Ao meu grande amigo C. B. de Lyra.

Rigidity of Hypersurfaces of Constant Scalar Curvature

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Grad. (University of Sao Paulo) 1959

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

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Degree conferred ....1969..........
Date

Introduction

In [7] S. Kobayashi proved that the only compact homogeneous hypersurfaces of an Euclidean space are the spheres.

This result was extended by T. Nagano and T. Takahashi ([9]) who proved that if a homogeneous Riemannian manifold has an isometric immersion in an Euclidean space of one diemnsion greater and such that the rank of the second fundamental form is distinct from two at some point, then it is isometric to the Riemannian product of a sphere by an Euclidean space.

The original purpose of this paper was to show that this fact remains true without the restriction on the second fundamental form.

In both [7] and [9], the concept of rigidity has an important role. In fact if M^n is assumed rigid (see preliminaries), the theorem is an immediate consequence of results of E. Cartan [3] and K. Nomizu and B. Smith [10].

For a homogeneous hypersurface of an Euclidean space, having non-zero constant scalar curvature, there are only two possibilities a priori; it is either rigid or contains no rigid open submanifold (see Corollary (1-8)).

The main result of this paper (Theorem 3-1) is that a hypersurface of an Euclidean space, having non-zero constant scalar curvature and containing no open rigid submanifold, is isometric to the product of a two dimensional sphere and an Euclidean space. This result with the remarks made above gives a proof of Nagano and Takahashi's theorem in the most general case.

The proofs contained in this paper rely heavily on methods developed by E. Cartan [2] and S. Dolbeaut Lemoine [5].

Finally, using very similar arguments, the following is proved.

If M^n is a hypersurface of a space form $\tilde{M}^{n+1}(K)$, $n \geq 4$, having constant scalar curvature and an isometric immersion with type number greater than one at all points, then M^n is rigid.

1. Preliminaries

All manifolds and maps considered in this work will be assumed of class C^∞.

Let M^n be an n-dimensional Riemannian manifold. Its tangent space at a point p will be denoted by T_pM^n and the scalar product given by the Riemannian structure by < , > .

Following [8], ∇ will be the covariant derivation of M^n.

An r-dimensional C^∞ distribution $\mathcal{H}$ is said to be parallel at a point $p \in M^n$ if for any vector field X belonging to $\mathcal{H}$ and any tangent vector $Y_p \in T_pM^n$, it holds

$$(\nabla_{Y_p} X)_p \in \mathcal{H}_p.$$

If this holds at all points p then $\mathcal{H}$ is said to be parallel on M^n.

On the other hand there is the notion of parallel translation of a vector along a path (see [8]).

The following proposition relates these two concepts.

1-1. Proposition. An r-dimensional distribution $\mathcal{H}$ is parallel on M^n if and only if parallel translate of a vector $Y_p \in \mathcal{H}_p$ along any path still belongs to $\mathcal{H}_p$.

For details see [1] and [8].

By means of the operator ∇, the curvature tensor of M^n can be expressed as

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

where X,Y,Z are vector fields on M^n.

The sectional curvature of the subspace π of T_pM^n, spanned by the vectors X,Y is

$$S(\pi) = \frac{\langle R(X,Y)Y\ ,\ X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2} .$$

A Riemannian manifold M^n has constant curvature K, if and only if

$$R(X,Y)Z = K(\langle Y,Z\rangle X - \langle X,Z\rangle Y),$$

for all vectors X,Y,Z, and at all points of M^n.

If $X_1,\dots,X_n$ is an orthonormal basis of T_pM^n then the scalar curvature of M^n at p is given by (see [8])

$$\mathrm{scal}(M^n) = \sum_{i \neq j} S(\pi_{ij}) ,$$

where π_{ij} denotes the plane in T_pM^n spanned by X_i, X_j.

1-2. Proposition. Let $\mathcal{D}$ be an n-r dimensional C^∞, involutive distribution on M^n, such that each leaf has constant curvature with respect to the Riemannian metric induced by M^n.

Then for each point $p \in M^n$ it is possible to find a coordinate system, $x^1, \ldots, x^n$ on M^n, defined around p in such a way that the vectors $\partial/\partial x^j$, $j > r$, form a basis for $\mathfrak{D}$ and furthermore

$$\text{(i)} \quad \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \left(1 + \frac{K}{4}\left(\sum_{k>r}(x^k)^2\right)\right)^2 \delta_{ij}, \quad i,j > r.$$

Indication of the Proof: Around p, there are coordinates $y^1, \ldots, y^n$ such that the vector fields $\partial/\partial y^{r+1}$, $\partial/\partial y^n$ form a basis of $\mathfrak{D}$ at each point. It may be assumed that $y^1(p) = y^n(p) = 0$.

These coordinates give a diffeomorphism of a neighborhood of p in M^n, onto an open subset of R^n containing the origin. If the first neighborhood is conveniently small it may be assumed that the second is of the form $U^r \times U^{n-r}$, where U^r and U^{n-r} are open neighborhoods of the origin in R^r and R^{n-r} respectively.

Consider the functions

$$\text{(i)} \quad g_{ij}(y^1, \ldots, y^r, y^{r+1}, \ldots, y^n) = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle; \quad i,j > r.$$

For each point $(y^1, \ldots, y^r)$ of U^r, they define a Riemannian metric on U^{n-r}, and it follows from the assumption made on the leaves of $\mathfrak{D}$ that this metric has constant curvature K.

On the other hand the metric given by

$$(2) \quad \tilde{g}_{ij} = \left(1 + \frac{K}{4} \sum_{k>r} (y^k)^2\right)^2 \delta_{ij} \qquad i,j > r,$$

has the same constant curvature.

The functions defining this diffoemorphism are solutions of a system of first order differential equations whose coefficients involve the g_{ij} and their derivatives. Hence this solution depends differentiably on the $y^1 \ldots y^r$.

With this in mind, the coordinates $y^{r+1},\ldots,y^n$ can be replaced by new functions $x^{r+1},\ldots,x^n$, such that (i) holds.

Q.E.D.

Remark. Let E_i be the vector fields,

$$E_i = \left(1 + \frac{K}{4} \sum_{k>r} (x^k)^2\right)^2 \frac{\partial}{\partial x^i} \quad , \quad i > r$$

and assume that the leaves of $\mathfrak{D}$ are totally geodesic submanifolds of M^n, then the vectors E_i form an orthonormal basis and

$$\nabla_{E_i} E_j = \frac{K}{2} \sum_{k>r} (\delta^i_j\, x^2 - \delta^k_j\, x^i)\, E_k, \qquad i,j > r.$$

The next fact is the local part of a **De Rham's theorem** and can be found in [8].

1-3. Proposition. Let $\mathcal{H}$ be a non-trivial parallel distribution on M^n and $\mathcal{H}^\perp$ its orthogonal complement. Then any point p of M^n has an open neighborhood $V \times V'$, where V and V' are open submanifolds of the leaves of $\mathcal{H}$ and $\mathcal{H}'$ respectively, and the Riemannian metric on $V \times V'$ is the direct product of the metrics of V and V'.

Isometric Immersions

Let M^n and $\tilde{M}^{n+r}$ be Riemannian manifolds of dimensions n and $n + r$ respectively. A differentiable map

$$f: M^n \to M^{n+r},$$

is an isometric immersion if for each $p \in M^n$, the differential f_* of f is a scalar product preserving isomorphism between $T_p M^n$ and a subspace of $T_{f(p)}\tilde{M}^{n+r}$.

Consider two vector fields X, Y defined in some neighborhood of a point $p \in M^n$. Since f is locally a diffeomorphism, it is possible to consider the vector fields f_*X, f_*Y on some submanifold of $\tilde{M}^{n+r}$.

If $\nabla, \tilde{\nabla}$ denote the covariant derivations of M^n and $\tilde{M}^{n+r}$ respectively, then

$$\tilde{\nabla}_{f_*X}(f_*Y) = f_*(\nabla_X Y) + \alpha(X,Y),$$

where $\alpha(X,Y)$ belongs to the orthogonal complement of $f_*(TM^n)$ in $T\tilde{M}^{n+r}$ (see [1],[8]).

When α vanishes at a point p, the immersion f is said to be totally geodesic at this point. If this holds for all points, f is called totally geodesic.

In case $r = 1$, M^n is usually called a hypersurface of $\tilde{M}^{n+1}$. Denote by ξ a local unit normal field to M^n in $\tilde{M}^{n+1}$, then

$$\alpha(X,Y) = \langle AX, Y\rangle \xi,$$

where A is the symmetric operator of TM^n given by,

$$AX = -f_*^{-1}(\tilde{\nabla}_{f_* X}\xi).$$

From now on the operator A will be called the second fundamental form of f with respect to ξ.

The rank of A at a point p is called the type number of f at this point and is commonly denoted by $t(p)$.

1-4. Proposition. If the type number of f is greater than one at a point p, then the kernel of A_p is given by:

$$\ker A_p = \{X \in T_pM^n \mid \tilde{R}(X,Y) = R(X,Y), \text{ for all } Y \in T_pM^n\},$$

R and $\tilde{R}$ denoting the curvature tensors of M^n and $\tilde{M}^{n+1}$ respectively. For a proof of this fact see [12].

