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Coherent states of finite-level systems

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Abstract A method for constructing coherent states (CS) of finite-level systems with a given angular momentum is proposed. To this end we generalize the known spin equation (SE) to an infinite-dimensional Fock space. The equation describes a special quadratic system in the latter space. Its projections on d-dimensional subspaces represent analogs of SE for d-dimensional systems in an external electromagnetic field which describe d-dimensional systems with a given angular moment. Using a modification of the Malkin-Manko method developed in our earlier work, we construct the corresponding CS for the total quadratic system. Projections of the later CS on finite-dimensional subspaces we call angular moment CS (AMCS) of finite-level systems. The AMCS have a clear physical meaning; they obey the Schrödinger equation for a d-dimensional system with a given angular moment $j = (d-1)/2$ in an external electromagnetic field. Their possible exact solutions are constructed via exact solutions of the SE in 2-dimensional space. The latter solutions can be found analytically and are completely described in our earlier works. One subset of AMCS can be related to Perelomov spinning CS (PSCS). This reflects the fact that the set of possible AMCS is wider than the set of PSCS. AMCS states in a constant magnetic field are constructed.

1 Introduction

Coherent states (CS) play an important role in modern quantum theory as states that provide a natural relation between quantum mechanical and classical or semi-classical descriptions. They have a number of useful properties and, as a consequence, a wide range of applications, e.g., in radiation theory, in quantization theory, in condensed matter physics, in quantum computations, and so on, see, e.g., Refs. [1–5].

Initially, CS were constructed for non-relativistic quantum systems such as oscillatory or quadratic systems in infinite-dimensional Hilbert spaces [6–10]. Such CS turned out to be orbits of the Heisenberg-Weyl group. That observation allowed one to formulate by analogy some general definition of CS for any Lie group [11–13] as orbits of the group factorized with respect to a stationary subgroup. In particular CS of the $SU(N)$ and $SU(N, 1)$ group were constructed in Refs. [14–16].

In the present article, we consider the problem of constructing CS of finite-level systems. Finite-level systems have always played an important role in quantum physics. For example, in the semiclassical theory of laser beams, in optical resonance, in problems of interaction of an assembly of two-level atoms with a transverse electromagnetic field, and so on [17–19]. 2- and 4-level systems have attract even more attention, due to their relationship to problems of quantum computations, see e.g. Refs. [20–24]. In particular, many problems in quantum physics which can be dealt with in terms of the 2-level systems were studied by many authors using different methods, see, for example, [25, 26]. In this problem, the computation is performed by the manipulation of the so-called one and 2-qubit gates [27].

In the description of an assembly of two-level atoms, atomic coherent states are defined which have properties analogous to those of the field coherent states [28]. In fact, to each property of the atomic coherent states there exists a corresponding property of the field coherent states. The Dicke states [29], that have been used in the study of superradiance, they are a close relationship to the Fock states of the free-field problem and by rotating this states through an angle (θ, ϕ) one obtain the atomic coherent states (or Bloch states).

The article is organized as follows. In Secs. 2.1 and 2.2 we discuss states of 2- and d-dimensional systems with the help of angular momentum operators that are acting in an infinite dimensional Fock space. In Sec. 3 we construct a generating spin equation (GSE) in the infinite dimensional Fock space generalizing a spin equation (SE) for a 2-dimensional system in an external electromagnetic

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field. By projecting the GSE on d-dimensional subspaces, we obtain an analog of SE for d-dimensional systems in an external electromagnetic field. The Hamiltonian of GSE turns out to be quadratic in some Bose creation and annihilation operators. This fact allows us to construct in Sec. 4 generalized CS using Malkin-Manko integral of motion method [30] and its development presented in Refs. [31]. Then, we study projections of the constructed CS on the d-dimensional subspaces. We demonstrate that a subset of these projections can be related to know spin CS of $SU(2)$ group derived by Perelomov, see e.g. Ref. [11]. The important case of 2-level system in a constant magnetic field is considered in detail in Sec. 5. In the last Sect. 6, we summarize and discuss the main results. Some auxiliary mathematical formulas are placed in Appendices.

2 Finite-level systems

2.1 2-level systems. Spin equation

We recall that states of a 2-level quantum system are two columns $|\boldsymbol{\Psi}\rangle^{(2)}$,

$$|\boldsymbol{\Psi}\rangle^{(2)} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{1}$$

The Hilbert space describing such a system is a complex 2-dimensional space $\mathcal{R}^{(2)} = \mathbb{C}^2$. As well known physical examples of 2-level systems are spin-1/2 particle, and polarization states of transversal photons.

A scalar product of two vectors $\left|\boldsymbol{\Psi}'\right\rangle^{(2)}$ and $|\boldsymbol{\Psi}\rangle^{(2)}$ in the space $\mathcal{R}^{(2)}$ reads: ${}^{(2)}\langle\boldsymbol{\Psi}'|\boldsymbol{\Psi}\rangle^{(2)} = \psi_1'^*\psi_1 + \psi_2'^*\psi_2$. An orthogonal basis $|a\rangle$, $(a = 1, 2)$ in $\mathcal{R}^{(2)}$ can be chosen as:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \langle a|a'\rangle = \delta_{aa'},\ \sum_{a=1}^{2} |a\rangle\langle a| = I_{2\times 2} , \tag{2}$$

where $I_{2\times 2}$ is 2×2 unit matrix.

A quantum dynamic of a 2-level system in a time-dependent background, given by a real 3-dimensional vector $\mathbf{F}(t) = (F_1(t),\ F_2(t),\ F_3(t))$, is described by the Schrödinger equation of the form[1]:

$$i\partial_t|\boldsymbol{\Psi}(t)\rangle^{(2)} = H^{(2)}(t)|\boldsymbol{\Psi}(t)\rangle^{(2)},\ H^{(2)}(t) = \mathbf{sF}(t) , \tag{3}$$

were $\mathbf{s} = (s_1,\ s_2,\ s_3)$ is the 1/2 angular momentum operator,

$$\left[s_i, s_j\right] = i\varepsilon_{ijk}s_k,\ \mathbf{s} = \frac{\boldsymbol{\sigma}}{2},\ \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) , \tag{4}$$

where σ_i are Pauli matrices[2] and ε_{ijk} is the fully antisymmetric tensor with the normalization $\varepsilon_{123} = 1$. The operators s_i implement a representation of the Lie algebra $su(2)$ in the space $\mathcal{R}^{(2)}$. Thus, the Hamiltonian $H^{(2)}(t)$ is given by the 2×2 Hermitian matrix:

$$H^{(2)}_{\alpha\beta}(t) = \frac{1}{2}\begin{pmatrix} F_3(t) & F_1(t) - iF_2(t) \\ F_1(t) + iF_2(t) & -F_3(t) \end{pmatrix},\ H^{(2)}(t) = H^{(2)}(t)^{\dagger} . \tag{7}$$

In particular, for a frozen in the space nonrelativistic electron of the mass m_e and of the charge $-e$ $(e > 0)$, (and spin 1/2) interacting with an external magnetic field $\mathbf{B}(t) = (B_1(t),\ B_2(t),\ B_3(t))$, the vector $\mathbf{F}(t)$ and the corresponding Hamiltonian $H^{(2)}(t)$ reads:

$$\mathbf{F}(t) = \frac{e}{m_e}\mathbf{B}(t),\ H^{(2)}(t) = \frac{e}{m_e}\mathbf{sB}(t) . \tag{8}$$

For this reason, equation (3) is often called the SE. Its possible exact solutions were studied in detail in Ref. [26].

2.2 d-level systems

States of a d-level quantum system are d-component columns $|\boldsymbol{\Psi}\rangle^{(d)}$,

$$|\boldsymbol{\Psi}\rangle^{(d)} = \left(\psi_1^{(d)}\ \psi_2^{(d)} \cdots \psi_d^{(d)}\right)^T,\ d \in . \tag{5}$$

The Hilbert space describing such a system is the complex d-dimensional space $\mathcal{R}^{(d)} = \mathbb{C}^d$.

[1] Throughout the text $\hbar = c = 1$, Latin indices are $i,\ j,\ k,\ ... = 1,\ 2,\ 3$ and Greek indices are $\alpha,\ \beta,\ \eta,\ ... = 0,\ 1$. Unless otherwise indicated, summation over repeated indices is implied.

