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MATHEMATISCHE NACHRICHTEN

Isomorphisms of $C_0(K, X)$ spaces with large distortion

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Abstract

Let K and S be locally compact Hausdorff spaces and let X be a strictly convex Banach space of finite dimension at least 2. In this paper, we prove that if there exists an isomorphism T from $C_0(K, X)$ onto $C_0(S, X)$ satisfying

$$\|T\| \left\|T^{-1}\right\| = \lambda(X),$$

then K and S are homeomorphic. Here $\lambda(X)$ denotes the Schäffer constant of X. Even for the classical cases $X = \ell_p^n$, $1 < p < \infty$ and $n \geq 2$, this result is the X-valued Banach–Stone theorem via isomorphism with the largest distortion that is known so far, namely $\lambda(X) = \min\left\{2^{1/p}, 2^{1-1/p}\right\}$. On the other hand, it is well known that this result is not true for $X = \mathbf{R}$, even though K and S are compact Hausdorff spaces.

KEYWORDS

$C_0(K, X)$ spaces, ℓ_p^n spaces, Schäffer constant, strictly convex spaces, vector-valued Banach–Stone theorems

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1 | INTRODUCTION

For a locally compact Hausdorff space K and a Banach space X we denote by $C_0(K, X)$ the space of X-valued continuous functions on K which vanish at infinity, provided with the supremum norm. If X is the scalar field $\mathbf{R}$ or $\mathbf{C}$, this space is denoted by $C_0(K)$.

The well-known Banach–Stone theorem states that the existence of an isometric isomorphism T from $C_0(K)$ onto $C_0(S)$ implies that K and S are homeomorphic [2,26]. Amir [1] and Cambern [6] independently extended the Banach–Stone theorem for certain isomorphisms T with distortion $\|T\| \left\|T^{-1}\right\|$ greater than 1. More precisely, they proved that if T is an isomorphism from $C_0(K)$ onto $C_0(S)$ satisfying

$$\|T\| \left\|T^{-1}\right\| < 2, \tag{1.1}$$

then K and S are homeomorphic. In [7] Cambern showed that the result of Amir–Cambern do not remain true if we replace (1.1) by

$$\|T\| \left\|T^{-1}\right\| = 2, \tag{1.2}$$

see also the Cohen result in [13].

Various authors, beginning with Jerison [22], have considered the problem of determining geometric properties of X which allow generalizations of the Banach–Stone theorem to the $C_0(K, X)$ spaces. We stress that Cambern [8] generalized the

Amir–Cambern theorem to the vector-valued case by proving that if X is a finite-dimensional Hilbert space and T is an isomorphism from $C_0(K, X)$ onto $C_0(S, X)$ satisfying

$$\|T\| \left\|T^{-1}\right\| < \sqrt{2}, \tag{1.3}$$

then K and S are homeomorphic.

Recently in [16] this last theorem has been improved showing that it remains true if we replace (1.3) by

$$\|T\| \left\|T^{-1}\right\| = \sqrt{2}. \tag{1.4}$$

These results naturally lead us to look for other vector-valued Banach–Stone theorems via isomorphisms with large distortion similar to that in (1.4). In other words, the following question arises naturally.

Problem 1.1. When do isomorphisms of $C_0(K, X)$ spaces with large distortion determine the locally compact spaces K?

In the present paper, we will consider this problem and turn our attention to a class of Banach spaces that contains the Hilbert spaces, i.e. the strictly convex spaces. Recall that a real normed linear space X is said to be strictly convex if for any given distinct vectors in the closed unit sphere S_X, the midpoint of the line segment joining them must not lie in the closed unit sphere.

Historically, the property of a space being strictly convex has been part of the hypotheses in many of the vector-valued Banach–Stone theorems, including that of Jerison mentioned above, see for instance [3,4,9–11,14,18,20–23,27].

