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Oscillating and Periodic Solutions of Lienard Equations.

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OSCILLATING AND PERIODIC SOLUTIONS OF LIENARD EQUATIONS

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Abstract

In this paper we consider the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

with $f, g : \mathbf{R} \to \mathbf{R}$, and we study its qualitative behavior from the point of view of oscillation and periodicity of the solutions. In this work, the positive definite function (α and β are real, with $\beta > 0$)

$$W_{\alpha,\beta}(x,y) = \int_0^{y/H_\beta(x)} \frac{s}{s^2 + \alpha s + 1} ds + \ell n[\beta^{-1/2} H_\beta(x)]$$

where $H_\beta(x) = [2\int_0^x g(t)dt + \beta]^{1/2}$, will play a fundamental role.

1. The Definite Positive Function $W_{\alpha,\beta}$. Auxilliary Lemmas.

Throughout this work we assume f and g as functions of $\mathbf{R}$ in $\mathbf{R}$ satisfying the following conditions:

a) f is continuous and g is of class C^1;

b) $xg(x) > 0$ for $x \neq 0$;

c) $\int_0^{+\infty} g(x)dx = +\infty = \int_0^{+\infty} g(x)dx$.

Let α and β real, with $\beta > 0$, and $H_\beta : \mathbf{R} \to \mathbf{R}$ given by

$$H_\beta(x) = [2\int_0^x g(t)dt + \beta]^{1/2}.$$

We indicate by $\Omega_{\alpha,\beta}$ the following open set:

$$\Omega_{\alpha,\beta} = \mathbf{R}^2 \qquad \text{for} \quad |\alpha| < 2,$$

$$\Omega_{\alpha,\beta} = \{(x,y) \in \mathbf{R}^2 \mid y > \rho_1 H_\beta(x),\ x \text{ real}\} \qquad \text{for} \quad \alpha \geq 2$$

and

$$\Omega_{\alpha,\beta} = \{(x,y) \in \mathbf{R}^2 \mid y < \rho_2 H_\beta(x),\ x \text{ real}\} \qquad \text{for} \quad \alpha \leq -2$$

where

$$\rho_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4}}{2} \qquad \text{and} \qquad \rho_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4}}{2}.$$

Consider the definite positive function

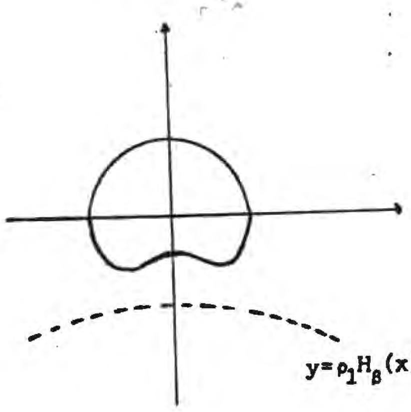
$$W_{\alpha,\beta} : \Omega_{\alpha,\beta} \to \mathbf{R}$$

given by

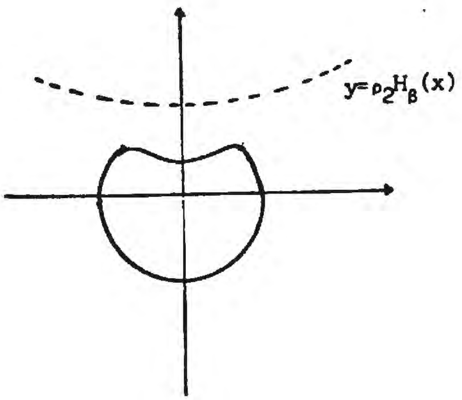
$$W_{\alpha,\beta}(x,y) = \int_0^{y/H_\beta(x)} \frac{s}{s^2 + \alpha s + 1}\, ds + \ell n[\beta^{-1/2} H_\beta(x)].$$

