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A Hierarchy for the Recognizable $\mathcal{M}$-subsets

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Abstract

We study some properties of recognizable $\mathcal{M}$-subsets of a free monoid A^* ($\mathcal{M}\,\mathrm{Rec}\,A^*$) and of two of its subfamilies: the simple $\mathcal{M}$-subsets ($\mathcal{M}\,\mathrm{SRec}\,A^*$) and the $\mathcal{M}$-subsets which are nondeterministic complexities of finite automata ($\mathcal{M}\,\mathrm{CRec}\,A^*$). At first, we study some necessary conditions for membership in each one of these families and we show that $\mathcal{M}\,\mathrm{CRec}\,A^* \subsetneq \mathcal{M}\,\mathrm{SRec}\,A^* \subsetneq \mathcal{M}\,\mathrm{Rec}\,A^*$. We also study the closure properties of these families under several operations and the existing relations among these families and the families $\mathcal{H}_p$ ($p \geq 0$) obtained by Simon; in particular, we study some properties of the limited $\mathcal{M}$-subsets. We also show that the equality problem for $\mathcal{M}\,\mathrm{CRec}\,A^*$ is undecidable.

1 Introduction

The study of recognizable subsets with multiplicities in a field had its origin in the fundamental works of M. P. Schützenberger [16, 17, 18] written in the

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beginning of the sixties. In the seventies, S. Eilenberg [4] systematized this theory for an arbitrary semiring K, paying special attention to the cases of the Boolean semiring and the semiring of natural numbers. A more algebraic treatment of recognizable K-subsets is given by J. Berstel and C. Reutenauer [2].

In this paper, we study some properties of the family of recognizable $\mathcal{M}$-subsets of A^*, $\mathcal{M}\,\mathrm{Rec}\,A^*$, where $\mathcal{M}$ denotes the tropical semiring, which consists of the natural numbers extended with ∞ and equipped with the operations of minimum and addition. An $\mathcal{M}$-subset of A^* is a function that associates a multiplicity in $\mathcal{M}$ to each word in A^*. An $\mathcal{M}$-A-automaton is a finite automaton in which one associates multiplicities in $\mathcal{M}$ to the initial states, final states and edges. This allows to associate a multiplicity in $\mathcal{M}$ to each word in A^* and one says that the resulting $\mathcal{M}$-subset of A^* is recognizable.

The semiring $\mathcal{M}$ is known in Operation Research [3], where it has been used in problems of cost minimization. In the Theory of Automata the study of the multiplicities in the semiring $\mathcal{M}$ was introduced by I. Simon [20], in 1978, to give a characterization of recognizable subsets of a free monoid which has the finite power property. An independent solution were also obtained by K. Hashiguchi [6]. In the last years, other important problems related with the semiring $\mathcal{M}$ were solved. For instance, K. Hashiguchi [7, 8] characterized the recognizable and limited $\mathcal{M}$-subsets through a great complexity reasoning; H. Leung [15] and I. Simon [22, 24] obtained, independently, other more algebraic solutions to decide whether a recognizable $\mathcal{M}$-subset is limited; K. Hashiguchi [9] solved the star height problem of recognizable sets. A survey of the most important results about recognizable $\mathcal{M}$-subsets was written by I. Simon [23]. More recently, D. Krob [14] showed that the equality problem for recognizable $\mathcal{M}$-subsets is undecidable.

In particular, we study two of the subfamilies of $\mathcal{M}\,\mathrm{Rec}\,A^*$: the family of simple $\mathcal{M}$-subsets, $\mathcal{M}\,\mathrm{SRec}\,A^*$, and the family of the $\mathcal{M}$-subsets which are nondeterministic complexities, $\mathcal{M}\,\mathrm{CRec}\,A^*$. An $\mathcal{M}$-subset of A^* is *simple* if it is recognized by an $\mathcal{M}$-A-automaton whose multiplicities belong to $\{0, 1, \infty\}$ and it is a nondeterministic complexity if it is recognized by an $\mathcal{M}$-A-automaton which can be obtained by taking a (nondeterministic) finite automaton and associating multiplicity 0 to its deterministic edges, 1 to its nondeterministic edges and 0 to its initial and final states.

At first, we study some necessary conditions for membership in each one of these three families and we show that

$$\mathcal{M}\,\mathrm{CRec}\,A^* \subsetneq \mathcal{M}\,\mathrm{SRec}\,A^* \subsetneq \mathcal{M}\,\mathrm{Rec}\,A^* \ ,$$

where A is an alphabet with at least two letters.

We present some properties of recognizable and limited $\mathcal{M}$-subsets and we study their relation with the families $\mathcal{M}\,\mathrm{Rec}\,A^*$, $\mathcal{M}\,\mathrm{SRec}\,A^*$, $\mathcal{M}\,\mathrm{CRec}\,A^*$ and the families $\mathcal{H}_p$ ($p \geq 0$) obtained by I. Simon [21].

We also study the closure properties of the families $\mathcal{M}\,\mathrm{Rec}\,A^*$, $\mathcal{M}\,\mathrm{SRec}\,A^*$ and $\mathcal{M}\,\mathrm{CRec}\,A^*$ under several operations.

In the last section, we show that the equality problem for $\mathcal{M}\,\mathrm{CRec}\,A^*$, when A has at least two letters, is undecidable. Our proof of this result uses the same ideas and constructions of Krob [14].

2 The semiring $\mathcal{M}$, $\mathcal{M}$-subsets and $\mathcal{M}$-A-automata

The *tropical semiring* $\mathcal{M}$ has as support $\mathbb{N} \cup \infty$ and as operations the minimum and the addition. The minimum plays the rôle of semiring addition and the addition plays the rôle of semiring multiplication. Note that $\mathcal{M}$ is a commutative semiring and the identities with respect to minimum and addition are ∞ and 0, respectively. Moreover, $\mathcal{M}$ is a positive and complete semiring in the sense of Eilenberg [4].

Let A be a finite alphabet. An *$\mathcal{M}$-subset* X of A^* is a function $X: A^* \to \mathcal{M}$. For each w in A^*, wX is called the multiplicity with which w belongs to X. If $1X = \infty$, we also say that X is an $\mathcal{M}$-subset of A^+.

The following operations are defined over $\mathcal{M}$-subsets of A^*, where $\{X_i : i \in I\}$ is a family of $\mathcal{M}$-subsets of A^* indexed by a set I (not necessarily finite), X and Y are $\mathcal{M}$-subsets of A^*, and $m \in \mathcal{M}$.

(a) $\forall w \in A^*, \quad w(\min_{i\in I} X_i) = \min_{i\in I}(wX_i)$ (minimum)

(b) $\forall w \in A^*, \quad w(\sum_{i\in I} X_i) = \sum_{i\in I}(wX_i)$ (addition)

(c) $\forall w \in A^*, \quad w(m + X) = m + wX$

(d) $\forall w \in A^*, \quad w(XY) = \min_{xy=w}(xX + yY)$ (concatenation)

(e) $\forall w \in A^*, \quad wX^+ = w(\min_{n\geq 1} X^n) = \min_{n\geq 1}(wX^n)$

(f) $X^* = \min(\mathbf{1}, X^+)$, where the $\mathcal{M}$-subset $\mathbf{1}$ is defined by $\forall w \in A^*$, $w\mathbf{1} = 0$ if $w = 1$ and $w\mathbf{1} = \infty$, otherwise.

Recall that, for any semiring K, one naturally has the operations of addition, intersection, and multiplication of K-subsets. For the semiring $\mathcal{M}$, these operations are, respectively, the ones given in (a), (b) and (d) above.

Note that the operations in (a) and (b) are well defined for any set I. In particular, if $I = \emptyset$, $\min_{i\in I}(m_i) = \infty$ and $\sum_{i\in I} m_i = 0$; if I is infinite and there are infinitely many elements $m_i \neq 0$, $\sum_{i\in I} m_i = \infty$. As a consequence of this, the $\mathcal{M}$-subsets X^+ and X^* are well defined even when $1X \neq \infty$.

The family $\mathcal{M} \ll A \gg$ of all $\mathcal{M}$-subsets of A^* with the minimum (a) and concatenation (d) operations constitutes a semiring, whose identities are, respectively, the $\mathcal{M}$-subset $\emptyset$ (where, for all $w \in A^*$, $w\emptyset = \infty$) and the $\mathcal{M}$-subset $\mathbf{1}$.

The operations in (a), (c), (d) and (f) are called *rational operations* in $\mathcal{M}\ll A\gg$ and we say that a set $\mathcal{F} \subseteq \mathcal{M}\ll A\gg$ is *rationally closed* if it is closed under the rational operations and it contains the identities $\emptyset$ and $\mathbf{1}$.

We denote by $\mathcal{M}\,\mathrm{Rat}\,A^*$ the smallest rationally closed subset of $\mathcal{M}\ll A\gg$, containing the single $\mathcal{M}$-subset $\mathbf{a}$, for each $a \in A$, such that $w\mathbf{a} = 0$ if $w = a$ and $w\mathbf{a} = \infty$, otherwise.

For a given subset $\mathcal{F}$ of $\mathcal{M}\ll A\gg$, we define the *rational closure* of $\mathcal{F}$ as being the smallest rationally closed subset of $\mathcal{M}\ll A\gg$, containing $\mathcal{F}$.

An $\mathcal{M}$-A-*automaton* $\mathcal{A} = (Q, I, T)$ is an automaton over A, with a finite set Q of states, two $\mathcal{M}$-subsets I and T of Q and an $\mathcal{M}$-subset $E_{\mathcal{A}}$ of $Q\times A\times Q$.

If $pI \neq \infty$ (resp. $pT \neq \infty$), we say that p is an *initial state* (resp. *final state*) of $\mathcal{A}$.

If (p, a, q) is an *edge* in $\mathcal{A}$, we say that its *label* is a and that its *multiplicity* is $(p, a, q)E_{\mathcal{A}}$. If $(p, a, q)E_{\mathcal{A}} \neq \infty$, the edge (p, a, q) is said to be a *useful edge* of $\mathcal{A}$.

If P is a *path* of length n in $\mathcal{A}$, with *origin* p_0 and *terminus* p_n, that is

$$P = (p_0, a_1, p_1)(p_1, a_2, p_2)\dots(p_{n-1}, a_n, p_n) ,$$

then its *label* is $|P| = a_1a_2...a_n$ and its *multiplicity* $\|P\|$ is the sum of the multiplicities of its edges, that is

$$\|P\| = \sum_{i=1}^{n}(p_{i-1}, a_i, p_i)E_{\mathcal{A}} \ .$$

For convenience, if P is the path above, we also write

$$P = (p_0, a_1a_2 \dots a_n, p_n) \quad \text{and} \quad P : p_0 \xrightarrow{a_1a_2\dots a_n} p_n \ .$$

Concatenations, factorizations and factors of paths are defined as usual.

A *path* P is *useful* if $\|P\| \neq \infty$. A useful path, whose origin i and terminus t satisfy $iI \neq \infty$ and $tT \neq \infty$, is called *successful.*

The *behavior* of $\mathcal{A}$ is the $\mathcal{M}$-subset $\|\mathcal{A}\|$ of A^* that associates a multiplicity to each word as follows. Let w be in A^* and let C be the set of successful paths P in $\mathcal{A}$ with label $|P| = w$. Then

$$w\|\mathcal{A}\| = \min_{P \in C}(iI + \|P\| + tT) \ ,$$

where i and t are the origin and the terminus of the path P, respectively.

A successful path P in $\mathcal{A}$, with label w, origin i and terminus t, is called *victorious*, if $iI + \|P\| + tT = w\|\mathcal{A}\|$.

The unique paths of length zero are the trivial paths $(q, 1, q)$, for every $q \in Q$. Their labels are the empty word and their multiplicities are equal to zero.

We say that an $\mathcal{M}$-A-automaton $\mathcal{A} = (Q, I, T)$ is *normalized* if $\mathcal{A}$ has a unique initial state i and a unique final state t, with $t \neq i$ and $iI = tT = 0$, and, moreover, there are neither useful edges with terminus i nor useful edges with origin t.

An $\mathcal{M}$-*subset* of A^* is *recognizable* if it is the behavior of some $\mathcal{M}$-A-automaton. It is well known that every recognizable $\mathcal{M}$-subset of A^+ is the behavior of a normalized $\mathcal{M}$-A-automaton. The family of all recognizable $\mathcal{M}$-subsets of A^* is denoted by $\mathcal{M}\,\mathrm{Rec}\,A^*$.

Let us denote by $\mathbf{A}^+$ the $\mathcal{M}$-subset of A^* such that

$$\forall w \in A^*, \quad w\mathbf{A}^+ = \begin{cases} \infty & \text{if } w = 1 \\ 0 & \text{otherwise} \ . \end{cases}$$

Then one can easily verify the following result.

Proposition 1 *For every recognizable $\mathcal{M}$-subset X of A^* there is a normalized $\mathcal{M}$-A-automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = X + \mathbf{A}^+$.* ■

Remark: In a normalized $\mathcal{M}$-A-automaton $\mathcal{A}$, every victorious path P with label w satisfies $\|P\| = w\|\mathcal{A}\|$ (because $QI, QT \subseteq \{0, \infty\}$) and every successful path P' with label w is such that $w\|\mathcal{A}\| \leq \|P'\|$ (because $\|P\| \leq \|P'\|$). These properties will be frequently used in the proofs and they are also valid for simple or type nc $\mathcal{M}$-A-automaton, which we present in the following sections.

3 Some necessary conditions for recognizable $\mathcal{M}$-subsets

In this section, we study some necessary conditions so that an $\mathcal{M}$-subset of A^* be recognizable.

The condition in the next proposition is valid for K-subsets, for every positive semiring K. (See Berstel and Reutenauer [2].)

