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ORIGINAL ARTICLE

The Metivier inequality and ultradifferentiable hypoellipticity

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Abstract
In 1980, Métivier characterized the analytic (and Gevrey) hypoellipticity of L^2-solvable partial linear differential operators by a priori estimates. In this note, we extend this characterization to ultradifferentiable hypoellipticity with respect to Denjoy–Carleman classes given by suitable weight sequences. We also discuss the case when the solutions can be taken as hyperfunctions and present some applications.

KEYWORDS
Denjoy–Carleman classes, ultradifferentiable hypoellipticity

INTRODUCTION

In this work, we study the regularity of solutions of linear partial differential equations within the framework of Denjoy–Carleman classes defined by appropriate weight sequences. In fact, our principal concern here is to what extend a result due to G. Métivier (cf. [8]), proved in the study of analytic and Gevrey regularity, could still be valid in this more general set up.

More precisely, in [8], a characterization of analytic hypoellipticity is presented for L^2-solvable linear partial differential equations in terms of a very precise a priori inequality. The author also mentions that a similar characterization for Gevrey hypoellipticity is also valid, and in [1] the result is extended for pseudodifferential operators.

In this work, we are able to extend this Métivier result for what we call here *admissible weight sequences* (see Definition 1). The corresponding Denjoy–Carleman classes contain the Gevrey classes of order $s \geq 1$ properly. It is important to note that it is irrelevant in our presentation if the classes are either quasi-analytic or non–quasi-analytic.

One of the main points in Métivier's argument is to give a positive answer to a question related to the concept of solvability: Assume that $P = P(x, D)$ is a real analytic, linear partial differential operator in an open set $\Omega \subseteq \mathbb{R}^n$, which is L^2-solvable and Gevrey hypoelliptic of some order $s \geq 1$. Take $f \in \mathcal{G}^s(\Omega)$, an open set $U \Subset \Omega$, and let $u \in L^2(U)$ solve $Pu = f$ in U with minimum L^2-norm. Automatically $u \in \mathcal{G}^s(U)$. Is it possible to bound the Sobolev norms of $u|_V$, where $V \Subset U$ is another open set, in terms of the Gevrey norms of $f|_U$?

This is what is achieved by Métivier when $s = 1$. The extension to the Gevrey case of arbitrary order is straightforward. Indeed, Métivier's argument in the analytic case is based on an interpolation method for which the choice of suitable subsequences of $(j!)^{1/j} \sim j$ (in this case the subsequences are explicitly described) is needed. The main difference in the Gevrey case is that now subsequences of $(j!^s)^{1/j} \sim j^s$ are needed, which can be obtained from the ones of the real analytic case after applying the uniform deformation $j \mapsto j^s$, cf. [1].

This deformation argument is no longer possible in the Denjoy–Carleman case. The situation is now much more delicate, and the determination of the class of weight sequences for which the result is valid is indeed one of the key points in our work (see Definition 1). Moreover the proof of our main result requires several new insights, which we believe justifies its publication.

After we discuss several results on weight sequences and Denjoy–Carleman classes in Section 1, we state our main result in Section 2 (Theorem 1) and prove some consequences of it. Section 3 is devoted to the proof of Theorem 1. Finally, in Section 4, we extend our result to the hyperfunction set up and discuss the case of Hörmander's sum of squares operators.

1 | PRELIMINARIES ON WEIGHT SEQUENCES AND THE CORRESPONDING DENJOY–CARLEMAN CLASSES

We say that a sequence of positive numbers $\mathbf{M} = (M_k)_k$ is a weight sequence if $M_0 = 1$, $M_1 \geq 1$, and $\mathbf{M}$ is logarithmic convex, that is,

$$M_k^2 \leq M_{k-1}M_{k+1}, \quad \forall k \in \mathbb{N}, \tag{1}$$

and

$$\lim_{k\to\infty} \sqrt[k]{M_k} = \infty. \tag{2}$$

If $\mathbf{M}$ is a weight sequence, the Denjoy–Carleman class $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open, of ultradifferentiable functions associated to $\mathbf{M}$, consists of all functions $f \in \mathcal{E}(\Omega)$, for which the following holds: For every compact set $K \subseteq \Omega$, there are constants $C, h > 0$ such that

$$\sup_{x\in K} |D^\alpha f(x)| \leq Ch^{|\alpha|}M_{|\alpha|}$$

for all $\alpha \in \mathbb{Z}_+^n$. It follows from (1) that $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$ is an algebra with respect to the pointwise addition and multiplication of functions. If $\mathbf{M} = (k!)_k$, then $\mathcal{E}^{\{\mathbf{M}\}}(\Omega) = \mathcal{A}(\Omega)$ is the space of analytic functions on Ω, More generally, the Gevrey classes $\mathcal{G}^s(\Omega)$, $s \geq 1$, are the Denjoy–Carleman classes associated to the weight sequences $\mathbf{G}^s = (k!^s)_k$.

In order to be able to impose further properties on the classes, we need to discuss some additional conditions on the weight sequences we shall deal with. On the set of weight sequences, we can establish the following relation: If $\mathbf{M}$ and $\mathbf{N}$ are two weight sequences, we set

$$\mathbf{M} \preceq \mathbf{N} \quad :\Longleftrightarrow \quad \exists\, C, h > 0: \ M_k \leq Ch^k N_k \quad \forall k \in \mathbb{Z}_+.$$

The relation $\preceq$ is both reflexive and transitive. When we also consider the equivalence relation $\approx$ given by

$$\mathbf{M} \approx \mathbf{N} \quad :\Longleftrightarrow \quad \mathbf{M} \preceq \mathbf{N} \ \wedge \ \mathbf{N} \preceq \mathbf{M}$$

and identify any pair $\mathbf{M}, \mathbf{N}$ with $\mathbf{M} \approx \mathbf{N}$, then $\preceq$ is also antisymmetric, that is, $\preceq$ is a partial order. We may write $\mathbf{M} \precnapprox \mathbf{N}$ if $\mathbf{M} \preceq \mathbf{N}$ and $\mathbf{N} \npreceq \mathbf{M}$.

It is easy to see that $\mathcal{E}^{\{\mathbf{M}\}}(\Omega) \subseteq \mathcal{E}^{\{\mathbf{N}\}}(\Omega)$ if $\mathbf{M} \preceq \mathbf{N}$. Thus, if $\mathbf{M}$ satisfies

$$\exists\, C, h > 0 \;:\; k! \le C h^k \mathbf{M}_k, \quad k \in \mathbb{Z}_+, \tag{3}$$

then $\mathcal{A}(\Omega) \subseteq \mathcal{E}^{\{\mathbf{M}\}}(\Omega)$.

We finally introduce a condition taken from [4]:

$$\exists A > 0 : \; \forall j, k \in \mathbb{Z}_+ : \quad M_{j+k} \le A^{j+k} M_j M_k. \tag{4}$$

Condition (4) implies in particular that $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$ is closed under differentiation and that $\mathbf{M} \preceq \mathbf{G}^s$ for some $s > 1$, see Matsumoto [7].