The following equations are basic in the study of hypersurfaces:

$$R(X,Y)Z = \mathrm{proj}_{TM_p}(\tilde{R}(X,Y)Z) + \langle AY,Z\rangle AX - \langle AX,Z\rangle AY$$

$$\mathrm{proj}_{TM_p}(\tilde{R}(X,Y)\xi) = \nabla_X(AY) - \nabla_Y(AX) - A[X,Y],$$

where R, $\tilde{R}$ denote the curvature tensors of M^n and $\tilde{M}^{n+1}$ respectively, ξ being a local unit normal field and A the second fundamental form of f with respect to ξ.

These relations are known as Gauss and Codazzi equations respectively.

If $\tilde{M}^{n+1}$ has constant curvature, the Codazzi equation becomes

$$\nabla_X(AY) - \nabla_Y(AX) = A[X,Y],$$

For details, see [1], [7], [10].

1-5. <u>Proposition.</u> <u>Let f be an isometric immersion of M^n in $\tilde{M}^{n+1}$, such that its type number is constant and greater than one.</u>

<u>If $\tilde{M}^{n+1}$ has constant curvature then the nullity distribution $\mathcal{N}$ of f is integrable and its leaves are totally geodesic both in M^n and $\tilde{M}^{n+1}$.</u>

<u>Proof.</u> The integrability of $\mathcal{N}$ follows from Proposition (1-4) and [6].

Next it will be shown that the restriction of f to each leaf of $\mathcal{N}$ is a totally geodesic immersion.

Let k be the dimension of $\mathcal{N}$ and consider an orthonormal frame field,

$$\xi_1, \ldots, \xi_{n-k}, \xi_{n-k+1}$$

in such a way that the first $n-k$ vectors are orthogonal to a given leaf, $\mathcal{N}_p$, in M^n and ξ_{n-k+1} is orthogonal to M^n in $\tilde{M}^{n+1}$.

The bilinear form $\alpha(X,Y)$ defined by the immersion

$$f: W_p \rightarrow \tilde{M}^{n+1},$$

can be written,

$$\alpha(X,Y) = \sum_{i=1}^{n-k+1} \langle H_i(X), Y \rangle \xi_i,$$

where

$$H_i(X) = -f_*^{-1}(\mathrm{proj}_{T\mathcal{N}_p} \tilde{\nabla}_{f_*X} \xi_i)$$

for any $X \in T\mathcal{N}_p$.

From the way the normal frame was chosen it follows that

$$H_i(X) = -\mathrm{proj}_{T\mathcal{N}_p}(\nabla_X \xi_i) \quad 1 \leq i \leq n-k$$

$$H_{n-k+1}(X) = -f_*^{-1}(\mathrm{proj}_{f_*(T\mathcal{N}_p)} \tilde{\nabla}_{f_*X} \xi_{n-k+1})$$

$$= -\mathrm{proj}_{T\mathcal{N}_p}(f_*^{-1}(\mathrm{proj}_{f_*(TM^n)} \tilde{\nabla}_{f_*X}))$$

$$= \mathrm{proj}_{T\mathcal{N}_p} A(X) = 0.$$

On the other hand, the vector fields

$$A\xi_1, \ldots, A\xi_{n-k},$$

form a basis for $(T\mathcal{N})^{\perp}$ (in M^n).

Let $Y \in T\mathcal{N}_p$ ($\iff AY = 0$), then from the Codazzi equation it follows

$$\langle \nabla_X A\xi_i, Y\rangle = \langle A[X,\xi_i], Y\rangle = 0,$$

which gives

$$\mathrm{proj}_{T\mathcal{N}_p} \nabla_X A\xi_i = 0 \qquad 1 \leq i \leq n-k .$$

and from this it follows that

$$\mathrm{proj}_{T\mathcal{N}_p} \nabla_X \xi_i = 0.$$

These relations prove that all the H_i vanish, or in other words, that f restricted to $\mathcal{N}_p$ is totally geodesic.

Q.E.D.

Rigidity. A Riemannian manifold M is called homogeneous if for any pair of points p,g there is an isometry ϕ of M such that $\phi(p) = q$.

A simply connected Riemannian manifold of constant curvature K is called a space form and will be denoted by $\tilde{M}(K)$.

It is well known that the space forms are homogeneous.

Let $\tilde{M}^{n+1}(K)$ denote an n+1 dimensional space form. A Riemannian manifold M^n is said to be rigid in $\tilde{M}^{n+1}(K)$ if for any pair of isometric immersions f, $\tilde{f}$ of M^n into $\tilde{M}^{n+1}(K)$, there is an isometry ϕ of $\tilde{M}^{n+1}(K)$ such that

$$\tilde{f} = \phi \circ f$$

The following result is basic:

1-6. Proposition. If the type number of an isometric immersion f of M^n in $M^{n+1}(K)(n \geq 3)$ is ≥ 3 at all points then M^n is rigid.

A simple proof is given in [12].

1-7. Proposition. Let f be an isometric immersion of M^n in $M^{n+1}(K)$ such that M^n contains no open subset on which f is totally geodesic.

Let U_α be open submanifolds which are rigid and form a covering of M^n. Then M^n is rigid.

Proof. Consider another isometric immersion $\tilde{f}$ of M^n in $\tilde{M}^{n+1}(K)$ and denote by f_α, $\tilde{f}_\alpha$ the restrictions of f, $\tilde{f}$ to U_α respectively.

Since U_α is assumed to be rigid there is an isometry ϕ_α of $\tilde{M}^{n+1}(K)$ such that

$$\tilde{f}_\alpha = \phi_\alpha \circ f_\alpha,$$

thus if α, β are such that $U_\alpha \cap U_\beta$ is non-void then

$$\phi_\alpha \circ f_\alpha = \phi_\beta \circ f_\beta,$$

at all points of $U_\alpha \cap U_\beta$. This means that $f(U_\alpha \cap U_\beta)$ is kept pointwise fixed by the isometry $\phi_\alpha^{-1} \cdot \phi_\beta$.

Now it is easy to show that if $\phi_\alpha \neq \phi_\beta$ then $f(U_\alpha \cap U_\beta)$ is contained in a totally geodesic submanifold of $\tilde{M}^{n+1}$, which is a contradiction.

By the connectedness of M^n it follows that all ϕ_α must coincide with an isometry ϕ, which gives

$$\overline{f} = \phi \cdot f,$$

thus proving the proposition.

Q.E.D.

1-8. Corollary. *If M^n is a homogeneous hypersurface of $M^{n+1}(K)$, with scalar curvature distinct from $n(n-1)K$, then it is either rigid or contains no rigid open submanifold.*

Proof. The assumption on the scalar curvature excludes the existence of points at which the given immersion is totally geodesic.

Q.E.D.

Complexification

The complex tangent space $T^c_x M$ of a manifold M is the complexification of the tangent space $T_x M$. A complex vector field (resp. complex differential form) is defined by assigning to each point x of M an element of $T^c_x M$ (resp. $T^{c^*}_x M$).

Any complex vector field Z can be written uniquely as $Z = Z' + iZ''$ where Z' and Z'' are realvector fields. By duality it follows that a complex differential form w can be expressed uniquely as $w = w' + iw''$, w' and w'' being real differential forms.

1-9. Proposition. Let $\mathcal{H}$ be a n-2 dimensional integrable distribution on an n-dimensional manifold M^n, $n \geq 3$. Let Z, W be two linearly independent complex vector fields satisfying:

i). Z, W and $\mathcal{H}^c$ span $T^c_x M^n$ at each point x.

ii) $[Z, \mathcal{H}^c] \subset (Z) \oplus \mathcal{H}^c$; $[W, \mathcal{H}^c] \subset (W) \oplus \mathcal{H}^c$

then there are locally defined, non-zero complex valued functions p, q such that:

$$[pZ, \mathcal{H}^c] \subset \mathcal{H}^c \text{ and } [qW, \mathcal{H}^c] \subset \mathcal{H}^c.$$

The proof is straightforward and will be amitted.

If M^n is a Riemannian manifold then the scalar product $\langle\ ,\ \rangle$ and the Riemannian connection can be extended to complex vector fields by linearity. The same notations will be used for these extensions.

Let P_x denote a two-dimensional subspace of T_xM^n and Z, W be a basis for P_x^c. The sectional curvature of P_x is given by:

$$S(P_x) = \frac{\langle R(Z,W)\,W,Z\rangle}{\langle Z,Z\rangle\langle W,W\rangle - \langle Z,W\rangle^2}$$

as it can be easily verified.

For an isometric immersion f of M^n in $\tilde{M}^{n+1}$, with second fundamental form A, the Gauss and Codazzi equations are valid for complex vector fields, provided A is extended to T^cM^n by linearity.

2.

An n-dimensional Riemannian manifold M^n, isometrically immersed in the n+1 dimensional space form $\tilde{M}^{n+1}(K)$ is called <u>deformable</u> in $\tilde{M}^{n+1}(K)$ if it contains no open rigid submanifold. If each point $x \in M^n$ has a deformable neighborhood then M^n is said to be <u>locally deformable</u> in $\tilde{M}^{n+1}(K)$.

It should be noticed that deformability implies local deformability but the converse is not true in general.

The following fact is basic and will be used without further mention.

If M^n is locally deformable in $\tilde{M}^{n+1}(K)$ with $n \geq 3$ and if the scalar curvature of M^n is distinct from n(n-1(K) at each point then the type number of any isometric immersion of M^n in $\tilde{M}^{n+1}(K)$ equals two at all points.