Proposition 2 *Let X be a recognizable $\mathcal{M}$-subset of A^*. Then* $\text{support}(X) = \{ w \in A^* : wX \neq \infty \}$ *is a recognizable subset of A^*.* ■

Proposition 3 *Let X be a recognizable $\mathcal{M}$-subset of A^*. Then there is a positive integer k such that for every $w \in A^+$, either $wX = \infty$ or $wX \leq k|w|$.* ■

Proposition 4 *Let X be a recognizable $\mathcal{M}$-subset of A^*. Then, for every $m \in \mathcal{M}$, mX^{-1} is a recognizable subset of A^*.*

Proof. As $\infty X^{-1} = A^* - \text{support}(X)$, the result follows from Proposition 2.

Let $m \in \mathcal{M} - \{\infty\}$. As X is a recognizable $\mathcal{M}$-subset of A^*, by Proposition 1 there exists a normalized $\mathcal{M}$-A-automaton $\mathcal{A} = (Q, I, T)$ such that $\|\mathcal{A}\| = X + \mathbf{A}^+$. From $\mathcal{A}$, let us construct an $\mathcal{M}$-A-automaton $\mathcal{B}$, which accepts only the words that are recognized by $\mathcal{A}$ with multiplicity at most m. We define $\mathcal{B} = (Q', I', T')$ as follows:

$Q' = Q \times ([0, m] \cup \{\infty\})$;

the $\mathcal{M}$-subset I' of Q' is given by

$$\forall q \in Q, \quad (q,i)I' = \begin{cases} qI & \text{if } i = 0 \\ \infty & \text{if } i \in [1,m] \cup \{\infty\} \end{cases}$$

and the $\mathcal{M}$-subset T' of Q' is given by

$$\forall q \in Q, \quad (q,i)T' = \begin{cases} qT & \text{if } i \in [0,m] \\ \infty & \text{if } i = \infty\ . \end{cases}$$

For each useful edge (p,a,q) of $\mathcal{A}$,

- if $(p,a,q)E_{\mathcal{A}} = 0$, then, for each $i \in [0,m] \cup \{\infty\}$, $((p,i),a,(q,i))E_{\mathcal{B}} = 0$;
- if $(p,a,q)E_{\mathcal{A}} = k$, with $0 < k < \infty$, then, for each $i \in [0,m] \cup \{\infty\}$, $((p,i),a,(q,j))E_{\mathcal{B}} = k$, with $j = i + k$, if $i + k \leq m$, and $j = \infty$, otherwise.

One can verify that $|\mathcal{B}| = [0,m]\|\mathcal{A}\|^{-1}$. Hence, $[0,m]\|\mathcal{A}\|^{-1}$ is a recognizable subset of A^*. In a similar way, we have that $[0,m-1]\|\mathcal{A}\|^{-1}$ is a recognizable subset of A^*.

Therefore, $m\|\mathcal{A}\|^{-1} = [0,m]\|\mathcal{A}\|^{-1} - [0,m-1]\|\mathcal{A}\|^{-1}$ is a recognizable subset of A^*.

Thus, $mX^{-1} = m\|\mathcal{A}\|^{-1} \cup \{1\}$, if $1X = m$, and $mX^{-1} = m\|\mathcal{A}\|^{-1}$, otherwise. Therefore, mX^{-1} is a recognizable subset of A^*. ■

Lemma 5 *The conditions of Propositions 2, 3 and 4 are not sufficient for a given $\mathcal{M}$-subset to be recognizable.*

Proof. Let $A = \{a,b\}$ and let X be the $\mathcal{M}$-subset of A^* defined by

$$1X = \infty \quad \text{and} \quad \forall w \in A^+, \quad wX = \max\{|w|_a, |w|_b\}\ .$$

It is easy to verify that X satisfies the conditions of Propositions 2, 3 and 4. Let us show that X is not a recognizable $\mathcal{M}$-subset of A^*.

Suppose that X is a recognizable $\mathcal{M}$-subset of A^*. Then, by Proposition 1, there is a normalized $\mathcal{M}$-A-automaton $\mathcal{A}$ with n states such that $\|\mathcal{A}\| = X$.

Consider the word $w = a^n b^n$. Then $wX = n$.

Let P be a victorious path in $\mathcal{A}$ with label w. Then $||P|| = w||\mathcal{A}|| = wX = n$ and there are naturals r, s and t, with $s > 0$ and $r+s+t = n$, such that the path P can be factorized as follows:

$$P : i \xrightarrow{a^n} q_1 \xrightarrow{b^r} q_2 \xrightarrow{b^s} q_2 \xrightarrow{b^t} f \ .$$

If $||(q_2, b^s, q_2)|| = 0$, then there is a successful path P' in $\mathcal{A}$,

$$P' : i \xrightarrow{a^n} q_1 \xrightarrow{b^r} q_2 \xrightarrow{b^s} q_2 \xrightarrow{b^s} q_2 \xrightarrow{b^t} f \ ,$$

spelling the word $w' = a^n b^{n+s}$ such that $||P'|| = ||P|| = n$. Hence, $w'X = w'||\mathcal{A}|| \leq ||P'|| = n$. This is a contradiction because $w'X = \max\{n, n+s\} \geq n+1$.

If $||(q_2, b^s, q_2)|| > 0$, then there is a successful path P'' in $\mathcal{A}$,

$$P'' : i \xrightarrow{a^n} q_1 \xrightarrow{b^r} q_2 \xrightarrow{b^t} f \ ,$$

spelling the word $w'' = a^n b^{r+t}$ such that $||P''|| < ||P|| = n$. So, $w''X = w''||\mathcal{A}|| \leq ||P''|| < n$. This is a contradiction because $w''X = \max\{n, r+t\} = n$.

Therefore, X is not a recognizable $\mathcal{M}$-subset of A^*. ■

The technique used in the proof of the previous lemma, that is, to iterate or to remove a factor of a given path, is frequent in all this work.

Another necessary condition for a given $\mathcal{M}$-subset to be recognizable looks like the 'Pumping Lemma' for the regular languages; more precisely, with the Ogden's Iteration Lemma [1].

Let $x \in A^*$ such that $x = x_1 \ldots x_n$, with $x_i \in A$ $(1 \leq i \leq n)$. A *position* in x is any integer i, $1 \leq i \leq n$. Given a subset I of $[1, n]$, we say that a position i is *fixed* with respect to I if and only if $i \in I$.

Lemma 6 *Let X be a recognizable $\mathcal{M}$-subset of A^*. Then there is a positive integer m such that for every word x in A^* with $xX < \infty$ and, for every choice of at least m fixed positions in x, the word x admits a factorization of the form $x = uvw$, in such a way that*
(i) *v contains at least one and at most m fixed positions;*
(ii) *there exists $c \geq 0$ such that for every $k \geq 0$, $(uv^kw)X \leq xX + (k-1)c$.*

Proof. Let X be a recognizable $\mathcal{M}$-subset of A^*. Then there is a normalized $\mathcal{M}$-A-automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = X + \mathbf{A}^+$.

Let m be the number of states of $\mathcal{A}$ and let $x \in A^*$ such that $xX < \infty$ and $x = x_1 \ldots x_n$, with $x_l \in A$ $(1 \leq l \leq n)$.

We consider the subset I of $[1, n]$ as being a choice of at least m positions in x. As $|I| \geq m$, it follows that $n \geq m$.

Let $i_1, \ldots, i_m$ be the m smallest elements of I, with $1 \leq i_1 < \ldots < i_m \leq n$. We define the following factorization for x,

$$x = y_0 y_1 y_2 \cdots y_m y_{m+1} \ ,$$

with

$$\begin{cases} y_0 &= x_1 \ldots x_{i_1 - 1} \\ y_1 &= x_{i_1} \\ y_l &= x_{i_{l-1}+1} \ldots x_{i_l}, \quad \text{for } 2 \leq l \leq m \\ y_{m+1} &= x_{i_m + 1} \ldots x_n \ . \end{cases}$$

Then, for each l, $1 \leq l \leq m$, y_l contains exactly one fixed position.

Let P be a victorious path in $\mathcal{A}$ with label x. Consider the following factorization:

$$P : p \xrightarrow{y_0} q_0 \xrightarrow{y_1} q_1 \xrightarrow{y_2} \ldots \xrightarrow{y_{m-1}} q_{m-1} \xrightarrow{y_m} q_m \xrightarrow{y_{m+1}} r \ .$$

Then there are h and j, $0 \leq h < j \leq m$, such that $q_h = q_j$. We define

$$u = y_0 y_1 \cdots y_h, \quad v = y_{h+1} \cdots y_j \quad \text{and} \quad w = y_{j+1} \cdots y_{m+1} \ .$$

Then $x = uvw$, $uw \neq 1$, and v contains exactly $j - h$ fixed positions, with $0 < j - h \leq m$.

Let us consider the words

$$uv^k w = y_0 y_1 \ldots y_h (y_{h+1} \cdots y_j)^k y_{j+1} \cdots y_{m+1}, \quad \text{for } k \geq 0$$

and the factor $P_1 = (q_h, v, q_j)$ of P.

As $\|P_1\| \geq 0$, by considering $c = \|P_1\|$, it results that

$$\forall k \geq 0, \quad (uv^k w)X \leq \|P\| + (k-1)\|P_1\| = xX + (k-1)c \ .$$

■

Observe that the proof of the following lemma uses a different strategy to prove that an $\mathcal{M}$-subset is not recognizable.

Lemma 7 *The condition of Lemma 6 is also not sufficient for a given $\mathcal{M}$-subset to be recognizable.*

Proof. Let $A = \{a, b, c\}$ and let X be the $\mathcal{M}$-subset of A^* defined by

$$wX = \begin{cases} |w|_a + |w|_b & \text{if } w \in c^+\{a^n b^n : n \geq 0\} \\ \min\{|w|_a, |w|_b\} & \text{otherwise} . \end{cases}$$

One can verify that X satisfies the conditions of Propositions 2, 3 and 4, and of Lemma 6. Let us show that X is not a recognizable $\mathcal{M}$-subset.

Suppose that X be a recognizable $\mathcal{M}$-subset of A^*. Then there is a normalized $\mathcal{M}$-A-automaton $\mathcal{A} = (Q, I, T)$ such that $\|\mathcal{A}\| = X + \mathbf{A}^+$. Denote by p the initial state of $\mathcal{A}$ and by r the final state of $\mathcal{A}$.

Let m be a positive integer and define, for each natural l, the subset Q_l of Q, as follows:

$$Q_l = \{q : \text{there is a victorious path } p \xrightarrow{c^m a^l} q \xrightarrow{b^h} r, \text{ for some } h \neq l\} .$$

Then there are naturals i and j such that $i < j$ and $Q_i = Q_j$. We observe that Q_l is not empty, for every l.

As $c^m a^i b^j \in |\mathcal{A}|$, there is a victorious path P in $\mathcal{A}$, spelling $c^m a^i b^j$, with the following factorization:

$$P : p \xrightarrow{c^m a^i} q \xrightarrow{b^j} r ,$$

for some $q \in Q$. But, as $j \neq i$, we conclude that $q \in Q_i = Q_j$. Then there is a victorious path P' in $\mathcal{A}$, spelling $c^m a^j b^k$, for some $k \neq j$, with the following factorization:

$$P' : p \xrightarrow{c^m a^j} q \xrightarrow{b^k} r .$$

However, since the following equalities are true,

$$\|P\| = (c^m a^i b^j)\|\mathcal{A}\| = (c^m a^i b^j)X = \min\{i, j\} = i$$

$$\text{and} \quad \|P'\| = (c^m a^j b^k)\|\mathcal{A}\| = (c^m a^j b^k)X = \min\{j, k\} ,$$

we conclude that the factors $P_1 = (p, c^m a^j, q)$ of P' and $P_2 = (q, b^j, r)$ of P satisfy $\|P_1\| \leq j$ and $\|P_2\| \leq i$.

Hence, the path $P_1 P_2 = (p, c^m a^j, q)(q, b^j, r)$ satisfies

$$\|P_1 P_2\| = \|P_1\| + \|P_2\| \leq j + i < 2j .$$

Then
$$2j = (c^m a^j b^j)X = (c^m a^j b^j)\|\mathcal{A}\| \leq \|P_1 P_2\| < 2j \ ;$$
that is a contradiction.

Therefore, X is not a recognizable $\mathcal{M}$-subset of A^*. ∎

4 Simple $\mathcal{M}$-subsets

In this section, we study the family of simple $\mathcal{M}$-subsets of A^*, denoted by $\mathcal{M}\,\mathrm{SRec}\,A^*$.

An *$\mathcal{M}$-subset* of A^* is *simple* if it is the behavior of some simple $\mathcal{M}$-A-automaton. We say that an *$\mathcal{M}$-A-automaton* $\mathcal{A} = (Q, I, T)$ is *simple* if it satisfies

$$(Q \times A \times Q)E_{\mathcal{A}} \subseteq \{0, 1, \infty\}, \quad QI \subseteq \{0, \infty\} \quad \text{and} \quad QT \subseteq \{0, \infty\} \ .$$

Note that, by definition, if X is a simple $\mathcal{M}$-subset, then $1X \in \{0, \infty\}$.

A necessary condition for an $\mathcal{M}$-subset to be simple is given in the following proposition whose proof is immediate.

Proposition 8 *Let X be a recognizable $\mathcal{M}$-subset of A^*. If X is simple, then for every $w \in A^*$, either $wX = \infty$ or $wX \leq |w|$.* ∎

A consequence of the previous proposition is that the simple $\mathcal{M}$-subsets form a proper subfamily of all recognizable $\mathcal{M}$-subsets.

Corollary 9 $\mathcal{M}\,\mathrm{SRec}\,A^* \subsetneq \mathcal{M}\,\mathrm{Rec}\,A^*$.

Proof. Let X be the $\mathcal{M}$-subset of A^* defined by $wX = 2|w|$, for every $w \in A^*$. It is clear that X is a recognizable $\mathcal{M}$-subset; however, by Proposition 8, X is not simple.

