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Anti-Ramsey threshold of cycles[☆]

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ABSTRACT

For graphs G and H, let $G\xrightarrow{\text{rb}}H$ denote the property that for every *proper* edge colouring of G there is a *rainbow* copy of H in G. Extending a result of Nenadov et al. (2017), we determine the threshold for $G(n,p)\xrightarrow{\text{rb}}C_\ell$ for cycles C_ℓ of any given length $\ell \geq 4$.

1. Introduction

In this paper we investigate an *anti-Ramsey* property of random graphs. Given graphs G and H, we denote by $G\xrightarrow{\text{rb}}H$ the following anti-Ramsey property: for every *proper* edge colouring of G there is a *rainbow* copy of H in G, i.e. a subgraph of G isomorphic to H in which all edges have distinct colours.

In 1992, Rödl and Tuza [12] proved the following result, which answered affirmatively a question raised by Spencer (see [4, p. 29]) asking whether there are graphs of arbitrarily large girth containing a rainbow cycle in every proper edge colouring.

Theorem 1 (*[12]*)**.** *For every positive integer t and every positive δ with $\delta < 1/(2t+1)$ there exists n_0 such that for every $n \geq n_0$ there exists an n-vertex graph G with girth at least $t+2$ having the property $G\xrightarrow{\text{rb}}C_\ell$, for $2t+1 \leq \ell \leq n^\delta$, where C_ℓ is an ℓ-vertex cycle.*

In their proof, Rödl and Tuza showed that $G(n,p)\xrightarrow{\text{rb}}C_\ell$ holds a.a.s.[1] for a small p. Note that since $G\xrightarrow{\text{rb}}H$ is an increasing property, there exists a threshold[2] $p_H^{\text{rb}} = p_H^{\text{rb}}(n)$ for any fixed graph H (see [2]). In [6], Kohayakawa, Konstadinidis

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1 A property in $G(n,p)$ holds asymptotically almost surely (a.a.s.) if the probability tends to one as n tends to infinity.

2 The threshold for a property is a function $\hat{p} = \hat{p}(n)$ such that $G(n,p)$ a.a.s. has this property if $p \gg \hat{p}$ and a.a.s. does not have it if $p \ll \hat{p}$.

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and Mota obtained an upper bound for the threshold p_H^{rb} for any fixed graph H in terms of the maximum 2-density $m_2(H) = \max\{(e(J)-1)/(v(J)-2) : J \subseteq H, v(J) \geq 3\}$.

Theorem 2 ([6]). *Let H be a fixed graph. Then there exists a constant $C > 0$ such that a.a.s. $G(n,p) \xrightarrow{\mathrm{rb}} H$ whenever $p = p(n) \geq Cn^{-1/m_2(H)}$. In particular, $p_H^{\mathrm{rb}} \leq n^{-1/m_2(H)}$.*

A classical result in Ramsey Theory obtained by Rödl and Ruciński [11] implies that $n^{-1/m_2(H)}$ is the threshold for the following Ramsey property, as long as H contains a cycle: every colouring of $E(G(n,p))$ with r colours contains a monochromatic copy of H. In view of this result, it is plausible to conjecture that $n^{-1/m_2(H)}$ is also the threshold for the anti-Ramsey property, for any fixed graph H. However, as proved in [7], there are infinitely many graphs H for which the threshold p_H^{rb} is asymptotically smaller than $n^{-1/m_2(H)}$. Recently, this result was extended to a larger family of graphs (see [1]). On the other hand, Nenadov, Person, Škorić and Steger [10] proved that at least for sufficiently large cycles and complete graphs H the lower bound for p_H^{rb} matches the upper bound $n^{-1/m_2(H)}$ of Theorem 2.

Theorem 3 ([10]). *If H is a cycle on at least 7 vertices or a complete graph on at least 19 vertices, then $p_H^{\mathrm{rb}} = n^{-1/m_2(H)}$.*

In [8], Kohayakawa, Mota, Parczyk and Schnitzer extended Theorem 3, by showing that for all complete graphs K_ℓ with $\ell \geq 5$ the threshold $p_{K_\ell}^{\mathrm{rb}}$ is in fact $n^{-1/m_2(K_\ell)}$, and for K_4 we have $p_{K_4}^{\mathrm{rb}} = n^{-7/15} \ll n^{-1/m_2(K_4)}$. Our result determines the threshold $p_{C_\ell}^{\mathrm{rb}}$ for every cycle C_ℓ on $\ell \geq 4$ vertices.

Theorem 4. *If $\ell \geq 5$ then $p_{C_\ell}^{rb} = n^{-1/m_2(C_\ell)} = n^{-(\ell-2)/(\ell-1)}$. Furthermore, $p_{C_4}^{rb} = n^{-3/4}$.*

In Section 2 we prove Theorem 4 for cycles with at least 5 vertices. Similarly to what happens with complete graphs, the situation for C_4 is different: For $p = n^{-3/4} \ll n^{-1/m_2(C_4)}$ the random graph $G(n,p)$ a.a.s. contains a small graph F such that $F \xrightarrow{\mathrm{rb}} C_4$. In Section 3 we prove that $n^{-3/4}$ is the threshold for C_4[3] and we finish with some concluding remarks in Section 4. We use standard notation and terminology (see e.g. [3,5]). In particular, given a subgraph H of a graph G, we write $G - H$ for the graph obtained from G by removing all vertices that belong to H and all edges incident with these vertices.

