) = \int_a^{t\,\cdot} d\alpha(s) \cdot f_{d,\xi^\cdot}(s).$$

For $(\overline{d},\overline{\xi}^\cdot) \in \mathcal{D}^\cdot$ we have

$$\left\|\int_a^{b\,\cdot} d_t\left[\int_a^{t\,\cdot} d\alpha(s) \cdot f(s)\right] g(t) - \int_a^{b\,\cdot} d_t\left[\int_a^{t\,\cdot} d\alpha(s) \cdot f_{d,\xi^\cdot}(s)\right] g(t)\right\|$$

$$= \left\|\int_a^{b\,\cdot} dI(t) \cdot g(t) - \int_a^{b\,\cdot} dI_{(d,\xi^\cdot)}(t) \cdot g(t)\right\|$$

$$\leq \left\| \int_a^{b\bullet} dI_{(d,\xi^\bullet)}(t) \cdot g(t) - \int_a^{b\bullet} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(t) \right\|$$

$$+ \left\| \int_a^{b\bullet} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(t) - \sum_{j=1}^{|\bar{d}|} \int_{\bar{t}_{j-1}}^{\bar{t}_j \bullet} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(\bar{\xi}_j^\bullet) \right\|$$

$$+ \left\| \sum_{j=1}^{|\bar{d}|} \int_{\bar{t}_{j-1}}^{\bar{t}_j \bullet} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(\bar{\xi}_j^\bullet) \right.$$

$$\left. - \sum_{j=1}^{|\bar{d}|} \left[I_{(d,\xi^\bullet)}(\bar{t}_j) - I_{(d,\xi^\bullet)}(\bar{t}_{j-1}) \right] g(\bar{\xi}_j^\bullet) \right\|$$

$$+ \left\| \sum_{j=1}^{|\bar{d}|} \left[(I_{(d,\xi^\bullet)}(\bar{t}_j) - I(\bar{t}_j)) - (I_{(d,\xi^\bullet)}(\bar{t}_{j-1}) - I(\bar{t}_{j-1})) \right] g(\bar{\xi}_j^\bullet) \right\|$$

$$+ \left\| \sum_{j=1}^{|\bar{d}|} \left[I(\bar{t}_j) - I(\bar{t}_{j-1}) \right] g(\bar{\xi}_j^\bullet) - \int_a^{b\bullet} dI(t) \cdot g(t) \right\|.$$

By the definition of $I_{(d,\xi^\bullet)}$ the third summand is zero; by Lemma 5.5$^\bullet$ the first summand is zero because $f_{d,\xi^\bullet} \in E([a, b], E_2) \subset S_\alpha^\bullet([a, b], E_2) = D_\alpha^\bullet([a, b], E_2)$. The second summand is $\leq \|f\| \Sigma_{j=1}^{|\bar{d}|} V_j[\alpha]\, \omega_j^\bullet(g)$ *for all* $(d, \xi^\bullet) \in \mathcal{D}^\bullet$, and so is the fifth summand by Theorem 5.2$^\bullet$ and by Theorem 5.1$^\bullet$ (c). Hence, given $\varepsilon > 0$, there is a $d_\varepsilon \in D$ such that for $\bar{d} \geq d_\varepsilon$ the second and fifth summands are $< \varepsilon/3$ for all $d \in D$, we fix such a $\bar{d}$; then by Proposition 1.3$^\bullet$ the fourth summand becomes $< \varepsilon/3$ for all d sufficiently "large."

<u>PROPOSITION 5.6</u>. Under the hypothesis of Theorem 5.8 we have

$$\int_a^b d_t\left[\int_a^t d\alpha(s) \cdot f(s)\right] g(t) = \lim_{\Delta d \to 0} \int_a^b d_t\left[\int_a^t d\alpha(s) \cdot f_{d,\xi}(s)\right] g(t).$$

Proof. The proof follows the steps of the proof of Proposition 5.6$^\bullet$.

LEMMA 5.7•. Under the hypothesis of Theorem 5.8• we have

$$\left\| \int_a^{\bullet b} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(t) - \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i^\bullet)\, g(\xi_i^\bullet) \right\|$$

$$\leq \|f\| \sum_{i=1}^{|d|} v_i[\alpha]\, \omega_i^\bullet(g).$$

The proof is obvious.