∎

The next theorem shows that the converse of Proposition 8 is not valid.

Theorem 10 *There is a recognizable $\mathcal{M}$-subset X of A^* such that for each $w \in A^*$, either $wX = \infty$ or $wX \leq |w|$, but X is not simple.*

Proof. Let $A = \{a, b\}$ and let X be the $\mathcal{M}$-subset of A^* defined by

$$\forall w \in A^*, \quad wX = 2\min\{|w|_a, |w|_b\} \ .$$

It is clear that X is a recognizable $\mathcal{M}$-subset of A^* and X satisfies $wX \leq |w|$, for every $w \in A^*$.

Let us suppose that X is a simple $\mathcal{M}$-subset. In this case, there is a simple $\mathcal{M}$-A-automaton $\mathcal{A} = (Q, I, T)$ such that $\|\mathcal{A}\| = X$.

Let $n = |Q|$ and let us consider the word

$$w = a^n b^m, \qquad \text{with } m = 2n + 1 \ .$$

Then there is a victorious path P in $\mathcal{A}$, with $|P| = w$ and $\|P\| = w\|\mathcal{A}\| = wX = 2n$. Moreover, there are naturals r, s and t, with $s > 0$ and $r+s+t = n$, such that the path P can be decomposed in

$$P : i \xrightarrow{a^r} p \xrightarrow{a^s} p \xrightarrow{a^t} q \xrightarrow{b^m} f \ .$$

Let $w' = a^{n+s} b^m$. Then

$$w'X = 2\min\{\, |w'|_a, |w'|_b \,\} = 2\min\{n + s, m\} \ .$$

As $n+s \leq 2n$ and $m = 2n+1$, we have that $n+s < m$. Thus, $w'X = 2n+2s$.

Let us consider the factor $P_1 = (p, a^s, p)$ of P. Then, by inserting another factor P_1 in P, the resulting path is

$$P' : i \xrightarrow{a^r} p \xrightarrow{a^s} p \xrightarrow{a^s} p \xrightarrow{a^t} q \xrightarrow{b^m} f \ .$$

Since $\mathcal{A}$ is a simple $\mathcal{M}$-A-automaton, $0 \leq \|P_1\| \leq s$. Then we have that

$$w'\|\mathcal{A}\| \leq \|P'\| = \|P\| + \|P_1\| \leq \|P\| + s = 2n + s < 2n + 2s = w'X \ ,$$

contradicting that $X = \|\mathcal{A}\|$.

Therefore, X is not a simple $\mathcal{M}$-subset. ■

5 $\mathcal{M}$-subsets which are nondeterministic complexities

In this section, we study another subfamily of recognizable $\mathcal{M}$-subsets of A^*, that is, the family of nondeterministic complexities, denoted by $\mathcal{M}\,\mathrm{CRec}\,A^*$.

The idea of the nondeterministic complexity of a finite automaton consists in to associate, for each word, the minimum number of decisions which are necessary to spell it in a nondeterministic finite automaton. This idea appeared, by the first time, for the Turing machines and was formalized by Kintala and Fischer in 1977 [10]. In 1980, Kintala and Wotschke [11] considered this idea for the finite automata. Recently, Goldstine, Leung and Wotschke [5] related the ambiguity and the non-determinism in finite automata.

Let $\mathcal{A} = (Q, I, T)$ be a finite automaton (not necessarily deterministic) over an alphabet A. We say that an edge (p, a, q) of $\mathcal{A}$ is *non-deterministic* if there is another edge (p, a, q') in $\mathcal{A}$, with $q' \neq q$ and, is *deterministic*, otherwise. From $\mathcal{A}$, we construct an $\mathcal{M}$-A-automaton $\mathcal{B} = (Q, I_{\mathcal{B}}, T_{\mathcal{B}})$, defining the $\mathcal{M}$-subsets $I_{\mathcal{B}}$ and $T_{\mathcal{B}}$ of Q by

$$\forall q \in Q, \quad qI_{\mathcal{B}}(qT_{\mathcal{B}}) = \begin{cases} 0 & \text{if } q \text{ is an initial (final) state of } \mathcal{A} \\ \infty & \text{otherwise} \end{cases}$$

and the $\mathcal{M}$-subset $E_{\mathcal{B}}$ of $Q \times A \times Q$ by

$$(p, a, q)E_{\mathcal{B}} = \begin{cases} 0 & \text{if } (p, a, q) \text{ is a deterministic edge of } \mathcal{A} \\ 1 & \text{if } (p, a, q) \text{ is a non-deterministic edge of } \mathcal{A} \\ \infty & \text{if } (p, a, q) \text{ is not an edge of } \mathcal{A}\ . \end{cases}$$

Then, for each $w \in A^*$, $w\|\mathcal{B}\|$ is exactly the minimum number of non-deterministic edges necessary to spell w in $\mathcal{A}$ from some initial state to some final state.

Now, let $\mathcal{C}$ be a simple $\mathcal{M}$-A-automaton such that for each useful edge (p, a, q) of $\mathcal{C}$,

$$(p, a, q)E_{\mathcal{C}} = \begin{cases} 0 & \text{if there is no other useful edge } (p, a, q') \text{ in } \mathcal{C} \text{ with } q' \neq q \\ 1 & \text{otherwise}\ . \end{cases}$$

Then we say that the $\mathcal{M}$-A-automaton $\mathcal{C}$ is of *type nc*. The $\mathcal{M}$-A-automaton $\mathcal{B}$ previously constructed is also of type nc.

An $\mathcal{M}$-subset X of A^* is a *nondeterministic complexity* if it is the behavior of some $\mathcal{M}$-A-automaton $\mathcal{A}$ which is of type nc. Indeed, it is enough that $\mathcal{A}$ be a simple $\mathcal{M}$-A-automaton such that for each useful edge (p, a, q) in $\mathcal{A}$, with multiplicity zero, there is no other useful edge (p, a, r) in $\mathcal{A}$ with $r \neq q$.

Note that every nondeterministic complexity is a simple $\mathcal{M}$-subset.

Before stating a necessary condition for an $\mathcal{M}$-subset to be a nondeterministic complexity, we give a definition.

We say that a recognizable $\mathcal{M}$-subset X is of *differentiable multiplicity* if there exist words x, y, u and v in A^* such that for each $k \geq 1$, there exists a word $z_k \in A^+$ satisfying

$$\forall l \geq 0 \quad \text{and} \quad \forall m > k, \quad w_{klk}X = w_{kkk}X < w_{kkm}X < \infty ,$$

where $w_{klm} = x(uz_k^k v)^l uz_k^m y$. That is, for each $k \geq 1$, the word $w_{kkk} = x(uz_k^k v)^k uz_k^k y$ has a factor z_k which occurs in two distinct contexts. In one of these contexts, the factor $uz_k^k v$ can be eliminated from w_{kkk} or can be infinitely iterated without to modify the multiplicity of the resulting word $w_{klk} = x(uz_k^k v)^l uz_k^k y$, $l \geq 0$. However, in another context, if the factor z_k is iterated m times, with $m > k$, the multiplicity of the resulting word $w_{kkm} = x(uz_k^k v)^k uz_k^m y$ is greater than the multiplicity of w_{kkk}.

Remarks: (1) Note that an $\mathcal{M}$-subset X being a nondeterministic complexity is a property which depends of the existence of a type nc $\mathcal{M}$-A-automaton with behavior X. But, X being of differentiable multiplicity is independent of any $\mathcal{M}$-A-automaton recognizing X.
(2) In this paper, in all proofs in which an $\mathcal{M}$-subset is shown to be a nondeterministic complexity (except in the proof of Theorem 23), it is possible to consider the same word z, for every $k \geq 1$.

The following lemma presents a necessary condition for an $\mathcal{M}$-subset to be a nondeterministic complexity.

Lemma 11 *If an $\mathcal{M}$-subset X is a nondeterministic complexity then X is not of differentiable multiplicity.*

Before proving this lemma, we state some properties of paths in an $\mathcal{M}$-A-automaton which is of type nc. One of these properties is in the following proposition whose proof is immediate.

Proposition 12 *Let $\mathcal{A}$ be a type nc $\mathcal{M}$-A-automaton. Let P and P' be useful and distinct paths in $\mathcal{A}$ with the same labels. If P and P' have the same origin, then their multiplicities are different of zero.* ■

Lemma 13 *Let $\mathcal{A}$ be a type nc $\mathcal{M}$-A-automaton. Let P be a path in $\mathcal{A}$ spelling w^n, for some $w \in A^+$ and $n > 0$,*

$$P : q_0 \xrightarrow{w} q_1 \xrightarrow{w} q_2 \xrightarrow{w} \cdots \xrightarrow{w} q_{n-1} \xrightarrow{w} q_n \ .$$

If there are j and k, $0 \le j < k \le n$, such that $q_j = q_k$ and the factor (q_j, w^{k-j}, q_k) of P has multiplicity zero, then $q_n \in \{ q_i : 0 \le i \le n-1 \}$, and the factor (q_l, w^{n-l}, q_n) of P, with $l = \min\{ i : 0 \le i \le n-1$ and $q_i = q_n \}$, has multiplicity zero.

Proof. Let $\mathcal{A}$ be a type nc $\mathcal{M}$-A-automaton and let $w \in A^+$.

Consider a useful path P in $\mathcal{A}$, spelling w^n, for some $n > 0$,

$$P : q_0 \xrightarrow{w} q_1 \xrightarrow{w} q_2 \xrightarrow{w} \cdots \xrightarrow{w} q_n \ .$$

Suppose that there are j and k, $0 \le j < k \le n$, such that $q_j = q_k$ and the factor (q_j, w^{k-j}, q_k) of P has multiplicity zero. Then we can determine the maximum m of the following set:

$$m = \max\{ i : 1 \le i \le n \text{ and there exists } h,\ 0 \le h < i \text{ such that } q_h = q_i$$
$$\text{and the factor } (q_h, w^{i-h}, q_i) \text{ of } P \text{ has multiplicity zero} \} \ .$$

Let t, $0 \le t \le m-1$, such that $q_t = q_m$ and the factor $P_1 = (q_t, w^{m-t}, q_m)$ of P has multiplicity zero.

If $m \neq n$, by the choice of m, we conclude that the path P has two factors

$$P_2 = (q_t, w, q_{t+1}) \quad \text{and} \quad P_3 = (q_m, w, q_{m+1}) \ ,$$

with $q_t = q_m$ and $q_{t+1} \neq q_{m+1}$. As $\mathcal{A}$ is of type nc, from Proposition 12 it results that P_2 and P_3 must have positive multiplicities. But, P_2 is also factor of P_1; then $||P_2|| \le ||P_1||$. Hence, we have that $0 < ||P_2|| \le ||P_1|| = 0$; this is a contradiction. Therefore, $m = n$.

Let l be the minimum of the following set:

$$l = \min\{ i : 0 \le i \le n-1 \text{ and } q_i = q_n \} \ .$$

Then $l \le t$.

If $l = t$, we know that the factor $P_1 = (q_l, w^{n-l}, q_n)$ of P has multiplicity zero.

If $l < t$, we consider the paths

$$P_4 = (q_l, w^{t-l}, q_t) \quad \text{and} \quad P_5 = (P_1)^s (q_t, w^r, q_{t+r})$$

such that P_4 is a factor of P, (q_t, w^r, q_{t+r}) is a factor of P_1 and, s and r are naturals satisfying $t-l = s(n-t)+r$ and $0 \leq r < n-t$. As $||P_1|| = 0$, it follows that $||P_5|| = 0$. Then, as $\mathcal{A}$ is of type nc, $q_t = q_n = q_l$, $|P_4| = |P_5| = w^{t-l}$ and $||P_5|| = 0$, by Proposition 12 it results that P_4 coincides with P_5. Hence, $||P_4|| = 0$.

Thus, the factor $P_4 P_1 = (q_l, w^{n-l}, q_n)$ of P has multiplicity zero. ∎

Proof of Lemma 11. Let X be an $\mathcal{M}$-subset of A^* which is a nondeterministic complexity. Let $\mathcal{A}$ be a type nc $\mathcal{M}$-A-automaton such that $||\mathcal{A}|| = X$.

Suppose that X is of differentiable multiplicity. Then there are words x, y, u and v in A^* such that for every $k \geq 1$, there is a word z_k in A^+ in such a way that for every $l \geq 0$ and for every $m > k$, $w_{klk}X = w_{kkk}X < w_{kkm}X < \infty$, where $w_{klm} = x(uz_k^k v)^l uz_k^m y$.

Let k be the number of states of $\mathcal{A}$ and let us consider the word

$$w = w_{kkk} = x(uz_k^k v)^k uz_k^k y$$

and a victorious path P in $\mathcal{A}$, spelling w. To simplify the notation, let us use z, instead of z_k, throughout this proof.

The path P can be decomposed as follows:

$$P : p_0 \xrightarrow{x} q_0 \xrightarrow{uz^k v} q_1 \xrightarrow{uz^k v} q_2 \cdots \xrightarrow{uz^k v} q_k \xrightarrow{uz^k} q_{k+1} \xrightarrow{y} q_{k+2} .$$

Then there are integers j and h, $0 \leq j < h \leq k$ such that $q_j = q_h$.

Consider the factor $P_1 = (q_j, (uz^k v)^{h-j}, q_h)$ of P. If $||P_1|| \neq 0$, the word

$$w' = w_{k,j+k-h,k} = x(uz^k v)^{j+k-h} uz^k y$$

can be spelled in $\mathcal{A}$ by the following successful path

$$P' : p_0 \xrightarrow{x} q_0 \xrightarrow{(uz^k v)^j} q_j = q_h \xrightarrow{(uz^k v)^{k-h}} q_k \xrightarrow{uz^k} q_{k+1} \xrightarrow{y} q_{k+2} .$$

Hence, it results that $w'||\mathcal{A}|| \leq ||P'|| < ||P|| = w||\mathcal{A}|| = wX$.

