WELL-POSEDNESS OF ABSTRACT INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

EDUARDO HERNÁNDEZ § ♮, DENIS FERNANDES ‡, AND JIANHONG WU

Abstract. We study existence, uniqueness and well-posedness for a general class of abstract integro-differential equations with state-dependent delay. In the last section, some examples concerning partial integro-differential equations with state-dependent delay are presented.

1. Introduction

In this work, we continue our studies in [13, 15] on abstract differential equations with state-dependent delay. Specifically, we study the global existence and uniqueness of strict solutions and well-posedness for a general class of abstract integro-differential equations with state-dependent delay of the form

$$(1.1)\quad u'(t) = Au(t) + F\left(t, u(t), \int_0^t K(t,\tau)u(\tau - \sigma(\tau, u(\tau)))d\tau\right), \qquad t \in [0,a],$$

$$(1.2)\quad u_0 = \varphi \in C_{Lip}([-p,0];X),$$

where $A : D(A) \subset X \to X$ is the generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$, $K(\cdot)$ is an operator valued map and $F(\cdot), \sigma(\cdot)$ are suitable continuous functions.

The theory of differential equations with state-dependent delay is field of intensive research because applications and the fact that the qualitative theory is quite different from the theories of equations with constant and time-dependent delay. Concerning the literature, for ODEs on finite dimensional spaces we cite the early papers by Driver [5, 6], the survey by Hartung, Krisztin, Walther & Wu [10], the works by Aiello, Freedman & Wu [1], the paper by Walther [23] and the references therein. For abstract and partial differential equations, we mention the early paper by Hernandez, Prokopczyk & Ladeira [12] and the recent interesting papers [13, 14, 15, 16, 17, 18, 20, 21].

As pointed, the theory of SDD differential equations is different to the usual ones on differential equations with memory. In particular, partial and abstract differential equations with SDD are (in general) not well-possed in spaces of continuous functions since in this case, functions of the form $u \to u(\cdot, \sigma(\cdot, u(\cdot)))$ are (in general) not Lipschitz. In addition, the nonlinearity of the function $t \to u(t - \sigma(t, u(t)))$ has

2000 *Mathematics Subject Classification.* 34J, 34K30, 47D06 .

Key words and phrases. State dependent delay, strict solutions, analytic semigroup.

§ Corresponding author. ♮ The work of this author is supported by Fapesp 2017/13145-8.

‡ The work of this author is supported by Fapesp 2017/01776-3 .

natural implications concerning the global existence of solutions. From the above, the global existence and the long time behavior of solutions, are relevant and hard problems in the general theory of SDD differential equations.

The study of the model (1.1)-(1.2) is motivated by an extensive literature on ordinary and partial integro-differential equations arising in the theory of population dynamics, see in particular, the influent works by Briton [3], the paper [7, 8, 9] and the references therein. As motivation, we also mention the work by Cooke & Huang [4] and Alt [2] on ordinary integro-differential equations with state-dependent delay and by Zhang & Vandewalle [25] on integro-differential equations with memory.

In our studies, we establish non-restrictive conditions under which the problem (1.1)-(1.2) is well-posed related the spaces

$$\mathcal{B}_{Lip,A} = \{\varphi \in C_{Lip}([-p,0];X) : \varphi(0) \in D(A)\},$$

endowed with the norm $\| \varphi \| = \| \varphi \|_{C([-p,0];X)} + \| A\varphi(0) \|$, and $C([-p,a];X)$, (see Theorem 2.1). From Theorem 2.1 we obtain that $\| u_t(\cdot,\varphi) - u_s(\cdot,\varphi) \|_{C([-p,0];X)} \to 0$ as $s \to t$ and $\| u_t(\cdot,\varphi) - u_t(\cdot,\psi) \|_{C([-p,0];X)} \to 0$ as $\| \psi - \varphi \|_{\mathcal{B}_{Lip,A}} \to 0$, where $u(\cdot,\psi)$ denotes the unique strict solution of (1.1) with initial condition ψ. Similar results are proved related to spaces of C^1 functions.

To prove our results, we use some of the ideas in [13, 15]. Using the developments in [15], we establish the "global" existence and "uniqueness" of strict solutions for (1.1)-(1.2), an unconsidered problem in [15], working in $\mathcal{B}_{Lip,A}$. In addition, from [15] we assume a natural and non-restrictive integrability condition on the operator valued function $K(\cdot)$ (the condition $\mathcal{L}_\alpha$), which allows us to obtain useful estimates for $[u_K(\cdot,\varphi) - v_K(\cdot,\psi)]_{C^\alpha([0,a];X)}$ and $[F(\cdot,u(\cdot),u_K(\cdot)) - F(\cdot,v(\cdot),v_K(\cdot))]_{C^\alpha([0,a];X)}$, where $z_K : [0,a] \to X$ is the function given by $z_K(t) = \int_0^t K(t,s)z(s-\sigma(s,z(s)))ds$. This estimates are fundamentals to prove our results.

This work have three sections. Our abstract results are presented in Section 2. In Proposition 2.2 and Corollary 2.1 we prove the global existence and uniqueness of strict solutions for (1.1)-(1.2). This result extend those in [13, 15] on local existence and uniqueness of solutions. In the same section are established different estimates for $u(\cdot,\varphi)$, $u(\cdot,\varphi) - u(\cdot,\psi)$ and $F(\cdot,u(\cdot),u_K(\cdot)) - F(\cdot,v(\cdot),v_K(\cdot))$, which are necessaries to prove our mains results, Theorem 2.1 and Proposition 2.4.

We include now some notations and results. Let $(Z, \| \cdot \|_Z)$ be Banach space. For $r > 0$ and $z \in Z$, $B_r(z,Z) = \{x \in Z : \| x - z \|_Z \leq r\}$. The norms of the spaces $C([b,c];Z)$, $C^\gamma([b,c];Z)$, $\gamma \in (0,1)$, and $C_{Lip}([b,c];Z)$ are denoted by $\| \cdot \|_{C([b,c];Z)}$, $\| \cdot \|_{C^\gamma([b,c];Z)}$ and $\| \cdot \|_{C_{Lip}([b,c];Z)}$. We only remark that $\| \cdot \|_{C^\gamma([b,c];Z)} = \| \cdot \|_{C([b,c];Z)} + [\cdot]_{C^\gamma([b,c];Z)}$ and $\| \cdot \|_{C_{Lip}([b,c];Z)} = \| \cdot \|_{C([b,c];Z)} + [\cdot]_{Lip([b,c];Z)}$ where $[\xi]_{C^\gamma([b,c];Z)} = \sup_{t,s\in[b,c],t\neq s} \frac{\|\xi(s)-\xi(t)\|_Z}{|t-s|^\gamma}$ and $[\zeta]_{C_{Lip}([b,c];Z)} = \sup_{t,s\in[b,c],t\neq s} \frac{\|\zeta(s)-\zeta(t)\|_Z}{|t-s|}$.

In this work we assume that $0 \in \rho(A)$ and C_i $(i = 0,1)$ are constants such that $\| A^i T(t) \|_{\mathcal{L}(X)} \leq \frac{C_i}{t^i}$ for all $t \in (0,a]$. We include now some remarks on the problem

$$u'(t) = Au(t) + \xi(t), \quad t \in [0,a], \quad u(0) = x \in X. \tag{1.3}$$

The function $u \in C([0,b];X)$, $0 < b \leq a$, given by $u(t) = T(t)x + \int_0^t T(t-s)\xi(s)ds$, is called a mild solution of (1.3) on $[0,b]$. A function $v \in C([0,b];X)$ is said to be a strict solution of (1.3) on $[0,b]$ if $v \in C^1([0,b];X)$, $v(t) \in D(A)$ for all $t \in [0,b]$, $Av \in C([0,b];X)$ and $v(\cdot)$ satisfies (1.3) on $[0,b]$.

From [19], we note the followings results. In the next lemmas, $u \in C([0,b];X)$ is the mild solution of (1.3).