The main aim of this work is to prove Theorem 1.3. It is a stronger vector-valued Banach–Stone theorem for all finite-dimensional strictly convex spaces. Before establishing it we recall the following parameter introduced by Schäffer [17,25] for Banach spaces X:

$$\lambda(X) = \inf\{\max\{\|x - y\|, \|x + y\|\} \ : \ \|x\| = 1 \text{ and } \|y\| = 1\}.$$

Note that by the parallelogram law it follows that if X is a Hilbert space of finite dimension at least 2, then $\lambda(X) = \sqrt{2}$. Furthermore, $\lambda(X) \leq \sqrt{2}$ for every Banach space with dimension greater than or equal to 2 [17, Theorem 2.5] and $\lambda(\mathbf{R}) = 2$.

So, in contrast with the result quoted in (1.2) we have the following extension of the result named in (1.4).

Theorem 1.2. *Let X be a strictly convex space of finite dimension at least 2. Suppose that K and S are locally compact Hausdorff spaces and that T is an isomorphism from $C_0(K, X)$ onto $C_0(S, X)$ satisfying*

$$\|T\| \left\|T^{-1}\right\| = \lambda(X).$$

Then K and S are homeomorphic.

Theorem 1.2 is an immediate consequence of our main result, namely.

Theorem 1.3. *Let X be a strictly convex space of finite dimension at least 2 and let $M = \lambda(X)^{1/2}$. If K and S are locally compact Hausdorff spaces and there exists an isomorphism T from $C_0(K, X)$ onto $C_0(S, X)$ satisfying*

$$\frac{\|f\|}{M} \leq \|T(f)\| \leq M\|f\|,$$

for every $f \in C_0(K, X)$, then K and S are homeomorphic.

Proof of Theorem 1.2. Assume that τ is an isomorphism satisfying the hypotheses of Theorem 1.2. Then putting $T = \tau\|\tau^{-1}\|\lambda(X)^{-1/2}$ and $M = \lambda(X)^{1/2}$, it is easy to check that T satisfies the hypotheses of Theorem 1.3. □

Observe that Theorem 1.2 is also an improvement of [11, Main Theorem] in the case where X is a strictly convex space of finite dimension greater than or equal to 2. Indeed, in that theorem the hypothesis on the isomorphism T is

$$\|T\| \left\|T^{-1}\right\| < \lambda(X).$$

Notice also that even in the case where $X = \ell_p^n$, the real n-dimensional l_p spaces, $1 < p < \infty$ and $n \geq 2$, the following immediate corollary of Theorem 1.2 was only known when $p = 2$, i.e. the result mentioned in (1.4).

Corollary 1.4. *Let $1 < p < \infty$ and $n \geq 2$. Suppose that K and S are locally compact Hausdorff spaces and that T is an isomorphism from $C_0(K, \ell_p^n)$ onto $C_0(S, \ell_p^n)$ satisfying*

$$\|T\| \left\|T^{-1}\right\| = \min\{2^{1/p}, 2^{1-1/p}\},$$

then K and S are homeomorphic.

Proof. It is enough to recall that according to [17, Theorem 3.1], for every $1 < p < \infty$ and $n \geq 2$, we know that

$$\lambda\left(\ell_p^n\right) = \min\{2^{1/p}, 2^{1-1/p}\}.$$

□

The result cited in (1.4) was the first vector-valued Banach–Stone theorem obtained via isomorphism with distortion $\sqrt{2}$. However, by using Theorem 1.2 we can provide new vector-valued Banach–Stone theorems via isomorphism with that large distortion. Indeed, in view of Theorem 1.2 it suffices to show that there is a finite-dimensional strictly convex space E with $\lambda(E) = \sqrt{2}$ which is not a Hilbert space. Indeed, let U be the operator of rotation by 45° on $\mathbf{R}^2$. Fix $p > 2$ and denote by $\|\cdot\|_p$ the norm of ℓ_p^2. Define E to be $\mathbf{R}^2$ endowed with the norm