However, as X is of differentiable multiplicity, it follows that $wX = w_{kkk}X = w_{k,j+k-h,k}X = w'X$. So, $w'||\mathcal{A}|| < w'X$, contradicting that $X = ||\mathcal{A}||$. Therefore, $||P_1|| = 0$.

Thus, by Lemma 13, there is an integer i, $0 \leq i \leq k-1$, such that $q_i = q_k$ and the factor $P_2 = (q_i, (uz^k v)^{k-i}, q_k)$ of P has multiplicity zero.

Consider, now, the factor $P_3 = (q_i, uz^k v, q_{i+1})$ of P_2, with the following factorization:

$$P_3 : q_i \xrightarrow{u} r_0 \xrightarrow{z} r_1 \xrightarrow{z} r_2 \xrightarrow{z} \cdots \xrightarrow{z} r_k \xrightarrow{v} q_{i+1} \ .$$

As $||P_3|| = 0$ and k is the number of states of $\mathcal{A}$, it results that P_3 has a factor $(r_{i_1}, z^{i_2-i_1}, r_{i_2})$ with multiplicity zero such that $0 \leq i_1 < i_2 \leq k$ and $r_{i_1} = r_{i_2}$. Then, by Lemma 13, there is an integer l, $0 \leq l \leq k-1$, such that $r_l = r_k$.

As $\mathcal{A}$ is of type nc, $||P_3|| = 0$ and $q_i = q_k$, we conclude that the factor (q_k, uz^k, q_{k+1}) of P coincides with the following factor of P_3:

$$q_i \xrightarrow{u} r_0 \xrightarrow{z^l} r_l \xrightarrow{z^{k-l}} r_k \ ;$$

hence, $r_k = r_l = q_{k+1}$. Then the word

$$w'' = w_{k,k,k+k-l} = x(uz^k v)^k uz^{k+k-l} y$$

can be spelled in $\mathcal{A}$ by the following successful path

$$P'' : p_0 \xrightarrow{x} q_0 \xrightarrow{uz^k v} q_1 \cdots \xrightarrow{uz^k v} q_k \xrightarrow{uz^k} q_{k+1} = r_l \xrightarrow{z^{k-l}} r_k = q_{k+1} \xrightarrow{y} q_{k+2} \ .$$

So, $w''||\mathcal{A}|| \leq ||P''|| = ||P|| = w||\mathcal{A}|| = wX$.

But, as X is of differentiable multiplicity, $wX = w_{kkk}X < w_{k,k,k+k-l}X = w''X$. Therefore, $w''||\mathcal{A}|| < w''X$, contradicting that $X = ||\mathcal{A}||$. Thus, X is not of differentiable multiplicity. ∎

The condition presented in Lemma 11 is useful to give examples of simple $\mathcal{M}$-subsets which are not nondeterministic complexities.

Theorem 14 $\mathcal{M}\,\mathrm{CRec}\,A^* \subsetneq \mathcal{M}\,\mathrm{SRec}\,A^*$ *for an alphabet A with at least two letters.*

Proof. Let $A = \{a, b\}$. The $\mathcal{M}$-subset X of A^* defined by

$$wX = \begin{cases} n & \text{if } w = ua^n, \ \text{ with } \ u \in (aa^*b)^* \\ \infty & \text{otherwise} \end{cases}$$

is simple but it is not a nondeterministic complexity.

It is easy to see that X is a simple $\mathcal{M}$-subset of A^* and we can verify that X satisfies the following conditions:

$\forall k \geq 1, \quad ((a^k b)^k a^k)X = k;$

$\forall k \geq 1, \quad \forall m > k, \quad ((a^k b)^k a^m)X = m > k;$

$\forall k \geq 1, \quad \forall l \geq 0, \quad ((a^k b)^l a^k)X = k.$

Then we conclude that,

$$\forall k \geq 1, \forall l \geq 0, \forall m > k, \; ((a^k b)^l a^k)X = ((a^k b)^k a^k)X < ((a^k b)^k a^m)X < \infty \; .$$

Therefore, by considering the words $x = y = 1$, $u = 1, v = b$ and $z_k = a$, for every $k \geq 1$, it results that X is of differentiable multiplicity. Thus, by Lemma 11, X is not a nondeterministic complexity. ■

6 Closure properties under the basic operations

In this section, we present the closure properties of the families $\mathcal{M}\,\mathrm{Rec}\,A^*$, $\mathcal{M}\,\mathrm{SRec}\,A^*$ and $\mathcal{M}\,\mathrm{CRec}\,A^*$ under the basic operations. These properties are summarized in Table 1 and their proofs can be found in [12].

As $\mathcal{M}$ is a commutative semiring, the majority of these properties for $\mathcal{M}\,\mathrm{Rec}\,A^*$ follows from the corresponding properties showed by Eilenberg [4] for the family of recognizable K-subsets of A^*, where K is an arbitrary commutative semiring.

For the operations under which the families $\mathcal{M}\,\mathrm{SRec}\,A^*$ and $\mathcal{M}\,\mathrm{CRec}\,A^*$ are closed, either the proofs follow from the respective proofs for $\mathcal{M}\,\mathrm{Rec}\,A^*$, or it is necessary to use different constructions to maintain the property of being simple or a nondeterministic complexity. And, for the operations under which some family is not closed, the idea is to obtain $\mathcal{M}$-subsets which do not satisfy one of the necessary conditions seen in the Sections 3, 4 and 5.

One knows by the Kleene-Schützenberger Theorem that for every finite alphabet A, $\mathcal{M}\,\mathrm{Rec}\,A^* = \mathcal{M}\,\mathrm{Rat}\,A^*$. That is, $\mathcal{M}\,\mathrm{Rec}\,A^*$ is the rational closure of $\mathcal{M}\,\mathrm{CRec}\,A^*$.

In Table 1, we observe that $\mathcal{M}\,\mathrm{CRec}\,A^*$ is not closed under concatenation and star. Investigating the closure of $\mathcal{M}\,\mathrm{CRec}\,A^*$ under these operations, we showed in [12] the following result, whose proof is based on the proof of the Kleene Theorem given by McNaughton and Yamada.

Operation	$\mathcal{M}$ Rec	$\mathcal{M}$ SRec	$\mathcal{M}$ CRec
$\min(X, Y)$	yes	yes	yes
$m + X$, $0 < m < \infty$	yes	no	no
$m + X$, $m = \infty$ or $0 \leq m \leq \min\{\,\lvert w\rvert - wX : wX < \infty\,\}$	yes	yes	yes
$X + Y$	yes	no	no
$X + Y$, $\max\{\,wY : wY < \infty\,\} \leq \min\{\,\lvert w\rvert - wX : wX < \infty\,\}$	yes	yes	yes
$X\rho$, ρ is the reverse function	yes	yes	no
XY $\quad X^*$ $\quad X^+$	yes	yes	no
$X \sqcup\!\sqcup Y$, $\sqcup\!\sqcup$ is the shuffle	yes	yes	yes
Xf, f is a morphism	yes	no	no
Xf, f is a morphism s.t. $1f^{-1} = 1$	yes	yes	no
Xf, f is a fine and injective morphism	yes	yes	yes
Xf^{-1}, f is a morphism	yes	no	no
Xf^{-1}, f is a fine morphism	yes	yes	yes

Table 1: Closure properties of $\mathcal{M}$ Rec, $\mathcal{M}$ SRec and $\mathcal{M}$ CRec under the basic operations.

Theorem 15 *For every finite alphabet A, the closure of $\mathcal{M}$ CRec A^* under the minimum, concatenation and star operations is exactly $\mathcal{M}$ SRec A^*.* ∎

7 Limited $\mathcal{M}$-subsets

We say that an $\mathcal{M}$-subset X of A^* is *limited* if A^*X is a finite subset of $\mathcal{M}$.

Let us first consider some limited $\mathcal{M}$-subsets X of A^* such that, either $A^*X = \{\,m, 0\,\}$ or $A^*X = \{\,m, \infty\,\}$, for some $m \in \mathcal{M}$.

Let R be a subset of A^* and $m \in \mathcal{M}$. We define two $\mathcal{M}$-subsets of A^*, $\lfloor R, m\rfloor$ and $\lceil R, m\rceil$, as follows.

$$\forall w \in A^*, \quad w\lfloor R, m\rfloor = \begin{cases} m & \text{if } w \in R \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and} \quad w\lceil R, m\rceil = \begin{cases} m & \text{if } w \in R \\ \infty & \text{otherwise .} \end{cases}$$

The single $\mathcal{M}$-subsets $\mathbf{w}$, for every $w \in A^*$, are particular cases of the $\mathcal{M}$-subsets we just define; that is, $\mathbf{w} = \lceil \{w\}, 0 \rceil$. Thus, for every $\mathcal{M}$-subset X of A^*, a expansion of X can be given by

$$X = \min_{w \in A^*}(wX + \lceil \{w\}, 0 \rceil) = \min_{w \in A^*} \lceil \{w\}, wX \rceil \ .$$

It is interesting to observe that, for every $\mathcal{M}$-subset X of A^*, there is another kind of expansion which uses the addition instead of the minimum.

$$X = \sum_{w \in A^*} (\sum_{i=1}^{wX} \lfloor \{w\}, 1 \rfloor) = \sum_{w \in A^*} \lfloor \{w\}, wX \rfloor \ .$$

We observe that if $wX = 0$, then $\sum_{i=1}^{wX} \lfloor \{w\}, 1 \rfloor = \lfloor A^*, 0 \rfloor$.

The following proposition verifies when $\lfloor R, m \rfloor$ and $\lceil R, m \rceil$ are recognizable $\mathcal{M}$-subsets.

Proposition 16 *Let $R \subseteq A^*$ and $m \in \mathcal{M}$. $\lfloor R, m \rfloor$ (resp. $\lceil R, m \rceil$) is a recognizable $\mathcal{M}$-subset of A^* if and only if either $m = 0$ (resp. $m = \infty$) or R is a recognizable subset of A^*.*

Proof. Suppose that R is a recognizable subset of A^*. It is easy to verify that $\lceil R, m \rceil$ is a recognizable $\mathcal{M}$-subset of A^*.

To show that $\lfloor R, m \rfloor$ is a recognizable $\mathcal{M}$-subset, one can consider the identity

$$\lfloor R, m \rfloor = \min(\lceil R, m \rceil, \lceil A^* - R, 0 \rceil)$$

and the property that $\mathcal{M}\,\mathrm{Rec}\,A^*$ is closed under the minimum operation.

Now, suppose that $\lfloor R, m \rfloor$ is a recognizable $\mathcal{M}$-subset of A^*. If $m \neq 0$, then $R = m(\lfloor R, m \rfloor)^{-1}$, and by Proposition 4, R is a recognizable subset of A^*. If R is not a recognizable subset of A^*, then $R \neq m(\lfloor R, m \rfloor)^{-1}$; however, this occurs only if $m = 0$.

The proof for $\lceil R, m \rceil$ is similar. ■

The following lemma shows that every (recognizable and) limited $\mathcal{M}$-subset is the sum of a finite number of $\mathcal{M}$-subsets of the form $\lfloor R, m \rfloor$ and the minimum of a finite number of $\mathcal{M}$-subsets of the form $\lceil R, m \rceil$.

Lemma 17 *An $\mathcal{M}$-subset X of A^* is (recognizable and) limited if and only if there is a positive integer n and, there are n (recognizable) subsets $X_1, \ldots, X_n$ of A^* and n elements $m_1, \ldots, m_n$ of $\mathcal{M}$ such that*

$$X = \sum_{i=1}^{n} \lfloor X_i, m_i \rfloor \qquad \text{and} \qquad X = \min_{1 \le i \le n} \lceil X_i, m_i \rceil \ .$$

Proof. Let X be a limited $\mathcal{M}$-subset of A^* and let $n = |A^*X|$. Denote the elements of A^*X by $m_1, \ldots, m_n$ and consider, for each i, $1 \le i \le n$, the set

$$X_i = \{w \in A^* : wX = m_i\} = m_i X^{-1} \ .$$

Then one can verify that

$$\forall w \in A^*, \quad wX = w \sum_{i=1}^{n} \lfloor X_i, m_i \rfloor \ .$$

Moreover, if X is a recognizable $\mathcal{M}$-subset of A^*, from Proposition 4 it results that for each i, $1 \le i \le n$, X_i is a recognizable subset of A^*.

The converse is immediate. ■

The following proposition shows that some $\mathcal{M}$-subsets can be defined from others, by using an $\mathcal{M}$-subset of the form $\lceil R, 0 \rceil$ (resp. $\lfloor R, \infty \rfloor$) and the addition (resp. minimum) operation.

Proposition 18 *Let R be a subset of A^* and X be an $\mathcal{M}$-subset of A^*. Then the $\mathcal{M}$-subsets Y_1 and Y_2 defined by*

$$\forall w \in A^*, \quad wY_1 = \begin{cases} wX & \text{if } w \in R \\ \infty & \text{otherwise} \end{cases}$$

$$\text{and} \quad wY_2 = \begin{cases} wX & \text{if } w \in R \\ 0 & \text{otherwise} \end{cases}$$

satisfy

$$Y_1 = X + \lceil R, 0 \rceil \qquad \text{and} \qquad Y_2 = \min(X, \lfloor R, \infty \rfloor) \ .$$

Moreover, if R is a recognizable subset of A^* and $X \in \mathcal{M}\,\mathrm{Rec}\,A^*$ (resp. $\mathcal{M}\,\mathrm{SRec}\,A^*$, $\mathcal{M}\,\mathrm{CRec}\,A^*$), then Y_1 and $Y_2 \in \mathcal{M}\,\mathrm{Rec}\,A^*$ (resp. $\mathcal{M}\,\mathrm{SRec}\,A^*$, $\mathcal{M}\,\mathrm{CRec}\,A^*$).