Lemma 1.1. *If $\xi \in C_{Lip}([0,b];X)$ and $x \in X_1$, then $u(\cdot)$ is a strict solution of (1.3) on $[0,b]$ and $\| Au \|_{C([0,b];X)} \leq C_0 \| Ax \| + (C_0+1) \| \xi \|_{C([0,b];X)} + C_1[\xi]_{C_{Lip}([0,b],X)} b$.*

Lemma 1.2. *If $\xi \in L^\infty([0,b];X)$, $\alpha \in (0,1)$ and $T(\cdot)x \in C^\alpha([0,a];X)$, then $u \in C^\alpha([0,b];X)$ and $[u]_{C^\alpha([0,b];X)} \leq \| \xi \|_{L^\infty([0,b];X)} (\frac{C_1}{\alpha(1-\alpha)} + C_0) b^{1-\alpha} + [T(\cdot)x]_{C^\alpha([0,a];X)}$.*

Lemma 1.3. *If $\alpha \in (0,1)$, $\xi \in C^\alpha([0,b];X)$ and $x \in X_1$, then $u(\cdot)$ is a strict solution of (1.3) on $[0,b]$ and $\| Au \|_{C([0,b];X)} \leq C_0 \| Ax \| + (C_0+1) \| \xi \|_{C([0,b];X)} + C_1[\xi]_{C^\alpha([0,b],X)} b^\alpha \alpha^{-1}$.*

2. Well-Posedness

In this section we study the global existence and uniqueness of strict solutions and the well-posedness of the problem (1.1)-(1.2). In the remainder of this work, $\mathcal{B}_{Lip,A}$ and $\mathcal{B}^1_{Lip,A}$ are the spaces

$$\begin{aligned} \mathcal{B}_{Lip,A} &= \{\varphi \in C_{Lip}([-p,0];X) : \varphi(0) \in D(A)\}, \\ \mathcal{B}^1_{Lip,A} &= \{\varphi \in C^1([-p,0];X) : \varphi(0) \in D(A),\ \varphi'(0^-) = A\varphi(0)\}, \end{aligned}$$

endowed with the norms $\| \varphi \|_{\mathcal{B}_{Lip,A}} = \| A\varphi(0) \| + \| \varphi \|_{C([-p,0];X)}$ and $\| \varphi \|_{\mathcal{B}^1_{Lip,A}} = \| A\varphi(0) \| + \| \varphi \|_{C^1([-p,0];X)}$ respectively. The choice of the spaces $\mathcal{B}_{Lip,A}$ and $\mathcal{B}^1_{Lip,A}$ is based in technical considerations on semigroup theory. The above spaces are early considered in [16, 20, 23].

Notations 1. For convenience, for a function $u \in C([-p,b];X)$, $b \in (0,a]$, we use the symbols u^σ and u_K for the functions $u^\sigma : [0,b] \to X$ and $u_K : [0,b] \to X$ defined by $u^\sigma(t) = u(t - \sigma(t,u(t)))$ and $u_K(t) = \int_0^t K(t,s) u(s - \sigma(s,u(s))) ds$.

Remark 2.1. For simplicity, we always assume that $F(\cdot)$ and $\sigma(\cdot)$ are Lipschitz and we write $[F]_{C_{Lip}}$ and $[\sigma]_{C_{Lip}}$ instead $[F]_{C_{Lip}([0,a]\times X\times X;X)}$ and $[\sigma]_{C_{Lip}([0,a]\times X;[0,p])}$. Similarly, for $v \in C_{Lip}([-p,b];X)$ we write $[v]_{C_{Lip}[-p,b]}$ in place $[v]_{C_{Lip}([-p,b];X)}$. Similar notations are adopted for another type of spaces of continuous functions.

Definition 2.1. *A function $u \in C([-p,b];X)$ is called a mild solution of (1.1)-(1.2) on $[-p,b]$ if $u_0 = \varphi$ and*

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s) F(s, u(s), u_K(s)) ds, \qquad \forall\ t \in [0,b]. \tag{2.1}$$

Definition 2.2. *A function $u \in C([-p,b];X)$ is called a strict solution of (1.1)-(1.2) on $[-p,b]$ if $u_{|_{[0,b]}} \in C^1([0,b];X)$, $u(t) \in D(A)$ for all $t \in [0,b]$, $Au_{|_{[0,b]}} \in C([0,b];X)$, $u_0 = \varphi$ and $u(\cdot)$ satisfies (1.1) on $[0,b]$.*

Concerning the above definitions, we note that a similar nomenclature is used for problems defined on $[-p,b)$. From [15], we include the following condition.

Condition $\mathfrak{L}_\alpha$: $\alpha \in (0,1]$, $K : [0,a] \times [0,a] \mapsto \mathcal{L}(X)$ is an integrable function, $K(t,\cdot) \in L^{\frac{1}{(1-\alpha)}}([0,t], \mathcal{L}(X,X))$ for all $t \in [0,a]$ (we take $L^{\frac{1}{(1-\alpha)}} = L^\infty$ for $\alpha = 1$); for all $b \in [0,a]$

$$\Theta(b) = \sup_{s\in[0,b]} (\int_0^s \| K(s,\tau) \|^{\frac{1}{1-\alpha}} d\tau)^{(1-\alpha)} < \infty, \text{ if } \alpha < 1, \tag{2.2}$$

$$\Theta(b) = \sup_{s\in[0,b]} \| K(s,\cdot) \|_{L^\infty([0,s];\mathcal{L}(Z,W))} < \infty, \quad \text{if } \alpha = 1, \tag{2.3}$$

and for all $s \in [0,a]$ there is a function $L_{K,\alpha,s} \in L^1([0,s];\mathbb{R}^+)$ such that

$$\| K(t,\tau) - K(s,\tau) \| \leq L_{K,\alpha,s}(\tau) \mid t-s \mid^\alpha, \ \ \forall \ 0 \leq \tau \leq s \leq t \leq a, \tag{2.4}$$

and $\Upsilon(b) = (\Theta(b) + \sup_{s\in[0,b]} \| L_{K,\alpha,s} \|_{L^1([0,s];\mathbb{R})}) < \infty$ for all $b \in [0,a]$.

We include now some useful Lemmas. The proof of Lemma 2.4 follows proceeding as in the proof of [15, Lemma 3.1]. For completeness, we include a short proof.

Lemma 2.4. *Let condition $\mathfrak{L}_\alpha$ hold, $u \in C([-p,b];X)$ and $v \in C_{Lip}([-p,b];X)$. Then $u_K \in C^\alpha([0,b];X)$, $[u_K]_{C[0,b]} \leq \Upsilon(b) \| u \|_{C[-p,b]}$ and*

$$\begin{aligned}
\| u^\sigma - v^\sigma \|_{C[0,b]} &\leq \| \varphi - \psi \|_{C[-p,0]} + \Psi(b,v,\psi) \| u - v \|_{C[0,b]}, \\
\| u_K - v_K \|_{C[0,b]} &\leq \Theta(b) b^\alpha (\| \varphi - \psi \|_{C[-p,0]} + \Psi(b,v,\psi) \| u - v \|_{C[0,b]}), \\
[u_K - v_K]_{C[0,b]} &\leq \Upsilon(b)(\| \varphi - \psi \|_{C[-p,0]} + \Psi(b,v,\psi) \| u - v \|_{C[0,b]}),
\end{aligned}$$

where $u_0 = \varphi$, $v_0 = \psi$ and $\Psi(b,v,\psi) = 1 + ([\psi]_{C_{Lip}[-p,0]} + [v]_{C_{Lip}[0,b]})[\sigma]_{C_{Lip}}$.

Proof: The inequality for $[u_K]_{C^\alpha([0,b];X)}$ is proved in [15, Lemma 3.1]. Related to the second one, we note that

$$\begin{aligned}
&\| u^\sigma - v^\sigma \|_{C[0,b]} \\
&\leq \| u(\cdot - \sigma(\cdot, u(\cdot))) - v(\cdot - \sigma(\cdot, u(\cdot))) \|_{C[0,b]} \\
&\quad + \| v(\cdot - \sigma(\cdot, u(\cdot))) - v(\cdot - \sigma(\cdot, v(\cdot))) \|_{C[0,b]} \\
&\leq \| u - v \|_{C[-p,b]} + [v]_{C_{Lip}[-p,b]} \mid \sigma(\cdot, u(\cdot)) - \sigma(\cdot, v(\cdot)) \mid_{C[0,b]} \\
&\leq \| u - v \|_{C[-p,b]} + [v]_{C_{Lip}[-p,b]} [\sigma]_{C_{Lip}} \| u - v \|_{C[0,b]} \\
&\leq \| \varphi - \psi \|_{C[-p,0]} + (1 + ([\psi]_{C_{Lip}[-p,0]} + [v]_{C_{Lip}[0,b]})[\sigma]_{C_{Lip}} \| u - v \|_{C[0,b]}.
\end{aligned}$$

Noting that $\| u_K - v_K \|_{C[0,b]} \leq \| u^\sigma - v^\sigma \|_{C[0,b]} \Theta(b) b^\alpha$ we obtain the third inequality. To prove the last inequality, for $0 \leq s < t \leq b$ we note that

$$\begin{aligned}
&\| u_K(t) - v_K(t) - (u_K(s) - v_K(s)) \| \\
&\leq (\int_0^s \| K(t,\tau) - K(s,\tau) \| d\tau + \int_s^t \| K(t,\tau) \| d\tau) \| u^\sigma - v^\sigma \|_{C[0,b]} \\
&\leq (\| L_{K,\alpha,s} \|_{L^1([0,s];\mathbb{R})} + \Theta(b)) \| u^\sigma - v^\sigma \|_{C[0,b]} (t-s)^\alpha,
\end{aligned}$$

which allows us to end the proof using the second inequality. ■

Next, in Proposition 2.1 and Proposition 2.1 we establish the global existence of strict solution, an unconsidered problem in [15].