$$\|x\|_E = \max\{\|x\|_p, \|U(x)\|_p\}.$$

Since $\|\cdot\|_p$ is invariant under rotation by 90°, it follows easily that $U(B_E) = B_E$, where B_E denotes the closed unit ball of E. Moreover, recall that a result due to Gao and Lau [17, Proposition 2.8] ensures that every $X = (\mathbf{R}^2, \|\cdot\|_X)$ such that $U(B_X) = B_X$ has $\lambda(X) = \sqrt{2}$. Therefore, $\lambda(E) = \sqrt{2}$. Since $\|\cdot\|_E$ is the maximum of two strictly convex norms, it is clear that $\|\cdot\|_E$ is strictly convex. Finally, it is easy to check that

$$\|e_1\|_E = \|e_2\|_E = 1 \text{ and } \|e_1 + e_2\|_E = \sqrt{2}, \tag{1.5}$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbf{R}^2$.

Suppose by contradiction that $\|\cdot\|_E$ is determined by an inner product $\langle\cdot,\cdot\rangle_E$. Then (1.5) implies that $\{e_1, e_2\}$ is an orthonormal basis with respect to $\langle\cdot,\cdot\rangle_E$. Therefore, $\langle\cdot,\cdot\rangle_E$ coincides with the usual inner product on $\mathbf{R}^2$ and consequently, $\|\cdot\|_E = \|\cdot\|_2$, but this is a contradiction.

In view of Theorem 1.2 we introduce the following definition.

Definition 1.5. Let X be a Banach space with $\lambda(X) > 1$. We will say that X is a *Banach–Stone–Schäffer space* whenever the existence of an isomorphism T from $C_0(K, X)$ onto $C_0(S, X)$ satisfying

$$\|T\| \left\|T^{-1}\right\| = \lambda(X)$$

implies that K and S are homeomorphic.

In opposition to Corollary 1.4, it follows from [11, Remark 4.7] that every infinite-dimensional l_p space, $2 \leq p < \infty$, is not a Banach–Stone–Schäffer space. Then, it would be interesting to know the solution to the following intriguing problem.

Problem 1.6. Suppose that X is a Banach–Stone–Schäffer space.

(1) Is it true that X is a finite-dimensional space?

(2) Does it follow that X is a strictly convex space?

2 | ON THE SCHÄFFER CONSTANT OF FINITE-DIMENSIONAL STRICTLY CONVEX SPACES

This section concerns the doubles of vectors x and y for which the infimum defining the Schäffer constant $\lambda(X)$ is reached. Proposition 2.3 will be fundamental in the proof of our results. In order to prove this proposition we initially recall two geometric properties of strictly convex spaces of dimension 2. The first one is a result due to Gao and Lau [17, Lemma 2.2.i] and the second one is part of the well-known Monotonicity lemma [24, Proposition 31].

Lemma 2.1. *Let X be a two-dimensional space and let $x \in X$. Then, there exists $y \in S_X$ such that*

$$\|x+y\| = \|x-y\| = \inf_{z \in S_X} \max\{\|x+z\|, \|x-z\|\}.$$

Proposition 2.2. *Let X be a two-dimensional strictly convex space and fix $x \in S_X$. Suppose that $x_1, x_2 \in S_X$ are in some closed halfspace H determined by the line $\{tx : t \in \mathbf{R}\}$ and $\|x+x_1\| = \|x+x_2\|$. Then $x_1 = x_2$.*

Proposition 2.3. *Let X be a strictly convex space of finite dimension at least 2.*

(1) *If $x, y \in S_X$ satisfy* $\max\{\|x+y\|, \|x-y\|\} = \lambda(X)$, *then*

$$\|x+y\| = \|x-y\| = \lambda(X).$$

(2) *Let X_2 be a two-dimensional subspace of X. For each $x \in S_{X_2}$ there exists at most one $y \in S_{X_2}$, up to sign, such that*

$$\|x \pm y\| = \lambda(X).$$

(3) *Suppose that $\{x_1, x_2, x_3\} \subset S_X$ satisfy*

$$\|x_i \pm x_j\| = \lambda(X), \quad \textit{for all} \quad i \neq j.$$

Then $\{x_1, x_2, x_3\}$ is linearly independent.