2. Cycles on at least five vertices

In [10], Nenadov, Person, Škorić, and Steger provide a general framework (see also the Meta-Theorem in [10, Section 1.3]) that reduces some Ramsey problems into deterministic problems for graphs with bounded maximum density, where the *maximum density* of a graph G is denoted by

$$m(G) = \max\left\{\frac{e(J)}{v(J)} : J \subseteq G, v(J) \geq 1\right\}.$$

The proof of Theorem 3 for cycles relies on the following lemma (see [10, Lemma 24]).

Lemma 5 ([10]). *Let $\ell \geq 7$ be an integer and G be a graph such that $m(G) < m_2(C_\ell)$. Then $G \stackrel{\mathrm{rb}}{\not\longrightarrow} C_\ell$.*

In fact they prove a slightly stronger statement for which they need a non-strict inequality relating the densities [10, Corollary 13]. The condition $\ell \geq 7$ in Theorem 3 is simply a consequence of the restriction on the cycle length imposed in Lemma 5, as observed by the authors [10]. We extend Lemma 5, proving the following result, where we note that $m_2(C_\ell) = (\ell-1)/(\ell-2)$.

Lemma 6. *Let $\ell \geq 5$ be an integer and G be a graph such that $m(G) < (\ell-1)/(\ell-2)$. Then, $G \stackrel{\mathrm{rb}}{\not\longrightarrow} C_\ell$.*

Theorem 4 thus follows immediately by replacing Lemma 5 with our Lemma 6 in the proof of Theorem 3 in [10]. We remark that the proof of Lemma 6 considers all the cycle lengths in the range $\ell \geq 5$, i.e. it is not a proof only for the cases $\ell = 5$ and $\ell = 6$.

Throughout this section let $\ell \geq 5$ be an integer and G be a graph with $m(G) < (\ell-1)/(\ell-2)$. We use the term *k-path* to refer to a path with k vertices. For the proof of Lemma 6, we will define a *partial* proper edge colouring of G such that every ℓ-cycle has two non-adjacent edges with the same colour. Clearly, having defined such a partial edge colouring, we can extend it to a proper edge colouring (for instance, the uncoloured edges may be assigned distinct colours).

[3] We remark that a sketch of the proof for C_4 was given in a short abstract of the fourth author [9].

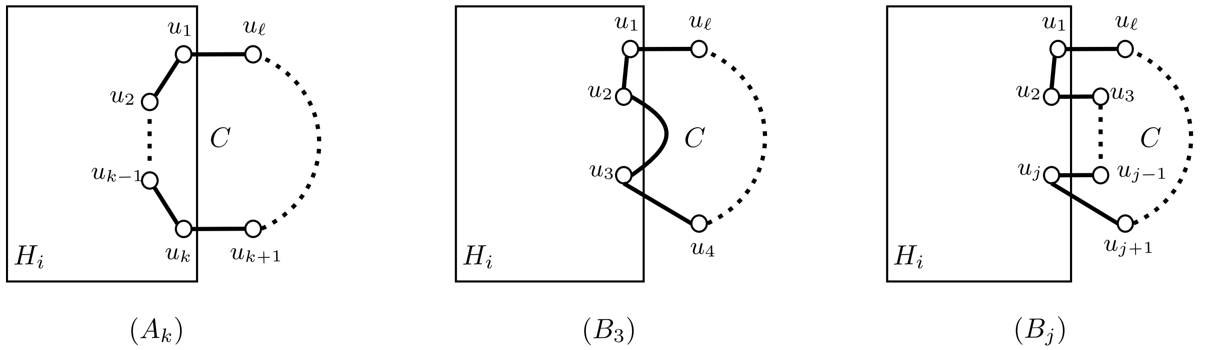

Fig. 1. Possible configurations of a C_ℓ added to H_i to form H_{i+1}.

2.1. Cycle components

Let $\mathcal{C}_\ell(G)$ be the set of all ℓ-cycles of G. We start by defining key concepts that we use throughout our proof. The *edge intersection graph* of $\mathcal{C}_\ell(G)$ is the graph whose vertex set is $\mathcal{C}_\ell(G)$ and whose edges correspond to pairs $\{C, C'\}$ such that $C \neq C'$ and $E(C) \cap E(C') \neq \varnothing$. A subgraph $H \subseteq G$ is a C_ℓ*-component* of G if it is the union of all ℓ-cycles corresponding to the vertices of some component of the edge intersection graph of $\mathcal{C}_\ell(G)$.

Let H_1 be an ℓ-cycle in G. A C_ℓ-component H of G containing H_1 can be constructed from H_1 as follows. Suppose we have defined $H_1 \subseteq \ldots \subseteq H_i$ for $i \geq 1$. If there is an ℓ-cycle C in G such that $C \not\subseteq H_i$ and $E(C) \cap E(H_i) \neq \varnothing$, then we put $H_{i+1} = H_i \cup C$; otherwise we terminate the construction and set $H = H_i$. Let t be such that $H = H_t$. We call $(H_1, \ldots, H_t)$ a *construction sequence* of H. For brevity, sometimes we will identify a C_ℓ-component with a construction sequence of it; for example, we will write "a C_ℓ-component $(H_1, \ldots, H_t)$".

Note that there can be multiple new ℓ-cycles appearing in H_{i+1} that were not present in H_i before; this will be the main problem to deal with when constructing the partial colouring. Also note that the process just described allows us to reconstruct a C_ℓ-component starting from any ℓ-cycle of it. Also note that two ℓ-cycles belonging to distinct C_ℓ-components may share vertices (obviously they do not share edges).

We start the colouring procedure in some C_ℓ-component H of G. Once we have coloured the edges of H avoiding a rainbow C_ℓ, we proceed to assign colours different from those used in H to edges of a C_ℓ-component of $G - E(H)$, using the same procedure. We continue colouring edges in this manner (taking an uncoloured C_ℓ-component, colouring it and removing its edges) until we have considered all the ℓ-cycles of G. Thus, our aim is to describe the colouring procedure of an arbitrary C_ℓ-component H of G.

