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Relatório Técnico

**RT-MAC-9401
An Algebraic View of
Combination Rules**

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Janeiro 94

An Algebraic View of Combination Rules

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Abstract. Combining information from multiple sources is a topic of central importance for Uncertain Reasoning. In this paper we approach the problem of formulating appropriate Combination Rules for Uncertain Reasoning from an algebraic, lattice-theoretical perspective. Our purpose is to state the problem of combination in a general, unifying language that can give us a "grand view" of the whole issue.

Keywords: representation languages for uncertain knowledge, pooling of uncertain evidence, summarisation of uncertain information.

1 Introduction

Uncertain Reasoning – the general denomination given to the problems of reasoning *with* and *about* uncertainty – is an interesting and challenging issue in Artificial Intelligence research. Among the many challenges offered by Uncertain Reasoning, the problem of how to formulate *Combination Rules* has shown to be of particular interest and difficulty. This problem can be described as follows:

Assume that we have a language to reason with, such that we can assign *degrees of belief* to well-formed sentences according to some uncertainty calculus. Typically, given a *Partial Assignment of Degrees of Belief to Sentences* (padb), we will be interested in *deriving* (or *propagating*) information about degrees of belief in other sentences of the language.

Now assume that we have *more than one* padb for the language. What is the "net" *derived* (or *propagated*) degree of belief in arbitrary sentences considering *all* padb's? In other words, what padb (if any) do we get as the result of *combining* all given padb's?

Intuitively, a *combination rule* must take into account the mutual influences among padb's. It can be stated as a *combination functional* that takes sequences of padb's to padb's. Assuming the availability of some such functional, a general two-step procedure to estimate degrees of belief in arbitrary sentences given a collection of padb's can become as follows:

1. find the "net" padb;
2. propagate the "net" padb to find the degrees of belief in the selected sentences.

In order to obtain computationally tractable combination functionals and padb propagation procedures, several simplifying assumptions have been proposed (see e.g. [Sha76, DP86, HF89, SS90, BL93]), none of which uncontroversial¹.

In the present article we *do not* propose any further set of assumptions *nor* a new combination rule. Rather, we present a simple exercise in mathematical abstraction, proposing what we believe to be a more adequate generic framework to describe the problem of Combination Rule formulation in Uncertain Reasoning, what – we hope – shall lead to a better understanding of this problem. In this sense, our goal is similar to Kennes' in [Ken91] and we urge the interested reader to contrast his paper with ours (Kennes proposes Category Theory as the generic framework for Uncertain Reasoning, whereas our proposal is more algebraic in nature).

Our paper is organised as follows: in section 2 we introduce our general framework. In section 3 we show how we can express various combination rules as special cases in this framework. Finally, in section 4 we present some further discussion.

2 The General Framework

Our purpose in this article is to abstract the essence of Uncertain Reasoning and the problem of formulating Combination Rules. Faithful to this purpose, we present here a mathematical framework as general as possible, with the minimal structuring necessary to express our problem.

First of all we need to specify a language to reason with. We choose the language of propositional logic². This language can be presented as follows [Men87]:

The *alphabet* of the language is formed by:

- a countable set of *basic propositions* $\Phi = \{p_1, p_2, \dots\}$;
- the *connectives* \neg, \rightarrow ;

The *expressions* of the language are called *formulae* and defined inductively as:

- the elements of Φ ;
- the empty formula $\{\}$;
- $\neg\varphi, \varphi \rightarrow \psi$, where φ and ψ are formulae.

¹ For example, the most widely discussed of these rules – Dempster-Shafer's combination rule [Sha76] – assumes every padb to be statistically independent from the others, thus presenting the features of being commutative and associative, together with the rather awkward property of being non-idempotent. Statistical independence has proved to be a very strong assumption to be made in many cases.