Proof. It is easy to verify that $Y_1 = X + \lceil R, 0 \rceil$ and $Y_2 = \min(X, \lfloor R, \infty \rfloor)$.

If R is a recognizable subset of A^*, from Proposition 16 it results that the $\mathcal{M}$-subsets $\lceil R, 0 \rceil$ and $\lfloor R, \infty \rfloor \in \mathcal{M}\,\mathrm{Rec}\,A^*$. Thus, if $X \in \mathcal{M}\,\mathrm{Rec}\,A^*$, we have that Y_1 and $Y_2 \in \mathcal{M}\,\mathrm{Rec}\,A^*$ since $\mathcal{M}\,\mathrm{Rec}\,A^*$ is closed under the addition and minimum operations.

Moreover, if R is a recognizable subset of A^*, it is clear that $\lceil R, 0 \rceil$ and $\lfloor R, \infty \rfloor = \lceil A^* - R, 0 \rceil$ are nondeterministic complexities. Then, if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, we have that Y_1 and $Y_2 \in \mathcal{M}\,\mathrm{SRec}\,A^*$, since $\mathcal{M}\,\mathrm{SRec}\,A^*$ is closed under the minimum operation and under the addition with $\lceil R, 0 \rceil$. If $X \in \mathcal{M}\,\mathrm{CRec}\,A^*$, the proof is similar. ∎

In the sequel, we study when the recognizable and limited $\mathcal{M}$-subsets are nondeterministic complexities.

Proposition 19 *Let R be a recognizable subset of A^* and $m \in \mathcal{M}$. If $m \leq \min\{\, |w| : w \in R \,\}$ then $\lfloor R, m \rfloor$ and $\lceil R, m \rceil$ are nondeterministic complexities.*

Proof. Let R be a subset of A^* and $m \in \mathcal{M}$. Then

$$\lceil R, m \rceil = m + \lceil R, 0 \rceil \quad \text{and} \quad \lfloor R, m \rfloor = \min(\lceil R, m \rceil, \lceil A^* - R, 0 \rceil)\ .$$

Let us suppose that R is recognizable. Then, from Proposition 16 it results that $\lceil R, 0 \rceil$ and $\lceil A^* - R, 0 \rceil$ are recognizable $\mathcal{M}$-subsets, and it is easy to verify that both are nondeterministic complexities. If $m \leq \min\{\, |w| : w \in R \,\}$, then $m + \lceil R, 0 \rceil \in \mathcal{M}\,\mathrm{CRec}\,A^*$. (See Table 1.) Therefore, $\lceil R, m \rceil \in \mathcal{M}\,\mathrm{CRec}\,A^*$. Moreover, as $\mathcal{M}\,\mathrm{CRec}\,A^*$ is closed under the minimum operation, we have that $\lfloor R, m \rfloor \in \mathcal{M}\,\mathrm{CRec}\,A^*$. ∎

Lemma 20 *Let X be a recognizable and limited $\mathcal{M}$-subset of A^* such that for each $w \in A^*$, either $wX = \infty$ or $wX \leq |w|$. Then $X \in \mathcal{M}\,\mathrm{CRec}\,A^*$.*

Proof. Let X be as in the statement of this lemma. Consider $n = |A^*X|$ and denote the elements of A^*X by $m_1, \ldots, m_n$. Then, from Lemma 17 (and its proof), there are n recognizable subsets $m_1X^{-1}, \ldots, m_nX^{-1}$ of A^* such that

$$X = \min_{1 \leq i \leq n} \lceil m_i X^{-1}, m_i \rceil\ .$$

Let us consider i, $1 \leq i \leq n$. If $m_i = \infty$, then $\lceil \infty X^{-1}, \infty \rceil = \emptyset$ is a nondeterministic complexity. If $m_i \neq \infty$, then for every $w \in m_iX^{-1}$,

$m_i = wX \leq |w|$. Thus, by Proposition 19 it follows that the $\mathcal{M}$-subset $[m_i X^{-1}, m_i]$ is a nondeterministic complexity. As $\mathcal{M}\,\mathrm{CRec}\,A^*$ is closed under the minimum operation, we have that $X \in \mathcal{M}\,\mathrm{CRec}\,A^*$. ■

In Proposition 8 we saw that every simple $\mathcal{M}$-subset X of A^* satisfies, for each $w \in A^*$, either $wX = \infty$ or $wX \leq |w|$. Then, by Lemma 20 we conclude that every limited and simple $\mathcal{M}$-subset is a nondeterministic complexity. Thus, we just prove the following corollary, where $\mathcal{H}_0$ denotes the family of the recognizable and limited $\mathcal{M}$-subsets.

Corollary 21 $\mathcal{M}\,\mathrm{SRec}\,A^* \cap \mathcal{H}_0 = \mathcal{M}\,\mathrm{CRec}\,A^* \cap \mathcal{H}_0$. ■

8 The Simon hierarchy for $\mathcal{M}\,\mathrm{Rec}\,A^*$ and its relation to $\mathcal{M}\,\mathrm{SRec}\,A^*$ and $\mathcal{M}\,\mathrm{CRec}\,A^*$

There exists a proper hierarchy for $\mathcal{M}\,\mathrm{Rec}\,A^*$ that was obtained by I. Simon [21] through the families $\mathcal{H}_p$ ($p \geq 0$) of recognizable $\mathcal{M}$-subsets of A^* defined by

$$\mathcal{H}_p = \{\, X \in \mathcal{M}\,\mathrm{Rec}\,A^* : \mathrm{sh}(X,m) \in O(m^p) \,\} ,$$

where $\mathrm{sh}(X,m) = \min\{\, |w| : w \in A^*, m \leq wX < \infty \}$; that is, $\mathrm{sh}(X,m)$ is the minimum length that a word needs to have so that its multiplicity be at least m.

Theorem 22 (I. Simon [21])
For an alphabet A with at least two letters, $\mathcal{M}\,\mathrm{Rec}\,A^* = \cup_{p\geq 0}\mathcal{H}_p$ and, for every $p \geq 1$, there is a nondeterministic complexity function in $\mathcal{H}_p - \mathcal{H}_{p-1}$. If the alphabet A has only one letter, $\mathcal{M}\,\mathrm{Rec}\,A^* = \mathcal{H}_0 \cup \mathcal{H}_1$.

We also studied in [12] the relations among the families $\mathcal{H}_p$ ($p \geq 0$) and the families $\mathcal{M}\,\mathrm{CRec}\,A^*$ and $\mathcal{M}\,\mathrm{SRec}\,A^*$, and showed the following result which uses a convenient extension of the Simon's nondeterministic complexity functions [21] and Lemma 11. It is easy to verify that a similar result holds for the families $\mathcal{H}_p$ ($p \geq 0$) restricted to the recognizable $\mathcal{M}$-subsets that are not simple.

Theorem 23 *For each $p \geq 1$, $(\mathcal{M}\,\mathrm{CRec}\,A^* \cap \mathcal{H}_p) \subsetneq (\mathcal{M}\,\mathrm{SRec}\,A^* \cap \mathcal{H}_p) \subsetneq \mathcal{H}_p$, where A is an alphabet with at least two letters.* ■

Let us represent in a diagram (Figure 1) the known relations for the families $\mathcal{M}\,\mathrm{Rec}\,A^*$, $\mathcal{M}\,\mathrm{SRec}\,A^*$, $\mathcal{M}\,\mathrm{CRec}\,A^*$ and $\mathcal{H}_p$ ($p \geq 0$), considering the alphabet A with at least two letters. These relations are described in Corollaries 9 and 21, Theorems 14 and 23, and Theorem 22 (Simon).

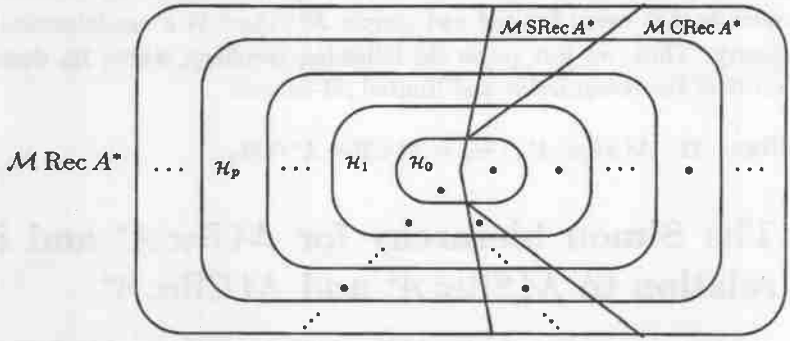

Figure 1: Relating the families $\mathcal{M}\,\mathrm{Rec}\,A^*$, $\mathcal{M}\,\mathrm{SRec}\,A^*$, $\mathcal{M}\,\mathrm{CRec}\,A^*$ and $\mathcal{H}_p$ ($p \geq 0$), where $|A| \geq 2$.

9 Closure properties under other operations

In this section, let us only show the results with respect to the closure properties of the families $\mathcal{M}\,\mathrm{Rec}\,A^*$, $\mathcal{M}\,\mathrm{SRec}\,A^*$ and $\mathcal{M}\,\mathrm{CRec}\,A^*$ under the maximum, remainder and minusp operations. These properties are summarized in Table 2. The results and the corresponding proofs about the closure of the families $\mathcal{M}\,\mathrm{Rec}\,A^*$ and $\mathcal{M}\,\mathrm{SRec}\,A^*$ under the div d operation can be found in [12, 13].

At first, consider the *maximum* operation (denoted by max) over the semiring $\mathcal{M}$. This operation can be extended to the $\mathcal{M}$-subsets of A^* as follows. Let X and Y be $\mathcal{M}$-subsets of A^*. The $\mathcal{M}$-subset $\max(X,Y)$ is defined by

$$\forall w \in A^*, \quad w(\max(X,Y)) = \max(wX, wY) \ .$$

A property that can be easily verified is the distributivity of the maximum with respect to the minimum. Let $X, Y_1, \ldots, Y_n$ be $\mathcal{M}$-subsets of A^*. Then

$$\max(X, \min_{1\leq i\leq n} Y_i) = \min_{1\leq i\leq n} (\max(X, Y_i)) \ .$$

Operation	$\mathcal{M}$ Rec	$\mathcal{M}$ SRec	$\mathcal{M}$ CRec
$\max(X, Y)$	no	no	no
$\max(X, Y)$, Y is limited	yes	yes	yes
$Y \bmod d$, $d > 1$	no	no	no
$Y \bmod d$, $d > 0$, Y is limited	yes	yes	yes
$X \dot{-} Y$, $\dot{-}$ is the minusp	no	no	no
$X \dot{-} Y$, Y is limited	yes	yes	yes
$Y \operatorname{div} d$, $d > 0$	yes	yes	?

Table 2: Closure properties of $\mathcal{M}$ Rec, $\mathcal{M}$ SRec and $\mathcal{M}$ CRec under other operations.

Proposition 24 *$\mathcal{M}$ Rec A^* is not closed under the maximum operation.*

Proof. Let $A = \{a, b\}$ and consider the $\mathcal{M}$-subsets X and Y of A^+ defined by

$$\forall w \in A^+, \quad wX = |w|_a \quad \text{and} \quad wY = |w|_b \ .$$

Then $X, Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$ and the $\mathcal{M}$-subset $\max(X, Y)$ is given by

$$\forall w \in A^+, \quad w(\max(X, Y)) = \max(wX, wY) = \max(|w|_a, |w|_b) \ .$$

Hence, by the proof of Lemma 5 it results that $\max(X, Y) \notin \mathcal{M}\,\mathrm{Rec}\,A^*$. Thus, $\mathcal{M}$ Rec A^* is not closed under maximum. ■

Now, let us see when the maximum of two recognizable $\mathcal{M}$-subsets is a recognizable $\mathcal{M}$-subset.

Proposition 25 *Let $X \in \mathcal{M}\,\mathrm{Rec}\,A^*$, $m \in \mathcal{M}$ and R be a recognizable subset of A^*. Then* $\max(X, \lceil R, m \rceil) \in \mathcal{M}\,\mathrm{Rec}\,A^*$.

Proof. Consider $X \in \mathcal{M}\,\mathrm{Rec}\,A^*$. Let $m \in \mathcal{M}$ and R be a recognizable subset of A^*. Then,

$$\forall w \in A^*, \quad w(\max(X, \lceil R, m \rceil)) = \begin{cases} \infty & \text{if } w \notin R \\ m & \text{if } w \in R \text{ and } wX \leq m \\ wX & \text{if } w \in R \text{ and } m < wX \ . \end{cases}$$

Now, we define some recognizable $\mathcal{M}$-subsets from which we can obtain the $\mathcal{M}$-subset $\max(X, \lceil R, m \rceil)$, using only operations under which $\mathcal{M}\,\mathrm{Rec}\,A^*$ is closed.

Consider the subsets

$$R_1 = \bigcup_{i=0}^{m} iX^{-1} \cap R = \{ w \in A^* : w \in R \text{ and } wX \leq m \}$$

and $$R_2 = (A^* - \bigcup_{i=0}^{m} iX^{-1}) \cap R = \{ w \in A^* : w \in R \text{ and } m < wX \} .$$

By Proposition 4, the subsets R_1 and R_2 are recognizable. From these subsets, we define the $\mathcal{M}$-subsets

$$X_1 = \lceil R_1, m \rceil \quad \text{and} \quad X_2 = X + \lceil R_2, 0 \rceil .$$

Then, from Propositions 16 and 18, it results that X_1 and $X_2 \in \mathcal{M}\,\mathrm{Rec}\,A^*$. But, as $R_1 \cap R_2 = \emptyset$, we can verify that $\max(X, \lceil R, m \rceil) = \min(X_1, X_2)$.