It can then be easily verified tha the level curves of $W_{\alpha,\beta}$ are all closed curves and that $W_{\alpha,\beta}(x,0)$ is strictly increasing in $[0,+\infty[$. For $|\alpha| < 2$, $\Omega_{\alpha,\beta} = \mathbf{R}^2$ and, for each $(x_0,y_0) \in \mathbf{R}^2$, $W_{\alpha,\beta}(x,y) = W_{\alpha,\beta}(x_0,y_0)$ is a closed curve. For $|\alpha| \geq 2$, such curves show the following aspect:

$\alpha \geq 2$



$\alpha \leq -2$



The equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1.1}$$

is equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x). \end{cases} \tag{1.2}$$

The condition a) ensures existence and uniqueness of solution for (1.2). The condition b) ensures that $(0,0)$ is the only point of equilibrium for system (1.2). It can be immediately verified that the derivative of $W_{\alpha,\beta}$ relative to the system (1.2) is:

$$\dot{W}_{\alpha,\beta}(x,y) = \frac{\alpha H'_\beta(x) - f(x)}{y^2 + \alpha y H_\beta(x) + H^2_\beta(x)} y^2 \tag{1.3}$$

where $H'_\beta = \frac{dH_\beta}{dx}$.

We observe that $\dot{W}_{\alpha,\beta}(x,y)$ has the same sign of $\alpha H'_\beta(x) - f(x)$, for $y \neq 0$.

Lemma 1. Assume there are $\beta > 0$ and $b > 0$ such that, for all $x \geq b$,

$$f(x) \geq 2H'_\beta(x).$$

Let $y_0 > 0$, $L = W_{2,\beta}(b, y_0)$ and

$$K = \{(x,y) \in \Omega_{2,\beta} \mid x \geq b \text{ and } W_{2,\beta}(x,y) \leq L\}.$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (1.2) so that $\gamma(t_0) = (b, y_1)$, with $0 < y_1 < y_0$. Then, there is $t_1 > t_0$ such that

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_1,$$

and

$$\gamma(t_1) = (b, y_2)$$

with

$$-H_\beta(b) < y_2 < 0.$$

Proof.

From $\dot{x}(t_0) = y_1 > 0$ it follows there is $t_2 > t_0$ so that

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_2.$$

On the other hand, being $\dot{x}(t) > 0$ on the half plane $y > 0$, $\dot{x}(t) < 0$ on the half plane $y < 0$, $\dot{y}(t) < 0$ on the positive half axis $0x$ and $(0,0)$ the only point of equilibrium, there must exist $t_3 > t_2$ such that

$$\gamma(t_3) \notin K.$$

Let

$$t_1 = \max\{u > t_0 \mid \gamma(t) \in K, \quad t_0 \leq t \leq u\}.$$

From the hypothesis

$$f(x) \geq 2H'_\beta(x), \quad x \geq b$$

and from (1.3) it follows that

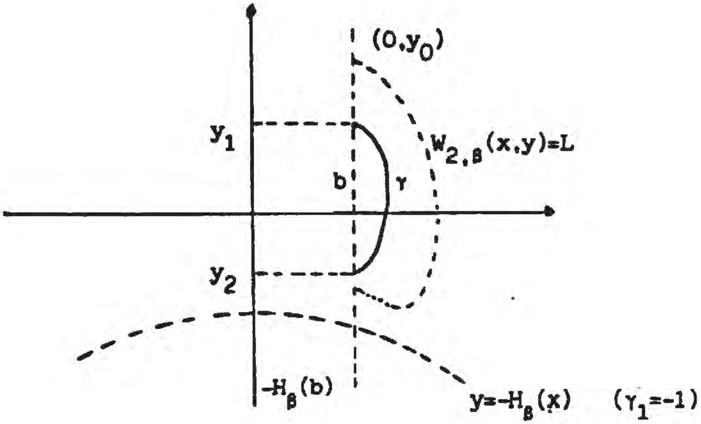
$$\dot{W}_{2,\beta}(\gamma(t)) \leq 0, \quad t_0 \leq t \leq t_1.$$