[2]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

Below, we consider a specific basis in d-dimensional space $\mathcal{R}^{(d)}$ and a possibility to construct an analog of SE (3) for d-level systems. To this end, it is convenient to consider an infinite-dimensional Fock space $\mathcal{R}^{(\infty)}$ (with vectors $|\Psi\rangle \in \mathcal{R}^{(\infty)}$) generated by two kinds of Bose annihilation and creation operators, $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$, $\alpha = 1, 2$,

$$\left[\hat{a}_\alpha, \hat{a}_\beta\right] = \left[\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger\right] = 0,\ \left[\hat{a}_\alpha, \hat{a}_\beta^\dagger\right] = \delta_{\alpha\beta},\ \alpha, \beta = 1, 2\ . \tag{9}$$

An occupation number basis $|n_1, n_2\rangle$, $n_1, n_2 \in \mathbb{N}$, in the space $\mathcal{R}^{(\infty)}$, is defined as:

$$\begin{aligned}
&|n_1, n_2\rangle = \frac{\left(\hat{a}_1^\dagger\right)^{n_1}\left(\hat{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0,0\rangle,\ \hat{a}_\alpha|0,0\rangle\ = 0,\ \alpha = 1,2\ ,\\
&\hat{a}_1|n_1,n_2\rangle = \sqrt{n_1}|n_1 - 1, n_2\rangle,\ \ \hat{a}_2|n_1,n_2\rangle = \sqrt{n_2}|n_1, n_2 - 1\rangle\ ,\\
&\hat{a}_1^\dagger|n_1,n_2\rangle = \sqrt{n_1+1}|n_1+1,n_2\rangle,\ \ \hat{a}_2^\dagger|n_1,n_2\rangle = \sqrt{n_2+1}|n_1,n_2+1\rangle\ ,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\hat{n}_\alpha|n_1,n_2\rangle = n_\alpha|n_1,n_2\rangle, \hat{n}_\alpha = \hat{a}_\alpha^\dagger\hat{a}_\alpha,\ \alpha = 1,2\ ;\\
&\langle m_2, m_1|n_1,n_2\rangle = \delta_{m_1n_1}\delta_{m_2n_2},\ \sum\lim_{n_1,n_2=0}^{\infty}|n_1,n_2\rangle\langle n_2,n_1| = 1\ ,
\end{aligned} \tag{11}$$

where $|0, 0\rangle$ is the vacuum vector in the space $\mathcal{R}^{(\infty)}$.

One can introduce angular momentum operators $\hat{\mathbf{S}} = \left(\hat{S}_1, \hat{S}_2, \hat{S}_3\right)$ acting in the space $\mathcal{R}^{(\infty)}$,

$$\hat{\mathbf{S}} = \hat{a}_\alpha^\dagger \mathbf{s}^{\alpha\beta}\hat{a}_\beta,\ \hat{S}_i = \frac{1}{2}\hat{a}_\alpha^\dagger\sigma_i^{\alpha\beta}\hat{a}_\beta\ , \tag{12}$$

where $\mathbf{s}$ are given by Eq. (4), see e.g. [32, 33]. These operators satisfy the following relations:

$$\begin{aligned}
&\left[\hat{S}_i, \hat{S}_j\right] = i\varepsilon_{ijk}\hat{S}_k,\ \left[\hat{S}_3, \hat{S}_\pm\right] = \pm\hat{S}_\pm,\ \left[\hat{S}_+, \hat{S}_-\right] = 2\hat{S}_3,\ \left[\hat{\mathbf{S}}^2, \hat{S}_3\right] = 0\ ,\\
&\hat{S}_+ = \hat{S}_1 + i\hat{S}_2 = \hat{a}_1^\dagger\hat{a}_2,\ \hat{S}_- = \hat{S}_1 - i\hat{S}_2 = \hat{a}_2^\dagger\hat{a}_1\ ,\\
&\hat{S}_3 = \frac{1}{2}\left(\hat{S}_+\hat{S}_- - \hat{S}_-\hat{S}_+\right) = \frac{1}{2}\left(\hat{n}_1 - \hat{n}_2\right),\ \hat{\mathbf{S}}^2 = \frac{\hat{N}}{2}\left(\frac{\hat{N}}{2} + 1\right),\\
&\hat{N} = \sum_\alpha \hat{a}_\alpha^\dagger\hat{a}_\alpha = \hat{n}_1 + \hat{n}_2\ ,\\
&\hat{S}_+|n_1,n_2\rangle = \sqrt{n_2(n_1+1)}|n_1+1,n_2-1\rangle\ ,\\
&\hat{S}_-|n_1,n_2\rangle = \sqrt{n_1(n_2+1)}|n_1-1,n_2+1\rangle\ .
\end{aligned} \tag{13}$$

Eigenvectors of the commuting operators $\hat{\mathbf{S}}^2$ and $\hat{S}_3$ we denote as $\overline{|s, m\rangle}$,

$$\begin{aligned}
&\hat{\mathbf{S}}^2\overline{|s,m\rangle}\ = s(s+1)\overline{|s,m\rangle},\ s = 0, 1/2, 1, 3/2, \cdots,\ \ \hat{N}\overline{|s,m\rangle}\ = 2s\overline{|s,m\rangle}\ ,\\
&\hat{S}_3\overline{|s,m\rangle}\ = m\overline{|s,m\rangle}, m = -s,\ -s+1,\ \cdots, s\ ,\\
&\hat{S}_+\ \overline{|s,m\rangle} = \sqrt{(s-m)(s+m+1)}\overline{|s,m\rangle},\\
&\hat{S}_-\overline{|s,m\rangle}\ = \sqrt{(s+m)(s-m+1)}\overline{|s,m\rangle}\ .
\end{aligned} \tag{14}$$

There is one-to-one correspondence between the vectors $\overline{|s, m\rangle}$ and the occupation number basis $|n_1, n_2\rangle$,

$$\begin{aligned}
&\overline{|s,m\rangle}\ = |s+m, s-m\rangle,\ |n_1,n_2\rangle = \overline{\left|\frac{n_1+n_2}{2}, \frac{n_1-n_2}{2}\right\rangle},\\
&\overline{|0,0\rangle} = |0,0\rangle\ .
\end{aligned} \tag{15}$$

The space $\mathcal{R}^{(\infty)}$ can be represented as a direct orthogonal sum[3] of the d-dimensional spaces $\mathcal{R}^{(d)}$,

$$\mathcal{R} = \bigoplus_{d=1}^{\infty}\mathcal{R}^{(d)}\ . \tag{16}$$

[3] Let H be a Hilbert space, and let H_1 and H_2 be its orthogonal closed subspaces, $H_1 \perp H_2$. If any vector $\xi \in H$ can be uniquely represented as a sum of two vectors $\xi_1 + \xi_2$, where $\xi_1 \in H_1$, and $\xi_2 \in H_2$, then $H = H_1 \oplus H_2$.

Thus, any vector $|\Psi\rangle \in \mathcal{R}^{(\infty)}$ can be represented as:

$$|\Psi\rangle = \sum_{d=1}^{\infty} |\Psi\rangle^{(d)},$$
$$|\Psi\rangle^{(d)} = P^{(d)}|\Psi\rangle = \sum_{m=-\frac{d-1}{2}}^{m=\frac{d-1}{2}} \psi_{\frac{d+1}{2}-m} \overline{\left|\frac{d-1}{2}, m\right\rangle} \in \mathcal{R}^{(d)}, \tag{17}$$

where ψ_k are some decomposition coefficients. By $P^{(d)}$ in Eq. (17) we denote the operator of projection of vectors from $\mathcal{R}^{(\infty)}$ onto the d-dimensional subspace $\mathcal{R}^{(d)}$. All the states that belong to $\mathcal{R}^{(d)}$ are eigenvectors of the operator $\hat{d} = \hat{N} + 1$,

$$\hat{d}|\Psi\rangle^{(d)} = d|\Psi\rangle^{(d)}, \ \ \hat{d} = \hat{N} + 1 . \tag{18}$$

It is convenient to introduce d-vectors $\overline{|a\rangle}$, $a = 1, \ldots, d$,

$$\overline{|a\rangle} = \overline{\left|\frac{d-1}{2}, \frac{d-1}{2} - a + 1\right\rangle} = |d-a, a-1\rangle . \tag{19}$$

As follows from Eq. (11) these vectors form an orthogonal basis in $\mathcal{R}^{(d)}$,

$$\overline{\langle a|a'\rangle} = \delta_{aa'}, \ \ \sum_{a=1}^{d} \overline{|a\rangle\langle a|} = 1 . \tag{20}$$

Basis (19) specifies $\overline{|a\rangle}$-representation of vectors in $\mathcal{R}^{(d)}$ as follows:

$$|\Psi\rangle^{(d)} = \sum_{a=1}^{d} \psi_a^{(d)} \overline{|a\rangle}, \ \ |\Psi\rangle^{(d)} \Longrightarrow \psi_a^{(d)} = \overline{\langle a|}\Psi\rangle^{(d)} . \tag{21}$$

The components $\psi_a^{(d)}$ form column (8). In the future, when it does not lead to misunderstanding, we will omit the superscript (d) and write $\psi_a^{(d)} = \psi_a$.