Let $(H_1, \ldots, H_t)$ be a C_ℓ-component of G. Since producing a colouring which avoids rainbow C_ℓ is a trivial task if the C_ℓ-component has only one cycle, we may assume $t \geq 2$. The following proposition is crucial in our proof and, given a C_ℓ-component $(H_1, \ldots, H_t)$, describes for any $1 \leq i \leq t-1$ the possible structure of an ℓ-cycle C which is added to H_i to form H_{i+1}, i.e. $C \subseteq H_{i+1}$, but $C \not\subseteq H_i$ and $E(C) \cap E(H_i) \neq \varnothing$. (see Fig. 1).

Proposition 7. *Let $\ell \geq 4$ be an integer, G be a graph with $m(G) < (\ell-1)/(\ell-2)$ and $(H_1, \ldots, H_t)$ be a C_ℓ-component of G. Then, the following holds for every $1 \leq i \leq t-1$.*

If C is an ℓ-cycle added to H_i to form H_{i+1}, then there exists a labelling $C = u_1u_2\cdots u_\ell u_1$ such that exactly one of the following occurs, where $2 \leq k \leq \ell$ and $3 \leq j \leq \ell-1$:

(A_k) $u_1u_2\cdots u_k$ is a k-path in H_i and $u_{k+1}, \ldots, u_\ell \notin V(H_i)$;
(B_j) $u_1u_2 \in E(H_i)$, $u_2u_3 \notin E(H_i)$, $\{u_3, \ldots, u_\ell\} \setminus \{u_j\} \subseteq V(H_{i+1}) \setminus V(H_i)$, $u_j \in V(H_i)$.

We refer to each of (A_k) and (B_j) as a *configuration* of H_{i+1}. Before proving Proposition 7, let us discuss some ideas used for this purpose. To show that some of the configurations are not possible or do not happen often during the construction of $(H_1, \ldots, H_t)$, we heavily use the fact that $m(G) < (\ell-1)/(\ell-2)$.

For any $1 \leq j \leq i$, define parameters e_j, v_j and c_j as follows: e_j is the number of edges in $E(H_{j+1}) \setminus E(H_j)$, while v_j stands for the number of vertices in $V(H_{j+1}) \setminus V(H_j)$. Lastly, let c_j be the number of components of $H_{j+1} - H_j$. Note that if $v_j = 0$, then $e_j \geq 1$, and if $v_j \geq 1$, then the components of $H_{j+1} - H_j$ are paths and we get $e_j \geq v_j + c_j \geq v_j + 1$. Therefore, we conclude that, for $1 \leq j \leq i$ we have $e_j \geq v_j + 1$ Also, since any ℓ-cycle added to H_j to form H_{j+1} contains at least one edge of H_j, for $1 \leq j \leq i$, we have $v_j \leq \ell - 2$. Note that we have

$$\frac{\ell-1}{\ell-2} > m(G) \geq \frac{e(H_{i+1})}{v(H_{i+1})} = \frac{\ell + \sum_{j=1}^{i} e_j}{\ell + \sum_{j=1}^{i} v_j}. \tag{1}$$

Using the bounds $e_j \geq v_j + 1$ and $v_j \leq \ell - 2$, we obtain

$$\frac{\ell-1}{\ell-2} > \frac{\ell + e_i + \sum_{j=1}^{i-1}(v_j+1)}{\ell + v_i + \sum_{j=1}^{i-1} v_j} \geq \frac{\ell + e_i + (i-1)(\ell-1)}{\ell + v_i + (i-1)(\ell-2)}, \tag{2}$$

which implies

$$e_i < \frac{(\ell-1)v_i + \ell}{\ell-2}. \tag{3}$$

We are ready to prove Proposition 7.

Proof of Proposition 7. We will prove the result for all possible values of v_i (i.e., $0 \leq v_i \leq \ell - 2$). If $v_i = \ell - 2$, then we have configuration (A_2).

Now let $v_i = \ell - 3$, which means that there are exactly three vertices of C in H_i. If these vertices form a path in H_i, then we have configuration (A_3). On the other hand, let u_1, u_2 and w be the vertices of C in H_i and let u_1u_2 be an edge of H_i. If there is an edge of C between w and $\{u_1, u_2\}$, then let w.l.o.g. u_2w be this edge. Then, we have configuration (B_3), where $u_3 = w$. If there is no edge of C between w and $\{u_1, u_2\}$, then w.l.o.g. C contains a path $P_1 = u_2, u_3, \ldots, u_{j-1}, w$ (with at least two edges) between u_2 and w with all edges outside H_i, and a path $P_2 = w, u_{j+1}, \ldots, u_\ell, u_1$ between w and u_1 with all edges outside H_i, such that w is the only common vertex of P_1 and P_2. Then, we have configuration (B_j), where $u_j = w$ and $4 \leq j \leq \ell - 1$ (as u_3 and u_ℓ are outside H_i).

Finally, let $0 \leq v_i \leq \ell - 4$. From (3) we have $e_i \leq v_i + 1$. Then, either $|E(H_{i+1}) \setminus E(H_i)| = 1$ or $H_{i+1} - H_i$ has only one component, which implies that the vertices of C in H_i form a $(\ell - v_i)$-path, where we have $4 \leq \ell - v_i \leq \ell$. Therefore, we have configuration (A_k) with $4 \leq k \leq \ell$. □

2.2. Proof of Lemma 6

Given a C_ℓ-component H described by a construction sequence $(H_1, \ldots, H_t)$, we will colour the edges of H_1, H_2 and so on iteratively, avoiding rainbow ℓ-cycles. For configurations (A_k) with $2 \leq k \leq \ell - 2$ we are always able to assign a new colour to two non-adjacent new edges. All other configurations may appear at most twice in $(H_1, \ldots, H_t)$, and in these cases we will colour all previous configurations carefully so that we are able to proceed.