Proposition 2.1. *Assume that the semigroup $(T(t))_{t\geq 0}$ is compact, the condition $\mathfrak{L}_\alpha$ is satisfied, $\alpha \in (0,1)$ and $\varphi \in \mathcal{B}_{Lip,A}$. Then there exists a unique strict solution $u(\cdot,\varphi) \in C_{Lip}([-p,b];X)$ of (1.1)-(1.2) on $[-p,b]$ for some $0 < b \leq a$.*

Proof: Let $R > C_0 \parallel \varphi \parallel_{C[-p,0]}$. We select now $0 < b \leq \min\{1,a\}$ such that $R > C_0(\parallel \varphi \parallel_{C[-p,0]} + \parallel F(\cdot,0,0) \parallel_{C[0,a]} b + L_F R(1+\Upsilon(a)a^\alpha)b)$.

Let $\mathcal{S}(R,\varphi) = \{u \in C([-p,b];X) : u_0 = \varphi, \parallel u \parallel_{C[-p,b]} \leq R\}$, endowed with the metric $d(u,v) = \parallel u - v \parallel_{C([0,b];X)}$ and $\Gamma : \mathcal{S}(R,\varphi) \mapsto C([-p,b];X)$ be the map defined by $\Gamma u = \varphi$ on $[p,0]$ and

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s,u(s),u_{_K}(s))ds, \qquad \text{for } t \in [0,b].$$

From Lemma 2.4, for $u \in \mathcal{S}(R,\varphi)$ and $t \in [0,b]$ we note that

$$\parallel \Gamma u(t) \parallel \leq C_0(\parallel \varphi \parallel_{C[-p,0]} + \parallel F(\cdot,0,0) \parallel_{C[0,a]} b + L_F R(1+\Upsilon(a)a^\alpha)b) \leq R,$$

which implies that $\Gamma\mathcal{S}(R,\varphi) \subset \mathcal{S}(R,\varphi)$. In addition to the above, by noting that $\widetilde{R} := \sup_{u\in\mathcal{S}(R,\varphi)} \parallel F(\cdot,u(\cdot),u_{_K}(\cdot)) \parallel_{C[0,b]} < \infty$, for $t \in [0,b)$ and $h > 0$ such that $t+h \in [0,b]$ we see that

$$\begin{aligned} &\parallel \Gamma u(t+h) - \Gamma u(t) \parallel \\ &\leq \ \parallel (T(t+h) - T(t))\varphi(0) \parallel + C_0\widetilde{R}h + \widetilde{R}\int_0^t \parallel T(t+h-s) - T(t-s) \parallel ds, \end{aligned}$$

which implies that $\{\Gamma u : u \in \mathcal{S}(R,\varphi)\}$ is right equicontinuous on $[0,b)$. A similar argument proves that $\{\Gamma u : u \in \mathcal{S}(R,\varphi)\}$ is left equicontinuous on $(0,b]$. In addition, for $u \in \mathcal{S}(R,\varphi)$, $t \in (0,b]$ and $0 < \varepsilon < t$ we have that

$$\begin{aligned} \Gamma u(t) &= T(t)\varphi(0) + T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)F(s,u(s),u_{_K}(s))ds \\ &\quad + \int_{t-\varepsilon}^t T(t-s)F(s,u(s),u_{_K}(s))ds \\ &\in \{T(t)\varphi(0)\} + T(\varepsilon)(t-\varepsilon)C_0\widetilde{R}B_1(0;X) + C_0\widetilde{R}\varepsilon\Lambda B_1(0;X), \end{aligned}$$

and hence, $\Gamma(\mathcal{S}(R,\varphi))(t) = \{\Gamma u(t) : u \in \mathcal{S}(R,\varphi)\} \subset K_\varepsilon + D_\varepsilon$, where K_ε is compact in X and the diameter of the set D_ε converge to zero as $\varepsilon \downarrow 0$. This proves that $\Gamma(\mathcal{S}(R,\varphi))(t)$ is relatively compact.

From the above remarks, Γ is a completely continuous map from $\mathcal{S}(R,\varphi)$ into itself and from the Schauder's fixed point theorem, there exits a mild solution $u(\cdot,\varphi) \in \mathcal{S}(R,\varphi)$ of (1.1)-(1.2) on $[-p,b]$.

From Lemma 1.2, $u \in C^\alpha([0,b];X)$ which implies that $F(\cdot,u(\cdot),u_{_K})$ belongs to $C^\alpha([0,b];X)$. Using now Lemma 1.3 we infer that $u(\cdot)$ is a strict solution on $[-p,b]$ and $u_{|_{[0,b]}} \in C^1([0,b];X)$, which shows that $u \in C_{Lip}([-p,b];X)$ since $\varphi(\cdot)$ is Lipschitz.

To finish, we prove the uniqueness. If $v \in C([0,b];X)$ is a mild solution of (1.1)-(1.2) on $[-p,b]$, by using Lemma 2.4 we see that

$$\parallel u(t) - v(t) \parallel \leq C_0 \int_0^t [F]_{C_{Lip}}(1+\Upsilon(a)a^\alpha\Psi(b,u(\cdot,\varphi),\varphi)) \parallel u - v \parallel_{C[0,s]} ds,$$

and hence,

$$\| u - v \|_{C[0,t]} \leq C_0 \int_0^t [F]_{C_{Lip}} (1 + \Upsilon(a) a^{\alpha} \Psi(b, u(\cdot, \varphi), \varphi)) \| u - v \|_{C[0,s]} \, ds,$$

which implies that $u(\cdot) = v(\cdot)$ on $[0, b]$. ■

Corollary 2.1. *Let the conditions in Proposition 2.1 be satisfied and assume that* $L_F(1 + \Upsilon(a)a^{\alpha})a < 1$. *Then there exists a unique strict solution* $u(\cdot, \varphi) \in C_{Lip}([-p, a]; X)$ *of (1.1)-(1.2) on* $[-p, a]$.

Proof: From the assumption, we can select $R > 0$ sufficiently large such that $R > C_0(\| \varphi \|_{C[-p,0]} + \| F(\cdot, 0, 0) \|_{C[0,a]} \, a + L_F R(1 + \Upsilon(a)a^{\alpha})a)$. The proof can be completed now using the argument in the proof of Proposition 2.1 with 'a' in place 'b'. ■

Arguing as in the proof of [15, Theorem 3.2], we can prove the next proposition on the local existence of solution.

Proposition 2.2. *Let condition* $\mathfrak{L}_{\mathbf{1}}$ *be holds and* $\varphi \in \mathcal{B}_{Lip,A}$. *Then there exists a unique strict solution* $u(\cdot, \varphi) \in C_{Lip}([-p, b]; X)$ *of (1.1)-(1.2) on* $[-p, b]$ *for some* $0 < b \leq a$.

Proposition 2.3. *Suppose that the conditions in Proposition 2.1 or Proposition 2.2 are satisfied. If* $\varphi \in \mathcal{B}_{Lip,A}$, *then there exists a unique strict solution* $u(\cdot, \varphi) \in C_{Lip}([-p, a]; X)$ *of (1.1)-(1.2) on* $[-p, a]$ *and positive constants* $\Lambda_1(a), \Lambda_2(a)$, *independents of* $\varphi \in \mathcal{B}_{Lip,A}$, *such that*

$$\| u(\cdot, \varphi) \|_{C[0,a]} \leq \Lambda_1(a) \| \varphi \|_{C[-p,0]} + \Lambda_2(a). \tag{2.5}$$

Proof: From Proposition 2.1 or Proposition 2.2, there exists $0 < b_1 \leq a$ and a unique strict solution $u^1 \in C_{Lip}([-p, b_1]; X)$ of (1.1)-(1.2) on $[-p, b_1]$. Assuming $b_1 < a$ and noting that $u^1_b(\cdot, \varphi) \in \mathcal{B}_{Lip,A}$, from Proposition 2.1 or Proposition 2.2 we infer that there exits $0 < b_2 \leq a$ and a unique strict solution $u^2(\cdot, u_{b_1}) \in C_{Lip}([b_1 - p, b_1 + b_2]; X)$ of the problem

$$\begin{aligned} w'(t) &= Aw(t) + F(t, w(t), w^1_{_K}(t)), \qquad t \in [b_1, b_1 + b_2], \\ w_{b_1} &= u_{b_1}(\cdot, \varphi), \end{aligned}$$

where $w^1_{_K}(t) = \int_0^{b^1} K(t, \tau) u^1(\tau - \sigma(\tau, u(\tau)) d\tau + \int_{b^1}^t K(t, \tau) w(\tau - \sigma(\tau, w(\tau)) d\tau$.

Defining $u : [-p, b_1 + b_2] \to X$ by $u(\cdot) = u^1(\cdot)$ on $[-p, b_1]$ and $u(\cdot) = u^2(\cdot)$ on $[b_1, b_2]$, we have that $u(\cdot)$ is a strict solution of (1.1)-(1.2) on $[-p, b_1 + b_2]$. Moreover, proceeding in a standard manner we obtain a maximal (strict) solution $u(\cdot, \varphi) \in C(I_{\max}; X)$ of (1.1)-(1.2). Next we prove that $I_{max} = [-p, a]$.