In fact since M^n contains no rigid submanifold, in view of proposition 1-6 the type number of any isometric immersion of M^n in $\tilde{M}^{n+1}(K)$ is at most two at all points. Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues (not necessarily distinct) of the second fundamental form of a given isometric immersion.

From the Gauss equation and the definition of the scalar curvature it follows that

$$\operatorname{scal}(M^n) = K\, n(n-1) + \sum_{i \neq j} \lambda_i \lambda_j$$

which shows that the type number has to be exactly 2.

The main objective in this section is to prove the following results.

2-1. *Theorem. Let $\bar{M}^n$ be an n-diemnsional Riemannian manifold with $n \geq 3$, having non-zero constant scalar curvature and being deformable in the Euclidean space E^{n+1}. Let $\bar{f}$ be an isometric immersion of $\bar{M}^n$ in the Euclidean space E^{n+1}.*

Then the relative nullity distribution $\bar{\mathcal{N}}$ of $\bar{f}$ is parallel on $\bar{M}^n$.

2-2. *Theorem. Let $\bar{M}^n$ be an n-dimensional Riemannian manifold with $n \geq 4$ and f an isometric immersion of $\bar{M}^n$ in the space form $\tilde{M}^{n+1}(K)$, $K \neq 0$. Assume further that the scalar curvature of $\bar{M}^n$ is constant and distinct from* $n(n-1)K$.

Then $\bar{M}^n$ is not deformable in $\tilde{M}^{n+1}(K)$. (i.e., $\bar{M}^n$ *contains an open rigid submanifold*.

The proofs of these theorems will depend on several lemmas In order to simplify the statements of these lemmas the following definition is useful.

Throughout this section it will be assumed that $n \geq 3$.

2-3. Let M^n be a Riemannian manifold. A *local isometric immersion* of M^n in $\tilde{M}^{n+1}(K)$ is a triple (h,H,U) where U is an open orientable submanifold, h an isometric immersion of

U in $\tilde{M}^{n+1}(K)$ and H is the second fundamental form operator of h.

2-4. Lemma. Let M' be an n-dimensional orientable Riemannian manifold and f an isometric immersion of M' in the space form $\tilde{M}^{n+1}(K)$, with second fundamental form A' and nullity distribution $\mathcal{N}'$.

Assume that there is an orthonormal frame

$$X, Y, E_3, \ldots, E_n,$$

defined on M' in such a way that the $E_i, \ldots, i \geq 3$, form a basis for $\mathcal{N}'$ and that for any local isometric immersion (h', H', U) of M' in $\tilde{M}^{n+1}(K)$ (see Definition 2-3) the equation

$$1) \quad \langle H'(X), X \rangle = 0$$

holds at all points of U.

Assume further that M' is deformable in $\tilde{M}^{n+1}(K)$. Then the following equations hold on M':

$$2) \quad \langle \nabla_{E_i} X, Y \rangle = 0, \text{ for all } i \geqslant 3.$$

$$3) \quad \langle \nabla_X E_i, Y \rangle = 0, \text{ for all } i \geqslant 3.$$

$$4) \quad \langle \nabla_X X, Y \rangle = 0, \text{ for all } i \geq 3.$$

<u>Proof</u>. The proofs of (2), (3), (4) follow the same pattern. They consist in showing that if some of these equations are not verified at a point of M' then this point is contained in an open rigid submanifold. This contradicts the deformability of M'.

Assume $\langle \nabla_{E_i} X, Y \rangle$ non-zero at a point p of M', for some index i. Thus it will be non-zero at all points of an open orientable submanifold U'.

Let h' be an isometric immersion of U' in $\tilde{M}^{n+1}(K)$ and denote by H' its second fundamental form. Then

$$\langle H'Y, \nabla_{E_i} X \rangle = \nabla_{E_i} \langle H'Y, X \rangle - \langle \nabla_{E_i} H'Y, X \rangle .$$

Since $\langle H'X, X \rangle$ and $H'E_i$ are zero at all points, the above relation can be written as

$$5) \quad \langle H'Y, \nabla_{E_i} X \rangle = \nabla_{E_i} \langle H'Y, X \rangle - \langle [E_i, Y], Y \rangle \langle Y, H'X \rangle .$$

A similar relation holds for the restriction of A' to U'.

From the Gauss equation it follows that,

$$\langle H'X, X \rangle \langle H'Y, Y \rangle - \langle H'X, Y \rangle^2 = \langle A'X, X \rangle \langle A'Y, Y \rangle - \langle A'X, Y \rangle^2 ,$$

this together with (1) gives

$$6) \quad \langle H'X, Y \rangle = e \langle A'X, Y \rangle ,$$

where e is a constant, either $+1$ or -1.

From (5) and (6):

$$\langle (H' - eA')Y, \nabla_{E_i} X \rangle = 0,$$

or

$$\langle (H' - eA')Y, Y \rangle \langle Y, \nabla_{E_i} X \rangle = 0,$$

or

$$7) \quad \langle H'Y, Y \rangle = e\langle A'Y, Y \rangle.$$

Now (1), (6) and (7) show that $H' = eA'$ and therefore U' is rigid. Since M' is assumed to be deformable, this is a contradiction. Thus (2) is proved.

By (1) it results that

$$0 = \nabla_{E_i} \langle A'X, X \rangle = \langle \nabla_{E_i} A'X, X \rangle + \langle A'X, \nabla_{E_i} X \rangle.$$

Using (1), (2) and noting that $A'E_i$ vanishes, this relation becomes

$$\begin{aligned} 0 &= \langle A'[E_i, X], X \rangle = \langle [E_i, X], A'X \rangle \\ &= \langle \nabla_{E_i} X, Y \rangle \langle Y, A'X \rangle - \langle \nabla_X E_i, Y \rangle \langle Y, A'X \rangle . \end{aligned}$$

Using (2) again it follows that

$$\langle \nabla_X E_i, Y \rangle \langle A'X, Y \rangle = 0 \quad \text{for all } i \geq 3.$$

By assumption $\mathcal{N}'$ is an $n-2$ dimensional distribution which means that $\langle A'X, Y \rangle$ is never zero. Thus,

$$\langle \nabla_X E_i, Y \rangle = 0, \quad \text{for all } i \geq 3$$

and (3) is proved.

The relation (4) is proved in a way similar to the proof of (2), replacing E_i by X. It is sufficient to start with:

$$\langle A'Y, \nabla_X X \rangle = \nabla_X \langle A'Y, X \rangle - \langle \nabla_X A'X, Y \rangle ,$$

to show that

$$\langle (H' - eA')Y, Y \rangle \langle \nabla_X X, Y \rangle = 0.$$

If the term $\langle \nabla_X X, Y \rangle$ does not vanish, U' must be rigid which again is a contradiction. Q.E.D.

The relation (2), (3) and (4) of Lemma 2-4 have a geometrical interpretation which will be stated next.

2-5. Corollary. Let M' and f' be as in Lemma 2-4. Then the $n-1$ dimensional distribution $\mathcal{N}' \oplus X$ is integrable, its leaves are totally geodesic submanifolds of M' and they are mapped by f' into totally geodesic submanifolds of $\tilde{M}^{n+1}(K)$.

Proof. The relations (2) and (3) of Lemma 2-4 show that

$$\langle [E_i, X], Y \rangle = 0 \quad i = 1, \ldots, n$$

on M', which means that $[E_i, X]$ belongs to $\mathcal{N}' \oplus X$. Since

$[E_i, E_j]$ belongs to $\mathcal{N}'$ it also belongs to $\mathcal{N}' \oplus X$, thus showing the integrability.

Next consider a leaf $\mathfrak{F}_0$. It will be shown that the inclusion map:

$$i: \mathfrak{F}_0 \to M',$$

considered as an isometric immersion is totally geodesic. The vector field Y may be viewed as a unit normal field to $\mathfrak{F}_0$ in M'. Thus it suffices to show that the covariant derivatives of Y with respect to tangent vectors to $\mathfrak{F}_0$ are orthogonal to $\mathfrak{F}_0$.

In fact, $\langle \nabla_X Y, E_i \rangle$ and $\langle \nabla_X Y, X \rangle$ vanish by (3) and (4) respectively. On the other hand, $\langle \nabla_{E_j} Y, E_i \rangle$ vanishes because $\mathcal{N}'$ is totally geodesic (see proposition 1-5) and $\langle \nabla_{E_j} Y, X \rangle$ is zero by (2).

To show the last part it has to be proved that the product of the isometric immersions $\bar{f}'$ and i is totally geodesic.

Let ξ be a unit normal field to M' with respect to the immersion f'. After suitable identifications, Y and ξ may be viewed as normal vectors to $\mathfrak{F}_0$ with respect to the immersion $f' \circ i$. With this in mind, the fact that $f' \circ i$ is totally geodesic is equivalent to the fact that the covariant derivatives of Y and ξ with respect to the tangent vectors to $\mathfrak{F}_0$, are orthogonal to $\mathfrak{F}_0$ in $\tilde{M}^{n+1}(K)$.

In fact

$$\langle \tilde{\nabla}_X, \xi \rangle = - \langle A'X, X \rangle = 0,$$

by using (1) of Lemma (2-4).