3 Angular momentum equations in spaces $\mathcal{R}^{(\infty)}$ and $\mathcal{R}^{(d)}$

Starting from the SE (3), defined for vectors $|\Psi(t)\rangle^{(2)}$ in the 2-dimensional Hilbert space $\mathcal{R}^{(2)}$, we construct the Schrödinger equation for vectors $|\Psi(t)\rangle \in \mathcal{R}^{(\infty)}$, with the Hamiltonian $\hat{H}$,

$$\hat{H} = \hat{\mathbf{S}}F(t) = \hat{a}_\alpha^\dagger \left(H^{(2)}\right)^{\alpha\beta} \hat{a}_\beta , \tag{22}$$

in fact, substituting the spin operators $\mathbf{s}$ in $H^{(2)}$ given by Eq. (3) by the angular momentum operators $\hat{\mathbf{S}}$. In such a way we obtain an analog of the SE (3) in the infinite-dimensional Hilbert space $\mathcal{R}^{(\infty)}$,

$$i\partial_t|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle, \ \ |\Psi(t)\rangle \in \mathcal{R}^{(\infty)} . \tag{23}$$

In what follows, we call equation (23) the angular momentum equation in the infinite-dimensional Hilbert space $\mathcal{R}^{(\infty)}$, or simply the generating spin equation (GSE) in $\mathcal{R}^{(\infty)}$. Obviously, that $\hat{H}$ is a quadratic in annihilation and creation operators, $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$, $\alpha = 1$, 2, defined by Eqs. (9).

Solutions $|\Psi(t)\rangle^{(d)}$ of Eq. (23) that belong to the space $\mathcal{R}^{(d)}$, have to satisfy the equation $\hat{d}|\Psi(t)\rangle^{(d)} = d|\Psi(t)\rangle^{(d)}$ and the corresponding Schrödinger equation, i.e.

$$i\partial_t|\Psi(t)\rangle^{(d)} = \hat{H}|\Psi(t)\rangle^{(d)} . \tag{24}$$

In what follows, we call Eq. (24) the angular momentum equation in $\mathcal{R}^{(d)}$.

Let us find an explicit form of Eq. (24) in representation (21).

It is demonstrated in the Appendix A that the space $\mathcal{R}^{(d)}$ is invariant under the angular momentum operators $\hat{\mathbf{S}}$ entering Hamiltonian (22). A restriction of the operators $\hat{\mathbf{S}}$ on the subspace $\mathcal{R}^{(d)}$ is given by operators $\mathbf{s}^{(d)} = \left(s_1^{(d)}, s_2^{(d)}, s_3^{(d)}\right)$ of the angular momentum $j = (d-1)/2$,

$$\left[s_i^{(d)}, s_j^{(d)}\right] = i\varepsilon_{ijk}s_k^{(d)}, \ \ s_i^{(d)} = \left(\hat{S}_i\right)_{ab}, \ i, j = 1, \ldots, d; \ \ a, b = 1, \ldots, d , \tag{25}$$

where explicit form of the matrices $\left(\hat{S}_i\right)_{ab}$ is presented by Eq. (6).

With help of Eqs. (5) and (6) one can find the action of the operator $\hat{H}$ on the vector $|\Psi\rangle^{(d)}$,

$$\hat{H}|\Psi\rangle^{(d)} = \sum_{a,b=1}^{d} H_{ab}^{(d)}(t)\psi_b(t)\overline{|a\rangle}\ ,\ \hat{d}\ \hat{H}|\Psi\rangle^{(d)} = d\ \hat{H}|\Psi\rangle^{(d)}\ ,$$

$$H_{ab}^{(d)}(t) = \sum_{i=1}^{3} \overline{\langle a|}\hat{S}_i\overline{|b\rangle}F_i(t) = \mathbf{s}^{(d)}\mathbf{F}(t),\ a,b = 1,\ldots,d\ . \tag{26}$$

Then, multiplying Eq. (24) from the left on the bra-vector $\overline{\langle a|} = \langle a-1,\, d-a|$, we obtain angular momentum equation in the $\overline{|a\rangle}$-representation,

$$i\dot{\psi}_a^{(d)}(t) = \sum_{b=1}^{d} H_{ab}^{(d)}(t)\psi_b^{(d)}(t),\ \psi_a^{(d)}(t) = \overline{\langle a|}\Psi\rangle^{(d)},\ a = 1,\ldots,d\ , \tag{27}$$

which reads in more detail as:

$$i\dot{\psi}_1^{(d)}(t) = (d-1)F_0(t)\psi_1^{(d)}(t) + \sqrt{d-1}F_{(-)}(t)\psi_2^{(d)}(t)\ , \tag{28}$$

$$\begin{aligned} i\dot{\psi}_a^{(d)}(t) &= \sqrt{(a-1)(d-a+1)}F_{(+)}(t)\psi_{a-1}^{(d)}(t) \\ &+ (d-2a+1)F_0(t)\psi_a^{(d)}(t) \\ &+ \sqrt{a(d-a)}F_{(-)}(t)\psi_{a+1}^{(d)}(t),\ a = 2,\ldots,d-1\ , \end{aligned} \tag{29}$$

$$\begin{aligned} i\dot{\psi}_d^{(d)}(t) &= -(d-1)F_0(t)\psi_d^{(d)}(t) + \sqrt{d-1}F_{(+)}(t)\psi_{d-1}^{(d)}(t)\ , \\ F_0(t) &= F_3(t)/2,\ \ F_{(\pm)}(t) = [F_1(t) \pm iF_2(t)]/2\ . \end{aligned} \tag{30}$$

Thus, a restriction of the angular momentum equation (23) in $\mathcal{R}^{(\infty)}$ to the subspace $\mathcal{R}^{(d)}$ induced the Schrödinger equation (27) for a d-level system in an external field. Eqs. (28)–(30) are ordinary differential equation in time.

The angular momentum equation (27) generalizes SE (3) to the case of higher dimension $d > 2$. Besides, the dynamic symmetry group of the d-dimensional system described by this equation is the $SU(2)$ group (which holds also true for the SE (3)). This follows from the fact that the corresponding Hamiltonian is linear in the generators $\mathbf{s}^{(d)}$ of the unitary irreducible representation of the Lie algebra $su(2)$ of the group $SU(2)$. It is important that the Hamiltonian (26) commutes with the operator of the square of the angular momentum operator

$$s_{(d)}^2 = \left[s_1^{(d)}\right]^2 + \left[s_2^{(d)}\right]^2 + \left[s_3^{(d)}\right]^2 .$$

Let us find conditions for the functions $F_0(t)$ and $F_{(\pm)}(t)$ that provide the existence of solutions $\psi_a^{(d)}(t)$ in each subspace $\mathcal{R}^{(d)}$.