If $\mathbf{M}$ is a weight sequence, then we are also going to use the following sequences:

$$\mu_k = \frac{M_k}{M_{k-1}}, \qquad \Lambda_k = \sqrt[k]{M_k}.$$

We note that (1) gives that $\Lambda_k \le \mu_k$ for all $k \in \mathbb{N}$ and therefore by (2), it follows that

$$\lim_{k\to\infty} \mu_k = \infty. \tag{5}$$

Furthermore, (3) is equivalent to the existence of $\delta > 0$ such that

$$k \le \delta \Lambda_k. \tag{6}$$

From (4), it follows that there is $\sigma > 0$ such that

$$\Lambda_{k+1} \le \sigma \Lambda_k. \tag{7}$$

It is easy to see that (7) is the condition (M2') of Komatsu [4] written in terms of the sequence $(\Lambda_k)_k$. This condition is sufficient to guarantee that $\mathcal{E}^{\{\mathbf{M}\}}$ is closed under differentiation.

Definition 1. A weight sequence satisfying properties (3) and (4) will be referred to as an *admissible weight sequence*.

Clearly the Gevrey sequences $\mathbf{G}^s$ are admissible weight sequences for any $s \ge 1$.

A more general family of admissible weight sequences is defined as follows: Let $s \ge 1$ and $\sigma \ge 0$. The weight sequence $\mathbf{N}^{s,\sigma}$ is given by $N_k^{s,\sigma} = (k!)^s(\log(k+e))^{\sigma k}$. It is easy to see that $\mathbf{N}^{s,\sigma}$ is admissible for any choice of $s \ge 1$ and $\sigma \ge 0$. Furthermore, $\mathbf{N}^{s,0} = \mathbf{G}^s$ for all $s \ge 1$ and we have for $s \ge 1$ fixed that

$$\mathbf{G}^s \precnapprox \mathbf{N}^{s,\sigma} \precnapprox \mathbf{G}^{s'}$$

for all $\sigma > 0$ and every $s' > s$.

In order to present a weight sequence that is not admissible, let $q > 1$ be a parameter. The sequence $\mathbf{L}^q$ given by $L_k^q = q^{k^2}$ is a weight sequence, which satisfies (3) and (7). However, since $\mathbf{G}^s \precnapprox \mathbf{L}^q$ for all $s, q > 1$, we conclude that $\mathbf{L}^q$ cannot satisfy (4) for any $q > 1$.

Lemma 1. *Let $\mathbf{M}$ be an admissible weight sequence. Then there is a constant $\sigma > 1$ such that the following holds: For each $k \in \mathbb{N}$, there is a sequence $(k_j)_j$ such that $\Lambda_{k_0} \le \Lambda_k$ and*

$$\sigma^{j-1}\Lambda_k \le \Lambda_{k_j} \le \sigma^j \Lambda_k, \quad j \in \mathbb{N}.$$

Proof. We note that due to (1) the sequence $(\Lambda_m)_m$ is increasing and that $\Lambda_m \to \infty$ for $m \to \infty$ by (2). Furthermore, we can assume that $\sigma > 1$ in (7).

Now fix $k \in \mathbb{N}$. We construct the sequence k_j iteratively. First set $k_0 = k$. Then $\Lambda_{k_0} = \sigma^0 \Lambda_k$. Since $\Lambda_m \to \infty$, there has to be some $m > k$ such that $\Lambda_k < \Lambda_m$. Now let m_0 be the smallest integer such that $\Lambda_k < \Lambda_{m_0}$, in particular this means $\Lambda_k = \Lambda_{m_0-1}$ since the sequence $(\Lambda_m)_m$ is nondecreasing. Then, we have $\Lambda_{m_0} \leq \sigma\Lambda_{m_0-1} = \sigma\Lambda_k$. It follows that the set $T_1^k = \{m \in \mathbb{N} : \Lambda_k < \Lambda_m \leq \sigma\Lambda_k\}$ is nonempty. We choose k_1 to be the greatest element of T_1^k.

Now assume that we have chosen k_j as the greatest number in $T_j^k = \{m \in \mathbb{N} : \sigma^{j-1}\Lambda_k < \Lambda_m \leq \sigma^j \Lambda_k\}$. There again has to be some $m > k_j$ such that $\Lambda_{k_j} < \Lambda_m$. Let m_j be the smallest integer $> k_j$ such that $\Lambda_{k_j} < \Lambda_{m_j}$, which also implies that $\Lambda_{k_j} = \Lambda_{m_j-1}$. It follows that $\Lambda_{m_j} \leq \sigma\Lambda_{m_j-1} = \sigma\Lambda_{k_j} \leq \sigma^{j+1}\Lambda_k$. Thence, $T_{j+1}^k = \{m \in \mathbb{N} : \sigma^j\Lambda_k < \Lambda_m \leq \sigma^{j+1}\Lambda_k\}$ is nonempty and we choose k_{j+1} to be the greatest element in T_{j+1}^k. □

Definition 2. Let $\mathbf{M}$ be a weight sequence. The weight function associated to $\mathbf{M}$ is given by

$$\omega_{\mathbf{M}}(t) = \sup_{k \in \mathbb{Z}_+} \log \frac{t^k}{M_k}, \qquad t \geq 0. \tag{8}$$

We recall that the associated weight function $\omega_{\mathbf{M}}$ is a continuous and increasing function on the positive real line, cf. [6].

Lemma 2. [1] *Let* $\mathbf{M}$ *be an admissible weight sequence. Then the associated weight function satisfies the following estimate:*

$$\omega_{\mathbf{M}}(\Lambda_k) \leq Hk, \qquad \forall k \in \mathbb{N}, \tag{9}$$

for some constant $H \geq 1$.

Proof. Since $\omega_{\mathbf{M}}$ is a continuous and increasing function, we have that $\omega_{\mathbf{M}}(\Lambda_k) \leq \omega_{\mathbf{M}}(\mu_k)$ for all $k \in \mathbb{N}$. Hence it is enough to show that $\omega_{\mathbf{M}}(\mu_k) \leq Hk$. By Mandelbrojt [6], we know the following fact:

$$\forall k \in \mathbb{N} : \qquad \omega_{\mathbf{M}}\left(\mu_k\right) = \log \frac{\mu_k^k}{M_k}.$$

According to Matsumoto [7], condition (4) is equivalent to

$$\exists D > 0\ \forall k \in \mathbb{N} : \quad \mu_k \leq D\Lambda_k.$$

Here, we can choose $D = 2A$, where A is the constant from (4). It follows that

$$\mu_k^k \leq D^k M_k,$$

which in turn implies that

$$\omega_{\mathbf{M}}(\mu_k) = \log \frac{\mu_k^k}{M_k} \leq Hk,$$

where $H = \log D$. Since we can assume without loss of generality that $D \geq e$, we have that $H \geq 1$. □

We need also to dwell a little bit on the functional analytic structure of Denjoy–Carleman classes, for more details see [4]. We shall use the following notation: If $U \subseteq \mathbb{R}^n$ is an open subset, then $\mathcal{B}(U)$ denotes the space of all bounded, smooth functions on U, which have all its derivatives also bounded. For each weight sequence $\mathbf{M}$ and all open sets $U \subseteq \mathbb{R}^n$, we can define a norm on $\mathcal{B}(U)$ by

$$\|f\|_{\mathbf{M},U,h} = \sup_{\alpha \in \mathbb{Z}_+^n} \frac{\|D^\alpha f\|_{L^\infty(U)}}{h^{|\alpha|} M_{|\alpha|}}.$$