On the other hand

$$\langle \tilde{\nabla}_X \xi, E_i \rangle = - \langle A'X, E_i \rangle = - \langle A'E_i, X \rangle = 0,$$

because $A'E_i = 0$.

Furthermore

$$\langle \tilde{\nabla}_X Y, X \rangle = \langle \nabla_X Y, X \rangle = 0,$$

by (4) of Lemma 2-4.

From (3) of Lemma 2-4 it follows that

$$\langle \tilde{\nabla}_X Y, E_i \rangle = \langle \nabla_X Y, E_i \rangle = 0.$$

Since $\mathcal{N}'$ is totally geodesic (see Proposition 1-5)

$$\langle \tilde{\nabla}_{E_j} \xi, E_i \rangle = -\langle A'E_j, E_i \rangle = 0$$

$$\langle \tilde{\nabla}_{E_j} \xi, X \rangle = -\langle A'E_j, X \rangle = 0$$

$$\langle \tilde{\nabla}_{E_j} Y, E_i \rangle = \langle \nabla_{E_j} Y, E_i \rangle = - \langle Y, \nabla_{E_j} E_i \rangle = 0.$$

Finally, by (2) of Lemma 2-4

$$\langle \tilde{\nabla}_{E_j} Y, X \rangle = \langle \nabla_{E_j} Y, X \rangle = 0,$$

and the proof of Corollary 2-5 is complete.

Q.E.D.

2-6. Corollary. Let M' and f' verify the conditions of Lemma 2-4, for $K = 0$ (i.e., M' is assumed to be deformable in the $n+1$ dimensional Euclidean space).

Then the scalar curvature of M' is not constant.

Proof. Since the dimension of $\mathcal{N}'$ is assumed to be $n-2$, the scalar curvature of M' has to be nonzero at each point. It will be shown that the assumption of constancy of the scalar curvature contradicts this fact.

By (1) of Lemma 2-4 it follows that:

$$\text{scal } M^n = -2\langle A'X,Y\rangle^2,$$

which shows that if scal M^n is constant so is $\langle A'X,Y\rangle$.

Hence:

$$(1) \quad 0 = \nabla_{E_i}\langle A'Y,X\rangle = \langle \nabla_{E_i} A'Y,X\rangle + \langle A'Y,\nabla_{E_i} X\rangle ,$$

By (2) of Lemma 2-4:

$$(2) \qquad \langle A'Y,\nabla_{E_i} X\rangle = \langle A'Y,Y\rangle\langle \nabla_{E_i} X,Y\rangle = 0.$$

On the other hand:

$$\langle \nabla_{E_i} A'Y,X\rangle = \langle \nabla_Y(A'E_i) + A[E_i,Y],X\rangle$$

$$= \langle [E_i,Y],A'X\rangle = \langle [E_i,Y],Y\rangle\langle A'X,Y\rangle ,$$

since $\langle A'X,X\rangle$ vanishes by (1) of Lemma 2-4.

Therefore:

$$(3) \qquad \langle \nabla_{E_i} A'Y, X\rangle = -\langle \nabla_Y E_i, Y\rangle \langle A'X, Y\rangle .$$

The relations (1), (2), (3) give

$$(4) \qquad \langle \nabla_Y E_i, Y\rangle = 0. \quad i = 3,\ldots,n.$$

Since $\langle A'X,Y\rangle$ is constant, it follows that

$$(5) \qquad 0 = \nabla_X \langle A'Y, X\rangle = \langle \nabla_X A'Y, X\rangle + \langle A'Y, \nabla_X X\rangle .$$

Making use of (4) of Lemma (2-4) one obtains:

$$\langle A'Y, \nabla_X X\rangle = \langle A'Y, Y\rangle \langle Y, \nabla_X X\rangle = 0,$$

hence (5) gives

$$(6) \qquad \langle \nabla_X A'Y, X\rangle = 0.$$

On the other hand

$$\langle \nabla_X A'Y, X\rangle = \langle \nabla_Y A'X + A'[X,Y], X\rangle =$$

$$= \langle \nabla_Y A'X, X\rangle + \langle [X,Y], A'X\rangle$$

$$= \langle \nabla_Y A'X, X\rangle - \langle \nabla_Y X, A'X\rangle + \langle \nabla_X Y, A'X\rangle .$$

Since $\langle A'X,X\rangle$ vanishes, the relation above can be written as

$$(7) \qquad \langle \nabla_X A'Y, X\rangle = -2\langle \nabla_Y X, A'X\rangle + \langle \nabla_X Y, A'X\rangle$$

$$= -2\langle \nabla_Y X, Y\rangle \langle A'X, Y\rangle .$$

The equations (6) and (7) give

$$(8) \qquad \langle \nabla_Y X, Y\rangle = 0 .$$

The parallelism of the distribution $\mathcal{N}\oplus X$ is a consequence of Lemma (2-4) and the relations (4) and (8) proved above.

Now, it follows from Proposition (1-3) and Corollary (2-5) that M' is locally flat, hence its scalar curvature is zero, which is the desired contradiction. This ends the proof of Corollary (2-6). Q.E.D.

Remark. The proof of this Corollary shows also that the relation (4) and (8) hold whether K is zero or not, thus the following is true:

2-7. Corollary. Let M', f' verify the conditions of Lemma (2-4) for $K \neq 0$. In addition, assume the scalar curvature of M' being constant.

Then the distribution $\mathcal{N}\oplus X$ is parallel on M'.

Proof. See Remark above. Q.E.D.

2-8. Corollary. Let M', f' verify the assumptions of Lemma 2-4 for $K \neq 0$ and assume further that the relations:

$$(1) \qquad \langle \nabla_X E_i, X \rangle = \langle \nabla_Y E_i, Y \rangle \qquad i = 3,\ldots,n$$

hold at all points of M'.

Then the scalar curvature of M' is not constant.

Proof. Assume the scalar curvature of M' constant. Using Corollary (2-7) one obtains

$$\langle \nabla_Y E_i, Y \rangle = 0,$$

and by (1), it follows,

$$(2) \qquad \langle \nabla_X E_i, X \rangle = 0, \qquad i = 3,\ldots n$$

The Gauss equation gives:

$$\langle R(X,E_i)E_i,X\rangle = K + \langle A'E_i,E_i\rangle X - \langle A'X,E_i\rangle E_i$$

and since $A'E_i$ vanishes, this yields:

$$(3) \qquad \langle R(X,E_i)E_i,X\rangle = K.$$

On the other hand:

$$(4) \qquad \langle R(X,E_i)E_i,X\rangle = \langle \nabla_X\nabla_{E_i}E_i,X\rangle - \langle \nabla_{E_i}\nabla_X E_i,X\rangle - \langle \nabla_{[X,E_i]}E_i,X\rangle.$$

The distributions $\mathcal{N}'$ and $\mathcal{N}' \oplus X$ are totally geodesic (see Proposition 1-5 and Corollary 2-5, respectively). This fact together with (2) and (4) gives,

$$\langle R(X,E_i)E_i,X\rangle = 0,$$

which by (3) implies $K = 0$. This is a contradiction since K is assumed non-zero and therefore the scalar curvature of M' is not constant. Q.E.D.

2-9. Lemma. Let M'' be an n-dimensional orientable Riemannian manifold and f'' an isometric immersion of M'' in the space form $\tilde{M}^{n+1}(K)$, with second fundamental form A'' and relative nullity distribution $\mathcal{N}''$.

Assume that $E_1,\ldots,E_n$ is an orthonormal frame defined on M'' such that the vector fields $E_3,\ldots,E_n$ form a basis for $\mathcal{N}''$.

Suppose that there are two complex vector fields Z *and* W, *belonging to the complexification of the vector space spanned by* E_1, E_2, *such that for any local isometric immersion* (h'', H'', U) *of* M'' *in* $\tilde{M}^{n+1}(K)$ (*see* 2-3), *the equation*

$$(1) \qquad \langle H''Z, W\rangle = 0,$$

holds at all points of U''.

Finally assume M'' *deformable in* $\tilde{M}^{n+1}(K)$.

Then the complex vector fields $\nabla_{E_i} Z, \nabla_Z E_i$ (*respectively* $\nabla_{E_i} W$, $\nabla_W E_i$) *have no* W*-component* (*respectively* Z*-component*) for all $i \geq 3$.

Proof. Denote by $(\nabla_Z E_i)_{(W)}$ $((\nabla_W E_i)_{(Z)})$ the W-component (Z-component) of $\nabla_Z E_i$ (resp. $\nabla_W E_i$).

Let p be a point of M'', and assume:

$$(\nabla_Z E_i)_{(W)} \neq 0, \quad \text{for some } i \geq 3,$$

at all points of an open orientable manifold $M''(p)$ containing p.