Let $u(\cdot) = u(\cdot, \varphi)$ and $b_{\varphi} = \sup I_{\max}$. For $t \in [0, b_{\varphi})$, it is easy to see that

$$\| u(t) \| \leq C_0 \| \varphi(0) \| + C_0 \int_0^t ([F]_{C_{Lip}} (\| u(s) \| + \| u_{_K}(s) \|) + \| F(s, 0, 0) \|) ds,$$

and using Lemma 2.4 we obtain that

$$\begin{aligned} \| u \|_{C[-p,t]} &\leq (1 + C_0)(\| \varphi \|_{C[-p,0]} + a \| F(\cdot, 0, 0) \|_{C[0,a]}) \\ &\quad + C_0 [F]_{C_{Lip}} \int_0^t (1 + \Upsilon(a) a^{\alpha}) \| u \|_{C[-p,s]} \, ds, \end{aligned} \tag{2.6}$$

which implies that $u(\cdot)$ and $F(\cdot, u(\cdot), u_K(\cdot))$ are bounded on $I_{\max}$. Moreover, from Lemma 1.2 we obtain that $u \in C^\alpha([0, b_\varphi); X)$ and that $\lim_{t\to b_\varphi} u(t,\varphi)$ exists. Defining $v : [-p, b_\varphi] \to X$ by $v(\cdot) = u(\cdot)$ on $[-p, b_\varphi)$ and $v(b_\varphi) := \lim_{t\to b_\varphi} u(t, \varphi)$, it is easy to show that $v(\cdot)$ is a mild solution of (1.1)-(1.2) on $[-p, b_\varphi]$, which implies that $I_{\max} = [-p, b_\varphi]$. Moreover, we also have that $u \in C^\alpha([0, b_\varphi]; X)$ and $F(\cdot, u(\cdot), u_K(\cdot)) \in C^\alpha([0, b_\varphi]; X)$.

Assume $b_\varphi < a$. From Lemma 1.3 and Lemma 2.4, $u(\cdot)$ is a strict solution on $[-p, b_\varphi]$ and

$$\begin{aligned}\| Au \|_{C[0,b_\varphi]} &\le C_0 \| A\varphi(0) \| + (C_0 + 1) \| F(\cdot, u(\cdot), u_K(\cdot)) \|_{C[0,b]} \\ &\quad + C_1 [F(\cdot, u(\cdot), u_K(\cdot))]_{C^\alpha[0,b_\varphi]} b_\varphi^\alpha \alpha^{-1} \\ &\le C_0 \| A\varphi(0) \| + (C_0 + 1) \| F(\cdot, u(\cdot), u_K(\cdot)) \|_{C[0,b]} \\ &\quad + C_1 b_\varphi^\alpha \alpha^{-1} [F]_{C_{Lip}} (a^{1-\alpha} + [u]_{C^\alpha[0,b_\varphi]} + \Upsilon(b) \| u \|_{C[-p,b_\varphi]}),\end{aligned} \tag{2.7}$$

which implies that $Au(\cdot)$ is bounded on $[0, b_\varphi]$. Thus, $u'(\cdot, \varphi)$ is also bounded on $[0, b_\varphi]$ and $u(\cdot, \varphi) \in C_{Lip}([-p, b_\varphi]; X)$ because $\varphi \in C_{Lip}([-p, 0]; X)$.

Noting now that $u_{|[b_\varphi - p, b_\varphi]} \in C_{Lip}([b_\varphi - p, b_\varphi]; X)$ and that $u(b_\varphi, \varphi) \in D(A)$, from Proposition 2.1 or Proposition 2.2 we infer that there exists $c > 0$ and a strict solution $v \in C([b_\varphi - p, b_\varphi + c], X)$ of the problem

$$w'(t) = Aw(t) + F(t, w(t), w_K(t))), \qquad t \in [b_\varphi, b_\varphi + c], \tag{2.8}$$

$$w_{b_\varphi} = u_{b_\varphi}(\cdot, \varphi), \tag{2.9}$$

where $w_K(t) = \int_0^{b_\varphi} K(t,\tau) u(\tau - \sigma(\tau, u(\tau))) d\tau + \int_{b_\varphi}^t K(t,\tau) w(\tau - \sigma(\tau, w(\tau))) d\tau$, which allows us to construct a strict solution of (1.1)-(1.2) on $[-p, b_\varphi + c]$. This implies that $b_\varphi = a$.

From the above, $u(\cdot, \varphi)$ is a strict solution (1.1)-(1.2) on $[-p, a]$. Moreover, arguing as in the last part of the proof of Proposition 2.1 we prove the uniqueness of $u(\cdot, \varphi)$ and from (2.6) we obtain (2.5). This completes the proof. ■

Remark 2.2. Next, we always assume that the conditions in Proposition 2.3 are satisfied. In addition, $u(\cdot, \psi)$ denotes the unique strict solution in $C_{Lip}([-p, a]; X)$ of (1.1) with initial condition $\psi \in \mathcal{B}_{Lip,A}$ and for $u \in C([-p, a]; X)$, $F_{u(\cdot)}$ is the function $F_{u(\cdot)} : [0, a] \to X$ given by $F_{u(\cdot)}(\cdot) = F(\cdot, u(\cdot), u_K(\cdot))$.

Lemma 2.5. There exists positive constants $\Lambda_i(a)$, $i = 3, 4$, independents of $\varphi \in \mathcal{B}_{Lip,A}$, such that

$$\max\{\| Au(\cdot, \varphi) \|_{C[0,a]}, [u(\cdot, \varphi)]_{C_{Lip}[0,a]}\} \le \Lambda_3(a) \| \varphi \|_{\mathcal{B}_{Lip,A}} + \Lambda_4(a). \tag{2.10}$$

Proof: Let $\varphi \in \mathcal{B}_{Lip,A}$ and $u(\cdot) = u(\cdot, \varphi)$. From Lemma 2.4 we note that

$$\begin{aligned}\| F_{u(\cdot,\varphi)} \|_{C[0,a]} &\le [F]_{C_{Lip}} (\| u \|_{C[0,a]} + \| u_K \|_{C[0,a]}) + \| F_0 \|_{C[0,a]} \\ &\le [F]_{C_{Lip}} (\| u \|_{C[0,a]} + \Upsilon(a) a^\alpha \| u \|_{C[-p,a]}) + \| F_0 \|_{C[0,a]} \\ &\le [F]_{C_{Lip}} (\| u \|_{C[0,a]} + \Upsilon(a) a^\alpha (\| \varphi \|_{C([-p,0])} + \| u \|_{C[0,a]}) \\ &\quad + \| F_0 \|_{C[0,a]} \\ &\le \alpha_1(a) \| \varphi \|_{C([-p,0])} + \alpha_2(a) \| u \|_{C[0,a]} + \alpha_3(a),\end{aligned} \tag{2.11}$$

where $\alpha_i(a)$, $i=1,\ldots,3$ are constants independents of $\varphi(\cdot)$. Defining $\alpha_4(a) = (\frac{C_1}{\alpha(1-\alpha)}+C_0)a^{1-\alpha}$, from Lemma 1.2 and the last inequality, we get

$$\begin{aligned} [u]_{C^\alpha[0,a]} &\leq [T(\cdot)\varphi(0)]_{C^\alpha[0,a]} + \| F_{u(\cdot,\varphi)} \|_{C[-p,a]} \alpha_4(a) \\ &\leq C_0 a^{1-\alpha} \| A\varphi(0) \| + \alpha_1(a) \| \varphi \|_{C[-p,0]} \\ &\quad + \alpha_2(a) \| u \|_{C[0,a]} + \alpha_3(a) \\ &\leq \alpha_5(a) \| \varphi \|_{\mathcal{B}_{Lip,A}} + \alpha_2(a) \| u \|_{C[0,a]} + \alpha_3(a), \end{aligned} \tag{2.12}$$

where $\alpha_5(a)$, is a constant independent of $\varphi(\cdot)$. From the above, (2.7), (2.5), (2.11) and (2.12), we obtain that

$$\| Au(\cdot) \|_{C[0,a]} \leq \alpha_6(a) \| \varphi \|_{\mathcal{B}_{Lip,A}} + \alpha_7(a),$$

where $\alpha_i(a)$, $i=6,7$, are constants independents of $\varphi(\cdot)$. Moreover, we can obtain a similar inequality for $[u]_{C_{Lip}[0,a]}$ observing that $u(\cdot,\varphi)$ is a strict solution and that $[u]_{C_{Lip}[0,a]} \leq \| u' \|_{C[0,a]} \leq \| Au \|_{C[0,a]} + \| F_{u(\cdot,\varphi)} \|_{C[0,a]}$. ■

Lemma 2.6. *Let $\varphi,\psi \in \mathcal{B}_{Lip,A}$, $u=u(\cdot,\varphi)$ and $v=u(\cdot,\psi)$. Then $F(\cdot,u(\cdot),u_{_K}(\cdot))$ belongs to $C^\alpha([0,a];X)$ and*

$$\begin{aligned} &[F_{u(\cdot,\varphi)}]_{C^\alpha[0,a]} \\ &\leq [F]_{C_{Lip}}(a^{1-\alpha} + \Upsilon(a) \| \varphi \|_{C[-p,0]}) + (1+\Upsilon(a)) \| u \|_{C^\alpha[0,a]}), \\ &\| F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)} \|_{C[0,a]} \\ &\leq [F]_{C_{Lip}}(\Upsilon(a)a^\alpha \| \varphi-\psi \|_{C[-p,0]} + (1+\Upsilon(a)a^\alpha\Psi(a,v,\psi)) \| u-v \|_{C[0,a]}). \end{aligned}$$