Because $W_{2,\beta}(\gamma(t_0)) = W_{2,\beta}(b, y_1) < L$, it follows that $W_{2,\beta}(\gamma(t_1)) < L$. So, $\gamma(t_1)$ does not belong to the arc given by

$$x \geq b \qquad \text{and} \qquad W_{2,\beta}(x,y) = L.$$

Because $\dot{x}(t) > 0$ on the $y > 0$ half plane, it follows that

$$\gamma(t_1) = (b, y_2), \qquad \text{with} \qquad -H_\beta(b) < y_2 < 0$$



In a similar way, we can demontrate the following lemmas:

Lemma 2. Assume there are $\beta > 0$ and $a < 0$ such that, for all $x \leq a$,

$$f(x) \geq -2H'_\beta(x).$$

Let $y_0 < 0$, $L = W_{-2,\beta}(a, y_0)$ and

$$K = \{(x, y) \in \Omega_{-2,\beta} \mid x \leq a \text{ and } W_{-2,\beta}(x, y) \leq L\}.$$

Also let $\gamma(t) = (x(t), y(t))$ be the solution of (1.2) such that $\gamma(t_0) = (a, y_1)$ with $y_0 < y_1 < 0$. Then there is $t_1 > t_0$ so that

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_1$$

and

$$\gamma(t_1) = (a, y_2) \quad \text{with} \quad 0 < y_2 < -H'_\beta(a).$$

Lemma 3. Assume there exist $\beta_1 > 0$, $\beta_2 > 0$, $-2 < \alpha_1 \le 0$, $0 \le \alpha_2 < 2$ and $b > 0$ such that, for all $x \ge b$,

$$\alpha_1 H'_{\beta_1}(x) \le f(x) \le \alpha_2 H'_{\beta_2}(x).$$

Let $y_0 > 0$, $L_1 = W_{\alpha_1,\beta_1}(b, y_0)$, $L_2 = W_{\alpha_2,\beta_2}(b, y_0)$ and

$$K = \{(x,y) \in \mathbf{R}^2 \mid x \ge b,\ W_{\alpha_1,\beta_1}(x,y) \le L_1 \text{ and } W_{\alpha_2,\beta_2}(x,y) \ge L_2\}.$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (1.2) so that $\gamma(t_0) = (b, y_0)$; then, there is $t_1 > t_0$ such that

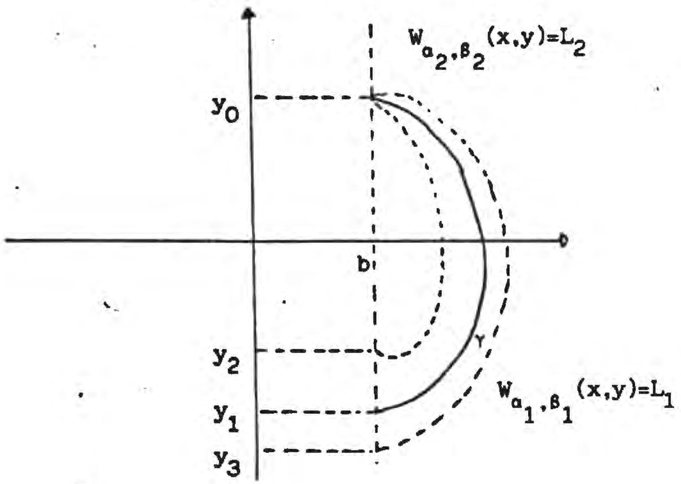
$$\gamma(t) \in K, \qquad t_0 \le t \le t_1$$

and

$$\gamma(t_1) = (b, y_1)$$

with $y_3 < y_1 < y_2 < 0$, where

$$W_{\alpha_1,\beta_1}(b, y_3) = L_1 \quad \text{and} \quad W_{\alpha_2,\beta_2}(b, y_2) = L_2$$