Consider a local isometric immersion $(h'', H'', M''(p))$ of M'' in $\tilde{M}^{n+1}(K)$ (see Definition 2-3). Since $H''(E_i)$ vanish for all $i \geq 3$, it follows that

$$\langle E_i, H''W\rangle = 0 \qquad i = 3, \ldots, n.$$

By covariant derivation with respect to Z, this relation yields:

$$(2) \qquad \langle \nabla_Z E_i, H''W\rangle + \langle E_i, \nabla_Z H''W\rangle = 0.$$

In view of (1) the first term of the left-hand side of (2) can be written as

$$(3) \qquad \langle \nabla_Z E_i, H''W \rangle = (\nabla_Z E_i)_{(W)} \langle W, H''W \rangle,$$

while for the second term

$$\langle E_i, \nabla_Z H''W \rangle = \langle E_i, \nabla_W H''Z + H''[Z,W] \rangle$$

$$= \langle E_i, \nabla_W H''Z \rangle = -\langle \nabla_W E_i, H''Z \rangle.$$

Again by (1) this equation becomes

$$(4) \qquad \langle E_i, \nabla_Z H''W \rangle = -(\nabla_W E_i)_{(Z)} \langle Z, H''Z \rangle.$$

The equations (2), (3), and (4) give

$$(5) \qquad (\nabla_Z E_i)_{(W)} \langle W, H''W \rangle = (\nabla_W E_i)_{(Z)} \langle Z, H''Z \rangle.$$

On the other hand, the extension of the Gauss equation to complex vector fields gives

$$\langle H''Z, Z \rangle \langle H''W, W \rangle - \langle Z, H''W \rangle^2$$

$$= \langle A''Z, Z \rangle \langle A''W, W \rangle - \langle Z, A''W \rangle^2,$$

and using (1) this gives

$$(6) \qquad \langle H''Z, Z \rangle \langle H''W, W \rangle = \langle A''Z, Z \rangle \langle A''W, W \rangle.$$

From (5)

$$(7) \qquad (\nabla_Z E_i)_{(W)} \langle W, H''W \rangle^2 = (\nabla_Z E_i)_{(W)} \langle Z, H''Z \rangle \langle W, H''W \rangle$$

$$(\nabla_Z E_i)_{(W)} \langle W, A''W \rangle^2 = (\nabla_Z E_i)_{(W)} \langle Z, A''Z \rangle \langle W, A''W \rangle,$$

which together with (6) yields

$$(8)\quad (\nabla_Z E_i)_{(W)}(\langle W,H''W\rangle^2 - \langle W,A''W\rangle^2) = 0$$

Since $(\nabla_Z E_i)_{(W)}$ is assumed non-zero, it follows from (8) that

$$\langle W,H''W\rangle^2 - \langle W,A''W\rangle^2 = 0$$

at all points of $M''(p)$, which means that

$$(9)\quad \langle H''W,W\rangle = e\langle A''W,W\rangle ,$$

where e is a constant either $+1$ or -1.

From (6) and (9)

$$(10)\quad \langle H''Z,Z\rangle = e\langle A''Z,Z\rangle .$$

Finally from (1), (9) and (10) it follows that

$$(11)\quad H'' = eA''.$$

Since (11) holds for any local immersion, it follows that $M''(p)$ is rigid in $\bar{M}^{n+1}(K)$, which contradicts the deformability of M''. Thus $(\nabla_Z E_i)_{(W)}$ vanishes at p. Since the above proof is symmetric in Z,W it results that $(\nabla_W E_i)_{(Z)}$ also vanishes on M''.

Next, denote by $(\nabla_{E_i} Z)_{(W)}$ (Resp. $(\nabla_{E_i} W)_{(Z)}$) the W-component (resp. Z-component) of $\nabla_{E_i} Z$ (resp. $\nabla_{E_i} W$).

Let p be a point of M'' and suppose:

$$(\nabla_{E_i} Z)_{(W)} \neq 0,$$

for some $i \geq 3$ and at all points of an open orientable submanifold $M''(p)$ containing p.

Consider a local isometric immersion $(h'',H'',M''(p))$.

By covariant derivation with respect to E_i, of both sides of the relation (1),

$$(12) \quad \langle \nabla_{E_i} Z, H''W \rangle + \langle Z, \nabla_{E_i} H''W \rangle = 0$$

In view of (1), the first term of the left-hand side of (12) becomes

$$(13) \quad \langle \nabla_{E_i} Z, H''W \rangle = (\nabla_{E_i} Z)_{(W)} \langle W, H''W \rangle$$

By (1) and the fact that $H''(E_i)$ is zero, the second term has the form

$$(14) \quad \langle Z, \nabla_{E_i} H''W \rangle = \langle Z, H''[E_i, W] \rangle$$

$$\langle H''Z, [E_i, W] \rangle = \langle H''Z, Z \rangle ((\nabla_{E_i} W)_{(Z)} - (\nabla_W E_i)_{(Z)}),$$

and since $(\nabla_W E_i)_{(Z)}$ vanishes, as it was shown above, the relation (14) simplifies to

$$(15) \quad \langle Z, \nabla_{E_i} H''W \rangle = (\nabla_{E_i} W)_{(Z)} \langle H''Z, Z \rangle$$

The relations (12), (13), and (15) give

$$(16) \quad (\nabla_{E_i} Z)_{(W)} \langle H''W, W \rangle + (\nabla_{E_i} W)_{(Z)} \langle H''Z, Z \rangle = 0$$

and of course

$$(17) \quad (\nabla_{E_i} Z)_{(W)} \langle A''W, W \rangle + (\nabla_{E_i} W)_{(Z)} \langle A''Z, Z \rangle = 0$$

By the same argument used before, it can be concluded from (17) that $M''(p)$ is rigid, which contradicts the deformability of M''. Hence the proof of Lemma 2-9 is complete.

Q.E.D.

2-10. Corollary. Assume the manifold M'', the immersion f'' and the vector fields Z, W satisfy the conditions of Lemma 2-9.

Then the following conclusions hold:

a) for $K \neq 0$ and $n \geq 4$, the scalar curvature of M'' cannot be constant.

b) for $K = 0$ and M'' with constant scalar curvature, the relative nullity distribution $\mathcal{N}''$ of f'' is parallel on M''.

Proof. From Lemma 2-9 it follows that

(1) $$[Z,E_i]_{(W)} = [W,E_i]_{(Z)} = 0 \quad i = 3,\dots,n$$

Consider a point $p \in M''$. From (1) and Proposition (1-9) there are two non-vanishing complex valued functions α,β defined in a neighborhood of p, in such a way that the new complex vector fields Z',W' defined by

(2) $$Z' = \alpha Z,\ W' = \beta W,$$

have the propriety

(3) $$[Z',E_i] = \sum_{k\geq 3} a_i^k E_k$$

$$[W',E_i] = \sum_{k\geq 3} b_i^k E_k,$$

where a_i^k, b_i^k are complex valued functions defined in a neighborhood of p.

Furthermore, by Propositions (1-5) and (1-9) the frame $E_1,\ldots,E_n$ may be assumed as verifying

$$(4) \qquad \nabla_{E_i} E_j = \frac{K}{2} \sum_{k=3}^{n} (\delta_j^i x^k - \delta_j^k x^i) E_k,$$

where the functions x^k are part of a suitable coordinate system of M'' at p. Finally, it is possible to assume the existence of an open orientable submanifold $M''(p)$, containing p and such that (3) and (4) hold at all of its points.

It follows from (2) that $M''(p)$ and the vector field Z', W' verify the assumption of Lemma (2-9). Hence

$$(5) \qquad (\nabla_{Z'} E_i)_{(W')} = (\nabla_{E_i} Z')_{(W')} = 0,$$

$$(\nabla_{W'} E_i)_{(Z')} = (\nabla_{E_i} W')_{(Z')} = 0.$$

On the other hand, since $\nabla_{E_i} E_j$ belong to $\mathcal{N}''$ and $\langle Z', E_j \rangle$ vanish, it follows that

$$(6) \qquad \nabla_{E_i} Z' = (\nabla_{E_i} Z')_{(Z')} Z',$$

$$\nabla_{E_i} W' = (\nabla_{E_i} W')_{(W')} W', \quad i = 3,\ldots,n,$$

at each point of $M''(p)$.

Recalling that A'' denotes the second fundamental form of f'', it may be written:

$$(7) \qquad \nabla_{E_i} \langle A''Z', Z' \rangle = \langle \nabla_{E_i} A''Z', Z' \rangle + \langle A''Z', \nabla_{E_i} Z' \rangle,$$

for all $i \geq 3$, on $M''(p)$.

Next it will be shown that the first term of the right-hand side of (7) vanishes.

In fact

$$\langle \nabla_{E_i} A''Z', Z' \rangle = \langle \nabla_{Z'} A''E_i + A''[E_i, Z'], Z' \rangle = 0,$$

as a consequence of (3).

For the second term, the relations (6) yield

$$(8) \quad \langle A''Z', \nabla_{E_i} Z' \rangle = (\nabla_{E_i} Z')_{(Z')} \langle A''Z', Z' \rangle$$

Combining (7) and (8) it follows that

$$(9) \quad \nabla_{E_i} \langle A''Z', Z' \rangle = (\nabla_{E_i} Z')_{(Z')} \langle A''Z', Z' \rangle,$$

and a similar relation holds for W'

$$(10) \quad \nabla_{E_i} \langle A''W', W' \rangle = (\nabla_{E_i} W')_{(W')} \langle A''W', W' \rangle .$$

Furthermore the relations below are also a consequence of (6)

$$(11) \quad \nabla_{E_i} \langle Z', Z' \rangle = 2(\nabla_{E_i} Z')_{(Z')} \langle Z', Z' \rangle ,$$

$$\nabla_{E_i} \langle W', W' \rangle = 2(\nabla_{E_i} W')_{(W')} \langle W', W' \rangle ,$$

$$\nabla_{E_i} \langle Z', W' \rangle = ((\nabla_{E_i} Z')_{(Z')} + (\nabla_{E_i} W')_{(W')}) \langle Z', W' \rangle ,$$

for all $i \geq 3$ and all points of $M''(p)$.