Proof: From Lemma 2.4 it is easy to see that

$$\begin{aligned} &[F_{u(\cdot,\varphi)}]_{C^\alpha[0,a]} \\ &\quad \leq [F]_{C_{Lip}}(a^{1-\alpha} + [u]_{C^\alpha[0,a]} + \Upsilon(a) \| u \|_{C[-p,a]}) \\ &\quad \leq [F]_{C_{Lip}}(a^{1-\alpha} + \Upsilon(a) \| \varphi \|_{C[-p,0]}) + (1+\Upsilon(a)) \| u \|_{C^\alpha[0,a]}). \end{aligned}$$

On the other hand, from Lemma 2.4 we have that

$$\begin{aligned} &\| F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)} \|_{C[0,a]} \\ &\quad \leq [F]_{C_{Lip}}(\| u-v \|_{C[0,a]} + \| u_{_K} - v_{_K} \|_{C[0,a]}) \\ &\quad \leq [F]_{C_{Lip}}(\| u-v \|_{C[0,a]} + a^\alpha [u_{_K} - v_{_K}]_{C^\alpha[0,a]}) \\ &\quad \leq [F]_{C_{Lip}} \| u-v \|_{C[0,a]} \\ &\qquad + [F]_{C_{Lip}} \Upsilon(a)a^\alpha(\| \varphi-\psi \|_{C[-p,0]} + \Psi(a,v,\psi) \| u-v \|_{C[0,a]}) \\ &\quad \leq [F]_{C_{Lip}}(\Upsilon(a)a^\alpha \| \varphi-\psi \|_{C[-p,0]} + (1+\Upsilon(a)a^\alpha\Psi(a,v,\psi)) \| u-v \|_{C[0,a]}), \end{aligned}$$

which allows us to end the proof. ■

Lemma 2.7. *There exists $\Lambda_5(a)>0$ such that*

$$\| u(\cdot,\varphi) - v(\cdot,\psi) \|_{C[-p,a]} \leq \Lambda_5(a) \| \varphi-\psi \|_{\mathcal{B}_{Lip,A}} e^{\Lambda_5(a)\Psi(a,u(\cdot,\psi),\psi))a}, \tag{2.13}$$

for all $\varphi,\psi \in \mathcal{B}_{Lip,A}$.

Proof: From Lemma 2.6, for $t \in [0,a]$ we have that

$$\begin{aligned}
&\| u(\cdot,\varphi) - u(\cdot,\psi) \|_{C[0,t]} \\
&\leq C_0 \| \varphi - \psi \|_{C[-p,0]} + \int_0^t C_0 [F]_{C_{Lip}} \Upsilon(a) a^\alpha \| \varphi - \psi \|_{C[-p,0]} \, ds \\
&\quad + \int_0^t C_0 [F]_{C_{Lip}} (1 + \Upsilon(a) a^\alpha \Psi(a, u(\cdot,\psi), \psi)) \| u - v \|_{C[0,s]} \, ds \\
&\leq C_0 (1 + [F]_{C_{Lip}} \Upsilon(a) a^{1+\alpha}) \| \varphi - \psi \|_{C[-p,0]} \\
&\quad + C_0 [F]_{C_{Lip}} (1 + \Upsilon(a) a^\alpha \Psi(a, u(\cdot,\psi), \psi)) \int_0^t \| u - v \|_{C[0,s]} \, ds,
\end{aligned}$$

and hence, $\| u(\cdot,\varphi) - v(\cdot,\psi) \|_{C[0,a]} \leq \alpha_1(a) \| \varphi - \psi \|_{\mathcal{B}_{Lip,A}} e^{\alpha_2(a)\Psi(a,u(\cdot,\psi),\psi))a}$, where $\alpha_1(a), \alpha_2(a)$ are constants independents of $\varphi(\cdot)$ and $\psi(\cdot)$. Using this inequality we obtain (2.13). ■

To estimate $\| Au(\cdot,\varphi) - Au(\cdot,\psi) \|$, from [15] we include the following condition.

Condition $\mathfrak{F}_\alpha$: $\alpha \in (0,1)$, $F(\cdot)$ is Frechet differentiable on $X \times X$, $D_2F(\cdot)$ is continuous on $[0,b] \times (X \times X)$ and there is $L_F > 0$ such that

$$\begin{aligned}
&\| F(t,(x,y)) - F(s,(x,y)) \| \\
&\quad + \| D_2F(t,(x,y)) - D_2F(s,(x,y)) \|_{\mathcal{L}(X\times X, X)} \leq L_F \mid t - s \mid^\alpha, \\
&\| D_2F(s,(x_1,y_1)) - D_2F(s,(x_2,y_2)) \|_{\mathcal{L}(X\times X, X)} \\
&\qquad\qquad \leq L_F(\| x_1 - x_2 \| + \| y_1 - y_2 \|),
\end{aligned}$$

for all $0 \leq s,t \leq b \leq a$, $x, x_i, y, y_i \in X$.

Arguing as in the proof of [11, Lemma 2.2], we can prove the next result.

Lemma 2.8. *Let condition $\mathfrak{F}_\alpha$ be satisfied, $\varphi, \psi \in \mathcal{B}_{Lip,A}$, $u = u(\cdot,\varphi)$ and $v = u(\cdot,\psi)$. Then*

$$\begin{aligned}
&[F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)}]_{C^\alpha[0,a]} \\
&\leq L_F \mathbf{B}_\alpha(u,v)(\| u - v \|_{C[0,a]} + \| u_K - v_K \|_{C[0,a]}) \\
&\quad + (L_F \mathbf{B}(u,v) + \| d_2F(\cdot,0,0) \|_{C[0,a]})([u - v]_{C^\alpha[0,a]} + [u_K - v_K]_{C^\alpha[0,a]}), \qquad (2.14)
\end{aligned}$$

where $\mathbf{B}_\alpha(u,v) = 1 + [u]_{C^\alpha[0,a]} + [u_K]_{C^\alpha[0,a]} + [v]_{C^\alpha[0,a]} + [v_K]_{C^\alpha[0,a]}$ *and* $\mathbf{B}(u,v) = \| u \|_{C[0,a]} + \| u_K \|_{C[0,a]} + \| v \|_{C[0,a]} + \| v_K \|_{C[0,a]}$.

Remark 2.3. It is convenient re-write the inequalities in Lemma 2.8 and Lemma 2.6. For $h \in C^\alpha([-p,a];X)$, we note that

$$\begin{aligned}
[h]_{C^\alpha[0,a]} + [h_K]_{C^\alpha[0,a]} &\leq [h]_{C^\alpha[0,a]} + \Upsilon(a) \| h \|_{C[-p,a]} \\
&\leq \Upsilon(a) \| h_0 \|_{C[-p,0]} + (1 + \Upsilon(a)) \| h \|_{C^\alpha[0,a]}, \\
\| h \|_{C[0,a]} + \| h_K \|_{C[0,a]} & \\
&\leq \| h \|_{C[0,a]} + \Upsilon(a) a^\alpha \| h \|_{C[-p,a]} \\
&\leq \Upsilon(a) a^\alpha \| h_0 \|_{C[-p,0]} + (1 + \Upsilon(a) a^\alpha) \| h \|_{C[0,a]} .
\end{aligned}$$

We also note that there exists $\alpha(a) > 0$ independent of $\varphi(\cdot)$ and $\psi(\cdot)$ such that $\mathbf{B}_\alpha(u,v) + \mathbf{B}(u,v) \leq \alpha(a)\mathcal{S}_1(u,v)$, where $\mathcal{S}_1(u,v) = 1 + \| \varphi \|_{C[-p,0]} + \| \psi \|_{C[-p,0]}$

$+ \| u \|_{C^\alpha[0,a]} + \| v \|_{C^\alpha[0,a]}$. From the above, we can re-write the inequalities in Lemma 2.8 and Lemma 2.6 in the form

$$\begin{aligned} & [F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)}]_{C^\alpha[0,a]} \\ & \quad \leq \Lambda_6(a)\mathcal{S}_1(u,v)(\| \varphi - \psi \|_{C[-p,0]} + \| u - v \|_{C[0,a]} + [u-v]_{C^\alpha[0,a]}), \\ & \| F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)} \|_{C[0,a]} \\ & \quad \leq \Lambda_6(a)(\| \varphi - \psi \|_{C[-p,0]} + \Psi(a,v,\psi) \| u - v \|_{C[0,a]}), \end{aligned}$$

where $\Lambda_6(a)$ is a positive constant independent of $\varphi(\cdot)$ and $\psi(\cdot)$.