Lemma 4. Assume there are $\beta_1 > 0$, $\beta_2 > 0$, $0 \le \alpha_1 < 2$, $-2 < \alpha_2 \le 0$ and $a < 0$

such that for all $x \leq a$,

$$\alpha_1 H'_{\beta_1}(x) \leq f(x) \leq \alpha_2 H'_{\beta_2}(x).$$

Let $y_0 < 0$, $L_1 = W_{\alpha_1,\beta_1}(a, y_0)$, $L_2 = W_{\alpha_2,\beta_2}(a, y_0)$ and

$$K = \{(x, y) \in \mathbf{R}^2 \mid x \leq a,\ W_{\alpha_1,\beta_1}(x, y) \geq L_1 \text{ and } W_{\alpha_2,\beta_2}(x, y) \leq L_2\}.$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (1.2) so that $\gamma(t_0) = (a, y_0)$. Then, there is $t_1 > t_0$ such that

$$\gamma(t) \in K, \qquad t_0 \leq t \leq t_1$$

and

$$\gamma(t_1) = (a, y_1)$$

with $0 < y_2 < y_1 < y_3$, where

$$W_{\alpha_1,\beta_1}(a, y_3) = L_1 \quad \text{and} \quad W_{\alpha_2,\beta_2}(a, y_2) = L_2.$$

2. The Equation $\ddot{x} + \alpha g(x)[H_\beta(x)]^{-1}\dot{x} + g(x) = 0$

Theorem 1. Consider the equation

$$\ddot{x} + \alpha g(x)[H_\beta(x)]^{-1}\dot{x} + g(x) = 0$$

where α and β are real, with $\beta > 0$, and

$$H_\beta(x) = [2\int_0^x g(t)dt + \beta]^{1/2}.$$

Assume that $xg(x) > 0$ for $x \neq 0$ and $\int_0^{\pm\infty} g(x)dx = +\infty$.

In these conditions, we have:

i) if $|\alpha| < 2$ then all solutions will be periodic;

ii) if $|\alpha| \geq 2$ then any solution passing by $(x_0, y_0) \in \Omega_{\alpha,\beta}$ will be periodic. Besides, any solution passing by $(x_0, y_0) \notin \Omega_{\alpha,\beta}$ will be non-periodic.

Proof.

It is enough to observe that

$$H'_\beta(x) = g(x)[2\int_0^x g(t)d(t) + \beta]^{-1/2}$$

and

$$\dot{W}_{\alpha,\beta}(x,y) = 0, \qquad (x,y) \in \Omega_{\alpha,\beta},$$

where $\dot{W}_{\alpha,\beta}$ is the derivative of $W_{\alpha,\beta}$ relative to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\alpha H'_\beta(x)y - g(x). \end{cases}$$

(Note that $g(x) = H_\beta(x)H'_\beta(x)$.) ■

Remark 1. In [5], the authors establish sufficient conditions for existence of center (global or local) for system of the type

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = g(x) \end{cases}$$

which is equivalent to the equation

$$\ddot{x} + F'(x)\dot{x} + g(x) = 0.$$

In that work, a fundamental role is played by the hypothesis

$$F(G^{-1}(-w)) = F(G^{-1}(w)), \quad w \geq 0$$

where $G(x) = \int_0^x |g(t)|dt$. The theorem we just proved establishes the existence of center (global if $|\alpha| < 2$) for a special class of system not using that hypothesis.

Remark 2. If the hypothesis $\int_0^{\pm\infty} g(x)dx = +\infty$ is not satisfied we can only assert that

$$W_{\alpha,\beta}(x,y) = c, \qquad (x,y) \in \Omega_{\alpha,\beta},$$

will be closed for small $c > 0$; that is, the origin will be a local center.

3. A Sufficient Condition for Oscillating Solutions

In this section we assume that all solutions of equation (1.1) and system (1.2) are continuable in the future.

A solution $x(t)$ of equation (1.1) is oscillating if there exists a sequence $(t_n)_{n\geq 1}$ tending monotonically to $+\infty$ such that $x(t_n) = 0$ for $n \geq 1$.