The scalar curvature of $M''(p)$ at each point if given by

$$(12)\quad \mathrm{scal}(M''(p)) = n(N-1)K + 2\frac{\langle A''Z',Z'\rangle\langle A''W',W\rangle}{\langle Z',Z'\rangle\langle W',W'\rangle-\langle Z',W'\rangle^2}.$$

Since $\mathrm{scal}(M''(p))$ is constant, it follows from (12)

$$(13)\quad (\langle Z',Z'\rangle\langle W',W'\rangle-\langle Z',W'\rangle^2)\,\nabla_{E_i}\left(\langle A''Z',Z'\rangle\langle A''W',W'\rangle\right)$$

$$= \nabla_{E_i}(\langle Z',Z'\rangle\langle W',W'\rangle-\langle Z',W'\rangle^2)(\langle A''Z',Z'\rangle\langle A''W',W'\rangle).$$

From (9), (10), (11), (13)

$$(14)\quad ((\nabla_{E_i}Z')_{(Z')}+(\nabla_{E_i}W')_{(W')})(\langle Z',Z'\rangle\langle W',W'\rangle-\langle Z',W'\rangle^2)\cdot$$

$$\cdot\langle A''Z',Z'\rangle\langle A''W',W'\rangle = 2((\nabla_{E_i}Z')_{(Z')}+(\nabla_{E_i}W')_{(W')})\cdot$$

$$(\langle Z',Z'\rangle\langle W',W'\rangle - \langle Z',W'\rangle^2)\langle A''Z',Z'\rangle\langle A''W',W'\rangle,$$

which gives

$$(15)\quad (\nabla_{E_i}Z')_{(Z')} + (\nabla_{E_i}W')_{(W')} = 0,$$

for all $i \geq 3$, at all points of $M''(p)$.

On the other hand, since $A''E_i$ vanishes for all $i \geq 3$, the Gauss equation gives

$$(16)\quad R(Z',E_i)E_i = \tilde{R}(Z',E_i)E_i = KZ'.$$

By the definition of curvature

$$(17)\quad R(Z',E_i)E_i = \nabla_{Z'}(\nabla_{E_i}E_i) - \nabla_{E_i}(\nabla_{Z'}E_i) - \nabla_{E_i}([Z',E_i]).$$

Using (4) it may be written

$$(18)\quad \nabla_{E_i}E_i = \sum_{j\neq i} \lambda^j E_j$$

and by covariant differentiation of (18) with respect to Z',

$$(19)\quad \nabla_{Z'}(\nabla_{E_i}E_i) = \sum_{j\neq i}(\lambda^j \nabla_{Z'}E_j + Z'(\lambda^j)E_j)$$

$$= \sum_{j\neq i}(\lambda^j(\nabla_{E_j}Z') + \lambda^j[Z',E_j] + Z'(\lambda^j)E_j).$$

From (3), (6) and (19) it follows that

$$(20)\quad \left[\nabla_{Z'}(\nabla_{E_i}E_i)\right]_{(Z')} = \sum_{j\neq i} \lambda^j(\nabla_{E_j}Z')_{(Z')},$$

and also

$$(21)\quad \left[\nabla_{W'}(\nabla_{E_i}E_i)\right]_{(W')} = \sum_{j\neq i} \lambda^j(\nabla_{E_j}W')_{(W')}$$

The Z'-component of $\nabla_{E_i}(\nabla_{Z'}E_i)$ is given by

$$(22)\quad \left[(\nabla_{E_i}(\nabla_{Z'}E_i)\right]_{(Z')} = E_i\left[(\nabla_{E_i}Z')_{(Z')}\right] + \left((\nabla_{E_i}Z')_{(Z')}\right)^2.$$

In fact,

$$(23) \qquad \nabla_{Z'} E_i = \nabla_{E_i} Z' + [Z', E_i],$$

by covariant differentiation of (23) and by (3) it follows

$$(24) \qquad \left(\nabla_{E_i}(\nabla_{Z'} E_i)\right)_{(Z')} = \left(\nabla_{E_i}(\nabla_{E_i} Z')\right)_{(Z')} ,$$

by (6), the right-hand side of (24) may be written,

$$(25) \qquad \nabla_{E_i}(\nabla_{E_i} Z')_{(Z')} = \nabla_{E_i}\left((\nabla_{E_i} Z')_{(Z')} Z'\right)_{(Z')}$$

$$= E_i\left((\nabla_{E_i} Z')_{(Z')}\right) + \left((\nabla_{E_i} Z')_{(Z')}\right)^2 ,$$

which proves (22).

The same relation holds for the W'-components.

Taking the Z'-components in (16) and using (17), (20) and (22) it follows that

$$(26) \qquad \sum_{j \neq i} \lambda^j (\nabla_{E_j} Z')_{(Z')} - E_i\left((\nabla_{E_i} Z')_{(Z')}\right) - \left((\nabla_{E_i} Z')_{(Z')}\right)^2 = K.$$

Adding (26) to its analog for W', and using (15), we obtain

$$(27) \qquad (\nabla_{E_i} Z')^2_{(Z')} + (\nabla_{E_i} W')^2_{(W')} = -2K,$$

and again by (15),

$$(28) \qquad (\nabla_{E_i} Z')^2_{(Z')} = (\nabla_{E_i} W')^2_{(W')} = -K; \quad i = 3, \ldots, n .$$

From (26) and (28) it follows

$$(29) \qquad \sum_{j\neq i} \lambda^j (\nabla_{E_j} Z')_{(Z')} = 0 \qquad i,j \geq 3,$$

and from (4),(18) and (28) we get respectively

$$(30) \qquad \lambda^j = \frac{K}{2} x^j \quad , \quad (\nabla_{E_j} Z')_{(Z')} = \sqrt{-K} \, , \quad j \geq 3.$$

The relations (29), (30) and (31) show that if $n \geq 4$, then $K = 0$, hence proving the statement (a) by contradiction.

In the case $K = 0$, (3) and (28) prove (b).

Q.E.D.

2-11. <u>Proof of Theorem 2-1</u>. Let p_o be a given point in $\bar{M}^n$ and consider an open neighborhood U_o of p_o on which there is an orthonormal frame

$$E_1, E_2, \ldots, E_n,$$

in such a way that $E_3, \ldots, E_n$ is a basis for $\bar{\mathcal{X}}$ and

$$(1) \qquad \nabla_{E_i} E_j = 0, \quad i,j \geq 3.$$

This is possible in view of Propositions (1-2),(1-5) and the fact that $\bar{M}^n$ is isometrically immersed in the Euclidean space E^{n+1}.

For any local isometric immersion (h, H, U_o) of U_o, it holds

$$(2) \qquad H[E_1, E_2], E_i = 0, \quad i = 3, \ldots, n,$$

on U_o. This relation and the Codazzi equations yield

$$(3)\quad \langle\nabla_{E_1}E_i,E_2\rangle\langle HE_2,E_2\rangle + [\langle\nabla_{E_1}E_i,E_1\rangle - \langle\nabla_{E_2}E_i,E_2\rangle]\langle HE_1,E_2\rangle$$

$$-\langle\nabla_{E_2}E_i,E_1\rangle\langle HE_1,E_1\rangle = 0,$$

for all $i \geq 3$, at all points of U_o.

The equation (2) will be used to define locally vector fields satisfying either the conditions of Lemma 2-4 or 2-9. Since this involves several discussions, it is convenient to consider the following subset of U_o,

P: Set of the points q of U_o such that

$$(4)\quad \begin{aligned} &\langle\nabla_{E_1}E_i,E_2\rangle_q = \langle\nabla_{E_2}E_i,E_1\rangle_q = 0, \\ &\langle\nabla_{E_1}E_i,E_1\rangle_q = \langle\nabla_{E_2}E_i,E_2\rangle_q, \quad i = 3,\ldots,n. \end{aligned}$$

This set has the following propriety:

(5) <u>The relative nullity distribution $\mathcal{N}$ is parallel at any point of the interior of</u> P.

In fact, consider a point $q \in \mathrm{Int}P$, locally it is possible to replace E_1, E_2 by unit vector fields X, Y, such that

(6) $\langle X,Y\rangle = 0$, $\langle \bar{A}X,Y\rangle = 0$,

on a neighborhood of q, provided the non-zero eigenvalues of $\bar{A}_q$ are distinct. A direct computation gives:

$$(7)\quad \langle \nabla_X E_i, Y\rangle = \langle \nabla_Y E_i, X\rangle = 0$$

$$\langle \nabla_X E_i, X\rangle = \langle \nabla_Y E_i, Y\rangle ,$$

for all $i \geq 3$ and at all points of a neighborhood of q_o.