The resulting Banach space is

$$\mathcal{B}_{\mathbf{M},h}(U) = \{f \in \mathcal{B}(U) : \|f\|_{\mathbf{M},U,h} < \infty\}.$$

Clearly we have $\mathcal{B}_{\mathbf{M},h_1}(U) \subseteq \mathcal{B}_{\mathbf{M},h_2}(U)$ if $h_1 \leq h_2$ and thus there is a continuous and injective map $\vartheta_{h_1}^{h_2} : \mathcal{B}_{\mathbf{M},h_1}(U) \longrightarrow \mathcal{B}_{\mathbf{M},h_2}(U)$, which is compact if $h_1 < h_2$ by [4, Proposition 2.2]. We can then introduce the classes of global ultradifferentiable functions on U:

$$\mathcal{B}^{\{\mathbf{M}\}}(U) = \operatorname{ind}_{h>0}\mathcal{B}_{\mathbf{M},h}(U) = \operatorname{ind}_{\ell\in\mathbb{N}}\mathcal{B}_{\mathbf{M},\ell}(U).$$

We observe that $\mathcal{B}^{\{\mathbf{M}\}}(U)$ is a (DFS)-space and thus, in particular, a webbed space [5, p. 63, (8)]. Notice also that

$$\mathcal{B}^{\{\mathbf{M}\}}(U) = \{f \in \mathcal{B}(U) : \exists h > 0 \text{ such that } \|f\|_{\mathbf{M},U,h} < \infty\}. \tag{10}$$

Next we localize the preceding concepts. The (local) Denjoy–Carleman class associated to the weight sequence $\mathbf{M}$ on the open set $\Omega \subseteq \mathbb{R}^n$ is defined as

$$\mathcal{E}^{\{\mathbf{M}\}}(\Omega) = \operatorname{proj}_{U\Subset\Omega}\mathcal{B}^{\{M\}}(U).$$

As before, we have

$$\mathcal{E}^{\{\mathbf{M}\}}(\Omega) = \left\{f \in \mathcal{E}(U) : \ \forall\, U \Subset \Omega\ \exists\, h > 0 \text{ such that } \|f\|_{\mathbf{M},U,h} < \infty\right\}.$$

It is easy to see that $\mathcal{B}^{\{\mathbf{M}\}}(U)$ and $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$ are algebras with respect to the pointwise operations. We are going to occasionally refer to f to be of class $\{\mathbf{M}\}$ in Ω if $f \in \mathcal{E}^{\{\mathbf{M}\}}(\Omega)$.

2 | THE CONCEPT OF {M}-HYPOELLIPTICITY: STATEMENT OF THE MAIN RESULTS

Definition 3. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\mathbf{M}$ be a weight sequence. If $P = P(x, D)$ is a linear partial differential operator with coefficients in $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$, then P is $\{\mathbf{M}\}$-hypoelliptic in U if given $u \in \mathcal{D}'(U)$ the following holds: If $V \subseteq U$ is open and if $Pu|_V \in \mathcal{E}^{\{\mathbf{M}\}}(V)$, then $u|_V \in \mathcal{E}^{\{\mathbf{M}\}}(V)$.

Furthermore, P is $\{\mathbf{M}\}$-hypoelliptic at $x_0 \in \Omega$ if there is a neighborhood U_0 of x_0 such that P is $\{\mathbf{M}\}$-hypoelliptic in U_0.

Métivier [8] gave a criterion for analytic (and Gevrey) hypoellipticity at a point in the case of differential operators P with analytic coefficients in Ω, which satisfy the following condition:

$$\text{There is a continuous operator } R \ : \ L^2(\Omega) \to L^2(\Omega) \text{ such that } PR = \mathrm{Id}. \tag{$\mathbf{H}$}$$

Bove, Mughetti, and Tartakoff [1] extended this characterization to Gevrey hypoellipticity of analytic pseudodifferential operators.

Our aim is to generalize Métivier's result to $\{\mathbf{M}\}$-hypoellipticity. In order to do so, we need to introduce a weighted Sobolev norm: For an open set $U \subseteq \mathbb{R}^n$, a weight sequence $\mathbf{M}$, and $k \in \mathbb{N}$, we set

$$|||u|||_{U,\mathbf{M},k} = \sum_{|\alpha|\leq k}\Lambda_k^{k-|\alpha|}\|D^\alpha u\|_{L^2(U)}.$$

Our main result is the following theorem:

Theorem 1. *Let $\mathbf{M}$ be an admissible weight sequence, $\Omega \subseteq \mathbb{R}^n$ be an open set, and P be a differential operator with ultradifferentiable coefficients of class $\{\mathbf{M}\}$ in Ω, which satisfies (**H**).*

Then, P is $\{\mathbf{M}\}$-hypoelliptic at a point $x_0 \in \Omega$ if and only if there is a neighborhood $U_0 \subseteq \Omega$ of x_0 such that for all open sets $V \Subset U \subseteq U_0$, there are constants $C, L > 0$ such that for all $\mathcal{D}'(U)$ and all $k \in \mathbb{Z}_+$, we have

$$Pu \in H^k(U) \Longrightarrow u \in H^k(V), \tag{11}$$

$$\|u\|_{H^k(V)} \leq CL^k\left(|||Pu|||_{U,\mathbf{M},k} + M_k\|u\|_{L^2(U)}\right). \tag{12}$$

It follows immediately from condition (11) that if a differential operator P is $\{\mathbf{M}\}$-hypoelliptic at x_0 for some admissible weight sequence $\mathbf{M}$, then P is smooth hypoelliptic at x_0. Furthermore we have the following corollaries.

Corollary 1. *Let $\mathbf{M}$ and $\mathbf{M}'$ be two admissible weight sequences such that $\mathbf{M} \preceq \mathbf{M}'$. Moreover, let $\Omega \subseteq \mathbb{R}^n$ be an open set and P be a differential operator with $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$-coefficients of class $\{\mathbf{M}\}$ such that (H) holds. If P is $\{\mathbf{M}\}$-hypoelliptic at $x_0 \in \Omega$, then P is $\{\mathbf{M}'\}$-hypoelliptic at x_0.*

Proof of Corollary 1. If we set $\Lambda_k = (M_k)^{1/k}$ and $\Lambda'_k = (M'_k)^{1/k}$, then $\mathbf{M} \preceq \mathbf{M}'$ implies that there are constants C_1, h such that $\Lambda_k \leq \sqrt[k]{C_1} h \Lambda'_k$ for all $k \in \mathbb{N}$. We can assume that $C_1, h \geq 1$. Thus, we conclude that

$$\begin{aligned} |||u|||_{U,\mathbf{M},k} &= \sum_{|\alpha|\leq k} \Lambda_k^{k-|\alpha|} \|D^\alpha u\|_{L^2(U)} \\ &\leq \sum_{|\alpha|\leq k} C_1^{(k-|\alpha|)/(k)} h^{k-|\alpha|} \left(\Lambda'_k\right)^{k-|\alpha|} \|D^\alpha u\|_{L^2(U)} \\ &\leq C_1 h^k |||u|||_{U,\mathbf{M}',k}. \end{aligned}$$