Theorem 2. Assume that

1) $xg(x) > 0$ for $x \neq 0$ and $\int_0^{\pm\infty} g(x)dx = +\infty$;

2) there are $r > 0$, $\beta_1 > 0$; $\beta_2 > 0$, $0 \leq \alpha_1 < 2$; and $-2 < \alpha_2 \leq 0$ such that

$$f(x) > \alpha_1 H'_{\beta_1}(x) \quad \text{for} \quad x \leq -r$$

and

$$f(x) > \alpha_2 H'_{\beta_2}(x) \quad \text{for} \quad x \geq r;$$

3) there are $\alpha_3 > 0$, $\beta_3 > 0$ and $r_1 > 0$ such that

$$f(x) < \alpha_3 |H'_{\beta_3}(x)| \quad \text{for} \quad 0 < |x| < r_1.$$

Then, any non-trivial solution of the equation (1.1) will be oscillating.

Proof.

The hypothesis $f(x) > \alpha_2 H'_{\beta_2}(x)$ for $x \geq r$, ensures (Lemma 4) that the solution starting at the point (r, y_0), with $y_0 > 0$, cannot leave the set

$$\{(x,y) \in \mathbb{R}^2 \mid W_{\alpha_2,\beta_2}(x,y) \leq W_{\alpha_2,\beta_2}(r,y_0),\ x \geq r\}$$

through the arc

$$W_{\alpha_2,\beta_2}(x,y) = W_{\alpha_2,\beta_2}(r,y_0), \quad x \geq r.$$

For the same reasons, the hypothesis

$$f(x) > \alpha_1 H'_{\beta_1}(x) \quad \text{for} \quad x \leq -r$$

ensures that the solution starting at $(-r, y_1)$, with $y_1 < 0$, cannot leave the set

$$\{(x,y) \in \mathbb{R}^2 \mid W_{\alpha_1,\beta_1}(x,y) \leq W_{\alpha_1,\beta_1}(-r,y_1), \quad x \leq -r\}$$

through the arc

$$W_{\alpha_1,\beta_1}(x,y) = W_{\alpha_1,\beta_1}(-r,y_1), \quad x \leq -r.$$

The hypothesis

$$f(x) < \alpha_3 H'_{\beta_3}(x) \quad \text{for} \quad 0 < x < r,$$

ensures that the solution starting at $(2x_1, 0)$, $0 < 2x_1 < r_1$, cannot enter the set

$$\{(x,y) \in \mathbb{R}^2 \mid W_{\alpha_3,\beta_3}(x,y) \leq W_{\alpha_3,\beta_3}(x_1,0), \quad x \geq 0 \text{ and } y \leq 0\}$$

through the arc

$$W_{\alpha_3,\beta_3}(x,y) = W_{\alpha_3,\beta_3}(x_1,0), \quad x \geq 0 \text{ and } y \leq 0.$$

For the same reason, the hypothesis

$$f(x) < -\alpha_3 H'_{\beta_3}(x), \quad -r_1 < x < 0,$$

ensures that the solution starting at $(2x_2, 0)$, $-r_1 < 2x_2 < 0$, cannot enter the set

$$\{(x,y) \in \mathbb{R}^2 \mid W_{-\alpha_3,\beta_3}(x,y) \leq W_{-\alpha_3,\beta_3}(x_2,0), \quad x \leq 0 \text{ and } y \geq 0\}$$

through the arc

$$W_{-\alpha_3,\beta_3}(x,y) = W_{-\alpha_3,\beta_3}(x_2,0), \quad x \leq 0 \text{ and } y \geq 0.$$

Due to the fact that no solution of

$$\frac{dy}{dx} = -f(x) - \frac{g(x)}{y}$$

can admit vertical asymptotic (see [1]), it follows that any non-trivial solution is oscillating.