From and the constancy of the scalar curvature

$$(8)\quad \nabla_{E_i} [\langle \bar{A}X, X\rangle\langle \bar{A}Y, Y\rangle] = 0 \qquad i = 3,\ldots,n ,$$

or

$$[\nabla_{E_i}\langle \bar{A}X, X\rangle]\langle \bar{A}Y, Y\rangle + \langle \bar{A}X, X\rangle [\nabla_{E_i}\langle \bar{A}Y, Y\rangle] = 0.$$

On the other hand

$$(9)\quad \nabla_{E_i}\langle \bar{A}X, X\rangle = \langle \nabla_{E_i} \bar{A}X, X\rangle + \langle \bar{A}X, \nabla_{E_i} X\rangle =$$

$$= \langle \bar{A}[E_i, X], X\rangle + \langle \bar{A}X, \nabla_{E_i} X\rangle$$

$$= \langle [E_i, X], X\rangle\langle \bar{A}X, X\rangle = -\langle \nabla_X E_i, X\rangle\langle \bar{A}X, X\rangle ,$$

similarly,

$$(10)\quad \nabla_{E_i}\langle \bar{A}Y, Y\rangle = -\langle \nabla_Y E_i, Y\rangle\langle \bar{A}Y, Y\rangle .$$

The relations (8), (9), and (10) give

$$(11)\quad [\langle \nabla_X E_i, X\rangle + \langle \nabla_Y E_i, Y\rangle] \langle \bar{A}X, X\rangle\langle \bar{A}Y, Y\rangle = 0,$$

which by (7) gives

$$(12)\quad \langle \nabla_X E_i, X\rangle = \langle \nabla_Y E_i, Y\rangle = 0 , \qquad \text{all } i \geq 3 .$$

Next assume that the non-zero eigenvalues of $\bar{A}_q$ coincide. If they coincide in a neighborhood of q, it is possible to find vector fields X, Y satisfying (6) and therefore possible to show that $\bar{\mathcal{N}}$ is parallel at q.

Finally assume that the non-zero eigenvalues coincide at q, but each neighborhood of q contains a point at which they are distinct. A simple continuity argument shows that in this case $\bar{\mathcal{N}}$ is also parallel at q.

On the other hand

(13) <u>The distribution</u> $\bar{\mathcal{N}}$ <u>is parallel at any point of</u> $U_0 - P$.

To show this, consider a point $q \in U - P_0$, this means that for some index $i_0 \geq 3$ the numbers

(14) $\langle \nabla_{E_1} E_{i_0}, E_2 \rangle_q, \langle \nabla_{E_1} E_{i_0}, E_1 \rangle_q - \langle \nabla_{E_2} E_{i_0}, E_2 \rangle_q, \langle \nabla_{E_2} E_i, E_1 \rangle_q$,

are not simultaneously zero.

For any $i \geq 3$, let Δ^i denote the function on U_0.

(15) $\Delta^i = [\langle \nabla_{E_1} E_i, E_1 \rangle - \langle \nabla_{E_2} E_i, E_2 \rangle]^2 + 4 \langle \nabla_{E_1} E_i, E_2 \rangle \langle \nabla_{E_2} E_i, E_1 \rangle$.

The distinct cases to be discussed can be indicated in the following way:

a) $\langle \nabla_{E_1} E_{i_0}, E_2 \rangle_q \neq 0$ (or $\langle \nabla_{E_2} E_{i_0}, E_1 \rangle_q \neq 0$)
- $\Delta^{i_0}(q) \neq 0$
- $\Delta^{i_0}(q) = 0$
  - $\Delta^{i_0} = 0$, on a nbhd. of q
  - Any nbhd. of q has a point at which Δ^{i_0} is non-zero

b) $\langle \nabla_{E_1} E_{i_o}, E_2 \rangle_q = 0$, $\langle \nabla_{E_2} E_{i_o}, E_1 \rangle_q = 0$

- $\Delta_q^{i_o} = 0$: The functions $\langle \nabla_{E_1} E_{i_o}, E_2 \rangle$, $\langle \nabla_{E_2} E_{i_o}, E_1 \rangle$, both vanish on a neighborhood of q.
- $\Delta_q^{i_o} \neq 0$: Any neighborhood of q contains a point at which either one of $\langle \nabla_{E_1} E_{i_o}, E_2 \rangle$, $\langle \nabla_{E_2} E_{i_o}, E_1 \rangle$, is non-zero at this point.

The proof of (13) consists in showing the parallelism of $\bar{\mathcal{J}}$ in each of the cases above.

a) Assume $\langle \nabla_{E_1} E_{i_o}, E_2 \rangle$, Δ^{i_o} non-zero at all points of a neighborhood V_q of q.

In view of the assumption made above, the quadratic equation

(16) $$\langle \nabla_{E_1} E_{i_o}, E_2 \rangle x^2 - [\langle \nabla_{E_1} E_{i_o}, E_1 \rangle - \langle \nabla_{E_2} E_{i_o}, E_2 \rangle]x - \langle \nabla_{E_2} E_{i_o}, E_1 \rangle = 0$$

defines two complex valued C^∞ functions α, β, such that

(17) $$\alpha\beta = - \frac{\langle \nabla_{E_2} E_{i_o}, E_1 \rangle}{\langle \nabla_{E_1} E_{i_o}, E_2 \rangle},$$

and

(18) $$\alpha + \beta = \frac{[\langle \nabla_{E_1} E_{i_o}, E_1 \rangle - \langle \nabla_{E_2} E_{i_o}, E_2 \rangle]}{\langle \nabla_{E_1} E_{i_o}, E_2 \rangle}$$

Consider the complex vector field Z, W on V_q defined by

$$Z = \alpha E_1 + E_2$$
$$W = \beta E_1 + E_2,$$

they are linearly independent at each point, since α and β are distinct (at each point).

Let (h,H,U) be any local isometric immersion of V_q in E^{n+1}, then

$$\langle HZ,W\rangle = \alpha\beta\langle HE_1,E_1\rangle + (\alpha+\beta)\langle HE_1,E_2\rangle + \langle HE_2,E_2\rangle$$

$$= \frac{1}{\langle\nabla_{E_1}E_{i_o},E_2\rangle}[\langle\nabla_{E_1}E_{i_o},E_2\rangle\langle HE_2,E_2\rangle + [\langle\nabla_{E_1}E_{i_o},E_1\rangle$$

$$- \langle\nabla_{E_2}E_{i_o},E_2\rangle]\langle HE_1,E_2\rangle - \langle\nabla_{E_2}E_{i_o},E_1\rangle\langle HE_1,E_1\rangle]$$

which is zero in view of (3). Thus Z,W satisfy the conditions of Lemma (2-9) and therefore by part b) of Corollary (2-10), the distribution $\overline{\mathcal{N}}$ is parallel on the neighborhood V_q.

The next case to be analyzed is that of

$$(19)\ \langle\nabla_{E_1}E_{i_o},E_2\rangle \neq 0,$$

in a neighborhood V_q of q and Δ^{i_o} vanishing at all points of V_q.

In this case the functions α,β coincide at each point and the vector field

$$X = \frac{\alpha E_1 + E_2}{||\alpha E_1 + E_2||},$$

satisfies the conditions of Lemma (2-4), therefore Corollary (2-6) shows that this case cannot occur.

The parallelsim of $\overline{\mathcal{N}}$ in the last subcase of a) is proved by using the reasoning of the proof of the first subcase and a simple continuity argument.

b) The first subcase cannot occur, for otherwise all functions listed in (14) would vanish at q.

Hence to study the next case it may be assumed that on a neighborhood V_q of q, the functions $\langle \nabla_{E_1} E_{i_o}, E_2 \rangle$, $\langle \nabla_{E_2} E_{i_o}, E_1 \rangle$ vanish, while Δ^{i_o} is never zero. Using again (3), it follows

$$[\langle \nabla_{E_1} E_{i_o}, E_1 \rangle - \langle \nabla_{E_2} E_{i_o}, E_2 \rangle] \langle HE_1, E_2 \rangle = 0,$$

which shows that the vector fields E_1, E_2 satisfy the conditions of Lemma (2-9), and again, by Corollary (2-10), the parallelsim of $\overline{\mathcal{N}}$ is established in V_q.

Finally, the last case of b) can be related to the first case of a) and as before, a continuity argument proves the parallelsim of $\overline{\mathcal{N}}$ at q, in this case.

From (5) and (13) it follows that $\overline{\mathcal{N}}$ is parallel at all points of U_o, and particularly at p_o. Since this point can be arbitrarily chosen, it follows that $\overline{\mathcal{N}}$ is parallel everywhere.

Thus Theorem (2-1) is proved.

Q.E.D.

2-12. Proof of Theorem 2-2. This proof presents a great analogy with the one given in (2-11). As before, let p_o be an arbitrarily chosen point of $\bar{M}^n$ and U_o be a neighborhood of it on which there are

(1) A coordinate system, $x',\ldots,x^n$

(2) An orthonormal frame $E_1,E_2,\ldots,E_n$, such that the vectors $E_3,\ldots,E_n$ form a basis of the relative nullity distribution $\mathcal{X}$, and satisfying

$$(3) \quad \nabla_{E_i} E_j = \frac{K}{2} \sum_{k=3}^{n} (\delta^i_j x^k - \delta^k_j x^i) E_k.$$

Again, Proposition (1-2) allows to make these assumptions.

Since the space $\tilde{M}^{n+1}(K)$ has constant curvature, the equation (3) of (2-10) holds in the present situation.

Further, the set P is defined exactly the same way, but its propriqties are drastically different in face of the assumption $K \neq 0$.