If P is $\{\mathbf{M}\}$-hypoelliptic at x_0, then there is a neighborhood U_0 of x_0 such that for all $V \Subset U \subseteq U_0$ the condition (11) holds and (12) is satisfied for $\mathbf{M}$ and for some constants C_2, L independent of k, and $u \in \mathcal{D}'(U)$. Hence

$$\begin{aligned} \|u\|_{H^k(V)} &\leq C_2 L^k \left(|||Pu|||_{U,\mathbf{M},k} + M_k \|u\|_{L^2(U)}\right) \\ &\leq C_2 L^k \left(C_1 h^k |||Pu|||_{U,\mathbf{M}',k} + C_1 h^k M'_k \|u\|_{L^2(U)}\right) \\ &= C_1 C_2 (hL)^k \left(|||Pu|||_{U,\mathbf{M}',k} + M'_k \|u\|_{L^2(U)}\right) \end{aligned}$$

and therefore P is $\{\mathbf{M}'\}$-hypoelliptic at x_0. □

As a special case we obtain the following:

Corollary 2. *Let P be a differential operator with analytic coefficients in $\Omega \subseteq \mathbb{R}^n$ satisfying (H). If P is analytic hypoelliptic at some point $x_0 \in \Omega$, then P is $\{\mathbf{M}\}$-hypoelliptic at x_0 for all admissible weight sequences $\mathbf{M}$.*

3 | PROOF OF THEOREM 1

We start by introducing the Ehrenpreis–Hörmander cut-off functions: For all pairs of open sets $V \Subset U \subseteq \mathbb{R}^n$, there exists a sequence $\chi_k \in \mathcal{D}(U)$ with $\chi_k|_V \equiv 1$ for all $k \in \mathbb{N}_0$ satisfying

$$|D^\alpha \chi_k(x)| \leq Q^{|\alpha|} k^{|\alpha|}, \qquad |\alpha| \leq k.$$

We will call such a sequence an Ehrenpreis–Hörmander cut-off sequence, which is supported in U and centered in V. Ehrenpreis–Hörmander cut-off sequences have been heavily used in local and microlocal regularity theory in the analytic and ultradifferentiable category.

Lemma 3. *Let $(\chi_k)_k$ be an Ehrenpreis–Hörmander cut-off sequence supported in an open set $U \subseteq \mathbb{R}^n$. If $\mathbf{M}$ is a weight sequence satisfying (3), then there is a constant $\gamma > 0$ such that for all $k \in \mathbb{N}$ and $u \in H^k(U)$, we have*

$$|||\chi_k u|||_{\mathbb{R}^n,\mathbf{M},k} \leq \gamma^k |||u|||_{U,\mathbf{M},k}.$$

Proof. Note first if $u \in H^k(U)$, then $\chi_k u$ can be extended to an element of $H^k(\mathbb{R}^n)$ by setting 0 outside U. We note also that (6) gives that there is some $\delta > 0$ such that $k \leq \delta \Lambda_k$.

The Leibniz rule gives

$$D^\alpha(\chi_k u) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta} D^\beta\chi_k D^{\alpha-\beta}u.$$

Thus, we obtain

$$\begin{aligned}\|D^\alpha(\chi_k u)\|_{L^2(\mathbb{R}^n)} &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta} Q^{|\beta|}k^{|\beta|}\|D^{\alpha-\beta}u\|_{L^2(U)}\\ &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta} Q^{|\beta|}\delta^{|\beta|}\Lambda_k^{|\beta|}\Lambda_k^{|\alpha|-|\beta|-k}|||u|||_{U,\mathbf{M},k}\\ &= \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(Q\delta)^{|\beta|}\Lambda_k^{|\alpha|-k}|||u|||_{U,\mathbf{M},k}.\end{aligned}$$

It follows that

$$\begin{aligned}|||\chi_k u|||_{\mathbb{R}^n,\mathbf{M},k} &= \sum_{|\alpha|\leq k}\Lambda_k^{k-|\alpha|}\|D^\alpha(\chi_k u)\|_{L^2(\mathbb{R}^n)}\\ &\leq \left[\sum_{|\alpha|\leq k}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(Q\delta)^{|\beta|}\right]|||u|||_{U,\mathbf{M},k},\end{aligned}$$

and we have proven the lemma since there is a constant $\gamma > 0$ such that

$$\sum_{|\alpha|\leq k}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(Q\delta)^{|\beta|}\leq\gamma^k.$$

□

Proposition 1. *Let* $\mathbf{M}$ *be an admissible weight sequence,* Ω *be a neighborhood of* x_0*, and let* E *be a Banach space continuously injected in* $L^2(\Omega)$*. If we suppose that there is an open set* $U_0 \subseteq \Omega$ *such that* $u|_{U_0} \in \mathcal{E}^{\{\mathbf{M}\}}(U_0)$ *for all* $u \in E$*, then for any* $V \Subset U \Subset U_0$ *and all Ehrenpreis–Hörmander cut-off sequences* $(\chi_k)_k$ *supported in* U *and centered on* V*, there exist constants* $C, \gamma > 0$ *such that*

$$|\xi|^k\chi_k u \in L^2(\mathbb{R}^n), \tag{13}$$

$$\left\||\xi|^k\widehat{\chi_k u}(\xi)\right\|_{L^2(\mathbb{R}^n)} \leq C\gamma^k M_k\|u\|_E \tag{14}$$

for all $k \in \mathbb{N}$ *and* $u \in E$*.*

Proof. We have that the restriction map

$$\begin{aligned}T_U : E &\longrightarrow \mathcal{B}^{\{\mathbf{M}\}}(U),\\ f &\longmapsto f|_U,\end{aligned}$$

for $U \Subset U_0$ is a well-defined mapping. If $(f_j)_j$ is a sequence, which converges in E to 0 and $f_j|_U \to g$ in $\mathcal{B}^{\{\mathbf{M}\}}(U)$, but this implies that $\|f_j - g\|_{L^2(U)} \to 0$ since $\|\,.\,\|_{L^2(U)} \leq C\sup_U|\,.\,|$ for a constant only depending on U. On the other hand, $\|f_j\|_{L^2(\Omega)} \to 0$ since E is continuously injected in $L^2(\Omega)$. It follows that $g = 0$ and thus the graph of T_U is closed. By the version of the closed graph theorem given in [5, p. 56, (1)], we have that T_U is continuous. If we denote the unit ball in E by B_1, then we deduce that $T_U(B_1)$ is bounded and thence there is some $h > 0$ such that $T_U(B_1) \subseteq \mathcal{B}_{\mathbf{M},h}(U)$, which gives that $T_U(E) \subseteq \mathcal{B}_{\mathbf{M},h}(U)$. The closed graph theorem for Banach spaces implies now that T_U is continuous from E to $\mathcal{B}_{\mathbf{M},h}(U)$.