Arguments involving calculations similar to those we did on (1) and (2) will be referred to as *density arguments*. For example, when H_{i+1} has configuration (A_ℓ), we have $v_i = 0$ and $e_i = 1$, which following the calculations in (1) and (2) implies that there cannot be a previous occurrence of (A_ℓ), as this would imply

$$\frac{\ell-1}{\ell-2} > m(G) \geq \frac{e(H_{i+1})}{v(H_{i+1})} = \frac{\ell + \sum_{j=1}^{i} e_j}{\ell + \sum_{j=1}^{i} v_j} \geq \frac{\ell + 2 + (i-2)(\ell-1)}{\ell + (i-2)(\ell-2)},$$

which gives the following contradiction, as $\ell \geq 5$:

$$\ell(\ell-1) > (\ell-2)(\ell+2).$$

Similarly, one can show that configuration $(A_{\ell-1})$, where $v_i = 1$ and $e_i = 2$, appears at most twice and any (B_j), where $v_i = \ell - 3$ and $e_i = \ell - 1$, at most once. Furthermore, when one of these configurations appears, the occurrence of (A_k) with $3 \leq k \leq \ell - 2$ is restricted, while only (A_2) can appear arbitrarily often.

Proof of Lemma 6. Let $\ell \geq 5$ be an integer and G be a graph such that $m(G) < (\ell-1)/(\ell-2)$. Choose an arbitrary ℓ-cycle H_1 in G and assign a colour c_1 to a pair of non-adjacent edges of H_1. Let $H = H_t$, with $t \geq 2$, be the C_ℓ-component of G obtained from a construction sequence $(H_1, \ldots, H_t)$.

Now we consider a few cases according to which configurations given by Proposition 7 occur in $(H_1, \ldots, H_t)$. For each $1 \leq i \leq t - 1$, note that there can be many cycles in H_{i+1} that are not in H_i. We will assign colours to the edges of $E(H_{i+1}) \setminus E(H_i)$ in such a way that in H_{i+1} any ℓ-cycle has two edges coloured with the same colour.

By Proposition 7, $H_{i+1} - H_i$ has at most two paths P and P' which are its connected components. In case each of these paths has at least two vertices, we can choose two new colours c and c', and assign c to two non-adjacent edges of P and c' to two non-adjacent edges of P'. Then any ℓ-cycle of H_{i+1} which contains these paths becomes non-rainbow (see Fig. 2(a)). If H_{i+1} has configuration (A_k) with $2 \leq k \leq \ell - 2$, this is how we proceed, unless stated otherwise. But it may be the case that an ℓ-cycle of H_{i+1} which is not in H_i does not contain such paths; it may be formed by the addition of an edge between vertices of H_i or each of the components of $H_{i+1} - H_i$ may have only one vertex (see Fig. 2(b)). We have to be more careful with the colouring in such a case.

Recall that by the density argument preceding this proof, configuration (A_ℓ) appears at most once, $(A_{\ell-1})$ at most twice, and any (B_j) at most once. As observed above, if for every $1 \leq i \leq t - 1$, the graph H_{i+1} has configuration (A_k) with $2 \leq k \leq \ell - 2$, we can easily avoid a rainbow C_ℓ by assigning, for each $1 \leq i \leq t - 1$, a new colour c_{i+1} to two non-adjacent edges of $E(H_{i+1}) \setminus E(H_i)$. Thus, from now on we assume that there exists at least one H_{i+1} $(1 \leq i \leq t - 1)$ with configuration $(A_{\ell-1})$, (A_ℓ), or (B_j) for some $3 \leq j \leq \ell - 1$. We split our proof into a few cases, depending on the occurrence of these configurations.

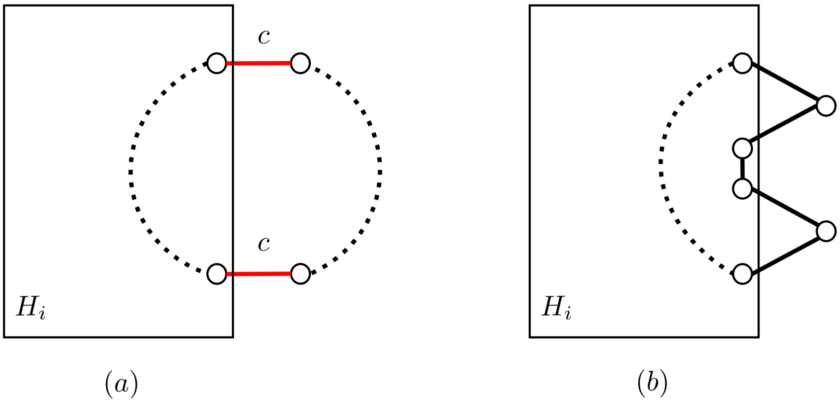

Fig. 2. Examples of cases to consider when colouring $E(H_{i+1}) \setminus E(H_i)$.