Lemma 2.9. *If the condition $\mathfrak{F}_\alpha$ is satisfied, then there exists $\Lambda_7(a) > 0$ such that*

$$\begin{aligned} & [u(\cdot,\varphi) - u(\cdot,\psi)]_{C^\alpha[-p,a]} \\ & \leq \Lambda_7(a)(\| \varphi - \psi \|_{\mathcal{B}_{Lip,A}} + \Psi(a,u(\cdot,\psi),\psi) \| u(\cdot,\varphi) - u(\cdot,\psi) \|_{C[0,a]}), \end{aligned} \tag{2.15}$$

for all $\varphi, \psi \in \mathcal{B}_{Lip,A}$.

Proof: Let $\alpha_1(a) = (\frac{C_1}{\alpha(1-\alpha)} + C_0)a^{1-\alpha}$. From Lemma 1.2 and Remark 2.3, we get

$$\begin{aligned} & [u(\cdot,\varphi) - u(\cdot,\psi)]_{C^\alpha[0,a]} \\ & \quad \leq a^{1-\alpha}C_0 \| A\varphi(0) - A\psi(0) \| + \alpha_1(a) \| F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)} \|_{C[0,a]} \\ & \quad \leq a^{1-\alpha}C_0 \| A\varphi(0) - A\psi(0) \| \\ & \qquad + \alpha_1(a)\Lambda_6(a)(\| \varphi - \psi \|_{C[-p,0]} + \Psi(a,u(\cdot,\psi),\psi) \| u - v \|_{C[0,a]}), \end{aligned}$$

which allows us to finish the proof. ■

Lemma 2.10. *There exists $\Lambda_8(a) > 0$ such that*

$$\begin{aligned} & \| Au(\cdot,\varphi) - Au(\cdot,\psi) \|_{C[0,a]} \\ & \quad \leq \Lambda_8(a)\mathcal{S}_1(u,v) \| \varphi - \psi \|_{\mathcal{B}_{Lip,A}} \\ & + \Lambda_8(a)(\mathcal{S}_1(u,v) + \mathcal{S}_1(u,v)\Psi(a,v,\psi) + \Psi(a,v,\psi)) \| u(\cdot,\varphi) - u(\cdot,\psi) \|_{C[0,a]}, \end{aligned} \tag{2.16}$$

for all $\varphi, \psi \in \mathcal{B}_{Lip,A}$, where $u = u(\cdot,\varphi)$ and $v = u(\cdot,\psi)$.

Proof: Noting that $\varphi(0) - \psi(0) \in D(A)$ and $F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)} \in C^\alpha([0,a];X)$ (see Lemma 2.8), we have that $u(\cdot) - v(\cdot)$ is a strict solution of

$$w'(t) = Aw(t) + F_{u(\cdot,\varphi)}(t) - F_{u(\cdot,\psi)}(t), \qquad t \in [0,a],$$

with initial condition $w(0) = \varphi(0) - \psi(0)$. From Lemma 1.3 we see that

$$\begin{aligned} & \| Au(\cdot,\varphi) - Au(\cdot,\psi) \|_{C[0,a]} \\ & \qquad \leq C_0 \| A\varphi(0) - A\psi(0) \| + (C_0+1) \| F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)} \|_{C[0,a]} \\ & \qquad\quad + C_1 \frac{a^\alpha}{\alpha}[F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)}]_{C^\alpha[0,a]}, \end{aligned} \tag{2.17}$$

and combining the estimates in Remark 2.3 and Lemma 2.9, we obtain (2.16). ■

To establish our next result, we include the next definition.

Definition 2.3. *Let $(\mathcal{S}, \| \cdot \|_S) \hookrightarrow C([-p,a];X)$ and $(\mathcal{W}, \| \cdot \|_W) \hookrightarrow C([-p,0];X)$ be normed spaces. We said the problem (1.1)-(1.2) is well-posed related $(\mathcal{W}, \| \cdot \|_S)$ and $(\mathcal{S}, \| \cdot \|_\mathcal{S})$, if for all $\varphi \in \mathcal{W}$ there exists a unique mild solution $u(\cdot,\varphi)$ of (1.1)-(1.2), $u(\cdot,\varphi) \in \mathcal{S}$ and $\| u(\cdot,\varphi) - u(\cdot,\psi) \|_\mathcal{S} \to 0$ as $\| \psi - \varphi \|_\mathcal{W} \to 0$.*

We can establish now our first theorem. In this result, $\mathcal{B}_{Lip,a,A}$ is the space

$$\mathcal{B}_{Lip,a,A} = \{u \in C([-p,a];X) : u_0 \in \mathcal{B}_{Lip,A},\, u_{|_{[0,a]}} \in C([0,a];X_1)\}$$

endowed with norm $\| \cdot \|_{\mathcal{B}_{Lip,a,A}} = \| \cdot \|_{C([-p,a];X)} + \| \cdot \|_{C([0,a];X_1)}$, where X_1 is the space $D(A)$ endowed with the norm $\| x \|_1 = \| Ax \|$.

Theorem 2.1. *Assume that the assumptions in Proposition 2.3 and the condition $\mathfrak{F}_\alpha$ are satisfied. Then the problem (1.1)-(1.2) is well-posed related the spaces $\mathcal{B}_{Lip,A}$ and $\mathcal{B}_{Lip,a,A}$.*

Proof: Let $\varphi, \psi \in \mathcal{B}_{Lip,A}$. The existence and uniqueness of a strict solution $u(\cdot,\psi) \in \mathcal{B}_{Lip,a,A}$ of (1.1) with initial condition ψ follows from Proposition 2.3. On the other hand, from Lemma 2.7 we have that

$$\| u(\cdot,\varphi) - u(\cdot,\psi) \|_{C[0,a]} \leq \Lambda_5(a) \| \varphi - \psi \|_{\mathcal{B}_{Lip,A}} e^{\Lambda_5(a)\Psi(a,u(\cdot,\varphi),\varphi)a} \to 0, \tag{2.18}$$

as $\| \psi - \varphi \|_{\mathcal{B}_{Lip,A}} \to 0$. In addition, by using the notation $u = u(\cdot,\varphi)$ and $v = u(\cdot,\psi)$, from Lemma 2.10 we infer that

$$\begin{aligned} &\| Au(\cdot,\varphi) - Au(\cdot,\psi) \|_{C[0,a]} \\ &\quad \leq \quad \Lambda_8(a)\mathcal{S}_1(u,v) \| \varphi - \psi \|_{\mathcal{B}_{Lip,A}} \\ &+\Lambda_8(a)(\mathcal{S}_1(u,v) + \mathcal{S}_1(u,v)\Psi(a,u,\varphi) + \Psi(a,u,\varphi)) \| u(\cdot,\varphi) - u(\cdot,\psi) \|_{C[0,a]} . \end{aligned} \tag{2.19}$$

From (2.5) and (2.10), it is easy to see that $\mathcal{S}_1(u,v)$ is bounded for $\psi(\cdot)$ in bounded sets of $\mathcal{B}_{Lip,A}$, which implies from (2.18) that $\| Au(\cdot,\varphi) - Au(\cdot,\psi) \|_{C([0,a];X)} \to 0$ as $\| \psi - \varphi \|_{\mathcal{B}_{Lip,A}} \to 0$. From the above we have that $\| u(\cdot,\varphi) - u(\cdot,\psi) \|_{\mathcal{B}_{Lip,a,A}} \to 0$ as $\| \psi - \varphi \|_{\mathcal{B}_{Lip,A}} \to 0$. ■

Next, for $\varphi \in \mathcal{B}_{Lip,A}$ and $t \geq 0$ we use the notation $S(t)$ for the map $S(t) : \mathcal{B}_{Lip,A} \mapsto C([-p,0];X)$ given by $S(t)\varphi = u_t(\cdot,\varphi)$. From Theorem 2.1, we have:

Corollary 2.2. *Assume that the conditions in Theorem 2.1 are satisfied and let $\varphi \in \mathcal{B}_{Lip,A}$. Then $\| S(t)\varphi - S(t)\psi \|_{C([-p,0];X)} \to 0$ as $\| \varphi - \psi \|_{\mathcal{B}_{Lip,A}} \to 0$ and $\| S(t)\varphi - S(s)\varphi \|_{C([-p,0];X)} \to 0$ as $s \to t$.*

Proof: The first assertion follows directly from Theorem 2.1. The second one follows noting that $u(\cdot,\varphi)$ is Lipschtz on $[-p,a]$. ■

In the next result, $\mathcal{B}^1_{Lip,A}$ is the space introduced at the beginning of this section.

Proposition 2.4. *Assume $F(0,\cdot,0) = 0$ and that the conditions in Theorem 2.1 are verified. Then the problem (1.1)-(1.2) is well-posed related the spaces $\mathcal{B}^1_{Lip,A}$ and $C^1([-p,a];X)$.*

Proof: Let $\varphi, \psi \in \mathcal{B}^1_{Lip,A}$. From Proposition 2.3, there exists a unique strict solution $u(\cdot,\psi)$ of (1.1) on $[-p,a]$ with initial condition ψ. Moreover, from the condition $F(0,\cdot,0) = 0$ we have that $u'(0^+,\psi) = A\psi(0)$, which implies that $u(\cdot,\psi) \in C^1([-p,a];X)$.