■

Remark 3. Note that our hypotheses allow $f(x) < 0$ for all x and, also,

$$\lim_{x \to +\infty} F(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} F(x) = +\infty$$

where $F(x) = \int_0^x f(t)dt$. In [3], Villari establishes a necessary and sufficient condition for the solution to be oscillating, working in the class of functions F satisfying the conditions: there is $c > 0$ such that

$$F(x) > -c \quad \text{for} \quad x > 0$$

and

$$F(x) < c \quad \text{for} \quad x < 0.$$

Remark 4. The hypothesis 3) can be replaced by: there are $\alpha_3 > 0$ and $r_1 > 0$ such that

$$f(x) < \alpha_3 |g(x)| \quad \text{for} \quad 0 < |x| < r_1.$$

In this case, we make use of the definite positive function

$$V_{\alpha_3}(x, y) = \int_0^y \frac{s}{\alpha_3 s + 1} ds + \int_0^x g(t)dt \quad \text{(see [1])}.$$

Theorem 3. Suppose that

1) $xg(x) > 0$ for $x \neq 0$ and $\int_0^{\pm\infty} g(x)dx = +\infty$;

2) there are α and $\beta > 0$, with $|\alpha| < 2$, such that, for $x \neq 0$,

$$f(x) > \alpha H'_\beta(x);$$

3) there are $\alpha_1 > 0$, $\beta_1 > 0$ and $r_1 > 0$ such that

$$f(x) < \alpha_1 |H'_{\beta_1}(x)|, \quad 0 < |x| < r_1.$$

Then any non-trivial solution of the equation (1.1) will be oscillating and approaches the origin when $t \to +\infty$.

Proof.

It is enough to observe that the origin is the largest invariant set included in

$$E = \{(x,y) \in \mathbb{R}^2 \mid \dot{W}_{\alpha,\beta}(x,y) = 0\}$$

and apply the La Salle Theorem (see [5]) to get to the conclusion that any solution approaches the origin when $t \to +\infty$. Proceeding as in theorem 2 demonstration, we conclude, based on hypothesis 3, that all solution starting at $(x_0, 0)$, $x_0 > 0$, crosses the negative y half-axis and that all solution starting at $(x_1, 0)$, $x_1 < 0$, crosses the positive y half-axis. Therefore, all non-trivial solution will be oscillating. ■

Remark 5. The hypothesis 3) can be replaced by: there are $\alpha_1 > 0$ and $r_1 > 0$ such that

$$f(x) < \alpha_1 |g(x)| \quad \text{for} \quad 0 < |x| < r_1.$$

Remark 6. If the hypothesis $|\alpha| < 2$ in 2) is replaced by $|\alpha| \geq 2$ we can assert only that any non-trivial solution passing by $(x_0, y_0) \in \Omega_{\alpha,\beta}$ is oscillating and approaches the origin when $t \to +\infty$.

4. A Sufficient Condition for Existence of Periodic Solution

Theorem 4. Suppose that

1) $xg(x) > 0$ for $x \neq 0$ and $\int_0^{\pm\infty} g(x)dx = +\infty$

2) the origin is repulsive

3) there are $b > 0$ and $\beta_1 > 0$ such that

$$f(x) \geq 2H'_{\beta_1}(x) \quad \text{for} \quad x \geq b$$

4) there are $a < 0$, $\beta_2 > 0$ and $0 < \alpha < 2$ such that

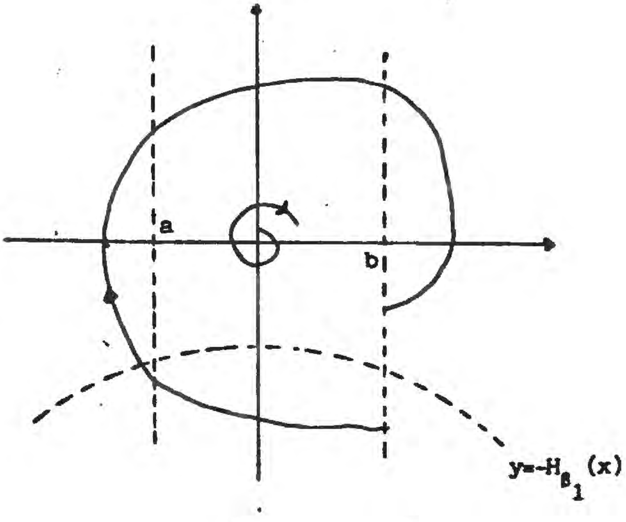
$$f(x) > \alpha H'_{\beta_2}(x) \quad \text{for} \quad x \leq a.$$

Then the equation (1.1) admits at least one non-trivial periodic solution.