Case 1. *There is an index $1 \leq i_1 \leq t-1$ such that H_{i_1+1} has configuration (A_ℓ).*

In this case, for all $i \neq i_1$, H_{i+1} has configuration (A_2) or (A_3), by the density argument:

$$\frac{\ell-1}{\ell-2} > \frac{\ell+1+e_i+(t-3)(\ell-1)}{\ell+v_i+(t-3)(\ell-2)}$$

is equivalent to $v_i > \ell-4$ when we substitute v_i+1 for e_i and contradicts $\ell \geq 5$ when we substitute $\ell-3$ for v_i and $\ell-1$ for e_i (so configuration (B_j) cannot occur for any j). Moreover, at most one H_{i+1} (for some $1 \leq i \leq t-1$) has configuration (A_3).

Let $C = u_1u_2\cdots u_\ell u_1$ be an ℓ-cycle added to H_{i_1} to form H_{i_1+1}, where $P = u_1u_2\cdots u_\ell$ is an ℓ-path in H_{i_1} and $u_\ell u_1 \notin E(H_{i_1})$. The number of ℓ-cycles in H_{i_1+1} which are not in H_{i_1} is exactly the number of ℓ-paths in H_{i_1} with endpoints u_1 and u_ℓ.

First suppose that P is the only ℓ-path between u_1 and u_ℓ in H_{i_1}. Let C' be an ℓ-cycle in H_{i_1} that contains the edge u_2u_3. W.l.o.g. we may assume that $H_1 = C'$. Then, give colour c_1 to two non-adjacent edges of C' that are not u_2u_3. For every H_{i+1} with $1 \leq i \leq i_1-1$ we assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$ (different from u_2u_3). Therefore, in step H_{i_1+1}, we can give a new colour c_{i_1+1} to u_1u_ℓ and u_2u_3. Note that this partial colouring of H_{i_1+1} gives two edges of the same colour in each C_ℓ.

Suppose that H_{i_1} contains more than one ℓ-path between u_1 and u_ℓ. Let $P' = u_1x_2\cdots x_{\ell-1}u_\ell$ with $P' \neq P$ be one of these paths. One can see by induction on the number of steps i_1 that the only possibilities for cycle lengths not greater than $2\ell-2$ in H_{i_1} are $2\ell-2$, $2\ell-4$, or ℓ, since each of $H_2, \ldots, H_{i_1}$ has either configuration (A_2) or (A_3). Thus we have that $P \cup P'$ contains a cycle of length $2\ell-2$, $2\ell-4$, or ℓ.

If $P \cup P'$ forms a $(2\ell-2)$-cycle C', then C' appears in H_{i_1} with configuration (A_2). W.l.o.g. we assume that $i_1 = 2$. Then, we colour alternately the edges of C' with a colour c_1, which implies that each of $E(P)$ and $E(P')$ contains at least two non-adjacent edges with the same colour. Note that H_2 may contain at most one other ℓ-path P'' between u_1 and u_ℓ, in which case ℓ must be even (and so $\ell \geq 6$). But such P'' contains at least two consecutive edges of P and two consecutive edges of P' and then it must contain two edges with colour c_1. Therefore, every ℓ-cycle in H_{i_1+1} is non-rainbow.

Suppose now that $P \cup P'$ contains a $(2\ell-4)$-cycle C'. Then, C' appears in H_{i_1} with configuration (A_3) (with two ℓ-cycles having exactly a 3-path in common). We may assume w.l.o.g. that $x_2 = u_2$, H_2 has configuration (A_3) and $(P \cup P') - u_1 \subseteq H_2$ (note that C' lies in $(P \cup P') - u_1$). We colour the edges of C' alternately with two colours c_1 and c_2. If ℓ is even, then there may be another $(\ell-1)$-path P'' between u_2 and u_ℓ in H_2 (other than $P-u_1$ and $P'-u_1$). One can easily check that P'' must contain two edges with the same colour (c_1 or c_2), by observing the colours given to the edges of C' which are adjacent to the endpoints of the 3-path xyz, where $x, z \in C'$ and y is the unique vertex in $((P \cup P') - u_1) - C'$.

Now consider that $P \cup P'$ contains an ℓ-cycle C'. W.l.o.g. assume that $H_1 = C'$ and $|E(P \cap C')| \leq |E(P' \cap C')|$. We assign a colour c_1 to two non-adjacent edges of $P' \cap C'$ (note that $|E(P' \cap C')| \geq \lceil \ell/2 \rceil \geq 3$). We cannot have $|E(P' \cap C')| = \ell-1$, because $u_\ell u_1 \notin E(H_{i_1})$. Hence $|E(P \cap C')| \geq 2$ and one of the edges of $P \cap C'$, say xy, satisfies $\{x, y\} \cap \{u_1, u_\ell\} = \varnothing$. So we assign a colour c_2 to xy and $u_\ell u_1$.

Case 2. *There are $1 \leq i_1 < i_2 \leq t-1$ such that H_{i_1+1} and H_{i_2+1} have configuration $(A_{\ell-1})$.*

By the density argument, this case occurs only if $\ell = 5$:

$$\frac{\ell-1}{\ell-2} > \frac{\ell+4+(t-3)(\ell-1)}{\ell+2+(t-3)(\ell-2)}$$

is equivalent to $6 > \ell$. A density argument also gives us that every H_{i+1} has configuration (A_2) for $i \neq i_1, i_2$:

$$\frac{\ell-1}{\ell-2} > \frac{5+4+(v_i+1)+(t-4)(\ell-1)}{5+2+v_i+(t-4)(\ell-2)}$$

is equivalent to $v_i > 2$.

Let $C = u_1u_2u_3u_4u_5u_1$ and $C' = v_1v_2v_3v_4v_5v_1$ be cycles where C is in H_{i_1+1} but not in H_{i_1} and C' is in H_{i_2+1} but not in H_{i_2}. W.l.o.g. let $P = u_1u_2u_3u_4$ and $P' = v_1v_2v_3v_4$ in H_{i_2} be 4-paths, and $u_5 \notin V(H_{i_1})$ and $v_5 \notin V(H_{i_2})$.