² This choice goes against our purpose of keeping the framework as general as possible. It is not a completely arbitrary choice, though, since more expressive languages (e.g. first-order logic) are much harder to treat algebraically. Moreover, propositional logic has been considered good enough by many other authors (see e.g. [Sha76, Bun85, Nil86, FH89]).

The connectives \vee and \wedge are introduced as *abbreviations* for particular expressions using the connectives presented above:

- $\varphi \vee \psi \equiv \neg\varphi \rightarrow \psi$.
- $\varphi \wedge \psi \equiv \neg(\varphi \rightarrow \neg\psi)$.

Interpretations are total functions $\pi : \mathcal{F} \rightarrow \{\perp, \top\}$ - where \mathcal{F} denotes the set of formulae - such that:

- $\pi(\neg\varphi) = \begin{cases} \top & \text{iff } \pi(\varphi) = \perp; \\ \perp & \text{otherwise} \end{cases}$
- $\pi(\varphi \rightarrow \psi) = \begin{cases} \perp & \text{iff } \pi(\varphi) = \top \text{ and } \pi(\psi) = \perp; \\ \top & \text{otherwise} \end{cases}$

The *Lindenbaum algebra* of a propositional logic P [BS71] is the boolean algebra \mathcal{L} in which:

- *objects* are given by equivalence classes of formulae: any two formulae φ and ψ belong to the same equivalence class whenever we have that $\pi_i(\varphi \rightarrow \psi) = \pi_i(\psi \rightarrow \varphi) = \top$ in all interpretations $\pi_i : \mathcal{F} \rightarrow \{\perp, \top\}$; and
- *ordering* is given by implication: a formula φ is said to be *below* a formula ψ ($\varphi \leq \psi$) whenever we have $\pi_i(\varphi \rightarrow \psi) = \top$ in all interpretations π_i .

Now we consider a sub-algebra \mathcal{L}_k of the Lindenbaum algebra \mathcal{L} - the sub-algebra of objects we know how to assign beliefs to. We say that an object L of \mathcal{L} is *approximated by the pair* (L_*, L°) of objects of \mathcal{L}_k if:

- L_* is the maximum element in \mathcal{L}_k such that $L_* \leq L$, and
- L° is the minimum element in \mathcal{L}_k such that $L \leq L^\circ$.

Next, we must characterise our degrees of belief in elements of \mathcal{L}_k . We say we believe in an object L to the extent that we can envisage a scenario in which a (any) sentence $\varphi \in L$ is true. Thus, we have a collection \mathcal{S} of *scenarios* and a mapping i from \mathcal{L}_k to \mathcal{S} . It seems natural to expect that this mapping is order-preserving, i.e. that \mathcal{S} has a partial order and that $i(L_1) \leq i(L_2)$ whenever $L_1 \leq L_2$. Moreover, following [Bir40], the least requirement for \mathcal{S} to be the domain of a measure of uncertainty (called "probability" in that reference) is that it be a complemented lattice³. This makes of i a *lattice homomorphism* between \mathcal{L}_k and \mathcal{S} .

Finally, we can consider a complemented sub-lattice \mathcal{S}_k of \mathcal{S} - the sub-lattice of scenarios we know how to evaluate the associated uncertainty measure. We say that a scenario S of \mathcal{S} is *approximated by the pair* (S_*, S°) of scenarios of \mathcal{S}_k if:

³ A complemented lattice \mathcal{S} is such that, for any $S \in \mathcal{S}$, there is an $\bar{S} \in \mathcal{S}$ such that $S \cup \bar{S} = 1$ and $S \cap \bar{S} = 0$, where 1 and 0 are respectively the maximum and minimum elements of \mathcal{S} .

- S_* is the maximum element in S_k such that $S_* \leq S$, and
- S^* is the minimum element in S_k such that $S \leq S^*$.

An *uncertainty measure* is any real-valued function $\mu : S_k \rightarrow \mathbb{R}$ which is *positive* and *submodular*⁴.