We denote the norm of this map by C_T and obtain for $|\alpha| = k$ that

$$\begin{aligned}\left\|D^\alpha(\chi_k u)\right\|_{L^2(\mathbb{R}^n)} &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(Qk)^{|\beta|}h^{|\alpha|-|\beta|}M_{|\alpha|-|\beta|}\|u\|_{U,\mathbf{M},h}\\ &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(Q\delta)^{|\beta|}\Lambda_k^{|\beta|}h^{|\alpha|-|\beta|}\Lambda_k^{|\alpha|-|\beta|}\|u\|_{U,\mathbf{M},h}\\ &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(Q\delta)^{|\beta|}h^{|\alpha|-|\beta|}M_{|\alpha|}C_T\|u\|_E\\ &\leq C_T(Q\delta+h)^k M_k\|u\|_E.\end{aligned}$$

Since Λ_k is increasing and by (6), there is some $\delta > 0$ such that $k \leq \delta\Lambda_k$. It follows that

$$\left\||\xi|^k\widehat{\chi_k u}\right\|^2_{L^2(\mathbb{R}^n)} \leq n^k C^2 h_1^{2k} M_k^2\|u\|_E^2,$$

where $h_1 = Q\delta + h$. □

If $\mathbf{M}$ is a weight sequence, it is easy to see that

$$\begin{aligned}\int(\Lambda_k+|\xi|)^{2k}|\hat{u}(\xi)|^2\,d\xi &= \int\left|(\Lambda_k+|\xi|)^k\hat{u}(\xi)\right|^2 d\xi\\ &\leq \sum_{\ell=0}^k\binom{k}{\ell}^2\Lambda_k^{2(k-\ell)}\int\left||\xi|^\ell\hat{u}(\xi)\right|^2 d\xi\\ &\leq 4^k\sum_{|\alpha|\leq k}\Lambda^{2(k-|\alpha|)}\|D^\alpha u\|^2_{L^2(\mathbb{R}^n)}\\ &\leq 4^k\left(|||u|||_{\mathbb{R}^n,\mathbf{M},k}\right)^2\end{aligned} \tag{15}$$

for every $u \in H^k(\mathbb{R}^n)$ and $k \in \mathbb{Z}_+$. We are also going to use the space

$$G_{\{\mathbf{M}\}} = \left\{u \in L^2(\mathbb{R}^n) \mid e^{\omega_{\mathbf{M}}(|\xi|)}\hat{u} \in L^2(\mathbb{R}^n)\right\}.$$

Then, $G_{\{\mathbf{M}\}} \subseteq \mathcal{E}^{\{\mathbf{M}\}}(\mathbb{R}^n)$ is a Hilbert space with respect to the topology inherited by $L^2(\mathbb{R}^n)$.

Lemma 4. *Let $\mathbf{M}$ be an admissible weight sequence and $k \in \mathbb{N}$. Then, every $u \in H^k(\mathbb{R}^n)$ can be written in the form $u = \sum_{j=0}^\infty u_j$ with the $u_j \in G_{\{\mathbf{M}\}}$ satisfying:*

$$\sum_{j=0}^\infty \Lambda_{k_j}^{2k}\left(\|u_j\|^2_{L^2(\mathbb{R}^n)} + e^{-2\omega_{\mathbf{M}}(\Lambda_{k_j})}\|u_j\|_G^2\right) \leq C\gamma^k\left(|||u|||_{\mathbb{R}^n,\mathbf{M},k}\right)^2, \tag{16}$$

where the constants $C, \gamma > 0$ are independent of k, j, and $(k_j)_j$ is the sequence from Lemma 1.

Proof. We may set $k_{-1} = 0$ and

$$u_j(x) = (2\pi)^{-n}\int_{\Lambda_{k_{j-1}}\leq|\xi|\leq\Lambda_{k_j}} e^{ix\xi}\hat{u}(\xi)\,d\xi.$$

For $|\xi| \leq \Lambda_{k_j}$, we conclude that

$$\|u_j\|_G \leq e^{\omega_{\mathbf{M}}(\Lambda_{k_j})}\|u_j\|_{L^2(\mathbb{R}^n)}.$$

On the other hand, in the case $\Lambda_{k_{j-1}} \leq |\xi|$, we note first that for $j \geq 2$ we have

$$\Lambda_{k_j} \leq \sigma^j \Lambda_k \leq \sigma^2 \Lambda_{k_{j-1}} \leq \sigma^2(\Lambda_k + |\xi|).$$

Moreover, $\Lambda_{k_1} \leq \sigma\Lambda_k \leq \sigma^2(\Lambda_k + |\xi|)$ and $\Lambda_{k_0} = \Lambda_k \leq \sigma^2(\Lambda_k + |\xi|)$ since $\sigma \geq 1$ and $|\xi| \geq 0$.

Hence, due to (15),

$$\sum_{j=0}^{\infty} \Lambda_{k_j}^{2k} \|u_j\|^2_{L^2(\mathbb{R}^n)} \leq \sigma^{4k} \int (|\xi| + \Lambda_k)^{2k} |\hat{u}(\xi)|^2 \, d\xi \leq \gamma^k \left(|||u|||_{\mathbb{R}^n,\mathbf{M},k} \right)^2$$

for some $\gamma > 0$. □

Lemma 5. *Assume that* $\mathbf{M}$ *is an admissible weight sequence and that* E *is a Banach space, which is continuously injected in* $L^2(\Omega)$. *If there is an open set* $U_0 \subseteq \Omega$ *such that* $u|_{U_0} \in \mathcal{E}^{\{\mathbf{M}\}}(U_0)$ *for every* $u \in E$, *then for all* $V \Subset U_0$, *there exists a constant* C *such that for all* $k \in \mathbb{N}$ *and every sequence* $u_j \in E$, $j \in \mathbb{Z}_+$, *satisfying*

$$\sum_{j=0}^{\infty} \left(\frac{|\xi|}{\Lambda_{k_j}} \right)^{2k} \left(\|u_j\|^2_{L^2(\Omega)} + e^{-2\omega_{\mathbf{M}}(\Lambda_{k_j})} \|u_j\|^2_E \right) = \Phi_k^2(\mathbf{u}) < \infty, \tag{17}$$

the series $u = \sum_j u_j$ *converges in* $L^2(\Omega)$ *and* $u|_V \in H^k(V)$ *with*

$$\|u|_V\|_{H^k(V)} \leq C^{k+1} \Phi_k(\mathbf{u}).$$

Proof. For $k = 1$, the condition (17) implies that u converges absolutely in $L^2(\Omega)$. By Proposition 1 we have that for all open sets $V \Subset U \Subset \Omega$ and for every Ehrenpreis–Hörmander cut-off sequence $\chi_k \in \mathcal{D}(U)$ centered in V, there are constants $C_0, \gamma > 0$ such that for all $k \in \mathbb{Z}_+$ and all $u \in E$:

$$\left\| \left(\frac{|\xi|}{\gamma \Lambda_k} \right)^k \widehat{\chi_k u}(\xi) \right\|_{L^2(\mathbb{R}^n)} \leq C_0 \|u\|_E. \tag{18}$$

We introduce the following functions:

$$\theta(j, \xi) = e^{-\omega_{\mathbf{M}}(\Lambda_{k_j})} \left(\frac{|\xi|}{\gamma \Lambda_{k_j}} \right)^{k_j},$$

$$g_j(\xi) = (1 + \theta(j, \xi)) \widehat{\chi_{k_j} u_j}.$$

By (18), we have that

$$\|g_j\|_{L^2(\mathbb{R}^n)} \leq \|u_j\|_{L^2(\Omega)} + C_0 e^{-\omega_{\mathbf{M}}(\Lambda_{k_j})} \|u_j\|_E,$$

which implies that

$$\sum_{j=0}^{\infty} \Lambda_{k_j}^{2k} \|g_j\|^2_{L^2(\mathbb{R}^n)} \leq \sum_{j=0}^{\infty} \Lambda_{k_j}^{2k} \left[\|u_j\|_{L^2(\Omega)} + C_0 e^{-\omega_{\mathbf{M}}(\Lambda_{k_j})} \|u_j\|_E \right]^2.$$