Note that P is the only 4-path between u_1 and u_4 in H_{i_1} and thus C is the only 5-cycle added to H_{i_1} to form H_{i_1+1}. However, it is possible that besides P' there exists one other 4-path P'' in H_{i_2} between v_1 and v_4. If this is the case, then $P' \cup P''$ contains either a 4-cycle or a 6-cycle. This information will be useful in what follows.

We divide this proof into three parts depending on the structure of P in H_{i_1}: (a) the three edges of P lie in the same 5-cycle, (b) exactly two consecutive edges of P lie in the same 5-cycle, or (c) any 5-cycle in H_{i_1} contains at most one edge of P.

(a) The three edges of P lie in the same 5-cycle.

W.l.o.g. assume that all the edges of P lie in H_1 and $i_1 = 1$. Hence H_1 is of the form $H_1 = u_1u_2u_3u_4x_5u_1$ for some x_5. Note that $C'' = u_1x_5u_4u_5u_1$ is a 4-cycle in H_2.

Suppose all the edges of P' lie in H_2. Then, w.l.o.g., we may assume $i_2 = 2$. If the endpoints of P' are u_1 and u_3, then there is another 4-path P'' between u_1 and u_3 in H_1, say w.l.o.g. $P' = u_1u_5u_4u_3$ and $P'' = u_1x_5u_4u_3$. We assign a colour c_1 to u_4x_5, u_1u_2 and u_3v_5, and a colour c_2 to u_2u_3, u_4u_5 and v_5u_1. In this way we make all 5-cycles in H_3 non-rainbow. The case in which the ends of P' are u_4 and u_2 is symmetric.

For all the remaining possibilities for the endpoints of P', we assign a colour c_1 to u_1u_2 and u_4u_3. If the endpoints of P' are two adjacent vertices in $V(C'')$, then we colour two non-adjacent edges of C' with a new colour c_2. If the ends of P' are u_1 and u_4, then the colouring we gave to u_1u_2 and u_4u_3 already makes every 5-cycle in H_3 non-rainbow. If the endpoints of P' are x_5 and a vertex in $\{u_2, u_3\}$, then we assign a new colour c_2 to v_5x_5 and u_2u_3. The case in which the ends of P' are u_5 and a vertex in $\{u_2, u_3\}$ is symmetric. Thus, we assume that there is no 4-path with endpoints v_1 and v_4 and all edges in H_2.

If at most two edges of P' are in H_2, then for any 4-path with endpoints v_1 and v_4 its edges in H_2 must be consecutive. Hence we may assume w.l.o.g. that, for P', the edge v_3v_4 is not in $E(H_2)$. There is no triangle in H_{i_2}, because H_{i_1+1} has none and triangles cannot be introduced in steps of configuration (A_2), so there can be no 6-cycle in H_{i_2}. As the unique 4-cycle in H_{i_2} has its edges in H_2, the 4-path P'' between v_1 and v_4 ($P'' \neq P'$), if it exists, contains the edge v_3v_4. Note that we can colour two non-adjacent edges of any H_i with configuration (A_2) avoiding colouring the edge v_3v_4. Thus, we assign a colour c_1 to u_1u_2 and u_3u_4, and a new colour c_2 to v_3v_4 and v_5v_1.

(b) Exactly two consecutive edges of P lie in the same 5-cycle.

W.l.o.g. H_1 contains the edges u_1u_2 and u_2u_3 but does not contain u_3u_4. Thus H_1 is of the form $H_1 = u_1u_2u_3x_4x_5u_1$ for some x_4 and x_5, and $C'' = u_1x_5x_4u_3u_4u_5u_1$ is a 6-cycle in H_{i_1+1}. We can assume w.l.o.g. that if P' lies in some 6-cycle then C'' is such cycle (by choosing H_1 appropriately among at most two possibilities). Note that H_{i_2} contains no 4-cycle. Hence, if there are two 4-paths between v_1 and v_4, they correspond to two internally disjoint paths in C''. Suppose that $E(P') \subseteq E(C'')$. In this case, alternately colour the edges of C'' with colours c_1 and c_2 and, for $1 \leq i \leq t-1$, with $i \neq i_1, i_2$, assign a new colour c_{i+2} to two non-adjacent edges in $E(H_{i+1}) \setminus (E(H_i) \cup \{u_3u_4\})$. Now we assume that $E(P') \nsubseteq E(C'')$. Thus P' is the only 4-path between v_1 and v_4 in H_{i_2}. If $E(P') \subseteq E(H_1)$ then $E(P') \cap \{u_1u_2, u_2u_3\} \neq \varnothing$ (since $E(P') \nsubseteq E(C'')$, the path P' cannot be $u_1x_5x_4u_3$), and we colour u_4u_5 and the two non-adjacent edges in $E(P')$ with c_1. Assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$, for $1 \leq i \leq t-1$, $i \neq i_1, i_2$. Now we assume that $E(P') \nsubseteq E(H_1)$ (possibly $P' = P$). Therefore, P' has an edge v_jv_{j+1} with $1 \leq j \leq 3$ which does not belong to $E(H_1)$. Colour u_2u_3, x_4x_5 and an edge in $\{u_5u_1, u_5u_4\} \setminus \{v_jv_{j+1}\}$ with c_1, and give a new colour c_{i_2+1} to v_jv_{j+1} and to some edge in $\{v_5v_1, v_5v_4\}$ not incident with v_j nor with v_{j+1}.