(4) *For each $x \in S_X$, the set*

$$\{y \in S_X : \|x \pm y\| = \lambda(X)\}$$

has empty interior with respect to the topology of S_X.

Proof.

(1) Let $X_2 \subset X$ be a plane containing x and y. By using Lemma 2.1, we fix $y' \in S_{X_2} = S_X \cap X_2$ such that

$$\|x+y'\| = \|x-y'\| = \lambda(X).$$

Since $\max\{\|x+y\|, \|x-y\|\}$ is assumed, we suppose without loss of generality that $\|x+y\| = \lambda(X)$. Then

$$\|x+y\| = \|x \pm y'\|.$$

It is easy to see that $\{y, y'\}$ or $\{y, -y'\}$ is contained in a closed halfspace determined by the line $\{tx : t \in \mathbf{R}\}$. Then, by Proposition 2.2, $y = y'$ or $y = -y'$, and this completes the proof of this item.

(2) It suffices to apply the Proposition 2.2.

(3) Suppose that $\{x_1, x_2, x_3\}$ is linearly dependent and fix a plane $X_2 \subset X$ such that $\{x_1, x_2, x_3\} \subset S_{X_2}$. By hypothesis, we have that

$$\|x_1 \pm x_2\| = \lambda(X) \quad \text{and} \quad \|x_1 \pm x_3\| = \lambda(X).$$

So, by the item 2 of the proposition, $x_2 = \pm x_3$. Therefore,

$$\{\|x_2 + x_3\|, \|x_2 - x_3\|\} = \{0, 2\},$$

this is a contradiction with $\|x_2 \pm x_3\| = \lambda(X)$.

(4) Put

$$P(x) = \{y \in S_X : \|x \pm y\| = \lambda(X)\}.$$

Pick $y \in P(x)$ and X_2 a two-dimensional subspace of X containing x and y. Then, by the item 2 of the proposition, $(X_2 \cap S_X) \cap P(x)$ contains at most y and $-y$. Therefore, y is not an interior point of $P(X)$, otherwise $(X_2 \cap S_X) \cap P(x)$ would have infinitely many points. □

3 | SPECIAL SETS ASSOCIATED TO ISOMORPHISMS BETWEEN $C_0(K, X)$ SPACES

Next we turn our attention to the proof of Theorem 1.3. So, from now on M is the positive number such that $M^2 = \lambda(X)$ and T is an isomorphism of $C_0(K, X)$ onto $C_0(S, X)$ satisfying

$$\frac{\|f\|}{M} \leq \|T(f)\| \leq M\,\|f\|, \tag{3.1}$$

for every $f \in C_0(K, X)$.

In a recent study of maps T satisfying (3.1) [16] it was introduced some classes of subsets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ and $\Gamma_{\mathrm{v}}(s, \mathrm{w})$ of S and K respectively, where $k \in K$, $s \in S$ and v and w are suitable elements of X.

In order to prove Theorem 1.3, we will need to state some new properties of these sets in the particular case where $M^2 = \lambda(X)$. Thus, in this short section we will remember some definitions and results concerning such sets.