Thus, as $\mathcal{M}\,\mathrm{Rec}\,A^*$ is closed under minimum, we have that $\max(X, \lceil R, m \rceil) \in \mathcal{M}\,\mathrm{Rec}\,A^*$. ∎

Proposition 26 *Let R be a recognizable subset of A^* and $m \in \mathcal{M}$ such that $\lceil R, m \rceil \in \mathcal{M}\,\mathrm{SRec}\,A^*$. If $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$ (resp. $\mathcal{M}\,\mathrm{CRec}\,A^*$), then $\max(X, \lceil R, m \rceil) \in \mathcal{M}\,\mathrm{SRec}\,A^*$ (resp. $\mathcal{M}\,\mathrm{CRec}\,A^*$).*

Proof. Consider $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$. Let R be a recognizable subset of A^* and $m \in \mathcal{M}$ such that $\lceil R, m \rceil \in \mathcal{M}\,\mathrm{SRec}\,A^*$. If $R = \emptyset$ or $m = \infty$, then $\lceil R, m \rceil = \emptyset$. And, in this case, $\max(X, \lceil R, m \rceil) = \emptyset \in \mathcal{M}\,\mathrm{CRec}\,A^*$. Then we can assume that $R \neq \emptyset$ and $m < \infty$. As $\lceil R, m \rceil$ is simple, we have that $m \leq \min\{ |w| : w \in R \}$.

Thus, the proof that $\max(X, \lceil R, m \rceil)$ is simple results from the proof of Proposition 25 and from the following remarks:

1. Since $R_1 \subseteq R$, we conclude that for every $w \in R_1$, $|w| \geq m$. Then, by Proposition 19, $X_1 = \lceil R_1, m \rceil \in \mathcal{M}\,\mathrm{CRec}\,A^*$.

2. By Proposition 18, if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, then $X_2 = X + \lceil R_2, 0 \rceil \in \mathcal{M}\,\mathrm{SRec}\,A^*$.

3. $\mathcal{M}\,\mathrm{SRec}\,A^*$ is closed under the minimum operation.

The proof is similar if $X \in \mathcal{M}\,\mathrm{CRec}\,A^*$. ■

By using Proposition 25 and one of the characterizations of limited $\mathcal{M}$-subsets, we can extend the subfamily of recognizable $\mathcal{M}$-subsets that is closed under the maximum operation.

Lemma 27 *Let X, $Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$. If Y is limited, then* $\max(X,Y) \in \mathcal{M}\,\mathrm{Rec}\,A^*$.

Proof. Let X, $Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$ and assume that Y is limited. Consider $n = |A^*Y|$ and denote the elements of A^*Y by $m_1, \ldots, m_n$. Then, by Lemma 17 (and its proof), there are n recognizable subsets $m_1Y^{-1}, \ldots, m_nY^{-1}$ such that

$$Y = \min_{1 \le i \le n} \lceil m_iY^{-1}, m_i \rceil \ .$$

Hence,

$$\max(X,Y) = \max(X, \min_{1\le i\le n} \lceil m_iY^{-1}, m_i\rceil) = \min_{1\le i\le n} (\max(X, \lceil m_iY^{-1}, m_i\rceil)) \ .$$

By Proposition 25, for each i, $1 \le i \le n$, $\max(X, \lceil m_iY^{-1}, m_i\rceil)$ is a recognizable $\mathcal{M}$-subset. As $\mathcal{M}\,\mathrm{Rec}\,A^*$ is closed under the minimum operation, we have that $\max(X,Y) \in \mathcal{M}\,\mathrm{Rec}\,A^*$. ■

Lemma 28 *Let X, $Y \in \mathcal{M}\,\mathrm{SRec}\,A^*$ (resp. $\mathcal{M}\,\mathrm{CRec}\,A^*$). If Y is limited, then* $\max(X,Y) \in \mathcal{M}\,\mathrm{SRec}\,A^*$ *(resp. $\mathcal{M}\,\mathrm{CRec}\,A^*$).*

Proof. If X, $Y \in \mathcal{M}\,\mathrm{SRec}\,A^*$ and Y is limited, the result follows from Lemma 27, by considering the following remarks:

1. By the proof of Lemma 20, for each i, $1 \le i \le n$, $\lceil m_iY^{-1}, m_i\rceil$ is a nondeterministic complexity.

2. By Proposition 26, for each i, $1 \le i \le n$, $\max(X, \lceil m_iY^{-1}, m_i\rceil) \in \mathcal{M}\,\mathrm{SRec}\,A^*$.

3. $\mathcal{M}\,\mathrm{SRec}\,A^*$ is closed under the minimum operation.

The proof is similar if $X \in \mathcal{M}\,\mathrm{CRec}\,A^*$. ■

Now, let us consider the *remainder* of the integer division of $m \in \mathcal{M}$ by a positive integer d, denoted by $m \bmod d$. For $m \in \mathbb{N}$, $m \bmod d$ is given by the usual definition and $\infty \bmod d = \infty$. This operation can be extended to the $\mathcal{M}$-subsets of A^* as follows.

Let X be an $\mathcal{M}$-subset of A^* and let d be a positive integer. We define the $\mathcal{M}$-subset $X \bmod d$ by

$$\forall w \in A^*, \quad w(X \bmod d) = wX \bmod d \ .$$

The following proposition states that the mod d operation is distributive with respect to the addition (resp. minimum) of $\mathcal{M}$-subsets of the type $\lfloor R, m \rfloor$ (resp. $\lceil R, m \rceil$).

Proposition 29 *Let $R_1, \ldots, R_k$ be recognizable subsets of A^* such that $R_i \cap R_j = \emptyset$, for every i, j, $1 \le i$, $j \le k$, and $i \ne j$. Let $m_1, \ldots, m_k$ be elements of $\mathcal{M}$ and d a positive integer. Then*

$$(\min_{1 \le i \le k} \lceil R_i, m_i \rceil) \bmod d = \min_{1 \le i \le k} (\lceil R_i, m_i \rceil \bmod d)$$

$$\textit{and} \qquad (\sum_{i=1}^{k} \lfloor R_i, m_i \rfloor) \bmod d = \sum_{i=1}^{k} (\lfloor R_i, m_i \rfloor \bmod d) \ .$$

■

Proposition 30 *Let d be an integer, $d > 1$. $\mathcal{M}\,\mathrm{Rec}\,A^*$ is not closed under the* mod d *operation.*

Proof. Let $A = \{ a, b \}$ and consider the $\mathcal{M}$-subset X of A^* defined by

$$\forall w \in A^*, \quad wX = \min\{ 2|w|_a, 2|w|_b + 1 \} \ .$$

It is clear that $X \in \mathcal{M}\,\mathrm{Rec}\,A^*$.

However, $1(X \bmod 2)^{-1} = \{ w \in A^* : |w|_b < |w|_a \}$ is not a recognizable subset of A^*. Then, from Proposition 4, it follows that $X \bmod 2 \notin \mathcal{M}\,\mathrm{Rec}\,A^*$.

Thus, $\mathcal{M}\,\mathrm{Rec}\,A^*$ is not closed under mod d, $d > 1$. ■

The following lemma presents a subfamily of recognizable $\mathcal{M}$-subsets which is closed under mod d, $d > 0$.

Lemma 31 *Let d be a positive integer. If $X \in \mathcal{M}\,\mathrm{Rec}\,A^*$ and is limited, then $X \bmod d \in \mathcal{M}\,\mathrm{Rec}\,A^*$. Moreover, if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, then $X \bmod d \in \mathcal{M}\,\mathrm{CRec}\,A^*$.*

Proof. Let X be a recognizable and limited $\mathcal{M}$-subset. If $d = 1$, then $X \bmod 1 = \lceil \mathrm{support}(X), 0 \rceil$ is a recognizable $\mathcal{M}$-subset.

Consider $d > 1$ and let $n = |A^*X|$. We denote the elements of A^*X by $m_1, \ldots, m_n$. Then, by Lemma 17 (and its proof), there are n recognizable subsets $m_1X^{-1}, \ldots, m_nX^{-1}$ such that

$$X = \min_{1 \le i \le n} \lceil m_iX^{-1}, m_i \rceil \ .$$

Hence, as $m_iX^{-1} \cap m_jX^{-1} = \emptyset$, for every $i,\ j$, $1 \le i,\ j \le n$, and $i \ne j$, by Proposition 29 it results that

$$X \bmod d = (\min_{1 \le i \le n} \lceil m_iX^{-1}, m_i \rceil) \bmod d = \min_{1 \le i \le n} (\lceil m_iX^{-1}, m_i \rceil \bmod d) \ .$$

However, for each i, $1 \le i \le n$,

$$\lceil m_iX^{-1}, m_i \rceil \bmod d = \lceil m_iX^{-1}, m_i \bmod d \rceil \ ,$$

which is a recognizable $\mathcal{M}$-subset by Proposition 16.

Thus,

$$X \bmod d = \min_{1 \le i \le n} \lceil m_iX^{-1}, m_i \bmod d \rceil$$

and, as $\mathcal{M}\,\mathrm{Rec}\,A^*$ is closed under minimum, $X \bmod d \in \mathcal{M}\,\mathrm{Rec}\,A^*$.

Moreover, if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, from Lemma 20 (and its proof) it follows that, for each i, $1 \le i \le n$, $\lceil m_iX^{-1}, m_i \bmod d \rceil \in \mathcal{M}\,\mathrm{CRec}\,A^*$. And, as $\mathcal{M}\,\mathrm{CRec}\,A^*$ is closed under minimum, we have that $X \bmod d \in \mathcal{M}\,\mathrm{CRec}\,A^*$. ∎

Now, we define over $\mathcal{M}$ a binary operation which is similar to the subtraction over the integer numbers. This operation is also extended to the family of $\mathcal{M}$-subsets.

Consider the *minusp* operation, $\dot{-}$: $\mathcal{M}^2 \to \mathcal{M}$, defined by

$$\forall m, n \in \mathbb{N}, \quad m \dot{-} n = \begin{cases} m - n & \text{if } m \ge n \\ 0 & \text{if } m < n \end{cases} ,$$

$$\infty \dot{-} n = \infty, \quad m \dot{-} \infty = 0 \quad \text{and} \quad \infty \dot{-} \infty = \infty \ .$$

For the $\mathcal{M}$-subsets X and Y of A^*, we define the $\mathcal{M}$-subset $X \dot{-} Y$ by

$$\forall w \in A^*, \quad w(X \dot{-} Y) = wX \dot{-} wY \ .$$

The following proposition states a property relating the minusp and the addition of $\mathcal{M}$-subsets.

Proposition 32 *Let X, $Y_1, \dots, Y_k$ be $\mathcal{M}$-subsets of A^*. Then*

$$X \dot{-} \sum_{i=1}^{k} Y_i = (((X \dot{-} Y_1) \dot{-} Y_2) \dot{-} \cdots) \dot{-} Y_k \ .$$

Proof. It is enough to use induction on k. ∎

Proposition 33 *$\mathcal{M}\,\mathrm{Rec}\,A^*$ is not closed under the minusp operation.*

Proof. Consider $A = \{a, b\}$.

Let X and Y be the $\mathcal{M}$-subsets of A^+ defined by

$$\forall w \in A^+, \quad wX = |w| \quad \text{and} \quad wY = \min\{|w|_a, |w|_b\} \ .$$

Then X and Y are recognizable $\mathcal{M}$-subsets and the $\mathcal{M}$-subset $X \dot{-} Y$ is given by

$$\forall w \in A^+, \quad w(X \dot{-} Y) = |w| - \min\{|w|_a, |w|_b\} = \max\{|w|_a, |w|_b\} \ .$$

Then, from the proof of Lemma 5, it follows that $X \dot{-} Y \notin \mathcal{M}\,\mathrm{Rec}\,A^*$. ∎

The following theorem states when the minusp of two recognizable $\mathcal{M}$-subsets is a recognizable $\mathcal{M}$-subset.

Theorem 34 *Let X, $Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$. If Y is limited, then $X \dot{-} Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$.*

Before to prove this theorem, we study the particular case in which the $\mathcal{M}$-subset Y is of the form $\lfloor R, m \rfloor$, for some $m \in \mathcal{M}$ and some recognizable subset R.

Proposition 35 *Let X be a recognizable $\mathcal{M}$-subset of A^+. Then $X \dot{-} \lfloor A^*, 1 \rfloor$ is a recognizable $\mathcal{M}$-subset of A^+. Moreover, if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, then $X \dot{-} \lfloor A^*, 1 \rfloor \in \mathcal{M}\,\mathrm{SRec}\,A^*$.*

Proof. Let X be a recognizable $\mathcal{M}$-subset of A^+. Then there is a normalized $\mathcal{M}$-A-automaton $\mathcal{A} = (Q, I, T)$ such that $\|\mathcal{A}\| = X$.