In this case the following holds

(4) The set P has no interior points.

First it should be noted that by the same argument used to prove (5) of (2-11), it turns out that $\bar{\mathcal{X}}$ is parallel, on the other hand:

$$(5) \quad \langle R(X,E_i)E_i,X\rangle = K,$$

where X is an unit vector field, orthogonal to the E_i.

On the other hand, the parallelsim of $\bar{\mathcal{N}}$ and the fact that its leaves are totally geodesic implies that

(6) $\langle R(X,E_i)E_i,X\rangle = 0.$

The relations (5) and (6) are contradictory, therefore (4) is proved.

Next it will be shown that;

(7) $U_o - P$ <u>has no interior points</u>. This can be proved by a series of discussions that follow the same pattern of a) and b) of (2-11), and using the first part of the conclusions of Corollary (2-10).

In order to eliminate the case,

(8) $\langle \nabla_{E_1} E_{i_o}, E_2\rangle_q \neq 0,$

$\Delta^{i_o} = 0$, on a neighborhood of q, some further information is needed.

As in (19) of (2-11), there is an orthonormal frame X,Y defined in a neighborhood V_q of q, which is orthogonal to the distribution $\bar{\mathcal{N}}$ and satisfies

(9) $\langle HX,X\rangle = 0,$

for any local isometric immersion (h,H,U) of V_q in $\tilde{M}^{n+1}(K)$.

From,

$$\langle \nabla_X E_{i_o}, Y\rangle\langle HY, Y\rangle + [\langle \nabla_X E_{i_o}, X\rangle - \langle \nabla_Y E_{i_o}, Y\rangle]\langle HX, Y\rangle$$
$$- \langle \nabla_Y E_{i_o}, X\rangle\langle HX, X\rangle = 0,$$

and (9) it follows

$$(10)\quad \langle \nabla_X E_{i_o}, Y\rangle\langle HY, Y\rangle + [\langle \nabla_X E_{i_o}, X\rangle - \langle \nabla_Y E_{i_o}, Y\rangle]\langle HX, Y\rangle = 0$$

If $\langle \nabla_X E_{i_o}, Y\rangle$ is non-zero at all points of some open subset $V'' \subset V_q$, the relations (9) and (10), with the Gauss equation imply the rigidity of V'', which is a contradiction. Thus

$$(11)\quad \langle \nabla_X E_{i_o}, Y\rangle = 0,$$

at all points of V_q.

In order to complete the discussion, Corollary (2-8) will be applied, but it requires that

$$(12)\quad \langle \nabla_X E_i, X\rangle - \langle \nabla_Y E_i, Y\rangle = 0$$

for all indices $i \geq 3$, and for all points of V_q.

The last relations are a consequence of

$$(13)\quad \langle \nabla_X E_i, Y\rangle = 0, \quad \Delta^i = 0,$$

for all $i \geq 3$, on V_q.

Proof of (13). Consider the linear system

(14) $$\langle \nabla_X E_1, Y \rangle a_1 + [\langle \nabla_X E_1, X \rangle - \langle \nabla_Y E_1, Y \rangle] a_2 = 0$$

For any local isometric immersion (h,H,U), it is known that

$$a_1 = \langle HY, Y \rangle, \quad a_2 = \langle HX, Y \rangle,$$

is a solution of (14) (see (30-(2-11).)

If at some point of V_q the matrix of (14) had rank greater than one, this system together with (9) and the Gauss equation would imply the existence of an open rigid submanifold of V_q, therefore contradicting the deformability of $\bar{M}^n$. Thus the rank is less than 2 at every point.

Now (13) follows from this fact and from (8), (11). Therefore Corollary (2-8) leads to another contradiction, showing that this case cannot occur, and (7) is proved.

The conclusions (4) and (7) are obviously incompatible. This proves that $\bar{M}^n$ is not deformable in $\tilde{M}^{n+1}(K)$.

Q.E.D.

3.

Theorem (2-1) in spite of being local, has an interesting global consequence.

3-1. Theorem. Let M^n, $n \geq 3$, be a complete Riemannian manifold, with non-zero constant scalar curvature and being locally deformable in E^{n+1}.

Then M^n is isometric to the Riemannian product $S^2 \times E^{n-2}$, of a two-dimensional sphere by a $n-2$ dimensional Euclidean space.

Proof. Let f be an isometric immersion of M^n in E^{n+1}. From the local deformability of M^n and from the fact that the scalar curvature is non-zero it follows that the type number of f is two everywhere.

Let $\mathcal{N}'$ be the relative nullity distribution of .
By Theorem (2-1) it follows that $\mathcal{N}$ is parallel on M^n (since it is parallel on a neighborhood of each point). Thus the universal covering $\hat{M}^n$ of M^n has the decomposition

$$(1) \qquad \hat{M}^n = \hat{M}^2 \times E^{n-2} .$$

by de Rham's theorem.

Since M^n is non-flat, $\hat{M}^2$ is necessarily irreducible.

Under these circumstances, it follows from a result of S. Alexander [1] that M^n itself is isometric to a Riemannian product

$$(2) \qquad M^n = M^2 \times E^{n-2},$$

and f immerses M^2 isometrically in a 3-dimensional Euclidean space.

It follows from (2) that the curvature of M^2 equals the scalar curvature of M^n and is therefore constant.

It follows from (2) that M^2 is complete. By a well-known theorem of Hilbert, the curvature of M^2 is positive, and therefore isometric to a sphere (see [7]).

Q.E.D.

3-2. Theorem. Let M^n be a homogeneous Riemannian manifold, having an isometric immersion f in the Euclidean space E^{n+1}, such that its type number is two everywhere.

Then M^n is isometric to the Riemannian product of a 2-sphere by a n-2 plane.

Proof. For $n = 2$, M^n is compact and the proof for this case is given in [7].

Next assume $n \geq 3$. Since M^n is homogeneous, by Proposition (1-7) it follows that the only possibilities to be discussed are

(1) M^n is locally deformable

(2) M^n is rigid.

From Theorem (3-1) it follows that in case (1) M^n is isometric to $S^2 \times E^{n-2}$.

To prove (2) the following result of K. Nomizu and B. Smyth [10] is used:

"Let M be a complete Riemannian manifold of dimension n with non-negative sectional curvature.

Let $\phi: M \to E^{n+1}$ be an isometric immersion with constant mean curvature.

If the trace of A^2 is constant, then M^n is isometric to $S^2 \times E^{n-2}$".

Next it will be shown that M^n and $\bar{f}$ satisfy the conditions of this theorem.

Let U be an open, orientable submanifold of M^n and denote by A a second fundamental form of f on U.

It is well-known (see [12]) that there are n continuous functions on U:

$$\lambda_1 \geq \lambda_2 \geq , \ldots \geq \lambda_n,$$

such that at each point $p \in U$, $\lambda_1(p_o), \ldots, \lambda_n(p)$ are the eigenvalues of A_p.

Since M^n is rigid, these functions are constant. By a theorem of E. Cartan [3], at most two of them can be distinct at each point. On the other hand, these are two non-zero eigenvalues λ_1, λ_2, and $n-2 \geq 1$, which vanish. Thus λ_1 and λ_2 must coincide at each point.

This means that

$$trA^2 = 2\lambda_1^2 = \text{constant},$$

$$trA = 2\lambda_1 = \text{constant},$$

and it is also clear that all sectional curvatures are non-negative.

Thus the above mentioned result gives the proof in case (2).

Q.E.D.

Regarding hypersurfaces of spaces on constant curvature, Theorem (2-2) gives

3-3. Theorem. A hypersurface of $\tilde{M}^{n+1}(K)$, $K \neq 0$, having constant scalar curvature distinct from $n(n-1)K$, is rigid, provided that $n \geq 4$.

Proof. Let M^n be such a hypersurface and consider two isometric immersions f, $\bar{f}$, with second fundamental forms A, $\bar{A}$ defined on some orientable open submanifold of M^n.

The assumption on the scalar curvature implies that A and $\bar{A}$ have rank ≥ 2 everywhere.

Let U be the subset of M^n consisting of those points which are contained in some rigid open neighborhood (this neighborhood may depend on the point).

It follows from Theorem (2-2) that $M^n - U$ has no interior points, i.e., that U is dense in M^n. Since U is covered by open rigid submanifolds, it follows that each connected component of U is rigid. Let p be a point of M^n and V an orientable neighborhood of p. It will be shown that there is a function e(q) defined on V, assuming only the values $+1$ or -1, and such that

$$(1) \qquad \bar{A}_q = e(q)\, A_q, \quad \text{for all } q \in V.$$

In fact, if $q \in U$, this follows from the rigidity of each component. On the other hand, if $q \notin U$, it can be approximated by points on which (1) holds, and by continuity it follows that (1) holds at q. Again the continuity of A and $\bar{A}$ give the continuity of e. Since V is assumed connected, e must be constant. But this means that V is rigid. This argument shows that M^n cam be covered by rigid neighborhoods, which implies the rigidity of M^n itself.

Q.E.D.

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ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor, Professor S. Kobayashi for suggesting the problem which motivated this work and for his constant interest and help.

Furthermore, I am very greatful to my friends I. Kupka and M. do. Carmo for many helpful suggestions.

This work was done under the support of the C. N. Pq Rio de Janeiro and U.S.P (Sao Paulo).