Let $k \in K$, $f \in C_0(K, X)$ and $\mathrm{v} \in X$. Following [16, p. 323] we set

$$\omega(k, f, \mathrm{v}) = \max\{\|f\|, \|f(k) - \mathrm{v}\|\}.$$

Moreover, if v and $\mathrm{w} \in X$ with $\mathrm{v} \neq 0$ satisfies $\|\mathrm{w}\| = \|\mathrm{v}\|/M$, following [16, p. 323], we also set

$$\Gamma_{\mathrm{w}}(k, \mathrm{v}) = \left\{s \in S \;:\; \|Tf(s) - \mathrm{w}\| \leq M\omega(k, f, \mathrm{v}), \text{ for all } f \in C_0(K, X)\right\}.$$

Analogously, for $s \in S$, w and $\mathrm{v} \in X$ with $\mathrm{w} \neq 0$ and $\|\mathrm{v}\| = \|\mathrm{w}\|/M$, we set

$$\Gamma_{\mathrm{v}}(s, \mathrm{w}) = \left\{k \in K \;:\; \left\|T^{-1}g(k) - \mathrm{v}\right\| \leq M\omega(s, g, \mathrm{w}), \text{ for all } g \in C_0(S, X)\right\}.$$

Let us recall the following basic properties of these sets. The proof of Proposition 3.1 is essentially the same proof of [15, Proposition 3.2] and [16, Proposition 2.1]. We leave it to the reader to transpose to our context.

Proposition 3.1. *Let $k \in K$ and let $\mathrm{v} \in X \setminus \{0\}$.*

(1) *There exists $\mathrm{w} \in X$ such that $\Gamma_{\mathrm{w}}(k, \mathrm{v}) \neq \emptyset$.*

(2) *Let $\mathrm{v}, \mathrm{w} \in X$. Then for all $t \neq 0$, we have $\Gamma_{\mathrm{w}}(k, \mathrm{v}) = \Gamma_{t\mathrm{w}}(k, t\mathrm{v})$.*

(3) *Suppose that $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v})$ and $\Gamma_{\mathrm{z}}(s, \mathrm{w}) \neq \emptyset$ for some $\mathrm{v}, \mathrm{w}, \mathrm{z} \in X$. Then $\Gamma_{\mathrm{z}}(s, \mathrm{w}) = \{k\}$.*

4 | THE SCHÄFFER CONSTANT $\lambda(X)$ AND THE SPECIAL SETS $\Gamma_{\mathrm{w}}(k, \mathrm{v})$

The objective of this section is to present a close relationship between the Schäffer constant $\lambda(X)$ and the special sets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ mentioned in the previous section. More precisely, we will prove the following proposition. For each $\mathrm{v} \in X \setminus \{0\}$ we will denote $\hat{\mathrm{v}} = \mathrm{v}/\|\mathrm{v}\|$.

Proposition 4.1. *Suppose that $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v}) \cap \Gamma_{\mathrm{w}'}(k', \mathrm{v}')$ for some v, v', w, $\mathrm{w}' \in X$, with $k \neq k'$. Then*

(1) $\left\|\widehat{\mathrm{w}} \pm \widehat{\mathrm{w}'}\right\| = \lambda(X)$*;*

(2) *Suppose that $\|\mathrm{v}\| = \|\mathrm{v}'\| = c$. Then, for all $f \in C_0(K, X)$ satisfying*

$$\omega(k, f, \mathrm{v}) = \omega(k', f, \mathrm{v}') = c/2,$$

we have

$$Tf(s) = (\mathrm{w} + \mathrm{w}')/2.$$

Proof. By Proposition 3.1.2 we may assume that $\|\mathrm{v}\| = \|\mathrm{v}'\| = c$. According to Urysohn's lemma, fix $f \in C_0(K, X)$ such that

$$\|f\| = c/2,\; f(k) = \mathrm{v}/2 \quad \text{and} \quad f(k') = \mathrm{v}'/2.$$

It is easy to check that

$$\omega(k, f, \mathrm{v}) = \omega(k', f, \mathrm{v}') = c/2. \tag{4.1}$$

Then, since $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v}) \cap \Gamma_{\mathrm{w}'}(k', \mathrm{v}')$, we have

$$\|Tf(s) - \mathrm{w}\| \leq Mc/2 \quad \text{and} \quad \|Tf(s) - \mathrm{w}'\| \leq Mc/2. \tag{4.2}$$