LEMMA 5.7. Under the hypothesis of Theorem 5.8 we have

$$\left\| \int_a^{\bullet b} d\alpha(t) \cdot f_{d,\xi}(t)\, g(t) - \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i)\, g(\xi_i) \right\|$$

$$\leq \|f\| \sum_{i=1}^{|d|} v_i[\alpha]\, \omega_i(g).$$

Proof of Theorem 5.8•. The first integral in (S•) exists by Theorem 5.2• and is, by Proposition 5.6•, equal to

$$\lim_{d\in D} \int_a^{\bullet b} d_t\left[\int_a^{\bullet t} d\alpha(s) \cdot f_{d,\xi^\bullet}(s)\right] g(t),$$

which by Lemma 5.5• is equal to

$$\lim_{d\in D} \int_a^{\bullet b} d\alpha(t) \cdot f_{d,\xi^\bullet}(t)\, g(t);$$

by Lemma 5.7• this limit is equal to

$$\lim_{d\in D} \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] \cdot f(\xi_i^\bullet)\, g(\xi_i^\bullet)$$

and by definition this is $\int_a^{\bullet b} d\alpha(t) \cdot f(t)\, g(t)$.

Proof of Theorem 5.8. The proof follows the steps of the proof of Theorem 5.8• (applying Theorem 5.2, Proposition 5.6, etc. instead of Theorem 5.2•, Proposition 5.6•, etc.).

COROLLARY 5.9'. Let (E, F, G) be a BT, $v \in BV([a, b], \mathbb{R})$, $f \in R_v^{\bullet}([a, b], E)$, and $g \in D_v^{\bullet}([a, b], F)$; we have

$$\int_a^{b\,\bullet} d_t\left(\int_a^{t\,\bullet} f(s)\, dv(s)\right) g(t) = \int_a^{b\,\bullet} f(t)\, g(t)\, dv(t).$$

COROLLARY 5.9. Let (E, F, G) be a BT, $v \in BV([a, b], \mathbb{R})$, $f \in R_v([a, b], E)$, and $g \in D_v([a, b], F)$; we have

$$\int_a^b d_t\left(\int_a^t f(s)\, dv(s)\right) g(t) = \int_a^b f(t)\, g(t)\, dv(t).$$

COROLLARY 5.10'. Let (E, F, G) be a BT, $v \in BV([a, b], \mathbb{R})$, $f \in R_v^{\bullet}([a, b], E)$, and $g \in D_v^{\bullet}([a, b], F)$; we have

$$\int_a^{b\,\bullet} f(t)\, d_t\left(\int_a^{t\,\bullet} g(s)\, dv(s)\right) = \int_a^{b\,\bullet} f(t)\, g(t)\, dv(t).$$

Proof. By Theorem 5.8' the second member exists, and by Theorem 5.3' it follows that the first one exists and that they are equal.

COROLLARY 5.10. Let (E, F, G) be a BT, $v \in BV([a, b], \mathbb{R})$, $f \in R_v([a, b], E)$, and $g \in D_v([a, b], F)$; we have

$$\int_a^b f(t)\, d_t\left(\int_a^t g(s)\, dv(s)\right) = \int_a^b f(t)\, g(t)\, dv(t).$$

Remark. Even in the particular case of Corollary 5.9, when we take $v(t) \equiv t$ the proof of $\int_a^b d_t[\int_a^t f(s)\, ds]\, g(t) = \int_a^b f(t)\, g(t)\, dt$ does not become essentially simpler then the proof of the general Theorem 5.8'!

Let E be a Banach space and $1 \leq p \leq \infty$; $L_p([a, b], E)$ denotes the Banach space of (equivalence classes of) functions

$f: [a, b] \to E$ that are p-integrable in the sense of Bochner and Lebesgue (see [8]). It is easy to prove that $S^-([a, b], E) \subset L_1([a, b], E)$, hence by Theorem 4.5^c we have $D([a, b], E) \subset L_1([a, b], E)$ and even $D([a, b], E) \subset L_\infty([a, b], E)$ (see also Corollary 4.15). We write $\alpha \in L_1^{(1)}([a, b], E)$ if there exists $\beta \in L_1([a, b], E)$ such that

$$\alpha(t) = \alpha(a) + {}_L\int_a^t \beta(s)\, ds$$

(where ${}_L\int$ denotes the integral of Bochner-Lebesgue), then α is absolutely continuous, differentiable a.e., and satisfies $\alpha' = \beta$ a.e. (see [8]); we have also

$$V[\alpha] = \|\alpha'\|_1 = {}_L\int_a^b \|\alpha'(t)\|\, dt.$$

Indeed: given $d \in D$ we have

$$\|\alpha(t_i) - \alpha(t_{i-1})\| = \left\| {}_L\int_{t_{i-1}}^{t_i} \alpha'(t)\, dt \right\| \le {}_L\int_{t_{i-1}}^{t_i} \|\alpha'(t)\|\, dt,$$

hence $V[\alpha] \le {}_L\int_a^b \|\alpha'(t)\|\, dt$. Reciprocally, it is immediate that we have $\|\alpha'(t)\| \le \hat{\alpha}'(t)$ a.e., hence

$${}_L\int_a^b \|\alpha'(t)\|\, dt \le \int_a^b \hat{\alpha}'(t)\, dt = \hat{\alpha}'(b) = V[\alpha].$$

<u>COROLLARY 5.11</u>. Let (E, F, G) be a BT, $\alpha \in L_1^{(1)}([a, b], E)$, and $f \in R([a, b], F)$; then there exists $\int_a^b d\alpha(t) \cdot f(t)$.