The sum inside the bracket on the right-hand side can be viewed as the inner product of the two vectors $(1, C_0)$ and $(\|u_j\|_{L^2(\Omega)}, e^{-\omega_{\mathbf{M}}(\Lambda_{k_j})} \|u_j\|_E)$. Then applying the Cauchy–Schwarz inequality gives

$$\sum_{j=0}^{\infty} \Lambda_{k_j}^{2k} \|g_j\|^2_{L^2(\mathbb{R}^n)} \leq \left(1 + C_0^2\right) \Phi_k(\mathbf{u})^2 < \infty. \tag{19}$$

It is clear that the function $v = \sum_{j=0}^{\infty} \chi_{k_j} u_j$ coincides with u on V. It suffices to show that $v \in H^k(\mathbb{R}^n)$ satisfies the estimate:

$$\|v\|_{H^k(\mathbb{R}^n)} \leq C^{k+1}\Phi_k(\mathbf{u}).$$

We have that

$$\|v\|_{L^2(\mathbb{R}^n)} \leq \sum_{j=0}^{\infty} \|u_j\|_{L^2(\Omega)} \leq 2\Phi_k(\mathbf{u}) \tag{20}$$

and thus it will be enough to show that $|\xi|^k \hat{v} \in L^2(\mathbb{R}^n)$ and

$$\left\| |\xi|^k \hat{v}(\xi) \right\|_{L^2(\mathbb{R}^n)} \leq C^{k+1}\Phi_k(\mathbf{u}) \tag{21}$$

holds, where $C > 0$ is independent of k. We write

$$|\xi|^k \hat{v}(\xi) = \sum_{j=0}^{\infty} \big(1 + \theta(j,\xi)\big)^{-1} g_j(\xi)|\xi|^k$$

and conclude that

$$|\xi|^{2k} \left|\hat{v}(\xi)\right|^2 \leq \left(\sum_{j=0}^{\infty} |g_j(\xi)|^2 \Lambda_{k_j}^{2k}\right)\Theta(\xi),$$

where

$$\Theta(\xi) = \sum_{j=0}^{\infty} \left(\frac{|\xi|}{\Lambda_{k_j}}\right)^{2k} \big(1 + \theta_j(j,\xi)\big)^{-2}.$$

Hence (21) (and thus the lemma) is proven by (19) and the estimate

$$\|\Theta(\xi)\|_{L^\infty(\mathbb{R}^n)} \leq C^{k+1}, \tag{22}$$

where C is some constant independent of k. In order to establish (22), we set

$$\Psi_j(\xi) = \left(\frac{|\xi|}{\Lambda_{k_j}}\right)^{2k} \big(1 + \theta_j(j,\xi)\big)^{-2}.$$

If $e^{2H}\gamma\Lambda_{k_j} \leq |\xi|$, where H is the constant from (9), then we have that

$$\begin{aligned}
\Psi_j(\xi) &\leq \left(\frac{|\xi|}{\Lambda_{k_j}}\right)^{2k} e^{2\omega_{\mathbf{M}}(\Lambda_{k_j})}\gamma^{2k_j}\left(\frac{|\xi|}{\Lambda_{k_j}}\right)^{-2k_j} \\
&\leq \gamma^{2k}\left(\frac{|\xi|}{\gamma\Lambda_{k_j}}\right)^{2(k-k_j)} e^{2\omega_{\mathbf{M}}(\Lambda_{k_j})} \\
&\leq (\gamma e^{2H})^{2k} \exp\left(2\omega_{\mathbf{M}}(\Lambda_{k_j}) - 4Hk_j\right) \\
&\leq (\gamma e^{2H})^{2k} \exp\left(2Hk_j - 4Hk_j\right) \\
&\leq (\gamma e^{2H})^{2k} e^{-2Hk_j}
\end{aligned}$$

by Lemma 2. This gives

$$\sum_{e^2\gamma\Lambda_{k_j}\leq|\xi|}\Psi_j(\xi)\leq\left(\gamma e^{2H}\right)^{2k}\sum_{j=0}^{\infty}e^{-2Hj}\leq C_1\left(\gamma e^{2H}\right)^{2k}.$$

On the other hand, if $e^{2H}\gamma\Lambda_{k_j}\geq|\xi|$, then we estimate

$$\Psi_j(\xi)\leq\left(\frac{|\xi|}{\Lambda_{k_j}}\right)^{2k}.$$

If we set $j_0=\min\{j\in\mathbb{Z}_+\mid\gamma e^{2H}\Lambda_{k_j}\geq|\xi|\}$ for a fixed ξ, then we have that

$$\begin{aligned}\Psi_j(\xi)\leq\left(\frac{|\xi|}{\Lambda_{k_j}}\right)^{2k}&\leq\left(\frac{|\xi|}{\Lambda_{k_{j_0}}}\right)^{2k}\left(\frac{\Lambda_{k_{j_0}}}{\Lambda_{k_j}}\right)^{2k}\\&\leq\left(\gamma e^{2H}\right)^{2k}\sigma^{j_0-j+1}\end{aligned}$$

and conclude that

$$\sum_{|\xi|\leq e^{2H}\gamma\Lambda_{k_j}}\Psi_j(\xi)\leq\left(\gamma e^{2H}\right)^{2k}\sigma\sum_{j=j_0}^{\infty}\sigma^{-j}\leq\left(\gamma e^{2H}\right)^{2k}\sigma\sum_{j=0}^{\infty}\sigma^{-j}\leq C_2\left(\gamma e^{2H}\right)^{2k}.$$

□

If P is a differential operator with smooth coefficients, then we set

$$\mathcal{P}_0(U)=\{u\in L^2(U)\mid Pu=0\}$$

for an open set U. Clearly $\mathcal{P}_0(U)$ is a closed subspace of $L^2(U)$.