First, we consider the case when $F_{(-)}(t)$ is not identically zero. In this case, the set of $d-1$ equations (28)–(29) turns out to be identity if the functions $\psi_2^{(d)}(t),\cdots,\ \psi_d^{(d)}(t)$ are expressed via the function $\psi_1^{(d)}(t)$ and its derivatives up to $d-1$ order due to the following recurrent conditions:

$$\psi_2^{(d)}(t) = \frac{1}{F_{(-)}(t)}\left[\frac{i\dot{\psi}_1^{(d)}(t)}{\sqrt{d-1}} - \sqrt{d-1}F_0(t)\psi_1^{(d)}(t)\right],$$

$$\psi_{a+1}^{(d)}(t) = \frac{1}{F_{(-)}(t)}\left[\frac{i\dot{\psi}_a^{(d)}(t)}{\sqrt{a(d-a)}} + \frac{(2a-d-1)}{\sqrt{a(d-a)}}F_0(t)\psi_a^{(d)}(t)\right. \tag{31}$$

$$\left. - \sqrt{\frac{(a-1)(d-a+1)}{a(d-a)}}F_{(+)}(t)\psi_{a-1}^{(d)}(t)\right],\ a = 2,\ldots,d-1\ . \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (30) we obtain a linear ordinary d-order differential equation for the function $\psi_1^{(d)}(t)$ with coefficients that depend on time via the functions $F_0(t)$ and $F_{(\pm)}(t)$ and their derivatives up to $d-1$ order. Thus, infinite differentiability of functions $F_0(t)$ and $F_{(\pm)}(t)$ in time provides the existence of solutions $\psi_a^{(d)}(t)$ in each subspace $\mathcal{R}^{(d)}$.

If $F_{(-)}(t) \equiv 0$, then $F_{(+)}(t) \equiv 0$ and the set of Eqs. (28) and (30) can be easily integrated,

$$\psi_a^{(d)}(t) = c_a^{(d)}e^{-i(d-2a+1)\Xi(t)},\ \ \Xi(t) = \int^t F_0(t')dt',\ \ a = 1,\ldots,d\ ,$$

where $c_a^{(d)}$ are integration constant. Here it is sufficient the existence of a primitive function $\Xi(t)$ for $F_0(t)$. We meet this case considering 2-level system placed in a constant magnetic field, see Sec. 5.

4 CS states of finite-level systems

4.1 Annihilation and creation operators-integrals of motion

As was already mentioned above, the Hamiltonian $\hat{H}$ is quadratic in the operators $\hat{a}$ (here and in what follows, we use condense notation: $\hat{a} = (\hat{a}_\alpha)$ and $\hat{a}^\dagger = (\hat{a}_\alpha^\dagger)$). The construction of CS of quadratic systems in infinite-dimensional Hilbert spaces has been considered in numerous works. In addition to the references given in the Introduction, the most relevant for the construction considered below are the pioneering works of Man'ko with coauthors [3, 4, 34, 35] and the Ref. [36].

In the beginning, we pass from the operators $\hat{a}$ and $\hat{a}^\dagger$ to new time-dependent operators $\hat{A} = (\hat{A}_\alpha(t))$ and $\hat{A}^\dagger = (\hat{A}_\alpha^\dagger(t))$ by the help of a linear canonical transformation:

$$\hat{A} = w\hat{a} + \upsilon\hat{a}^\dagger + \varphi, \ \hat{A}^\dagger = \hat{a}^\dagger w^\dagger + \hat{a}\upsilon^\dagger + \varphi^\dagger , \tag{33}$$

Here $w = (w_{\alpha\beta}(t))$, $\upsilon = (\upsilon_{\alpha\beta}(t))$ and $\varphi = (\varphi_\alpha(t))$ are some time dependent complex matrices and functions respectively. In the following we shall consider only linearly independent operators $\hat{A}_\alpha$. To this end one of the matrices w or υ must be nonsingular. For $\hat{A}$ and $\hat{A}^\dagger$ to be annihilation and creation operators,

$$\left[\hat{A}, \hat{A}^\dagger\right] = I, \ \left[\hat{A}, \hat{A}\right] = \left[\hat{A}^\dagger, \hat{A}^\dagger\right] = 0 , \tag{34}$$

the matrices w and υ have to satisfy the following conditions (see e.g. [37]):

$$\begin{aligned} &ww^\dagger - \upsilon\upsilon^\dagger = I, \ w^\dagger w - \upsilon^T\upsilon^* = I , \\ &\upsilon w^T = w\upsilon^T, \ w^\dagger\upsilon = \upsilon^T w^* , \end{aligned} \tag{35}$$

where by the upper sign T transpose matrices are denoted.

For operators $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ to be integrals of motion which is defined by Eq. (23), it is sufficient their commutativity with the operator $\hat{\Pi} = i\partial_t - \hat{H}$,

$$\left[\hat{\Pi}, \hat{A}\right] = \left[\hat{\Pi}, \hat{A}^\dagger\right] = 0, \ \hat{\Pi} = i\partial_t - \hat{H} . \tag{36}$$

If $\hat{U}(t)$ is an evolution operator for Eq. (23),

$$i\partial_t \hat{U}(t) = \hat{H}\hat{U}(t), \ \hat{U}(0) = 1 ,$$

then the integrals of motion $\hat{A}_\alpha(t)$ and $\hat{A}_\alpha^\dagger(t)$ satisfy the following relations:

$$\hat{A}_\alpha(t) = \hat{U}(t)\hat{A}_\alpha(0)\hat{U}^{-1}(t), \ \hat{A}_\alpha^\dagger(t) = \hat{U}(t)\hat{A}_\alpha^\dagger(0)\hat{U}^{-1}(t) . \tag{37}$$

Calculating the commutators $\left[\hat{\Pi}, \hat{A}\right]$, we obtain:

$$\begin{aligned} &\left[\hat{\Pi}, \hat{A}\right] = \left(i\dot{w} + wH^{(2)}\right)\hat{a} + \left(i\dot{\upsilon} - \upsilon H^{(2)*}\right)\hat{a}^\dagger + i\dot{\varphi} = 0 \\ &\Longrightarrow i\dot{w} = -wH^{(2)}, \ i\dot{\upsilon} = \upsilon H^{(2)*}, \ \dot{\varphi} = 0 . \end{aligned} \tag{38}$$

Without loss of the generality we can set $\varphi(t) = 0$.

Introducing spinors W_α and V_α,

$$W_\alpha = \begin{pmatrix} w_{\alpha 1} \\ w_{\alpha 2} \end{pmatrix}, \ V_\alpha = \begin{pmatrix} \upsilon_{\alpha 1} \\ \upsilon_{\alpha 2} \end{pmatrix} , \tag{39}$$

we can rewrite remaining equations for w and υ as follows:

$$\begin{aligned} &i\partial_t W_\alpha = -H^{(2)*} W_\alpha = \Gamma W_\alpha, \ \Gamma = -\frac{1}{2}\begin{pmatrix} F_3 & F_{(+)} \\ F_{(-)} & -F_3 \end{pmatrix} = (\boldsymbol{\sigma}\boldsymbol{\Omega}) , \\ &i\partial_t V_\alpha = H^{(2)}(t) V_\alpha = \Gamma^* V_\alpha = \left(\boldsymbol{\sigma}\tilde{\boldsymbol{\Omega}}\right) V_\alpha , \\ &\boldsymbol{\Omega} = -\frac{1}{2}(F_1, -F_2, F_3), \ \tilde{\boldsymbol{\Omega}} = -\frac{1}{2}(F_1, F_2, F_3) . \end{aligned} \tag{40}$$

Spin Eq. (40) for zero initial conditions $\upsilon(0) = 0$ has zero solution $V_\alpha(t) = 0$ for any external field (7). In what follows, we consider only the case $\upsilon(t) = 0$ and $\varphi(t) = 0$, which is sufficient for our purposes. Thus, integrals of motion $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ that are used in our further constructions have the form:

$$\hat{A}(t) = w(t)\hat{a}, \ \hat{A}^\dagger(t) = \hat{a}^\dagger w^\dagger(t) , \tag{41}$$

with matrices $w(t)$ that have to be found from Eqs. (38) under the condition $\det w(t) \neq 0$ that provides the linear independence of integrals of motion $\hat{A}_\alpha(t)$. We also impose the following initial condition for Eq. (40):

$$w(0) = I \Longrightarrow \hat{A}(0) = \hat{a} \ . \tag{42}$$

In such a case we have:

$$\hat{A}_\alpha(t) = U(t,0)\hat{a}U^{-1}(t,0),\ \hat{A}^\dagger_\alpha(t) = U(t,0)\hat{a}^\dagger U^{-1}(t,0) \ . \tag{43}$$