Proof.

Theorem 2 ensures that any non-trivial solution of the equation (1.1) is oscillating.

The hypothesis 3) ensures that the solution starting at (b, y_0), with $y_0 > 0$, crosses again the straight line $x = b$ at a point (b, y_1), with $-H_{\beta_1}(b) < y_1 < 0$ (Lemma 1).



From the theorem of Poincaré-Bendixson, the equation will admit at least one periodic solution. ■

Remark 7. The hypothesis 3) and 4) can be weakened. Let $y = y(x)$ be the solution of

$$\frac{dy}{dx} = -f(x) - \frac{g(x)}{y}$$

passing by $(b, -H_{\beta_1}(b))$. Assume that such solution crosses the straight line $x = a$ at the point (a, y_2) with $y_2 < -H_{\beta_1}(b)$. Then there will exist $a < x_0 < b$ such that $y(x) \leq -H_{\beta_1}(b)$ for $a \leq x \leq x_0$. The mean-value theorem ensures that

$$-H_{\beta_1}(b) - y_2 \leq (A + \frac{B}{H_{\beta_1}(b)})(x_0 - a)$$

and, therefore,

$$y_2 \geq -[(A + \frac{B}{H_{\beta_1}(b)})(b - a) + H_{\beta_1}(b)]$$

where $A = \max_{a\leq x\leq b}|f(x)|$ and $B = \max_{a\leq x\leq b}|g(x)|$.

Let $y_3 = -[(A + \frac{B}{H_{\beta_1}(b)})(b-a) + H_{\beta_1}(b)]$, $a_1 < a$ and $y_4 > 0$ such that

$$W_{\alpha,\beta_2}(a, y_3) = W_{\alpha,\beta_2}(a_1, 0) = W_{\alpha,\beta_2}(a, y_4).$$

Also let $y_5 = (A + \frac{B}{y_4})(b-a) + y_4$, and $b_1 > b$ such that

$$W_{2,\beta_1}(b, y_5) = W_{2,\beta_1}(b_1, 0).$$

Then the hypotheses 3) and 4) can be replaced respectively by

3_1) $f(x) \geq 2H'_{\beta_1}(x)$ for $b \leq x \leq b_1$

4_1) $f(x) > 2H'_{\beta_2}(x)$ for $a_1 \leq x \leq a$.

Note that $f(x)$ can have any behavior for a large $|x|$!

Remark 8.

The hypotheses 3) and 4) can be replaced respectively by

3_2) there are $b < 0$ and $\beta_1 > 0$ such that

$$f(x) \geq -2H'_{\beta_1}(x) \quad \text{for} \quad x < b;$$

4_2) there are $a > 0$, $\beta_2 > 0$ and $-2 < \alpha < 0$ such that

$$f(x) > \alpha H'_{\beta_2}(x) \quad \text{for} \quad x > a.$$

These hypotheses can be weakened the same way we did in remark 8.

5. On Periodic Solutions for the Case where f and g are Odd Functions

Lemma 5. Assume that

1) $g(x) > 0$ for $x > 0$ and $\int_0^{+\infty} g(x)dx = +\infty$

2) there are $\beta > 0$, $\beta_1 > 0$, $0 < \alpha < 2$, $-2 < \alpha_1 < 0$ and $r > 0$ such that, for all $x > r$,

$$\alpha_1 H'_{\beta_1}(x) < f(x) < \alpha H'_{\beta}(x).$$

Then, for any $y_1 < 0$, there is $y_0 > 0$ such that the solution of the system (1.2) starting at $(0, y_0)$ passes by $(0, y_1)$.

Proof.