Let $u = u(\cdot,\varphi)$ and $v = u(\cdot,\psi)$. Noting that $u(\cdot)$ and $v(\cdot)$ are strict solutions, we have that

$$\begin{aligned} &\| u'(\cdot,\varphi) - u'(\cdot,\psi) \|_{C[0,a]} \\ &\qquad \leq \quad \| Au(\cdot,\varphi) - Au(\cdot,\psi) \|_{C[0,a]} + \| F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)} \|_{C[0,a]} \end{aligned}$$

From Theorem 2.1, $\| Au(\cdot,\varphi) - Au(\cdot,\psi) \|_{C([0,a];X)} \to 0$ as $\| \psi - \varphi \|_{\mathcal{B}_{Lip,A}} \to 0$. Moreover, from the last inequality in Remark 2.3 and (2.13) we have that

$$\begin{aligned}
\| F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)} \|_{C[0,a]} & \\
&\leq \Lambda_6(a)(\| \varphi - \psi \|_{C[-p,0]} + \Psi(a,u,\varphi) \| u - v \|_{C[0,a]}) \\
&\leq \Lambda_6(a) \| \varphi - \psi \|_{C[-p,0]} \\
&\quad + \Lambda_6(a)\Lambda_5(a)\Psi(a,u,\varphi) \| \varphi - \psi \|_{\mathcal{B}_{Lip,A}} e^{\Lambda_5(a)\Psi(a,u(\cdot,\varphi),\varphi))a} \to 0,
\end{aligned}$$

as $\| \psi - \varphi \|_{\mathcal{B}_{Lip,A}} \to 0$.

From the above remarks, it is easy to see that $\| u(\cdot,\varphi) - u(\cdot,\psi) \|_{C^1([0,a];X)} \to 0$ as $\| \psi - \varphi \|_{\mathcal{B}_{Lip,A}} \to 0$, which allows us to end the proof. ■

We finish this section with the next corollary. We omit the proof.

Corollary 2.3. *If the conditions in Proposition 2.4 are satisfied and* $\varphi \in \mathcal{B}_{Lip,A}$*, then* $\| S(t)\varphi - S(s)\varphi \|_{\mathcal{B}^1_{Lip,A}} \to 0$ *as* $s \to t$ *and* $\| S(t)\varphi - S(t)\psi \|_{\mathcal{B}^1_{Lip,A}} \to 0$ *as* $\| \varphi - \psi \|_{\mathcal{B}^1_{Lip,A}} \to 0$.

3. Examples

In this section we study some examples of partial differential equations with state-dependent delay motivated by different works and applications, see for examples [2, 3, 4, 8, 7, 9, 13, 15, 21, 22, 24, 25].

Next, $\Omega \subset \mathbb{R}^N$, $N \in \{1,2,3\}$, is an open bounded set with regular boundary, $X = L^2(\overline{\Omega})$ and A is Laplacian operator with Dirichlet condition with domain $D(A) = \{u \in X : u = 0 \text{ on } \partial\Omega \text{ and } \Delta u \in X\}$. It is well known that A is the infinitesimal generator of an analytic compact semigroup $(T(t))_{t\geq 0}$ on X. Next, we adopt all the notations and properties considered in the introduction and we assume that $f \in C_{Lip}([0,a]\times\mathbb{R}^n;\mathbb{R}^n)$ and $\zeta \in C_{Lip}([0,a]\times X;[0,p])$.

To begin, we study some of the examples in [15]. Consider the problem

$$\begin{aligned}
(3.1)\qquad w'(t,\xi) &= \Delta w(t,\xi) + f(t, \int_0^t \beta(t,s) w(s-\zeta(s,w(s,\cdot)),\xi)ds), \\
(3.2)\qquad w(t,\cdot) &= 0, \text{ on } \partial\Omega, \\
(3.3)\qquad w(s,\xi) &= \varphi(s,\xi), \qquad s \in [-p,0],
\end{aligned}$$

for $\xi \in \Omega$, $t \in [0,a]$, where $\beta \in C([0,a]\times[0,a];\mathbb{R})$ and the family of maps $\{\beta(\cdot,\tau) : \tau \in [0,a]\}$ is bounded in $C^\alpha([0,a];\mathbb{R})$ for some $\alpha \in (0,1)$.

To apply our results, we define $F : [0,a]\times X \to X$, $K(t,s) : X \to X$ and $\sigma : [0,a]\times X \to \mathbb{R}$ by $F(t,x)(\xi) = f(t,x(\xi))$, $K(t,s)x(\xi) = \beta(t,s)x(\xi)$ and $\sigma(s,x) = \zeta(s,x)$. It is easy to see that F is Lipschitz and that the condition $\mathfrak{L}_\alpha$ is satisfied. In the next result, which follows from Proposition 2.3, we said that $u \in C([-p,a];X)$ is a strict solution of (3.1)-(3.3) on $[-p,a]$ if $u(\cdot)$ is a strict solution of the associated problem (1.1)-(1.2) on $[-p,a]$. We adopt a similar nomenclature in other examples.

Proposition 3.5. *Under the above conditions, for all* $\varphi \in \mathcal{B}_{Lip,A}$ *there exists a unique strict solution* $u(\cdot,\varphi) \in C_{Lip}([-p,a];X)$ *of (3.1)-(3.3) on* $[-p,a]$*. If, in* *addition,* $f \in C^2([0,a] \times \mathbb{R}^n; \mathbb{R}^n)$ *and there is* $L_f > 0$ *such that*

Condition $\mathfrak{f}_\alpha$: $\mid f(t,x) - f(s,x) \mid_{\mathbb{R}^N} + \mid D_2 f(t,x) - D_2 f(s,x) \mid \leq L_f \mid t-s \mid^\alpha,$
$$\mid D_2 f(t,x) - D_2 f(t,y) \mid \leq L_f \mid x-y \mid_{\mathbb{R}^n},$$

for all $0 \leq s,t \leq a$ *and* $x,y \in \mathbb{R}^N$, *then the problem is well-posed related the spaces* $\mathcal{B}_{Lip,A}$ *and* $\mathcal{B}_{Lip,a,A}$, $\parallel S(t)\varphi - S(t)\psi \parallel_{C([-p,0];X)} \to 0$ *as* $\parallel \varphi - \psi \parallel_{\mathcal{B}_{Lip,A}} \to 0$ *and* $\parallel S(t)\varphi - S(s)\varphi \parallel_{C([-p,0];X)} \to 0$ *as* $s \to t$.

Proof: We only note that the condition $\mathfrak{f}_\alpha$ implies that the condition $\mathfrak{F}_\alpha$ is verified.

We study now the problem

$$(3.4) \qquad w'(t,\xi) = \Delta w(t,\xi) + f(t, \int_0^t \frac{\varrho(s)}{(t-s)^\alpha} w(s - \zeta(s, w(s,\cdot)), \xi) ds),$$

$$(3.5) \qquad w(t,\cdot)_{|\partial\Omega} = 0,$$

$$(3.6) \qquad w(s,\xi) = \varphi(s,\xi) \qquad s \in [-p,0],$$

for $\xi \in \Omega$, $t \in [0,a]$, where $\alpha \in (0, \frac{1}{2})$ and $\varrho \in C([0,a];\mathbb{R})$.

Let $K(t,s)$ and $L_{K,\alpha,s} : [0,s] \to \mathbb{R}$ be the functions defined by $K(t,s)x(\xi) = \frac{\varrho(s)}{(t-s)^\alpha} x(\xi)$ and $L_{K,\alpha,s}(\tau) = \frac{\varrho(\tau)}{(s-\tau)^{2\alpha}}$. For $0 \leq \tau < s < t \leq b \leq a$, we have that

$$(\int_0^t \parallel K(t,s) \parallel^{\frac{1}{1-\alpha}} ds)^{1-\alpha} \leq \parallel \varrho \parallel_{C[0,a]} a^{1-2\alpha} (\frac{1-\alpha}{1-2\alpha})^{1-\alpha}$$

and $\parallel K(t,\tau) - K(s,\tau) \parallel_{\mathcal{L}(X)} \leq \frac{\|\varrho\|_{C[0,a]}}{(s-\tau)^{2\alpha}} \mid t-s \mid^\alpha$, which implies that the condition $\mathfrak{L}_\alpha$ is satisfied. From Theorem 2.1 we have the next result.

Proposition 3.6. *If the condition* $\mathfrak{f}_\alpha$ *is satisfied, then the problem (3.4)-(3.6) is well-posed related the spaces* $\mathcal{B}_{Lip,A}$ *and* $\mathcal{B}_{Lip,a,A}$, $\parallel S(t)\varphi - S(t)\psi \parallel_{C([-p,0];X)} \to 0$ *as* $\parallel \varphi - \psi \parallel_{\mathcal{B}_{Lip,A}} \to 0$ *and* $\parallel S(t)\varphi - S(s)\varphi \parallel_{C([-p,0];X)} \to 0$ *as* $s \to t$.