4.2 Solving defining equations

Now we define CS $|Z, t\rangle$ in $\mathcal{R}^{(\infty)}$ as solution of the Schrödinger equation (23) that is also eigenvectors of the annihilation operators $\hat{A}_\alpha(t)$ given by Eq. (37). Such a vector has to satisfy the following not contradictory (due to Eq. (38)) to each other equations:

$$\hat{A}_\alpha(t)|Z,t\rangle = Z_\alpha|Z,t\rangle \ , \tag{44}$$

$$\hat{\Pi}(t)|Z,t\rangle = 0 \ . \tag{45}$$

Taking into account Eq. (41), and the condition $ww^\dagger = I$, we can rewrite equations (44) as:

$$\hat{a}_\alpha|Z,t\rangle = \ \tilde{Z}_\alpha(t)|Z,t\rangle,\ \ \tilde{Z}_\alpha(t) = \left(w^{-1}\right)_{\alpha\beta} Z_\beta = Z_\beta w^*_{\beta\alpha} \ . \tag{46}$$

In each time instant t the state $|Z, t\rangle$ has the form of CS of the Heisenberg group,

$$|Z,t\rangle = \exp\left\{-\frac{|\tilde{Z}(t)|^2}{2}\right\}\exp\{Z_\beta w^*_{\beta\alpha}a^+_\alpha\}|0,0\rangle \ ,$$

$$|Z,0\rangle = |Z\rangle,\ \hat{a}_\alpha|Z\rangle = Z_\alpha|Z\rangle,\ \langle Z,t|Z,t\rangle = 1 \ . \tag{47}$$

We note also that

$$|\tilde{Z}(t)|^2 = |Z_\beta w^*_{\beta 1}|^2 + |Z_\beta w^*_{\beta 2}|^2 = Z^*_\alpha Z_\beta w_{\alpha\gamma} w^\dagger_{\gamma\beta} = Z^*_\alpha Z_\alpha = |Z|^2 \ .$$

Thus state $|Z, t\rangle$ takes the form:

$$|Z,t\rangle = \exp\left\{-\frac{|Z|^2}{2}\right\}\exp\{Z_\beta w^*_{\beta\alpha}a^+_\alpha\}|0,0\rangle \ , \tag{48}$$

On the other hand, Eqs. (43) imply:

$$\hat{a}_\alpha U^{-1}(t,0)|Z,t\rangle = Z_\alpha U^{-1}(t,0)|Z,t\rangle \Longrightarrow |Z,t\rangle = U(t,0)|Z\rangle \ . \tag{49}$$

Thus, the vector $|Z, t\rangle$ given by Eq. (47) satisfies also the Schrödinger equation (45).

In CS-representation, vector (47) reads:

$$\Psi_Z(z,t) = \langle z|Z,t\rangle = \exp\left\{-\frac{|z|^2 + |Z|^2}{2} + Z_\beta w^*_{\beta\alpha} z_\alpha\right\} \ . \tag{50}$$

Expanding the exponential in expression (48) into a series and acting on the vacuum vector $|0, 0\rangle$ by the creation operators, we obtain:

$$|Z,t\rangle = \sum_{d=1}^{\infty}|Z,t\rangle^{(d)},\ |Z,t\rangle^{(d)} = P^{(d)}|Z,t\rangle = \exp\left\{-\frac{|Z|^2}{2}\right\}$$

$$\times \sum_{a=1}^{d}\frac{\tilde{Z}_1^{d-a}\tilde{Z}_2^{a-1}}{\sqrt{(a-1)!\,(d-a)!}}\overline{|a\rangle} \in \mathcal{R}^{(d)},\ \left(\hat{N}+1-d\right)|Z,t\rangle^{(d)} = 0 \ . \tag{51}$$

The square of the norm of the vector $|Z, t\rangle^{(d)}$ in $\mathcal{R}^{(d)}$ does not depend on time and is determined by the modulus $|Z| = \sqrt{|Z_1|^2 + |Z_2|^2}$,

$$^{(d)}\langle Z,t|Z,t\rangle^{(d)} = e^{-|Z|^2}\frac{|Z|^{d-1}}{(d-1)!},\ \sum_{d=1}^{\infty}{}^{(d)}\langle Z,t|Z,t\rangle^{(d)} = 1 \ . \tag{52}$$

The action of the operators $\hat{A}_\alpha(t) = w_{\alpha\beta}(t)\hat{a}_\beta$ on the vectors $|Z, t\rangle^{(d)}$ can be easily find:

$$\hat{A}_\alpha(t)|Z,t\rangle^{(1)} = \exp\left\{-\frac{|Z|^2}{2}\right\}w_{\alpha\beta}(t)\hat{a}_\beta|0,0\rangle = 0 \ ,$$

$$
\begin{aligned}
\hat{A}_\alpha(t)|Z,t\rangle^{(d)} &= \exp\left\{-\frac{|Z|^2}{2}\right\}\sum_{a=1}^{d}\frac{\tilde{Z}_1^{d-a}\tilde{Z}_2^{a-1}}{\sqrt{(a-1)!\,(d-a)!}}w_{\alpha\beta}(t)\hat{a}_\beta\overline{|a\rangle} \\
&= \exp\left\{-\frac{|Z|^2}{2}\right\}\left\{\sum_{a=1}^{d-1}\sqrt{d-a}\frac{\tilde{Z}_1^{d-a}\tilde{Z}_2^{a-1}}{\sqrt{(a-1)!\,(d-a)!}}w_{\alpha 1}(t)\overline{|a\rangle}^{(d-1)}\right. \\
&\left.+\sum_{a=1}^{d-1}\sqrt{a}\frac{\tilde{Z}_1^{d-a-1}\tilde{Z}_2^{a}}{\sqrt{(a)!\,(d-a-1)!}}w_{\alpha 2}(t)\overline{|a\rangle}^{(d-1)}\right\} \\
&= \tilde{Z}_\beta w_{\alpha\beta}(t)\exp\left\{-\frac{|Z|^2}{2}\right\}\sum_{a=1}^{d-1}\frac{\tilde{Z}_1^{d-a-1}\tilde{Z}_2^{a-1}}{\sqrt{(a-1)!\,(d-a-1)!}}\overline{|a\rangle}^{(d-1)} \\
&= Z_\alpha|Z,t\rangle^{(d-1)},\;\; |Z,t\rangle^{(d-1)} = P^{(d-1)}|Z,t\rangle,\;\; d>1\,.
\end{aligned} \tag{53}
$$

Here

$$
\overline{|a\rangle}^{(d-1)} = \overline{|a\rangle} = |d-a-1,\, a-1\rangle \in \mathcal{R}^{(d-1)},\; a=1,\ldots,d-1
$$

and the relation $\tilde{Z}_\beta w_{\alpha\beta} = Z_\gamma\left(ww^\dagger\right)_{\alpha\gamma} = Z_\alpha$ was taken into account.

Thus, although the state $|Z,\, t\rangle$ is an eigenstate for the operator $\hat{A}_\alpha(t)$, see Eq. (44), its projection $|Z,\, t\rangle^{(d)}$ on the subspace $\mathcal{R}^{(d)}$ is not. Nevertheless, we will call the states $|Z,\, t\rangle^{(d)}$ angular momentum CS (AMCS) in the space $\mathcal{R}^{(d)}$.

4.3 Relation to Perelomov spin coherent states

Let us find a relation between the AMCS $|Z,\, t\rangle^{(d)}$ and Perelomov spinning CS (PSCS) of the group $SU(2)$ introduced in Ref. [11].