Proposition 2. *Let $\Omega\subseteq\mathbb{R}^n$ be open, $\mathbf{M}$ be an admissible weight sequence. Furthermore, assume that P is a linear differential operator with $\mathcal{E}^{\{\mathbf{M}\}}(\Omega)$-coefficients.*

If P is $\{\mathbf{M}\}$-hypoelliptic at some point $x_0\in\Omega$, then there is a neighborhood U_0 of x_0 such that for all open sets $V\Subset U\subseteq U_0$, there are constants $C,h>0$ such that for all $k\in\mathbb{Z}_+$ and all $u\in L^2(U)$ with $Pu=0$, we have

$$\|u\|_{H^k(V)}\leq Ch^kM_k\|u\|_{L^2(U)}. \tag{23}$$

Proof. If $V\Subset U$ are open sets and $h>0$, then we set

$$\|u\|^{\bullet}_{V,\mathbf{M},h}=\sup_{k\in\mathbb{N}_0}\frac{\|u\|_{H^k(V)}}{h^kM_k}$$

for $u\in\mathcal{E}(U)$. We define

$$H_{\mathbf{M},h}(V)=\{u\in\mathcal{E}(U)\mid\|u\|_{V,\mathbf{M},h}<\infty\}$$

and introduce

$$H^{\{\mathbf{M}\}}(V)=\operatorname{ind}_{h>0}H_{\mathbf{M},h}(V).$$

We note that if $u \in \mathcal{B}_{\mathbf{M},h}(V)$, then

$$\begin{aligned}
\|u\|^2_{H^k(V)} &= \sum_{|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(V)} \\
&\leq C^2 \sum_{|\alpha|\leq k} h^{2|\alpha|} M^2_{|\alpha|} \\
&\leq C^2 M_k^2 \sum_{j=0}^{k} h^{2k} \sum_{|\alpha|=j} 1 \\
&\leq C^2 M_k^2 \sum_{j=0}^{k} \binom{n+j-1}{j} h^{2j} \\
&\leq C^2 M_k^2 \sum_{j=0}^{k} \binom{k}{j} h^{2j} \frac{(n+j-1)!(k-j)!}{(n-1)!k!} \\
&\leq C^2 M_k^2 \binom{n+k-1}{k} \left(1+h^2\right)^k \\
&\leq C^2 M_k^2 2^{n-1} 2^k \left(1+h^2\right)^k.
\end{aligned}$$

Hence there is a constant A such that

$$\|u\|^{\bullet}_{V,\mathbf{M},h_1} \leq A\|u\|_{V,\mathbf{M},h},$$

where $h_1 = 2(1+h^2)$. Thus, we obtain that there is a continuous embedding

$$\iota_V : \mathcal{B}_{\mathbf{M},h}(V) \longrightarrow H_{\mathbf{M},h_1}(V)$$

and consequently

$$\mathcal{B}^{\{\mathbf{M}\}}(V) \longrightarrow H^{\{\mathbf{M}\}}(V).$$

Since P is $\{\mathbf{M}\}$-hypoelliptic at x_0, we know that there is a neighborhood $U_0 \subseteq \Omega$ of x_0 such that $u|_V \in \mathcal{B}^{\{\mathbf{M}\}}(V)$ for all $u \in \mathcal{P}_0(U)$ and all pairs $V \Subset U \subseteq U_0$. Then similarly to the proof of Proposition 1, we observe that the graph of the restriction map

$$T_V : \mathcal{P}_0(U) \longrightarrow \mathcal{B}^{\{\mathbf{M}\}}(V)$$

is closed. Hence the Closed Graph Theorem of De Wilde implies that T_V is continuous and therefore the map

$$S_V = \iota_V \circ T_V : \mathcal{P}_0(U) \longrightarrow H^{\{\mathbf{M}\}}(V)$$

is continuous. Again as before in the proof of Proposition 1, we thus can conclude that for all $V \Subset U$, there exists $h > 0$ such that the map $u \mapsto u|_V$ is continuous from $\mathcal{P}_0(U)$ to $H^{\{\mathbf{M}\}}(V)$, which is equivalent to the existence of some constant $C > 0$ such that

$$\|u\|^{\bullet}_{V,\mathbf{M},h} \leq C\|u\|_{L^2(U)} \qquad \forall u \in \mathcal{P}_0(U).$$ □

Proof of Theorem 1. Assume first P satisfies (11) and (12). If $u \in \mathcal{D}'(U)$, U being an open subset of U_0 is such that $Pu \in \mathcal{E}^{\{\mathbf{M}\}}(U)$, then by (11) we have that $u|_V \in \mathcal{E}(V)$ for any $V \Subset U$. In particular, $u \in H^k(V)$ for all $k \in \mathbb{N}_0$. We observe also that we have for a weight sequence $\mathbf{M}$ that

$$\begin{aligned}|||Pu|||_{U,\mathbf{M},k} &= \sum_{|\alpha|\le k} \Lambda_k^{k-|\alpha|}\|D^\alpha(Pu)\|_{L^2(U)}\\ &\le C_0 \sum_{|\alpha|\le k} \Lambda_k^{k-|\alpha|} h_0^{|\alpha|} M_{|\alpha|}\\ &\le C_0 \sum_{|\alpha|\le k} \Lambda_k^{k-|\alpha|}\Lambda_k^{|\alpha|} h_0^{k}\\ &\le C_0 M_k \sum_{|\alpha|\le k} h_0^{|\alpha|}\\ &\le C_1 h_1^k M_k\end{aligned}$$

for some constants $C_1, h_1 > 0$ independent of k. Thence, (12) implies that

$$\begin{aligned}\|u\|_{H^k(V)} &\le CL^k\left[|||Pu|||_{U,\mathbf{M},k} + M_k\|u\|_{L^2(U)}\right]\\ &\le CL^k\left[C_1 h_1^k M_k + C' M_k\right]\\ &\le C_2 h_2^k M_k\end{aligned}$$

for some $C_2, h_2 > 0$. Since $\|D^\alpha u\|_{L^2(V)} \le \|u\|_{H^{|\alpha|}(V)}$, it follows that u is ultradifferentiable of class $\{\mathbf{M}\}$ in V. Since $V \Subset U$ is arbitrary, we have actually that $u \in \mathcal{E}^{\{\mathbf{M}\}}(U)$.

On the other hand, assume now that P is a differential operator, which is $\{\mathbf{M}\}$-hypoelliptic in a neighborhood U_0 of and satisfies (**H**), that is, there is a continuous map $R : L^2(\Omega) \to L^2(\Omega)$ such that $PR = \mathrm{Id}$.

If

$$G_{\{\mathbf{M}\}} = \{u \in L^2(\mathbb{R}^n) \mid e^{\omega_{\mathbf{M}}(|\xi|)}\hat{u} \in L^2(\mathbb{R}^n)\}$$

is the space from Lemma 4, then we set $\tilde{G}_{\{\mathbf{M}\}} = \{f|_\Omega \mid f \in G_{\{\mathbf{M}\}}\}$. The map

$$G_{\{\mathbf{M}\}} \ni f \longmapsto R(f|_\Omega) \in L^2(\Omega)$$

is injective and

$$E = \{R(f) \mid f \in \tilde{G}_{\{\mathbf{M}\}}\}$$

is a Banach space with the norm inherited from $G_{\{\mathbf{M}\}}$. It is clear that the injection $E \to L^2(\Omega)$ is continuous and since P is $\{\mathbf{M}\}$-hypoelliptic in Ω, we can infer the following fact: For all $u \in E$, we have that $Pu \in \tilde{G}$ is of class $\{\mathbf{M}\}$ in U_0, but this implies u is of class $\{\mathbf{M}\}$ in U_0. Hence we can use Lemma 5.