The $\mathcal{M}$-subset $X \dot{-} \lfloor A^*, 1 \rfloor$ satisfies

$$\forall w \in A^*, \quad w(X \dot{-} \lfloor A^*, 1 \rfloor) = \begin{cases} \infty & \text{if } wX = \infty \\ 0 & \text{if } wX = 0 \\ wX - 1 & \text{if } 1 \leq wX < \infty \end{cases} .$$

From $\mathcal{A}$, let us construct an $\mathcal{M}$-A-automaton $\mathcal{B} = (Q_\mathcal{B}, I_\mathcal{B}, T_\mathcal{B})$ such that $\|\mathcal{B}\| = X \dot{-} \lfloor A^*, 1 \rfloor$, as follows:
$Q_\mathcal{B} = \{ q' \mid q \in Q \} \cup \{ q'' \mid q \in Q \}$,
$I_\mathcal{B}$ is the $\mathcal{M}$-subset of $Q_\mathcal{B}$ defined by

$$\forall q \in Q, \quad q' I_\mathcal{B} = qI \quad \text{and} \quad q'' I_\mathcal{B} = \infty$$

and $T_\mathcal{B}$ is the $\mathcal{M}$-subset of $Q_\mathcal{B}$ defined by

$$\forall q \in Q, \quad q' T_\mathcal{B} = q'' T_\mathcal{B} = qT .$$

The useful edges of $\mathcal{B}$ are defined as follows. For each useful edge (p, a, q) of $\mathcal{A}$,

- if $(p, a, q)E_\mathcal{A} = 0$, then (p', a, q') and (p'', a, q'') are useful edges of $\mathcal{B}$ and their multiplicities are equal to 0;
- if $(p, a, q)E_\mathcal{A} > 0$, then (p', a, q'') and (p'', a, q'') are useful edges of $\mathcal{B}$ and their multiplicities are given by

$$(p', a, q'')E_\mathcal{B} = (p, a, q)E_\mathcal{A} - 1 \quad \text{and} \quad (p'', a, q'')E_\mathcal{B} = (p, a, q)E_\mathcal{A} .$$

Thus, one can easily verify that $\|\mathcal{B}\| = X \dot{-} \lfloor A^*, 1 \rfloor$. Hence, $X \dot{-} \lfloor A^*, 1 \rfloor \in \mathcal{M}\,\mathrm{Rec}\,A^*$.

If $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, by Proposition 1 it follows that the $\mathcal{M}$-A-automaton $\mathcal{A}$ can be simple. Then, by the construction of $\mathcal{B}$, $Q_\mathcal{B} I_\mathcal{B}$ and $Q_\mathcal{B} T_\mathcal{B} \subseteq \{0, \infty\}$ and the multiplicities of the useful edges of $\mathcal{B}$ are 0 or 1. Hence, $\mathcal{B}$ is simple and we conclude that $X \dot{-} \lfloor A^*, 1 \rfloor \in \mathcal{M}\,\mathrm{SRec}\,A^*$. ∎

Proposition 36 *If* $X \in \mathcal{M}\,\mathrm{CRec}\,A^*$, *then* $X \dot{-} \lfloor A^*, 1 \rfloor \in \mathcal{M}\,\mathrm{CRec}\,A^*$.

Proof. If X is a nondeterministic complexity, the construction of the $\mathcal{M}$-A-automaton $\mathcal{B}$ given in the proof of the previous proposition does not guarantee that $X \dot{-} \lfloor A^*, 1 \rfloor$ is a nondeterministic complexity. Thus, we present a different construction.

Consider a type nc $\mathcal{M}$-A-automaton $\mathcal{A} = (Q, I, T)$ such that $\|\mathcal{A}\| = X$.

Let us construct an $\mathcal{M}$-A-automaton $\mathcal{B}$ such that $\|\mathcal{B}\| = X \dot{-} \lfloor A^*, 1 \rfloor$.

At first, we consider the $\mathcal{M}$-A-automaton $\mathcal{C} = (Q_\mathcal{C}, I_\mathcal{C}, T_\mathcal{C})$, which is the 0-accessible part of $\mathcal{A}$; that is,

$$Q_\mathcal{C} = \{\, q' : q \in Q \text{ and } q \text{ is accessible in } \mathcal{A} \text{ through a path } P, \text{ with } \|P\| = 0 \,\}.$$

Note that if q is an initial state of $\mathcal{A}$, then $q' \in Q_\mathcal{C}$.

$I_\mathcal{C}\colon\ Q_\mathcal{C} \to \mathcal{M}$, defined by $q' I_\mathcal{C} = qI$

and $T_\mathcal{C}\colon\ Q_\mathcal{C} \to \mathcal{M}$, defined by $q' T_\mathcal{C} = qT$.

For every $p', q' \in Q_\mathcal{C}$ and for every $a \in A$, if (p', a, q') is a useful edge of $\mathcal{C}$, then $(p, a, q)E_\mathcal{A} = 0$. Moreover, $(p', a, q')E_\mathcal{C} = 0$. It is clear that, for every $w \in A^*$, $w\|\mathcal{A}\| = 0$ if and only if $w\|\mathcal{C}\| = 0$.

Consider the following subset

$$R = \{\, \text{useful edges } \alpha = (p, a, q) \text{ of } \mathcal{A}, \text{ with } \|\alpha\| = 1 \text{ and } p' \in Q_\mathcal{C} \,\}\ .$$

Let $k = |R|$ and consider an arbitrary enumeration of its elements, say from 1 to k.

The $\mathcal{M}$-A-automaton $\mathcal{B}$ will be constructed from k 'copies' of the $\mathcal{M}$-A-automata $\mathcal{A}$ and $\mathcal{C}$. That is,

$Q_\mathcal{B} = (Q_\mathcal{C} \times [1, k]) \cup (Q \times [1, k])$;

$I_\mathcal{B}\colon\ Q_\mathcal{B} \to \mathcal{M}$, defined by

$$\forall q' \in Q_\mathcal{C},\ \forall i \in [1, k],\ (q', i)I_\mathcal{B} = qI \ \text{ and }\ \forall q \in Q,\ \forall i \in [1, k],\ (q, i)I_\mathcal{B} = \infty\ ;$$

$T_\mathcal{B}\colon\ Q_\mathcal{B} \to \mathcal{M}$, defined by

$$\forall q' \in Q_\mathcal{C},\ \forall i \in [1, k],\ (q', i)T_\mathcal{B} = qT \ \text{ and }\ \forall q \in Q,\ \forall i \in [1, k],\ (q, i)T_\mathcal{B} = qT\ .$$

As $\mathcal{A}$ is of type nc, we have that $Q_\mathcal{B} I_\mathcal{B}$ and $Q_\mathcal{B} T_\mathcal{B} \subseteq \{0, \infty\}$.

The useful edges of $\mathcal{B}$ are defined as follows:

- for each useful edge (p, a, q) of $\mathcal{A}$, $((p,i),a,(q,i))$ is a useful edge of $\mathcal{B}$, for every i, $1 \le i \le k$, and $((p,i),a,(q,i))E_{\mathcal{B}} = (p,a,q)E_{\mathcal{A}}$;

- for each useful edge (p', a, q') of $\mathcal{C}$, $((p',i),a,(q',i))$ is a useful edge of $\mathcal{B}$, for every i, $1 \le i \le k$, and $((p',i),a,(q',i))E_{\mathcal{B}} = 0$;

- for each i, $1 \le i \le k$, if $\alpha_i = (p,a,q)$ is the edge of number i in R, then $((p',i),a,(q,i))$ is a useful edge of $\mathcal{B}$ and $((p',i),a,(q,i))E_{\mathcal{B}} = 0$.

Thus, one can verify that $\mathcal{B}$ is of type nc and $\|\mathcal{B}\| = \|\mathcal{A}\| \dot{-} \lfloor A^*, 1 \rfloor$. Therefore, $X \dot{-} \lfloor A^*, 1 \rfloor \in \mathcal{M}\,\mathrm{CRec}\,A^*$. ■

Proposition 37 *Let X be a recognizable $\mathcal{M}$-subset of A^+ and $m \in \mathcal{M}$. Then $X \dot{-} \lfloor A^*, m \rfloor$ is a recognizable $\mathcal{M}$-subset of A^+. Moreover, if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, then $X \dot{-} \lfloor A^*, m \rfloor \in \mathcal{M}\,\mathrm{SRec}\,A^*$.*

Proof. Let X be a recognizable $\mathcal{M}$-subset of A^+ and $m \in \mathcal{M}$.

If $m = 0$, then $X \dot{-} \lfloor A^*, 0 \rfloor = X$.

If $m = \infty$, then $X \dot{-} \lfloor A^*, \infty \rfloor = \lceil \mathrm{support}(X), 0 \rceil$. Thus, by Propositions 2 and 16 it results that $X \dot{-} \lfloor A^*, \infty \rfloor$ is a recognizable $\mathcal{M}$-subset.

Now, consider $0 < m < \infty$. One can verify that

$$X \dot{-} \lfloor A^*, m \rfloor = \underbrace{(((X \dot{-} \lfloor A^*, 1 \rfloor) \dot{-} \lfloor A^*, 1 \rfloor) \dot{-} \cdots) \dot{-} \lfloor A^*, 1 \rfloor}_{m \text{ times}} .$$

Thus, by Proposition 35, we have that $X \dot{-} \lfloor A^*, m \rfloor$, with $0 < m < \infty$, is a recognizable $\mathcal{M}$-subset.

The proof is similar if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$. ■

Proposition 38 *Let $X \in \mathcal{M}\,\mathrm{CRec}\,A^*$ and $m \in \mathcal{M}$. Then $X \dot{-} \lfloor A^*, m \rfloor \in \mathcal{M}\,\mathrm{CRec}\,A^*$.*

Proof. It follows from Propositions 37 and 36, observing that $X \dot{-} \lfloor A^*, \infty \rfloor = \lceil \mathrm{support}(X), 0 \rceil \in \mathcal{M}\,\mathrm{CRec}\,A^*$. ■

Proposition 39 *Let X be a recognizable $\mathcal{M}$-subset of A^+. Let $m \in \mathcal{M}$ and let R be a recognizable subset of A^*. Then $X \dot{-} \lfloor R, m \rfloor$ is a recognizable $\mathcal{M}$-subset of A^+. Moreover, if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, then $X \dot{-} \lfloor R, m \rfloor \in \mathcal{M}\,\mathrm{SRec}\,A^*$.*

Proof. Let X be a recognizable $\mathcal{M}$-subset of A^+. Let $m \in \mathcal{M}$ and let R be a recognizable subset of A^*. If $R = A^*$, the result follows from Proposition 37. Then assume that $R \neq A^*$.

If $m = 0$, then $X \dot{-} \lfloor R, 0 \rfloor = X$, and there is nothing to prove.

Let us consider $0 < m < \infty$. Then

$$\forall w \in A^*, \quad w(X \dot{-} \lfloor R, m \rfloor) = \begin{cases} \infty & \text{if } wX = \infty \\ 0 & \text{if } wX < m \text{ and } w \in R \\ wX & \text{if } wX < \infty \text{ and } w \notin R \\ wX - m & \text{if } (m \leq wX < \infty \text{ and } w \in R) \ . \end{cases}$$

Now, we define some recognizable $\mathcal{M}$-subsets, from which it is possible to obtain $X \dot{-} \lfloor R, m \rfloor$, using only operations under which $\mathcal{M}\,\mathrm{Rec}\,A^*$ is closed.

Consider the subsets

$$R_1 = \bigcup_{i=0}^{m-1} iX^{-1} \cap R = \{ w \in A^* : wX < m \text{ and } w \in R \} \ ,$$

$$R_2 = (A^* - \infty X^{-1}) \cap (A^* - R) = \{ w \in A^* : wX < \infty \text{ and } w \notin R \} \quad \text{and}$$

$$R_3 = (A^* - (\bigcup_{i=0}^{m-1} iX^{-1} \cup \infty X^{-1})) \cap R = \{ w \in A^* : m \leq wX < \infty \text{ and } w \in R \} \ .$$

By Proposition 4 it follows that the subsets R_1, R_2 and R_3 are recognizable.

Define the $\mathcal{M}$-subsets X_1, X_2 and X_3 as follows:

$$X_1 = \lceil R_1, 0 \rceil, \quad X_2 = X + \lceil R_2, 0 \rceil \quad \text{and} \quad X_3 = (X \dot{-} \lfloor A^*, m \rfloor) + \lceil R_3, 0 \rceil \ .$$

Hence, from Propositions 16, 18 and 37, it results that X_1, X_2 and $X_3 \in \mathcal{M}\,\mathrm{Rec}\,A^*$.

Moreover, one can verify that $X \dot{-} \lfloor R, m \rfloor = \min(X_1, X_2, X_3)$.

Therefore, for $0 < m < \infty$, $X \dot{-} \lfloor R, m \rfloor \in \mathcal{M}\,\mathrm{Rec}\,A^*$, since $\mathcal{M}\,\mathrm{Rec}\,A^*$ is closed under the minimum operation.

We can observe that $X_1 \in \mathcal{M}\,\mathrm{CRec}\,A^*$, and if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, then, by Table 1 and Proposition 37, it results that X_2 and X_3 are also simple. And, as $\mathcal{M}\,\mathrm{SRec}\,A^*$ is closed under the minimum operation, $X \dot{-} \lfloor R, m \rfloor \in$

$\mathcal{M}\,\mathrm{SRec}\,A^*$, for $0 < m < \infty$.

Now, let us consider $m = \infty$. Then $X \dot{-} \lfloor R, \infty \rfloor = \min(X_1, X_2)$. Thus, $X \dot{-} \lfloor R, \infty \rfloor \in \mathcal{M}\,\mathrm{Rec}\,A^*$.

In a similar way, one can prove that if $X \in \mathcal{M}\,\mathrm{SRec}\,A^*$, then $X \dot{-} \lfloor R, \infty \rfloor \in \mathcal{M}\,\mathrm{SRec}\,A^*$. ■

Proposition 40 *Let* $X \in \mathcal{M}\,\mathrm{CRec}\,A^*$ *and* $m \in \mathcal{M}$. *Then* $X \dot{-} \lfloor R, m \rfloor \in \mathcal{M}\,\mathrm{CRec}\,A^*$.

Proof. The statement follows from Propositions 39 and 38. ■

Now, we can prove that if X and $Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$ and Y is limited, then $X \dot{-} Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$.

Proof of Theorem 34. Let $X \in \mathcal{M}\,\mathrm{Rec}\,A^*$. Consider $X' = X + \mathbf{A}^+$. By Table 1, it follows that $X' \in \mathcal{M}\,\mathrm{Rec}\,A^*$.