The (PSCS) are defined with the help of the operators $\mathbf{s}^{(d)} = \left(s_1^{(d)},\, s_2^{(d)},\, s_3^{(d)}\right)$ of the angular momentum $j = (d-1)/2$ in the following way. First, one chooses the fixed vector $|j,\, -j\rangle^{(d)} = \overline{|d\rangle}$ that minimizes the Casimir dispersion (7),

$$
\begin{aligned}
&\left\langle j,\,-j|\Delta s^{(d)^2}|j,\,-j\right\rangle = j(j+1) - j^2 = j\,, \\
&s^{(d)^2}|j,\,-j\rangle^{(d)} = j(j+1)|j,\,-j\rangle^{(d)}\,, \\
&s_3^{(d)}|j,\,-j\rangle^{(d)} = -j|j,\,-j\rangle^{(d)}\,.
\end{aligned} \tag{54}
$$

Parametrizing elements of the group $SU(2)$ by Euler angels, $g = (\phi,\, \theta,\, \psi)$, $0 \le \phi < 2\pi$, $0 \le \theta < \pi$, $0 \le \psi < 2\pi$, we consider the subgroup $U(1)$ of diagonal matrices,

$$
U(1) = \left\{\begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2}\end{pmatrix}\right\}. \tag{55}
$$

The homogeneous space $X = SU(2)/U(1)$ is isomorphic to the 2-dimensional unit sphere, that is, to the set of unit vectors $\mathbf{n} = (\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$. Using the stereographic projection of a given sphere onto the complex plane, we will parametrize the points of the homogeneous space X using a complex number $\zeta = -\tan(\theta/2)\exp(-i\phi)$.

The representation operator $T^j(g) = \exp\left(-i\phi s_1^{(d)}\right)\exp\left(-i\theta s_2^{(d)}\right)\exp\left(-i\psi s_3^{(d)}\right)$, $d = 2j+1$, of the group SU(2) in the space $\mathcal{R}^{(d)}$, admits the decomposition

$$
\begin{aligned}
&T^j(g) = D^{(d)}(\zeta)T^j(\psi),\;\; \zeta \in \mathbb{C},\quad 0 \le \psi < 2\pi\,, \\
&D^{(d)}(\zeta) = \exp\left(\zeta s_+^{(d)}\right)\exp\left[\ln\left(1+|\zeta|^2\right)s_3^{(d)}\right]\exp\left(-\zeta^* s_-^{(d)}\right), \\
&T^j(\psi) = \exp\left(-i\psi s_3^{(d)}\right),\;\; s_\pm^{(d)} = s_1^{(d)} \pm i s_2^{(d)}\,.
\end{aligned} \tag{56}
$$

Consider the action of the operators $T^j(g)$ on the fixed vector $|j,\, -j\rangle^{(d)}$,

$$
T^j(g)|j,\,-j\rangle^{(d)} = \exp[i(d-1)\psi/2]D^{(d)}(\zeta)|j,\,-j\rangle^{(d)}\,. \tag{57}
$$

Vectors (57) for different ψ, parameterizing the subgroup U(1), differ only in a phase and define the same state. Let us set $\psi = 0$ in Eq. (57). Then instantaneous PSCS are specified by a point of the homogeneous space X and have the form:

$$
|\zeta\rangle^{(d)} = D^{(d)}(\zeta)|j,\,-j\rangle^{(d)}\,. \tag{58}
$$

The instantaneous PSCS are parameterized by a complex number $\xi \in \mathbb{C}$ and form a complete set of states,

$$
\int|\zeta\rangle^{(d)\,(d)}\langle\zeta|\,d\mu_j(\zeta) = 1,\;\; d\mu_j(\zeta) = \frac{2j+1}{\pi}\frac{d\,\mathrm{Re}\,\zeta\;d\,\mathrm{Im}\,\zeta}{\left(1+|\zeta|^2\right)^2}\,. \tag{59}
$$

Note that the uncertainty relation is minimized for for instantaneous PSCS,

$$\left\langle \tilde{s}_1^{(d)2} \right\rangle \left\langle \tilde{s}_2^{(d)2} \right\rangle \geq \frac{1}{4} \left\langle \tilde{s}_3^{(d)} \right\rangle^2, \quad \tilde{s}_k^{(d)} = D^{(d)}(\zeta) s_k^{(d)} D^{(d)^{-1}}(\zeta) \,.$$

The states $|\zeta(t)\rangle^{(d)}$ are called time dependent PSCS. The function $\zeta(t)$ satisfy a special equation that includes the external field in which the system is placed.

It is shown in Appendix B that the action of the operator $\left[D^{(d)}(\zeta(t))\right]^{-1}$ for $\zeta(t) = \tilde{Z}_1(t)/\tilde{Z}_2(t)$ on the state $|Z, t\rangle^{(d)}$ yields a vector proportional to the vector $|j, -j\rangle^{(d)} = \overline{|d\rangle}$,

$$\begin{aligned} \left[D^{(d)}(\zeta(t))\right]^{-1} |Z, t\rangle^{(d)} &= \sum_{a,b=1}^{d} \overline{\langle a|} \left[D^{(d)}(\zeta(t))\right]^{-1} \overline{|b\rangle}\, \overline{\langle b|} Z, t\rangle\, \overline{|a\rangle} \\ &= \frac{|Z|^{d-1}}{\sqrt{(d-1)!}} e^{-|Z|^2/2} \left(\frac{\tilde{Z}_2(t)}{\tilde{Z}_2^*(t)}\right)^{\frac{d-1}{2}} |j, -j\rangle^{(d)}, \quad \zeta(t) = \frac{\tilde{Z}_1(t)}{\tilde{Z}_2(t)} \,. \end{aligned} \tag{60}$$

Eq. (60) allows us to find a relation between the AMCS and time-dependent PSCS,

$$|Z, t\rangle^{(d)} = \frac{|Z|^{d-1} e^{-|Z|^2/2}}{\sqrt{(d-1)!}} e^{i(d-1)\arg \tilde{Z}_2(t)} |\zeta(t)\rangle^{(d)}, \quad \zeta(t) = \frac{\tilde{Z}_1(t)}{\tilde{Z}_2(t)} \,. \tag{61}$$

Note that the average values of time-independent physical quantities L are proportional to their average values with respect to time-independent PSCS $|\zeta\rangle^{(d)}$,

$${}^{(d)}\langle Z, t|\hat{L}|Z, t\rangle^{(d)} = \frac{|Z|^{2(d-1)}}{(d-1)!} e^{-|Z|^2}\, {}^{(d)}\langle \zeta(t)|\hat{L}|\zeta(t)\rangle^{(d)} \,. \tag{62}$$

Constructed AMCS represent a non-trivial generalization of PSCS. The states $|\zeta(t)\rangle^{(d)}$ are set by one time-dependent parameter $\zeta(t)$, whereas the states $|Z, t\rangle^{(d)}$ by two parameters $\tilde{Z}_1(t)$ and $\tilde{Z}_2(t)$. The only one subset of AMCS can be related to PSCS. This reflects the fact that the set of possible AMCS is wider than the set of PSCS, this is feature of our method of constructing CS for quadratic systems noted in Ref. [36]. As was already said exact solutions for AMCS can be easily constructed on the base of exact solutions of SE in 2-dimensions that were found in Ref. [26]. The question of constructing exact solutions for PSCS remains open since it is connected with finding external fields that allow exact solutions for the parameter $\zeta(t)$ in the Perelomov approach.

5 CS of spin 1/2 in a constant magnetic field

As an example, let us consider the particular case of the moment $j = 1/2$ in a constant magnetic field[4] $\mathbf{B} = (0, 0, B)$ directed along the axis z. Then the corresponding vector $\mathbf{F}$ in Eq. (7) reads:

$$\mathbf{F} = (0, 0, F), \ F = 2\omega_0, \ \omega_0 = \frac{eB}{2m_e}. \tag{63}$$

According to Eq. (40), the corresponding equation for finding matrices w has the form:

$$\begin{aligned} &i\dot{W}_\alpha(t) = (\boldsymbol{\sigma}\boldsymbol{\Omega}) W_\alpha(t), \quad W_\alpha(t) = \begin{pmatrix} u_{\alpha 1}(t) \\ u_{\alpha 2}(t) \end{pmatrix}, \ \boldsymbol{\Omega} = -(0, 0, \omega_0) \\ &\Longrightarrow i\dot{W}_\alpha(t) = -\omega_0 \sigma_3 W_\alpha(t) \,. \end{aligned} \tag{64}$$

Its general solution reads:

$$W_\alpha(t) = \exp(i\omega_0 \sigma_3 t) W_\alpha(0) \,. \tag{65}$$

Representing the spinors $W_\alpha(0)$ as:

$$W_\alpha(0) = (c_\alpha)_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (c_\alpha)_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{66}$$

we obtain:

$$W_\alpha(t) = (c_\alpha)_1 \exp(i\omega_0 t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (c_\alpha)_2 \exp(-i\omega_0 t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,. \tag{67}$$

[4] We not that coherent spin states in constant magnetic field were considered in frame work of different approaches in a number of works, see, e.g. Refs. [13, 38–41].