The hypothesis 2) ensures that for any $y_3 < 0$ there is $y_2 > 0$ such that the solution starting at (r, y_2) crosses again the straight line $x = r$ at (r, y_3). On the other hand, there is $y_4 < 0$ such that the solution starting at (r, y_4) crosses the y axis at a point $(0, y_5)$, with $y_5 < 0$. Then, for any $y_1 < 0$ there is $y_0 > 0$ such that the solution starting at $(0, y_0)$ passes by $(0, y_1)$. ∎

Theorem 5. Assume that

1) f and g are odd;

2) $g(x) > 0$ for $x > 0$ and $\int_0^{+\infty} g(x)dx = +\infty$;

3) there are $\beta > 0$, $\beta_1 > 0$, $0 < \alpha < 2$, $-2 < \alpha_1 < 0$, and $r > 0$ such that

$$\alpha_1 H'_{\beta_1}(x) < f(x) < \alpha H'_{\beta}(x) \quad \text{for} \quad x \geq r.$$

Then there is $s > 0$ such that any solutions of (1.2) passing by $(0, y_0)$, with $y_0 > s$, is periodic; besides, any solution passing by $(0, y_1)$ with $y_1 < 0$, is periodic.

Proof.

It is enough to note that the trajectories of the system (1.2) passing by (a, b) and $(-a, b)$, for any a and b, are simmetrical relative to the y axis and apply the previous lemma. ■

Remark 9. If hypothesis:

4) there are $\alpha_2 > 0$ and $s > 0$ such that $f(x) < \alpha_2 g(x)$, $\quad 0 < x < s$

is also verified, then any non-trivial solution is periodic. (See demonstration of the theorem 2 and remark 4).

Theorem 6. Assume that

1) $g(x) > 0$ for $x > 0$ and $\int_0^{+\infty} g(x)dx = +\infty$;
2) g is odd;
3) there are $\alpha_1, \alpha_2, \beta_1 > 0$, $\beta_2 > 0$ and $r > 0$, with $-2 < \alpha_1 < 0$ and $0 < \alpha_2 < 2$, such that

$$\alpha_1 H'_{\beta_1}(x) < f(x) < \alpha_2 H'_{\beta_2}(x), \quad x \geq r;$$

4) $f(0) = 0$;
5) there is $b < 0$ such that

$$f(x) > -f(-x), \quad b < x < 0.$$

Then, any solution of (1.2) passing by $(x_0, 0)$ with $b < x_0 < 0$, will be non-periodic.

Proof.

Consider the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f_1(x)y - g(x) \end{cases} \tag{5.1}$$

where

$$f_1(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(-x) & \text{if } x < 0. \end{cases}$$

The function f_1 is evidently odd and continuous.

Due to the previous theorem, any solution of (5.1) passing by $(0, y_0)$ with $y_0 < 0$, is periodic.

Consider the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \tag{5.2}$$

and consider the point $(x_0, 0)$, with $b < x_0 < 0$. Assume there is $y_1 < 0$ such that the solution of (5.2) passing by $(0, y_1)$ passes also by $(x_0, 0)$. (If such y_1 did not exist, the solution passing by $(x_0, 0)$ could not be periodic). Due to the lemma, there is $y_2 > 0$ such that the solution of (5.2) passing by $(0, y_2)$ passes also by $(0, y_1)$. The solution of (5.1) that passes by $(0, y_1)$ passes also by $(0, y_2)$ and is periodic. Hypothesis 5) ensures that the solution of (5.2) starting at $(0, y_1)$ will cross the y positive half-axis at a point $(0, y_3)$, with $0 < y_3 < y_2$ (see [1]). Then, the solution of (5.2) passing by $(x_0, 0)$ is not periodic. ■

Remark 10. Following the same reasoning, we can prove that the theorem remains valid if hypothesis 5) is replaced by:

5_1) there is $b < 0$ such that $f(x) < -f(-x)$ for $b < x < 0$.

Remark 11. If hypothesis 5) is replaced by

$$f(x) > -f(-x), \quad x < 0,$$

or by

$$f(x) < -f(-x), \quad x < 0,$$

then any non-trivial solution will be non-periodic.

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