(c) Any 5-cycle in H_{i_1} contains at most one edge of P.

In H_{i_2} there are neither 4-cycles nor 6-cycles, and therefore P' is the only 4-path between v_1 and v_4. We may assume w.l.o.g. that H_1 contains u_2u_3. If $P' = P$, then we assign a colour c_1 to the edges u_2u_3, u_5u_1 and v_5u_4, and assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$, for $1 \leq i \leq t-1$, $i \neq i_1, i_2$. Now we assume that $P' \neq P$. Since P' is the only 4-path in H_{i_2} between v_1 and v_4, we know that P' and P cannot have both endpoints in common. Therefore, w.l.o.g., we may assume that $v_1 \notin \{u_1, u_2, u_4\}$. We assign a new colour c_1 to the edges u_2u_3 and u_5u_1. If $v_2v_3 \in \{u_2u_3, u_5u_1\}$ then colour v_5v_1 with c_1, otherwise, colour v_2v_3 and v_5v_1 with a new colour c_2. Then, we assign a new colour c_{i+2} to two non-adjacent edges in $E(H_{i+1}) \setminus (E(H_i) \cup \{v_2v_3\})$, for $1 \leq i \leq t-1$, $i \neq i_1, i_2$.

Case 3. *There is exactly one $1 \leq i_1 \leq t-1$ such that H_{i_1+1} has Configuration $(A_{\ell-1})$.*

By the density argument, H_{i+1} has Configuration (A_k) with $2 \leq k \leq 4$ for all $i \neq i_1$:

$$\frac{\ell-1}{\ell-2} > \frac{\ell+2+e_i+(t-3)(\ell-1)}{\ell+1+v_i+(t-3)(\ell-2)}$$

is equivalent to $v_i > \ell - 5$ when we substitute $v_i + 1$ for e_i and contradicts $\ell \geq 5$ when we substitute $\ell - 3$ for v_i and $\ell - 1$ for e_i (so configuration (B_j) cannot occur for any j).

Let $C = u_1u_2 \cdots u_\ell u_1$ be a cycle where C is in H_{i_1+1} but not in H_{i_1} and let $P = u_1 \cdots u_{\ell-1}$ be an $(\ell-1)$-path in H_{i_1}. The number of ℓ-cycles in H_{i_1+1} which are not in H_{i_1} is exactly the number of $(\ell-1)$-paths in H_{i_1} with endpoints u_1 and $u_{\ell-1}$. The remainder of the proof of Case 3 is similar to the proof of Case 1, but we include it here for completeness.

First, suppose that P is the only $(\ell-1)$-path between u_1 and $u_{\ell-1}$ in H_{i_1} Let C' be an ℓ-cycle in H_{i_1} that contains the edge u_2u_3. W.l.o.g. $H_1 = C'$. Then, give colour c_1 to two non-adjacent edges of C' that are not u_2u_3. For every H_{i+1} with $1 \leq i \leq i_1 - 1$ we assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$ different from u_2u_3. Therefore, in step H_{i_1+1}, we give a new colour c_{i_1+1} to u_1u_ℓ and u_2u_3. Note that in this partial colouring of H_{i_1+1} every copy of C_ℓ has two non-adjacent edges of the same colour.

Suppose that H_{i_1} contains more than one $(\ell-1)$-path between u_1 and $u_{\ell-1}$. Let $P' = u_1x_2 \cdots x_{\ell-2}u_{\ell-1}$ with $P' \neq P$ be one of these paths. Since there is no configuration (A_k) with $k \geq 5$, one can see that $P \cup P'$ contains a cycle of length $2\ell-4$, $2\ell-6$, or ℓ.

If $P' \cup P$ forms a $(2\ell-4)$-cycle C' in H_{i_1} (P' and P are internally disjoint), then we may assume w.l.o.g. that $i_1 = 2$, that H_2 has configuration (A_3), and $P \cup P' \subseteq H_2$. Then, we assign alternately colours c_1 and c_2 to the edges of C'. Note that if ℓ is even then H_2 may contain another $(\ell-1)$-path P'' between u_1 and $u_{\ell-1}$. But then it is not hard to see that P'' contains two edges of C' with the same colour.

Suppose now that $P \cup P'$ contains a $(2\ell-6)$-cycle C'. We may assume w.l.o.g. that $x_2 = u_2$, H_2 has configuration (A_4), and $(P \cup P') - u_1 \subseteq H_2$. We colour the edges of C' alternately with two colours c_1 and c_2, and colour the two non-adjacent edges of $C' \cap H_1$ with a new colour c_3. If ℓ is even, then there may be another path P'' between u_2 and $u_{\ell-1}$ in H_2. Such path contains the edges of $C' \cap H_1$, and therefore have two edges with the same colour.

Now consider that $P \cup P'$ contains an ℓ-cycle C'. W.l.o.g. $H_1 = C'$. Clearly $|E(P \cap C')|, |E(P' \cap C')| \geq 3$. Thus, we just colour the edges of C' alternately with two colours c_1 and c_2.

Case 4. *There is $1 \leq i_1 \leq t-1$ such that H_{i_1+1} has configuration (B_j) for some $3 \leq j \leq \ell - 1$.*

By the density argument, H_{i+1} has configuration (A_2) for all $i \neq i_1$:

$$\frac{\ell-1}{\ell-2} > \frac{\ell + (\ell-1) + (v_i+1) + (t-3)(\ell-1)}{\ell + (\ell-3) + v_i + (t-3)(\ell-2)}$$

is equivalent to $v_i > \ell - 3$.