<u>Proof</u>. Let us take a sequence $\beta_n \in C([a, b], E)$ such that $\|\beta_n - \alpha'\|_1 \to 0$ and define $\alpha_n(t) = \alpha(a) + \int_a^t \beta_n(s)\, ds$, $t \in [a, b]$. Given $d, \bar{d} \in D$ with $\bar{d} \ge d$ we write $\bar{I}_j = [\bar{t}_{j-1}, \bar{t}_j]$ and $I_i = [t_{i-1}, t_i]$ and we have

$$\|\sigma_{\bar{d},\bar{\xi}}(f; \alpha) - \sigma_{d,\xi}(f; \alpha)\| = \left\| \sum_{i=1}^{|d|} \sum_{\bar{I}_j \subset I_i} \left(\alpha(\bar{t}_j) - \alpha(\bar{t}_{j-1})\right) \times \left(f(\bar{\xi}_j) - f(\xi_i)\right) \right\|$$

$$\leq \left\| \sum_{i=1}^{|d|} \sum_{\overline{I}_j \subset I_i} \left\{ \left[\alpha(\overline{t}_j) - \alpha_n(\overline{t}_j)\right] - \left[\alpha(\overline{t}_{j-1}) - \alpha_n(\overline{t}_{j-1})\right] \right\} \times \left[f(\overline{\xi}_j) - f(\xi_i)\right] \right\|$$

$$+ \left\| \sum_{i=1}^{|d|} \sum_{\overline{I}_j \subset I_i} \left[\alpha_n(\overline{t}_j) - \alpha_n(\overline{t}_{j-1})\right] \left[f(\overline{\xi}_j) - f(\xi_i)\right] \right\|$$

$$\leq V[\alpha - \alpha_n]\; 2\|f\| + \|\sigma_{\overline{d},\overline{\xi}}(f;\alpha_n) - \sigma_{d,\xi}(f;\alpha_n)\|;$$

since $V[\alpha - \alpha_n] = \|\alpha' - \alpha_n'\|_1 = \|\alpha' - \beta_n\|_1$, given $\varepsilon > 0$ there exists an $n_\varepsilon \in N$ such that for $n \geq n_\varepsilon$ we have $\|\alpha' - \beta_n\|_1 \leq \varepsilon/2$, and if we fix such an n by Corollary 5.9 there exists $\int_a^b d\alpha_n(t) \cdot f(t) = \int_a^b \alpha_n'(t)\, f(t)\, dt$. Hence there exists $\delta > 0$ such that for $\Delta d \leq \delta$ we have

$$\|\sigma_{\overline{d},\overline{\xi}}(f;\alpha_n) - \sigma_{d,\xi}(f;\alpha_n)\| \leq \frac{\varepsilon}{2}. \quad \text{Q.E.D.}$$

<u>COROLLARY 5.12</u>. Let (E, F, G) be a BT, $\alpha \in L_1^{(1)}([a, b], E)$, and $f \in D([a, b], F)$; then there exists $\mathcal{L}\int_a^b \alpha'(t)\, f(t)\, dt = \int_a^b d\alpha(t) \cdot f(t)$.

<u>Proof</u>. Since $D([a, b], F) \subset L_\infty([a, b], F)$, the first integral exists. For every $d \in D$ we have

$$\left\| \mathcal{L}\int_a^b \alpha'(t)\, f(t)\, dt - \sum_{i=1}^{|d|} \left[\alpha(t_i) - \alpha(t_{i-1})\right] f(\xi_i) \right\|$$

$$= \left\| \sum_{i=1}^{|d|} \mathcal{L}\int_{t_{i-1}}^{t_i} \alpha'(t) \left[f(t) - f(\xi_i)\right] dt \right\|$$

$$\leq \sum_{i=1}^{|d|} \mathcal{L}\int_{t_{i-1}}^{t_i} \|\alpha'(t)\| \|f(t) - f(\xi_i)\|\, dt$$

$$\leq \sum_{i=1}^{|d|} V_i[\alpha]\; \omega_i(f)$$

that goes to zero when $\Delta d \to 0$ since by Corollary 4.9 we have $f \in D_\alpha([a, b], F)$.

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