Let Y be a recognizable and limited $\mathcal{M}$-subset of A^*. Consider $n = |A^*Y|$ and denote the elements of A^*Y by $m_1, \ldots, m_n$. Then, by Lemma 17 (and its proof), there are n recognizable subsets $m_1Y^{-1}, \ldots, m_nY^{-1}$ such that

$$Y = \sum_{i=1}^{n} \lfloor m_iY^{-1}, m_i \rfloor \; .$$

Thus, from Proposition 32, it follows that

$$X' \dot{-} Y = (((X' \dot{-} \lfloor m_1Y^{-1}, m_1 \rfloor) \dot{-} \lfloor m_2Y^{-1}, m_2 \rfloor) \dot{-} \cdots) \dot{-} \lfloor m_nY^{-1}, m_n \rfloor \; .$$

Let us denote $X_0 = X'$ and, for each i, $1 \leq i \leq n$,

$$X_i = ((X' \dot{-} \lfloor m_1Y^{-1}, m_1 \rfloor) \dot{-} \cdots) \dot{-} \lfloor m_iY^{-1}, m_i \rfloor \; .$$

By Proposition 39, for each i, $1 \leq i \leq n$, $X_{i-1} \dot{-} \lfloor m_iY^{-1}, m_i \rfloor$ is a recognizable $\mathcal{M}$-subset of A^*. Therefore, $X' \dot{-} Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$ and $1(X' \dot{-} Y) = 1X' \dot{-} 1Y = \infty \dot{-} 1Y = \infty$.

However, $X \dot{-} Y = \min(X' \dot{-} Y, (1X \dot{-} 1Y) + 1)$. And, as $\mathcal{M}\,\mathrm{Rec}\,A^*$ is closed under scalar addition and minimum, we have that $X \dot{-} Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$. ■

Theorem 41 *Let X, $Y \in \mathcal{M}\,\mathrm{SRec}\,A^*$ (resp. $\mathcal{M}\,\mathrm{CRec}\,A^*$). If Y is limited, then $X \dot{-} Y \in \mathcal{M}\,\mathrm{SRec}\,A^*$ (resp. $\mathcal{M}\,\mathrm{CRec}\,A^*$).*

Proof. If X, $Y \in \mathcal{M}\,\mathrm{SRec}\,A^*$ and Y is limited, then the result follows from Theorem 34 and Table 1, observing that $1X \dot{-} 1Y \in \{0, \infty\}$.

If X, $Y \in \mathcal{M}\,\mathrm{CRec}\,A^*$ and Y is limited, then the result follows from Theorem 34, Proposition 40 and Table 1, considering the above observation. ■

10 Some undecidable problems for $\mathcal{M}\,\mathrm{CRec}\,A^*$

We start this section by describing four problems studied by Krob [14].

Let K be a totally ordered semiring. Let us consider the following problems for every X and Y in the family of recognizable K-subsets of A^*:

- equality problem: $X = Y$?
- inequality problem: $X \leq Y$?
- local equality problem: there exists w in A^* such that $wX = wY$?
- local inequality problem: there exists w in A^* such that $wX \leq wY$?

Krob [14] showed that if A is an alphabet with at least two letters, the four problems above are undecidable for the families $\mathcal{Z}\,\mathrm{Rec}\,A^*$, $\mathcal{M}\,\mathrm{Rec}\,A^*$, $\mathcal{M}\,\mathrm{SRec}\,A^*$ and $\mathcal{H}_i\,(i \geq 1)$. In his paper, Krob also showed that these problems are decidable when A has only one letter.

By the other hand, it is easy to prove that the equality problem for $\mathcal{H}_0$, the family of recognizable and limited $\mathcal{M}$-subsets of A^*, is decidable. Hence, with respect to the diagram in the Section 8, we only need to verify if the problems mentioned above are decidable to $\mathcal{M}\,\mathrm{CRec}\,A^*$.

Let A be an alphabet and let n be a positive integer. Consider the substitution $\sigma_n : A^* \to A^*$ defined by $a\sigma_n = a^n$, for every $a \in A$.

Proposition 42 *Let $\mathcal{A}$ be a normalized $\mathcal{M}$-A-automaton such that the multiplicities of its edges are positive. Let m be the maximum value of the multiplicities of its useful edges. Then, for every $n \geq m$, there is a type nc $\mathcal{M}$-A-automaton $\mathcal{A}_n$ such that*

$$\forall w \in A^*, \quad w\|\mathcal{A}_n\| = \begin{cases} u\|\mathcal{A}\| & \text{if } w = u\sigma_n, \text{ with } u \in A^* \\ \infty & \text{if } w \notin A^*\sigma_n \ . \end{cases}$$

Proof. Let $\mathcal{A} = (Q, I, T)$ be a normalized $\mathcal{M}$-A-automaton such that for each useful edge α of $\mathcal{A}$, $\|\alpha\| > 0$. Let $m = \max\{\, \|\alpha\| : \alpha \text{ is a useful edge of } \mathcal{A}\}$ and consider an integer $n \geq m$.

We construct an $\mathcal{M}$-A-automaton $\mathcal{A}_n = (Q', I', T')$ from $\mathcal{A}$ as follows:
$Q' = Q \cup R$, where R is the set of new states;
the $\mathcal{M}$-subset I' of Q' is given by

$$\forall q \in Q', \quad qI' = \begin{cases} qI & \text{if } q \in Q \\ \infty & \text{if } q \in R \end{cases}$$

and the $\mathcal{M}$-subset T' of Q' is given by

$$\forall q \in Q', \quad qT' = \begin{cases} qT & \text{if } q \in Q \\ \infty & \text{if } q \in R \ . \end{cases}$$

For each useful edge (p, a, q) of $\mathcal{A}$, let us consider n edges in $\mathcal{A}_n$:

$$(p, a, r_1),\ (r_1, a, r_2),\ \ldots,\ (r_{n-2}, a, r_{n-1}) \text{ and } (r_{n-1}, a, q) \ ,$$

where $r_1, \ldots, r_{n-1} \in R$ are new states and the multiplicities of these edges are defined as follows:

$$\|(p, a, r_1)\| = 1,$$

$$\|(r_i, a, r_{i+1})\| = 1, \text{ if } i \in [1, k-1],$$

$$\|(r_i, a, r_{i+1})\| = 0, \text{ if } i \in [k, n-2] \text{ and}$$

$$\|(r_{n-1}, a, q)\| = 0,$$

where k is the multiplicity of (p, a, q) in $\mathcal{A}$.

It is easy to verify that $\mathcal{A}_n$ is a simple $\mathcal{M}$-A-automaton. Moreover, we can observe that if (p, a, q) is an edge in $\mathcal{A}_n$ with $\|(p, a, q)\| = 0$, then $p \in R$ and there is no other edge in $\mathcal{A}_n$ with origin p. Hence, $\mathcal{A}_n$ can be easily extended to a type nc $\mathcal{M}$-A-automaton.

By construction, it is also clear that $\mathcal{A}_n$ satisfies

$$\forall w \in A^*, \quad w\|\mathcal{A}_n\| = \begin{cases} u\|\mathcal{A}\| & \text{if } w = u\sigma_n, \text{ with } u \in A^* \\ \infty & \text{if } w \notin A^*\sigma_n \ . \end{cases}$$

■

Theorem 43 *The equality problem, the inequality problem, the local equality problem or the local inequality problem for $\mathcal{M}\,\mathrm{Rec}\,A^*$ is decidable if and only if the same problem is decidable for $\mathcal{M}\,\mathrm{CRec}\,A^*$.*

Proof. Let us only show the equivalence between the decidability of the equality problems for $\mathcal{M}\,\mathrm{Rec}\,A^*$ and $\mathcal{M}\,\mathrm{CRec}\,A^*$. The proofs of the other equivalences are similar.

It is clear that it is enough to prove that the decidability of the equivalence problem for $\mathcal{M}\,\mathrm{CRec}\,A^*$ implies the decidability of the same problem for $\mathcal{M}\,\mathrm{Rec}\,A^*$.

Let X and $Y \in \mathcal{M}\,\mathrm{Rec}\,A^*$ such that $1X = 1Y$. Let $\mathcal{A}$ and $\mathcal{B}$ be normalized $\mathcal{M}$-A-automata such that $\|\mathcal{A}\| = X + A^+$ and $\|\mathcal{B}\| = Y + A^+$.

Let k be a positive integer and consider the $\mathcal{M}$-A-automata $\mathcal{A}'$ and $\mathcal{B}'$ obtained from $\mathcal{A}$ and $\mathcal{B}$, respectively, by adding k in the multiplicity of each one of their useful edges. It is easy to see that, for every $w \in A^*$, $w\|\mathcal{A}'\| = w\|\mathcal{A}\| + k|w|$ and $w\|\mathcal{B}'\| \doteq w\|\mathcal{B}\| + k|w|$. Therefore, $\|\mathcal{A}\| = \|\mathcal{B}\|$ if and only if $\|\mathcal{A}'\| = \|\mathcal{B}'\|$.

However, by Proposition 42, from the $\mathcal{M}$-A-automata $\mathcal{A}'$ and $\mathcal{B}'$, there are $\mathcal{M}$-A-automata $\mathcal{A}''$ and $\mathcal{B}''$ which are of type nc and satisfy $\|\mathcal{A}'\| = \|\mathcal{B}'\|$ if and only if $\|\mathcal{A}''\| = \|\mathcal{B}''\|$.

Thus, the decidability of the equality problem for the $\mathcal{M}$-subsets which are nondeterministic complexities implies the decidability of the same problem for the recognizable $\mathcal{M}$-subsets. ■

By the undecidability of the four problems showed by Krob [14] and from the statement in the previous theorem, we conclude the following result.

Corollary 44 *Let A be an alphabet with at least two letters. The equality problem, the inequality problem, the local equality problem and the local inequality problem are undecidable for $\mathcal{M}\,\mathrm{CRec}\,A^*$.* ■

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References

[1] J. Berstel, *Transductions and Context-Free Languages,* B.G. Teubner, Stuttgart, 1979.

[2] J. Berstel and C. Reutenauer, *Les Séries Rationnelles et Leurs Langages,* Masson, Paris, 1984.

[3] R. Cuninghame-Green, Minimax Algebra, *Lecture Notes in Econom. and Math. Systems,* 166, Springer-Verlag, 1979.

[4] S. Eilenberg, *Automata, Languages and Machines,* vol. A, Academic Press, New York, 1974.

[5] J. Goldstine, H. Leung and D. Wotschke, On the Relation Between Ambiguity and Nondeterminism in Finite Automaton, *Technical Report* TR–89–CS–02, Department of Computer Science, New Mexico State University.

[6] K. Hashiguchi, A Decision Procedure for the Order of Regular Events, *Theoret. Comput. Sci.,* 8 (1979), 69–72.

[7] K. Hashiguchi, Limitedness Theorem on Finite Automata with Distance Functions, *J. Comput. System Sci.,* 24:2 (1982), 233–244.

[8] K. Hashiguchi, Improved Limitedness Theorem on Finite Automata with Distance Functions, *Theoret. Comput. Sci.,* 72 (1990), 27–38.

[9] K. Hashiguchi, Algorithms for Determining Relative Star Height and Star Height, *Inform. and Comput.,* 78 (1988), 124–169.

[10] C. M. R. Kintala and P. Fisher, Computations with a Restricted Number of Nondeterministic Steps, in *Proc. of the Ninth Annual ACM Symposium on Theory of Computing,* (1977), 178–185.

[11] C. M. R. Kintala and D. Wotschke, Amounts of Nondeterminism in Finite Automaton, *Acta Inform.,* 13 (1980), 199–204.

[12] N. Kobayashi, Propriedades de $\mathcal{M}$-subconjuntos Reconhecíveis de um Monóide Livre, *Tese de Doutorado,* Departamento de Ciência da Computação, Instituto de Matemática e Estatística da Universidade de São Paulo, 1991.

[13] N. Kobayashi, Properties of Recognizable $\mathcal{M}$-subsets of a Free Monoid, *Lecture Notes in Comput. Sci.,* 583, I. Simon (Ed.), Springer-Verlag, Proceedings of LATIN'92, 1st Latin American Symposium on Theoretical INformatics, São Paulo, Brazil, (1992), 314 – 328.

[14] D. Krob, The Equality Problem for Rational Series with Multiplicities in the Tropical Semiring is Undecidable, *Internat. J. Algebra and Comput.,* 4:3 (1994), 405–425.

[15] H. Leung, Limitedness Theorem on Finite Automata with Distance Functions: an Algebraic Proof, *Theoret. Comput. Sci.,* 81 (1991), 137–145.

[16] M. P. Schützenberger, On the Definition of a Family of Automata, *Inform. and Control,* 4 (1961), 245–270.

[17] M. P. Schützenberger, On a Theorem of R. Jungen, in *Proc. Amer. Math. Soc.,* 13 (1961), 885–889.

[18] M. P. Schützenberger, Certain Elementary Families of Automata, in *Proc. Symposium on Mathematical Theory of Automata,* Polytechnic Institute Brooklyn, (1962), 139–153.

[19] I. Simon, Caracterizações de Conjuntos Racionais Limitados, *Tese de Livre Docência,* Instituto de Matemática e Estatística da Universidade de São Paulo, 1978.

[20] I. Simon, Limited Subsets of a Free Monoid, in *Proc. 19th. Annual Symposium on Foundations of Computer Science,* Institute of Electrical and Electronics Engineers, Piscataway, N.J., (1978), 143–150.

[21] I. Simon, The Nondeterministic Complexity of a Finite Automaton, in M. Lothaire(ed.), *Mots - mélanges offerts à M. P. Schützenberger,* 384–400, Hermes, Paris, 1990.

[22] I. Simon, Factorization Forests of Finite Height, *Theoret. Comput. Sci.,* 72 (1990), 65–94.

[23] I. Simon, Recognizable Sets with Multiplicities in the Tropical Semiring, *Lecture Notes in Comput. Sci.*, 324, Springer-Verlag, (1988), 107–120.

[24] I. Simon, On Semigroups of Matrices over the Tropical Semiring, RAIRO *Inform. Théor. Appl.*, 28 (1994), 277–294.

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