Taking into account the initial conditions (42) for the matrices w, i.e., $w_{\alpha\beta}(0) = \delta_{\alpha\beta}$, we obtain: $(c_\alpha)_\beta = \delta_{\alpha\beta}$. Then:

$$W_\alpha(t) = \begin{pmatrix} \delta_{\alpha 1}\exp(i\omega_0 t) \\ \delta_{\alpha 2}\exp(-i\omega_0 t) \end{pmatrix}, \tag{68}$$

which implies:

$$\begin{aligned} &w_{\alpha 1}(t) = \delta_{\alpha 1}\exp(i\omega_0 t),\ w_{\alpha 2}(t) = \delta_{\alpha 2}\exp(-i\omega_0 t) \\ &\Longrightarrow w_{\alpha\beta}(t) = \begin{pmatrix} \exp(i\omega_0 t) & 0 \\ 0 & \exp(-i\omega_0 t) \end{pmatrix}. \end{aligned} \tag{69}$$

Thus, the AMCS of spin 1/2 in a constant magnetic field has the form:

$$\begin{aligned} |Z,t\rangle^{(2)} &= \exp\left\{-\frac{|Z_1|^2+|Z_2|^2}{2}\right\}\left[Z_1\exp(-i\omega_0 t)\overline{|1\rangle} + Z_2\exp(i\omega_0 t)\overline{|2\rangle}\right], \\ |\mathbf{Z,t}\rangle^{(2)} &= \exp\left\{-\frac{|Z_1|^2+|Z_2|^2}{2}\right\}\begin{pmatrix} Z_1\exp(-i\omega_0 t) \\ Z_2\exp(i\omega_0 t) \end{pmatrix}. \end{aligned} \tag{70}$$

Then Eq. (61) takes the form:

$$\begin{aligned} &|Z,\, t\rangle^{(2)} = \exp\left\{-\frac{|Z_1|^2+|Z_2|^2}{2}\right\}\sqrt{\frac{|Z_1|^2+|Z_2|^2}{2}}\exp\left[i(\arg Z_2 + \omega_0 t)\right]|\zeta(t)\rangle^{(2)}, \\ &\zeta(t) = \frac{Z_1}{Z_2}\exp(-2i\omega_0 t), \end{aligned}$$

where the PSCS $|\zeta\rangle^{(2)}$ are:

$$|\zeta\rangle^{(2)} = \sqrt{\frac{2}{1+|\zeta|^2}}\left[\zeta\overline{|1\rangle} + \overline{|2\rangle}\right]. \tag{71}$$

We note that in Ref. [28] CS for a 2-level system are constructed as CS of angular momentum, sometimes they are called atomic CS. The evolution of these states is described by a point on a unit two-dimensional sphere (on the Bloch sphere).

6 Summary

A way of introducing CS of finite-level systems with a given angular momentum in an external electromagnetic field is proposed. First we consider an analogue of the spin equation in 2-dimensional space in an infinite-dimensional Hilbert space $\mathcal{R}^{(\infty)}$. Such a space is constructed with the help of two kinds of Schwinger creation and annihilation operators. The introduced equation is an equation for the angular momentum in $\mathcal{R}^{(\infty)}$. One other side we treat it as a generating spin equation (GSE). By projecting the GSE on d-dimensional subspaces, we obtain an analog of SE for d-dimensional systems in an external electromagnetic field. The Hamiltonian of GSE turns out to be quadratic in the Schwinger creation and annihilation operators. This fact allows us to use a modification of Malkin-Manko integral of motion method developed in our earlier work for constructing the corresponding time-dependent generalized CS for the quadratic system in $\mathcal{R}^{(\infty)}$. Projections of the later CS on finite-dimensional subspaces describe d-dimensional system with a given angular moment, we call AMCS of finite-level systems. The new AMCS for d-dimensional systems in an external electromagnetic field admit the dynamic symmetry group $SU(2)$. We note that the projection operation can always be performed if the external field is sufficiently smooth. The set AMCS is complete in $\mathcal{R}^{(d)}$. The AMCS $|Z,\, t\rangle^{(d)}$ have a clear physical meaning, they obey the Schrödinger for a d-dimensional system with a given angular moment $j = (d-1)/2$ in an external electromagnetic field. Their possible exact solutions are constructed via exact solutions of the SE in 2-dimensional space. The latter solutions can be found analytically and are completely described in our earlier works. The only one subset of AMCS can be related to PSCS. This reflects the fact that the set of possible AMCS is wider than the set of PSCS, this is feature of our method of constructing CS for quadratic systems noted in Ref. [36]. The question of constructing exact solutions for PSCS remains open since it is connected with finding external fields that allow exact solutions for the parameter $\zeta(t)$ in the Perelomov approach.

The results obtained in the present work may be interesting in the context of revealing a connection between two different methods of constructing CS. According to Perelomov, in the case of the $SU(2)$ group, spin CS are introduced by the group-theoretic method, namely, by the action of representation operators of the group on a fixed vector. In our construction CS of angular momentum j are given by projections of a CS constructed, in fact, by Malkin-Manko method, in an infinite-dimensional Fock space onto its $(2j+1)$-dimensional subspace. The above mentioned relation between PSCS and AMCS establishes, in a sense, a relation between the Perelomov and the Malkin-Manko methods. AMCS states in a constant magnetic field are constructed.

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Declarations

Conflicts of Interest The authors have no relevant financial or non-financial interests to disclose.

Appendix A

Let us demonstrate that the subspace $\mathcal{R}^{(d)}$ is invariant under the action of the angular momentum operators (12) and find their matrix elements in $\mathcal{R}^{(d)}$.

It follows from Eq. (21) that

$$\hat{a}_\alpha^\dagger \hat{a}_\beta |\Psi\rangle^{(d)} = \sum_{a=1}^{d} \psi_a^{(d)}\, \hat{a}_\alpha^\dagger \hat{a}_\beta \overline{|a\rangle}\,. \tag{1}$$

The operators $\hat{a}_\alpha^\dagger \hat{a}_\beta$ act on the ket-vectors (19) in the following way:

$$\begin{aligned}\hat{a}_\alpha^\dagger \hat{a}_\beta \overline{|a\rangle} &= \hat{a}_\alpha^\dagger \hat{a}_\beta |d-a, a-1\rangle = \delta_{\alpha,\beta}\big[(d-a)\delta_{\alpha,1} + (a-1)\delta_{\alpha,2}\big]\overline{|a\rangle} \\ &+ \delta_{\alpha,2}\delta_{\beta,1}\sqrt{a(d-a)}\overline{|a+1\rangle} + \delta_{\alpha,1}\delta_{\beta,2}\sqrt{(a-1)(d-a+1)}\overline{|a-1\rangle}\,.\end{aligned} \tag{2}$$

As it follows from Eq. (2) resulting vectors remain in the same space,

$$\begin{aligned}\hat{a}_\alpha^\dagger \hat{a}_\beta |\Psi\rangle^{(d)} &= \sum_{a=1}^{d}\Big\{\delta_{\alpha,\beta}\big[(d-a)\delta_{\alpha,1} + (a-1)\delta_{\alpha,2}\big]\psi_a^{(d)} \\ &+ \delta_{\alpha,2}\delta_{\beta,1}\sqrt{(a-1)(d-a+1)}\psi_{a-1}^{(d)} + \delta_{\alpha,1}\delta_{\beta,2}\sqrt{a(d-a)}\psi_{a+1}^{(d)}\Big\}\overline{|a\rangle}\,.\end{aligned} \tag{3}$$

From here follows the invariance of operators (12).