Let $C = u_1u_2 \cdots u_\ell u_1$ be an ℓ-cycle added to H_{i_1} to form H_{i_1+1}, where $u_1u_2 \in E(H_{i_1})$, $u_2u_3 \notin E(H_{i_1})$, $(\{u_3, \ldots, u_\ell\} \setminus \{u_j\}) \subseteq V(H_{i_1+1}) \setminus V(H_{i_1})$, and $u_j \in V(H_{i_1})$. If there is a path P in H_{i_1} between u_1 and u_j such that $V(P) \cup \{u_{j+1}, \ldots, u_\ell\}$ induces an ℓ-cycle in H_{i_1+1} or there is a path P' in H_{i_1} between u_2 and u_j such that $V(P') \cup \{u_3, \ldots, u_{j-1}\}$ induces an ℓ-cycle in H_{i_1+1}, then H_{i_1+1} can be constructed with a construction sequence in which the last two steps have configuration (A_j) and $(A_{\ell-j+3})$, respectively, and therefore we have a construction sequence that we already know how to colour (see Cases 1, 2, and 3). So we may suppose that H_{i_1} contains none of these paths, and thus we assign a new colour c_{i_1+1} to u_2u_3 and $u_\ell u_1$. $\square$

3. Cycle on four vertices

In this section we prove that $p_{C_4}^{\mathrm{rb}} = n^{-3/4}$. By a classical result of Bollobás (see [5]), we know that if $p \gg n^{-3/4}$, then a.a.s. $G(n,p)$ contains a copy of $K_{2,4}$. It is not hard to see that in any proper colouring of the edges of $K_{2,4}$ there is a rainbow copy of C_4, which implies that $p_{C_4}^{\mathrm{rb}} \leq n^{-3/4} = n^{-1/m(K_{2,4})}$.

Let $G = G(n,p)$ where $p \ll n^{-3/4}$. To prove that $p_{C_4}^{\mathrm{rb}} \geq n^{-3/4}$, we must show that a.a.s. there exists a proper colouring of G that contains no rainbow copy of C_4. For that, it is sufficient to show that we can colour the edges of an arbitrary C_4-component of G in a way that all the C_4-copies in it are non-rainbow.

The following property holds: a.a.s. G does not contain any graph H with $m(H) \geq 4/3$ and $|V(H)| \leq 12$. Indeed, let H' be a fixed graph with $m(H') \geq 4/3$ and $|V(H')| \leq 12$, $H'' \subseteq H'$ be a subgraph with $e(H'')/v(H'') = m(H')$, and $X_{H''}$ be the number of copies of H'' in G. Clearly, $\mathbb{P}(H' \subseteq G) \leq \mathbb{P}(X_{H''} \geq 1)$, and Markov's inequality gives us that $\mathbb{P}(X_{H''} \geq 1) \leq \mathbb{E}(X_{H''}) \leq n^{v(H'')}p^{e(H'')} = o(1)$. Finally, we conclude that the probability that $H \subseteq G$ for some H with $m(H) \geq 4/3$ and $|V(H)| \leq 12$ is $o(1)$ by the union bound.

Let $F = (F_1, \ldots, F_t)$ be a C_4-component of G with $m(F) \geq 4/3$. Let $2 \leq i \leq t$ be the smallest index such that $m(F_i) \geq 4/3$. Then, the density argument allows us to conclude that $|V(F_i)| \leq 12$:

$$\frac{4}{3} > m(F_{i-1}) = \frac{4 + \sum_{j=1}^{i-2} e_j}{4 + \sum_{j=1}^{i-2} v_j} \geq \frac{4 + \sum_{j=1}^{i-2}(v_j+1)}{4 + \sum_{j=1}^{i-2} v_j} \geq \frac{4 + 3(i-2)}{4 + 2(i-2)}$$

implies $i < 6$, so $|V(F_i)| \leq 4 + 2(i-1) \leq 12$. Therefore, a.a.s. every C_4-component F of G satisfies $m(F) < 4/3$.

Let $F = (F_1, \ldots, F_t)$ be an arbitrary C_4-component of G with $m(F) < 4/3$. By Proposition 7 (note that $m(F) < 3/2$), each F_i, for $2 \leq i \leq t$, has one of the configurations (A_2), (A_3), (A_4) or (B_3). By density arguments (using the bound $m(F) < 4/3$), neither (A_4) nor (B_3) can occur, and if (A_3) occurs, then all other configurations are (A_2). If F_i has configuration (A_2) for every $2 \leq i \leq t$, then we give a new colour to two non-adjacent edges of $H_i - H_{i-1}$, avoiding rainbow copies of C_4 at each step. On the other hand, if some step of construction has configuration (A_3), then we may assume w.l.o.g. that F_2 has configuration (A_3) and we see that it is easy to colour F_2 and all the subsequent F_i's avoiding rainbow copies of C_4.

4. Concluding remarks

The problem of determining the threshold p_H^{rb} for the anti-Ramsey property $G(n,p)\xrightarrow{\mathrm{rb}}H$ for graphs H is far from being completely solved. We believe that an adaptation of the framework developed in [10] and the ideas described in this paper could be useful to prove that $n^{-1/m_2(H)}$ is in fact the threshold for other classes of graphs, for example, not so small bipartite graphs H (note that this is not the case for C_4). One of the main directions for future research is to solve the following problem.

Problem 8. Determine all graphs H such that $p_H^{\mathrm{rb}} = n^{-1/m_2(H)}$.

We remark that the only graphs H for which the threshold is known and is not $n^{-1/m_2(H)}$ are cycles and complete graphs on four vertices. Thus, to determine the threshold for a large family of graphs for which it is not given by the maximum 2-density is also an interesting problem.

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