(1) It follows by (4.2) that $\|\mathrm{w} - \mathrm{w}'\| \leq Mc$ and since

$$\|\mathrm{w}\| = \|\mathrm{w}'\| = c/M \quad \text{and} \quad M = \lambda(X)^{1/2},$$

we obtain

$$\left\|\widehat{\mathrm{w}} - \widehat{\mathrm{w}'}\right\| \leq \lambda(X).$$

On the other hand, by Proposition 3.1.2 we have

$$s \in \Gamma_{\mathrm{w}}(k, \mathrm{v}) \cap \Gamma_{-\mathrm{w}'}(k', -\mathrm{v}').$$

Then, repeating the argument replacing v' and w' by $-\mathrm{v}'$ and $-\mathrm{w}'$, respectively, we deduce that

$$\left\|\widehat{\mathrm{w}} + \widehat{\mathrm{w}'}\right\| \leq \lambda(X).$$

So, it follows that

$$\max\left\{\left\|\widehat{\mathrm{w}} + \widehat{\mathrm{w}'}\right\|, \left\|\widehat{\mathrm{w}} - \widehat{\mathrm{w}'}\right\|\right\} \leq \lambda(X).$$

But then by the definition of $\lambda(X)$ we infer that

$$\max\left\{\left\|\widehat{\mathrm{w}} + \widehat{\mathrm{w}'}\right\|, \left\|\widehat{\mathrm{w}} - \widehat{\mathrm{w}'}\right\|\right\} = \lambda(X).$$

By Proposition 2.3.1 this implies that $\left\|\widehat{\mathrm{w}} \pm \widehat{\mathrm{w}'}\right\| = \lambda(X)$, as we wished.

(2) Since $\|\mathrm{w}\| = \|\mathrm{w}'\| = c/M$, $M = \lambda(X)^{1/2}$ and by the item 1 of the proposition, $\left\|\widehat{\mathrm{w}} - \widehat{\mathrm{w}'}\right\| = \lambda(X)$, we conclude that

$$Mc/2 = \|\mathrm{w} - \mathrm{w}'\|/2.$$

Notice that (4.1) is a sufficient condition for (4.2) to hold. Then, by using the equation above in (4.2) we see that

$$\|Tf(s) - \mathrm{w}\| \leq \|\mathrm{w} - \mathrm{w}'\|/2 \quad \text{and} \quad \|Tf(s) - \mathrm{w}'\| \leq \|\mathrm{w} - \mathrm{w}'\|/2.$$

Thus, since X is strictly convex it follows that $Tf(s) = (\mathrm{w} + \mathrm{w}')/2$. □

5 | ON THE SUBSETS $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ OF K CONTAINING IRREGULAR POINTS

The aim of this section is to establish Proposition 5.1. It allows us to relate the vectors v and w involved in the construction of certain special sets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$.

Following [16, p. 325] a point $s \in S$ will be said irregular if there exist two different points k and $k' \in K$ such that $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v}) \cap \Gamma_{\mathrm{w}'}(k', \mathrm{v}')$ for some $\mathrm{v}, \mathrm{w}, \mathrm{v}'$ and $\mathrm{w}' \in X$. Analogously, we will say that a point $k \in K$ is irregular if $k \in \Gamma_{\mathrm{v}}(s, \mathrm{w}) \cap \Gamma_{\mathrm{v}'}(s', \mathrm{w}')$ for some different points s and $s' \in S$ and $\mathrm{v}, \mathrm{w}, \mathrm{v}'$ and $\mathrm{w}' \in X$.

Recall that since X is strictly convex, for all $x, y \in X$ with $y \neq 0$, if

$$\|x + y\| = \|x\| + \|y\|, \tag{5.1}$$

then $x = ty$ for some $t \in \mathbf{R}$ [12, p. 404].