To finish this section, we consider the problem,

$$(3.7) \quad w'(t,\xi) = \Delta w(t,\xi) + \mu(t) \int_0^t \int_\Omega \gamma(t,s,y-\xi) w(s - \zeta(s,w(s)), y) dy ds,$$

$$(3.8) \quad w(t,\cdot)_{|\partial\Omega} = 0,$$

$$(3.9) \quad w(s,\xi) = \varphi(s,\xi), \quad s \in [-p,0],$$

for $(t,\xi) \in [0,a] \times \Omega$, where $\gamma \in C([0,a] \times [0,a] \times \mathbb{R}^N; \mathbb{R})$ and $\mu \in C^\alpha([0,a];\mathbb{R})$ for some $\alpha \in (0,1)$.

To study this problem, we define $K : [0,a] \times [0,a] \to \mathcal{L}(X;X)$ and $F : [0,a] \times X \to X$ by $K(t,s)x(\xi) = \int_\Omega \gamma(t,s,y-\xi)x(y)dy$ and $F(t,x)(\xi) = \mu(t)x(\xi)$, and we assume that there is $\chi \in C([0,a] \times \mathbb{R}^n; \mathbb{R}^+)$ such that

$$\mid \gamma(t,s,x) - \gamma(t',s,x) \mid \leq \chi(s,x) \mid t - t' \mid^\alpha, \quad \forall\, t,t',s \in [0,a],\, x \in \mathbb{R}^N,$$

and that $\widetilde{\chi}(\cdot) = (\int_\Omega \int_\Omega \mid \chi(\cdot, x-y)^2 dydx)^{\frac{1}{2}}$ belongs to $L^1([0,a])$.

From the above, the condition $\mathfrak{L}_\alpha$ is satisfied with $L_{K,\alpha,s}(\cdot) = \chi(\cdot)$ and $\Theta(b) = \sup_{t \in [0,b]} (\int_0^t (\int_\Omega \int_\Omega \gamma(t,s,x-y)^2 dydx)^{\frac{1}{2}})^{\frac{1}{1-\alpha}} dt$. Noting that $F(\cdot)$ is “not Lispchitz”, to establish our next result, we need include some observations.

The proof of Theorem 2.1 follows from some inequalities presented in different propositions and lemmas. Noting that $F(\cdot)$ is linear, we have that the unique (possible) "qualitative" differences related to these inequalities can be appear in the estimates of $[F_{u(\cdot,\varphi)}]_{C^\alpha([0,a];X)}$ and $[F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)}]_{C^\alpha([0,a];X)}$ (see Lemma 2.6 and Lemma 2.8). In the current case, the estimate of $[F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)}]_{C^\alpha([0,a];X)}$ (which permit also estimate $[F_{u(\cdot,\varphi)}]_{C^\alpha([0,a];X)}$) can be obtained without to use [11, Lemma 2.2]. Using the last inequalities in Lemma 2.4, we obtain the inequality

$$\begin{aligned}
[F_{u(\cdot,\varphi)} - F_{u(\cdot,\psi)}]_{C^\alpha[0,a]} &\leq \| \mu \|_{C^\alpha} \| u_K(\cdot,\varphi) - u_K(\cdot,\psi) \|_{C^\alpha[0,a]} \\
&\leq \| \mu \|_{C^\alpha} (\Theta(a)a^\alpha + \Upsilon(a))(\| \varphi - \psi \|_{C[-p,0]} + \Psi(a,v,\psi) \| u - v \|_{C[0,a]}),
\end{aligned}$$

which is "qualitatively" simpler than the inequality in Lemma 2.8. From the above remarks and Theorem 2.1, we infer the next result.

Proposition 3.7. *The problem (3.7)-(3.9) is well-posed related* $\mathcal{B}_{Lip,A}$ *and* $\mathcal{B}_{Lip,a,A}$.

References

[1] Aiello, W., Freedman, H. I., Wu, J. Analysis of a model representing stage-structured population growth with state-dependent time delay. *SIAM J. Appl. Math.* 52 (1992), no. 3, 855-869.

[2] Alt, W. *Periodic solutions of some autonomous differential equations with variable time delay,* Lecture Notes in Mathematics, Vol. 730, Springer-Verlag, 1979. *integro ordinary*

[3] Britton, N. F. Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model. *SIAM J. Appl. Math.* 50 (1990), no. 6, 1663-1688.

[4] Cooke, K.L., Huang, W. On the problem of linearization for state-dependent delay differential equations *Proceedings of the American Mathematical Society* Vol. 124, No. 5 (1996), 1417-1426. *integro ordinary*

[5] Driver, R.D., *A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics,* in: J. LaSalle, S. Lefschtz (Eds.), International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, New York, 1963, pp. 474-484.

[6] Driver, R.D., A neutral system with state-dependent delay. *J. Differential Equations* 54 (1984) 73-86.

[7] Gopalsamy, K. Pursuit-evasion wave trains in prey-predator systems with diffusionally coupled delays. *Bull. Math. Biol.* 42 (1980), no. 6, 871-887

[8] Gourley, S. A., Britton, N. F. A predator-prey reaction-diffusion system with nonlocal effects. *J. Math. Biol.* 34 (1996), no. 3, 297-333.

[9] Gourley, S. A. Instability in a predator-prey system with delay and spatial averaging. *IMA J. Appl. Math.* 56 (1996), no. 2, 121-132.

[10] Hartung, F., Krisztin, T., Walther, Hans-Otto., Wu, J. *Functional differential equations with state-dependent delays: theory and applications.* Handbook of differential equations: ordinary differential equations. Vol. III, 435-545, Handb. Differ. Equ.,

[11] Hernández, E., O'Regan, D., Ponce, R. On C^α-Hölder classical solutions for non-autonomous neutral differential equations: the nonlinear case. *J. Math. Anal. Appl.* 420 (2014), no. 2, 1814-1831.

[12] Hernández, E., Prokopczyk, A., Ladeira, L. A note on partial functional differential equations with state-dependent delay. *Nonlinear Anal. Real World Appl.* 7 (2006), no. 4, 510-519.

[13] Hernández, E., Pierri., M., Wu, J. $\mathbf{C^{1+\alpha}}$-strict solutions and wellposedness of abstract differential equations with state dependent delay. *J. Differential Equations* 261, (2016) 12, 6856-6882.

[14] Hernández, E., Wu, J. Existence, uniqueness and qualitative properties of global solutions of abstract differential equations with state dependent delay. *Proceedings of the Edinburgh Mathematical Society, 1-18.* doi:10.1017/S001309151800069X.

[15] Hernández, E., Wu, J. Existence and uniqueness of $\mathbf{C^{1+\alpha}}$-strict solutions for integro-differential equations with state-dependent delay. *Differential and Integral Equations,* Vol. 32, 5-6, May/June 2019.

[16] Krisztin, T., Rezounenko, A. Parabolic partial differential equations with discrete state-dependent delay: Classical solutions and solution manifold. *Journal of Differential Equations* Vol. 260, (5) (2016), 4454-4472.

[17] Kosovalic, N., Magpantay, F. M. G., Chen, Y., Wu, J. Abstract algebraic-delay differential systems and age structured population dynamics. *J. Differential Equations* 255 (2013), no. 3, 593-609.

[18] Kosovalic, N., Chen, Y., Wu, J. Algebraic-delay differential systems:C^0-extendable submanifolds and linearization. *Trans. Amer. Math. Soc.* 369 (2017), no. 5, 3387-3419.

[19] Lunardi, A. *Analytic semigroups and optimal regularity in parabolic problems*, PNLDE Vol. *16*, Birkhäauser Verlag, Basel, 1995.

[20] Lv, Y., Rong, Y., Yongzhen, P. Smoothness of semiflows for parabolic partial differential equations with state-dependent delay. *J. Differential Equations* 260 (2016) 6201-6231.

[21] Rezounenko, A., Wu, J. A non-local PDE model for population dynamics with state-selective delay: local theory and global attractors. *J. Comput. Appl. Math.* 190 (2006), no. 1-2, 99-113.

[22] Shakourifar, M. Wayne H. E. Reliable approximate solution of systems of Volterra integro-differential equations with time-dependent delays. *SIAM Journal on Scientific Computing* 33 (3), (2011) 1134-1158.

[23] Walther, H.-O. The solution manifold and C^1-smoothness for differential equations with state-dependent delay. *J. Differential Equations* 195 (2003), no. 1, 46-65.

[24] Yi, T., Chen, Y., Wu, J. Global dynamics of delayed reaction-diffusion equations in unbounded domains. *Z. Angew. Math. Phys.* 63 (2012), no. 5, 793-812.

[25] Zhang, C., Vandewalle, S. General linear methods for Volterra integro-differential equations with memory. *SIAM J. Sci. Comput.* 27 (2006), no. 6, 2010-2031.

Departamento de Computação e Matemática, Faculdade de Filosofia Ciências e Letras de Ribeirão Preto Universidade de São Paulo, CEP 14040-901 Ribeirão Preto, SP, Brazil. e-mail:lalohm@ffclrp.usp.br

Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, CEP 13566-590 São Carlos, SP, Brazil. e-mail:denisfer@usp.br

Department of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3, Canada. ne e-mail:wujh@mathstat.yorku.ca