Given an algebra \mathcal{L} and a lattice S , we define a *Partial Assignment of Degrees of Belief* padb as a quadruple $[\mathcal{L}_k, i, S_k, \mu]$.

This completes the essential tools we need to specify Combination Rules. If we consider the set \mathcal{P} of all padb's given a Lindenbaum algebra \mathcal{L} and a complemented lattice S , a *Combination Rule* is any functional $\mathbb{K} : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$, i.e. any mapping from pairs of padb's to padb's. Any further refinement on the definition of \mathbb{K} shall be motivated by the (computational) difficulties the general definition may present.

Typically, these refinements have taken form of further assumptions of algebraic properties that ensure some form of tractability (e.g. if we assume \mathbb{K} to be commutative and associative we can easily generalise it to combine more than two padb's). In the next section we discuss a few (very) simple examples.

3 Some (very) Simple Examples

In this section we present three particular Combination Rules:

3.1 Fagin-Halpern's Combination Rule

Fagin-Halpern Combination Rule was presented in [FH89]. It can be presented using our framework as follows:

- $\mathbb{K}_{FH} : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$;
- given $\text{padb}_1 = [\mathcal{L}, i, S_{k1}, \mu_1]$ and $\text{padb}_2 = [\mathcal{L}, i, S_{k2}, \mu_2]$, we define

$$S_k = \{S_1 \cap S_2 . S_1 \in S_{k1}, S_2 \in S_{k2}\}$$

$$\mu : S_k \rightarrow \mathbb{R}$$

$$\mu(S_1 \cap S_2) = c\mu_1(S_1)\mu_2(S_2), \text{ where } c \text{ is a normalising constant.}$$

$$\mathbb{K}_{FH}(\text{padb}_1, \text{padb}_2) = [\mathcal{L}, i, S_k, \mu].$$

This rule assumes that padb_1 and padb_2 are statistically independent. As can be easily checked, Fagin-Halpern's Combination Rule is commutative and associative, but non-idempotent. Observe also that the whole Lindenbaum algebra \mathcal{L} belongs to the definition of padb_1 and padb_2 .

⁴ A measure μ is *positive* if $\mu(S_1) \leq \mu(S_2)$ whenever $S_1 \leq S_2$. It is *submodular* if $\mu(S_1) + \mu(S_2) \leq \mu(S_1 \cup S_2) + \mu(S_1 \cap S_2)$, $S_1, S_2 \in S_k$.

3.2 Perfect Dominance Combination Rule

A "Perfect Dominance Combination Rule" can be defined as follows:

$$- \mathbb{K}_{pd} : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P};$$

$$\mathbb{K}_{pd}(\text{padb}_1, \text{padb}_2) = \text{padb}_2.$$

Intuitively, this rule dictates that "a second opinion is always preferable to a first one". This very simple rule is idempotent and associative, but it is not commutative.

3.3 Skeptical Default Combination Rule

A "Skeptical Default Combination Rule" can be defined as follows:

$$- \mathbb{K}_{sd} : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P};$$

$$\mathbb{K}_{sd}(\text{padb}_1, \text{padb}_2) = \begin{cases} \text{padb}_1, & \text{if } \text{padb}_1 = \text{padb}_2 \\ \text{padb}_{default}, & \text{otherwise} \end{cases}$$

Intuitively, this rule dictates that "all opinions shall be discarded in favour of a standard ("default") opinion whenever there is disagreement" among opinions. It is idempotent, associative and commutative.

4 Conclusion

In this paper we presented a simple reconstruction of Combination Rules based on algebraic lattice theory. Our goal was to characterise the essentials of the problem of formulation of Combination Rules, to form the basis of a Unified Theory of Combination, of which existing rules would become particular cases.

Immediate future work includes a more thorough analysis of formerly proposed Combination Rules, as well as the formulation and analysis of general classes of Combination Rules that present nice computability features.

Acknowledgements: the author is partially supported by FAPESP grant 93/0603-01, and CNPq grant 300041/93-4.

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