Let C_0 be the constant appearing in Lemma 5 and $V \Subset U \subseteq U_0$ be arbitrary open sets. We choose an Ehrenpreis–Hörmander cut-off sequence $\chi_j \in \mathcal{D}(U)$ centered in V. Let $u \in \mathcal{D}'(U)$ be such that $Pu \in H^k(U)$. We set $f = \chi_k Pu$. By Lemma 3, we have that

$$|||f|||_{\mathbb{R}^n,\mathbf{M},k} \le \gamma^k |||Pu|||_{U,\mathbf{M},k} \tag{24}$$

for some constant γ independent of k. According to Lemma 4, we can write

$$f = \sum_{j=0}^{\infty} f_j$$

with $f_j \in G_{\{\mathbf{M}\}}$ and

$$\sum_{j=0}^{\infty} \Lambda_{k_j}^{2k}\left(\|f_j\|^2_{L^2(\mathbb{R}^n)} + e^{-2\omega_{\mathbf{M}}(\Lambda_{k_j})}\|f_j\|_G^2\right) \le 2\gamma_1^k\left(|||f|||_{\mathbb{R}^n,\mathbf{M},k}\right)^2.$$

Setting $v_j = R(f_j|_\Omega) \in E$, we obtain that

$$\sum_{j=0}^{\infty} \Lambda_{k_j}^{2k}\left(\|v_j\|_{L^2(\Omega)}^2 + e^{-2\omega_{\mathbf{M}}(\Lambda_{k_j})}\|v_j\|_E^2\right) \le 2(1+\|R\|^2)\gamma_1^k\left(|||f|||_{\mathbb{R}^n,\mathbf{M},k}\right)^2,$$

where $\|R\|$ is the norm of R in $\mathcal{L}(L^2(\Omega))$. The series $v = \sum_{j=0}^\infty v_j$ converges in $L^2(\Omega)$ by Lemma 5 and furthermore we have that $v|_V \in H^k(V)$ and

$$\|v|_V\|_{H^k(V)} \le C_1^{k+1}|||f|||_{\mathbb{R}^n,\mathbf{M},k} \tag{25}$$

for some constant C_1 independent of k.

We obtain actually that

$$P((u-v)|_V) = 0.$$

Since P is $\{\mathbf{M}\}$-hypoelliptic in U_0, it follows that $u - v \in \mathcal{E}^{\{\mathbf{M}\}}(V)$. Applying Proposition 2, we infer that for each open set $W \Subset V$, there is a constant $C_2 > 0$ independent of u (and v) such that

$$\forall k \in \mathbb{Z}_+ : \quad \|(u-v)|_W\|_{H^k(W)} \le C_2^{k+1}M_k\|(u-v)|_V\|_{L^2(V)}. \tag{26}$$

We note also that $v = R(f|_\Omega)$ and therefore

$$\|v\|_{L^2(\Omega)} \le \|R\|\|f\|_{L^2(\mathbb{R}^n)} \le \|R\|\|Pu\|_{L^2(U)}. \tag{27}$$

Furthermore, it is easy to see that

$$M_k\|Pu\|_{L^2(U)} \le |||Pu|||_{U,\mathbf{M},k}. \tag{28}$$

We are now able to finish the proof:

$$\begin{aligned}
\|u\|_{H^k(W)} &\le \|v\|_{H^k(W)} + \|u-v\|_{H^k(W)} \\
&\le C_1^{k+1}|||f|||_{\mathbb{R}^n,\mathbf{M},k} + C_2^{k+1}M_k\|u-v\|_{L^2(V)} \\
&\le (C_1\gamma)^{k+1}|||Pu|||_{U,\mathbf{M},k} + C_2^{k+1}M_k\left[\|u\|_{L^2(V)} + \|v\|_{L^2(V)}\right]
\end{aligned}$$

by (24), (25), and (26). Applying (27) and (28), we see that

$$M_k\|v\|_{L^2(V)} \le \|R\|\ |||Pu|||_{U,\mathbf{M},k}.$$

Hence we have shown that there are constants $C, h > 0$ such that for all $k \in \mathbb{Z}_+$ and every $u \in \mathcal{D}'(U)$ with $Pu \in H^k(U)$, the following estimate holds:

$$\|u\|_{H^k(V)} \le Ch^k\left[|||Pu|||_{U,\mathbf{M},k} + M_k\|u\|_{L^2(U)}\right]. \qquad \square$$

4 | THE HYPERFUNCTION CASE—FINAL REMARKS

Denote by B the sheaf of (germs of) hyperfunctions in $\mathbb{R}^n$. We strength the definition of hypoellipticity in the following way:

Definition 4. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{M}$ be a weight sequence. If $P = P(x, D)$ is a linear differential operator with real-analytic coefficients, then P is $\{\mathbf{M}\}$-hypoelliptic in U in the hyperfunction sense if given $u \in \mathrm{B}(U)$, the following holds: If $V \subseteq U$ is open and if $Pu|_V \in \mathcal{E}^{\{\mathbf{M}\}}(V)$, then $u|_V \in \mathcal{E}^{\{\mathbf{M}\}}(V)$.

Furthermore, P is $\{\mathbf{M}\}$-hypoelliptic at $x_0 \in U$ in the hyperfunction sense if there is a neighborhood U_0 of x_0 such that P is $\{\mathbf{M}\}$-hypoelliptic in U_0 in the hyperfuntion sense.

A close look at the proof of Theorem 1 leads to the following result:

Theorem 2. *Let* $\mathbf{M}$ *be an admissible weight sequence and* P *be a differential operator with real analytic coefficients in* Ω *satisfying (**H**).*

Then, P *is* $\{\mathbf{M}\}$*-hypoelliptic at a point* $x_0 \in \Omega$ *in the hyperfunction sense if and only if there is a neighborhood* $U_0 \subseteq \Omega$ *of* x_0 *such that for all open sets* $V \Subset U \subseteq U_0$*, there are constants* $C, L > 0$ *such that for all* $\mathrm{B}(U)$ *and all* $k \in \mathbb{Z}_+$*, we have*

$$Pu \in H^k(U) \Longrightarrow u \in H^k(V), \tag{29}$$

$$\|u\|_{H^k(V)} \leq CL^k\big(|||Pu|||_{U,\mathbf{M},k} + M_k\|u\|_{L^2(U)}\big). \tag{30}$$

An operator $P = P(x, D)$ in an open set $W \subseteq \mathbb{R}^n$ will be said to belong to the Hörmander class $\mathfrak{H}(W)$ if it can be written in the form

$$P = \sum_{j=1}^{\nu} X_j^2,$$

where each X_j is a real-valued, real analytic vector field defined in W and the following condition is satisfied: The Lie algebra spanned by $X_1, \ldots, X_\nu$ has rank equal to n at any point of W.

It can be proved that every point in W has an open neighborhood Ω such that if $P \in \mathfrak{H}(W)$, then P (and also its transpose!) satisfy (**H**). In particular, Theorem 2 applies to $P \in \mathfrak{H}(W)$ restricted to Ω. It is also well known (cf. [3]) that given $P \in \mathfrak{H}(W)$ and $x_0 \in W$, then there is $s_0 \geq 1$ so that P is $\{\mathbf{G}^{s_0}\}$-hypoelliptic at x_0. According to Cordaro–Hanges [2], it follows that P is $\{\mathbf{G}^{s_0}\}$-hypoelliptic at x_0 in the hyperfunction sense. By an elementary extension of Corollary 1 to this hyperfunction set up, it follows that P is then $\{\mathbf{M}\}$-hypoelliptic at x_0 in the hyperfunction sense for every admissible weight sequence $\mathbf{G}^{s_0} \preceq \mathbf{M}$.

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ENDNOTE

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