The matrix elements of the operators $\hat{a}_\alpha^\dagger \hat{a}_\beta$ are:

$$\begin{aligned}\overline{\langle a|}\hat{a}_\alpha^\dagger \hat{a}_\beta \overline{|b\rangle} &= \big[(d-a)\delta_{\alpha,1} + (a-1)\delta_{\alpha,2}\big]\delta_{a,b}\delta_{\alpha,\beta} \\ &+ \sqrt{a(d-a)}\delta_{a+1,b}\delta_{\alpha,2}\delta_{\beta,1} + \sqrt{(a-1)(d-a+1)}\delta_{a-1,b}\delta_{\alpha,1}\delta_{\beta,2}\,.\end{aligned} \tag{4}$$

From Eq. (4) it is easy to obtain matrix elements of angular momentum operators (12),

$$\Big(\hat{S}_i\Big)_{ab} = \overline{\langle a|}\hat{S}_i\overline{|b\rangle} = (\sigma_i^{\alpha\beta}/2)\overline{\langle a|}\hat{a}_\alpha^\dagger \hat{a}_\beta \overline{|b\rangle} \tag{5}$$

$$\begin{aligned}\Big(\hat{S}_1\Big)_{ab} &= \frac{1}{2}\Big[\sqrt{a(d-a)}\delta_{a+1,b} + \sqrt{b(d-b)}\delta_{a,b+1}\Big]\,, \\ \Big(\hat{S}_2\Big)_{ab} &= \frac{i}{2}\Big[\sqrt{b(d-b)}\delta_{a,b+1} - \sqrt{a(d-a)}\delta_{a+1,b}\Big]\,, \\ \Big(\hat{S}_3\Big)_{ab} &= \frac{1}{2}(d-2a+1)\delta_{a,b}\,,\end{aligned} \tag{6}$$

such that

$$\Big(\hat{\mathbf{S}}^2\Big)_{ab} = \Big(\hat{S}_1^2\Big)_{ab} + \Big(\hat{S}_2^2\Big)_{ab} + \Big(\hat{S}_3^2\Big)_{ab} = j(j+1)\delta_{a,b},\quad j = \frac{d-1}{2}\,. \tag{7}$$

The following matrix elements are used in our constructions:

$$\begin{aligned}\overline{\langle a|}\exp\Big(\zeta\hat{S}_+\Big)\overline{|b\rangle} &= \sum_{k=0}^{d-1}\sqrt{\frac{(a+k-1)!\,(d-a)!}{(d-k-a)!\,(a-1)!}}\frac{\zeta^k}{k!}\delta_{a+k,b}, \\ \overline{\langle a|}\exp\Big(\hat{S}_3\ln\zeta\Big)\overline{|b\rangle} &= \zeta^{\frac{d-2a+1}{2}}\delta_{a,b}\,, \\ \overline{\langle a|}\exp\Big(\zeta\hat{S}_-\Big)\overline{|b\rangle} &= \sum_{l=0}^{d-1}\sqrt{\frac{(b+l-1)!\,(d-b)!}{(d-l-b)!\,(b-1)!}}\frac{\zeta^k}{k!}\delta_{a,b+l},\quad \zeta\in\mathbb{C}\,.\end{aligned} \tag{8}$$

Appendix B

Let us demonstrate that the action of the operator (with $\zeta(t) = \tilde{Z}_1(t)/\tilde{Z}_2(t)$)

$$\Big[D^{(d)}(\zeta(t))\Big]^{-1} = \exp\Big[-\zeta^*(t)s_-^{(d)}\Big]\exp\Big[\ln\big(1+|\zeta(t)|^2\big)s_3^{(d)}\Big]\exp\Big[\zeta(t)s_+^{(d)}\Big] \tag{9}$$

on the state $|Z,t\rangle^{(d)}$ (see Eq. (51)) produces vector proportional to the vector $|j,-j\rangle^{(d)} = \overline{|d\rangle}$.

To this end we first study matrix elements of operator (9):

$$\begin{aligned}\overline{\langle a|}\Big[D^{(d)}(\zeta)\Big]^{-1}\overline{|b\rangle} &= \sum_{c,e=1}^{d}\overline{\langle a|}\exp\Big[-\zeta^*(t)s_-^{(d)}\Big]\overline{|c\rangle}\\ &\quad\times\overline{\langle c|}\exp\Big[\ln\big(1+|\zeta(t)|^2\big)s_3^{(d)}\Big]\overline{|e\rangle\langle e|}\exp\Big[\zeta(t)s_+^{(d)}\Big]\overline{|b\rangle}\\ &= \sum_{c,e=1}^{d}\overline{\langle a|}\exp\Big[-\zeta^*(t)\hat{S}_-\Big]\overline{|c\rangle}\\ &\quad\times\overline{\langle c|}\exp\Big[\ln\big(1+|\zeta(t)|^2\big)\hat{S}_3\Big]\overline{|e\rangle\langle e|}\exp\Big[\zeta(t)\hat{S}_+\Big]\overline{|b\rangle}\,.\end{aligned} \tag{10}$$

With the help of Eq. (8), we obtain:

$$\begin{aligned}\overline{\langle a|}\Big[D^{(d)}(\zeta)\Big]^{-1}\overline{|b\rangle} &= \big(1+|\zeta|^2\big)^{-(d+1)/2}\sum_{c=1}^{d}\frac{(d-c)!}{(c-1)!}\sqrt{\frac{(a-1)!\,(b-1)!}{(d-a)!\,(d-b)!}}\\ &\times\sum_{k,l=0}^{d-1}\left(-\frac{1+|\zeta|^2}{|\zeta|^2}\right)^{c}\frac{(\zeta^*)^a(-\zeta)^b}{(a-c)!\,(b-c)!}\delta_{a-c,l}\delta_{b-c,k}\,.\end{aligned} \tag{11}$$

Then we consider the following equation:

$$\begin{aligned}\overline{\langle a|}\Big[D^{(d)}(\zeta(t))\Big]^{-1}|Z,t\rangle^{(d)} &= \sum_{b=1}^{d}\overline{\langle a|}\Big[D^{(d)}(\zeta(t))\Big]^{-1}\overline{|b\rangle}\,\overline{\langle b|}Z,t\rangle^{(d)}\\ &= \frac{\big[\zeta^*(t)\big]^a e^{-|Z|^2/2}}{\big(1+|\zeta(t)|^2\big)^{(d+1)/2}}\frac{\tilde{Z}_1^d(t)}{\tilde{Z}_2(t)}\sqrt{\frac{(a-1)!}{(d-a)!}}\\ &\times\sum_{c=1}^{d}\frac{(d-c)!}{(a-c)!\,(c-1)!}\left(-\frac{1+|\zeta(t)|^2}{|\zeta(t)|^2}\right)^{c}\\ &\times\sum_{b=1}^{d}\frac{(-1)^b}{(b-c)!\,(d-b)!}\sum_{k,l=0}^{d-1}\delta_{a-c,l}\delta_{b-c,k}\,.\end{aligned} \tag{12}$$

Taking into account that

$$\sum_{k,l=0}^{d-1}\delta_{a-c,l}\delta_{b-c,k} = \begin{cases}1, & a\ge c,\quad b\ge c\\ 0, & otherwise\end{cases}, \tag{13}$$

we obtain

$$\sum_{b=1}^{d}\frac{(-1)^b}{(b-c)!\,(d-b)!}\sum_{k,l=0}^{d-1}\delta_{a-c,l}\delta_{b-c,k} = (-1)^d\delta_{a,d}\delta_{c,d}\,. \tag{14}$$

Substituting Eq. (14) into Eq. (12), we find:

$$\overline{\langle a|}\Big[D^{(d)}(\zeta(t))\Big]^{-1}|Z,t\rangle^{(d)} = \frac{|Z|^{d-1}}{\sqrt{(d-1)!}}e^{-|Z|^2/2}\left(\frac{\tilde{Z}_2(t)}{\tilde{Z}_2^*(t)}\right)^{\frac{d-1}{2}}\delta_{a,d}\,. \tag{15}$$

Finally, Eq., (15) allows us to obtain the following result that is used in our constructions:

$$\Big[D^{(d)}(\zeta(t))\Big]^{-1}|Z,t\rangle^{(d)} = \sum_{a,b=1}^{d}\overline{\langle a|}\Big[D^{(d)}(\zeta(t))\Big]^{-1}\overline{|b\rangle}\,\overline{\langle b|}Z,t\rangle\overline{|a\rangle}$$

$$= \frac{|Z|^{d-1} e^{-|Z|^2/2}}{\sqrt{(d-1)!}} \left(\frac{\tilde{Z}_2(t)}{\tilde{Z}_2^*(t)} \right)^{\frac{d-1}{2}} |j, -j\rangle^{(d)}, \quad \zeta(t) = \frac{\tilde{Z}_1(t)}{\tilde{Z}_2(t)}. \tag{16}$$

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