Proposition 5.1. *Suppose that $k \in K$ and that s is an irregular point of S. If $s \in \Gamma_{\mathrm{w}_1}(k, \mathrm{v}_1) \cap \Gamma_{\mathrm{w}_2}(k, \mathrm{v}_2)$ for some $\mathrm{v}_1, \mathrm{v}_2, \mathrm{w}_1, \mathrm{w}_2 \in X$, then*

$$\|\widehat{\mathrm{w}_1} \pm \widehat{\mathrm{w}_2}\| \le \|\hat{\mathrm{v}}_1 \pm \hat{\mathrm{v}}_2\|.$$

Proof. In virtue of Proposition 3.1.2 we may assume that $\|\mathrm{v}_1\| = \|\mathrm{v}_2\| = 1$. Hence $\|\mathrm{w}_1\| = \|\mathrm{w}_2\| = 1/M$. Since s is irregular, there exists $k' \in K$, $k' \neq k$ and vectors v' and $\mathrm{w}' \in X$ with $\|\mathrm{v}'\| = 1$ and $\|\mathrm{w}'\| = 1/M$ such that $s \in \Gamma_{\mathrm{w}'}(k', \mathrm{v}')$. According to Proposition 4.1.1 we have

$$\left\|\widehat{\mathrm{w}'} \pm \widehat{\mathrm{w}_i}\right\| = \lambda(X), \quad \text{for all} \quad i \in \{1, 2\}. \tag{5.2}$$

Since $k \neq k'$ by Urysohn's lemma there exist f and $f' \in C_0(K)$ satisfying:

(i) $f(K), f'(K) \subset [0, 1]$;
(ii) $f(k) = f'(k') = 1$;
(iii) The supports of f and f' are disjoint.

Put

$$h_1 = f \cdot (\mathrm{v}_1/2), \quad h_2 = f \cdot (\mathrm{v}_2/2) \quad \text{and} \quad h' = f' \cdot (\mathrm{v}'/2).$$

It is easy to check that

$$\omega\big(k, h' + h_1, \mathrm{v}_1\big) = \omega\big(k', h' + h_1, \mathrm{v}'\big) = 1/2.$$

Thus, by Proposition 4.1.2 we have

$$T(h' + h_1)(s) = \frac{\mathrm{w}' + \mathrm{w}_1}{2}. \tag{5.3}$$

In the same way we obtain

$$T(h' - h_1)(s) = \frac{\mathrm{w}' - \mathrm{w}_1}{2}, \tag{5.4}$$

and

$$T(h' + h_2)(s) = \frac{\mathrm{w}' + \mathrm{w}_2}{2}. \tag{5.5}$$

By combining (5.3), (5.4) and (5.5) we infer that

$$Th_1(s) = \mathrm{w}_1/2, \quad Th_2(s) = \mathrm{w}_2/2 \quad \text{and} \quad Th'(s) = \mathrm{w}'/2.$$

Hence we have that

$$\left\|\frac{\mathrm{w}_1}{2} + \frac{\mathrm{w}_2}{2} \pm \|\hat{\mathrm{v}}_1 + \hat{\mathrm{v}}_2\| \frac{\mathrm{w}'}{2}\right\| = \|T(h_1 + h_2 \pm \|\hat{\mathrm{v}}_1 + \hat{\mathrm{v}}_2\| h')(s)\| \le M \left\|h_1 + h_2 \pm \|\hat{\mathrm{v}}_1 + \hat{\mathrm{v}}_2\| h'\right\| = \frac{M\|\hat{\mathrm{v}}_1 + \hat{\mathrm{v}}_2\|}{2},$$

where the last equality is due to (iii). Since

$$\|\mathrm{w}_1\| = \|\mathrm{w}_2\| = \|\mathrm{w}'\| = 1/M,$$

we infer that

$$\left\|\frac{\widehat{\mathrm{w}_1} + \widehat{\mathrm{w}_2}}{\|\hat{\mathrm{v}}_1 + \hat{\mathrm{v}}_2\|} \pm \widehat{\mathrm{w}'}\right\| \le M^2 = \lambda(X). \tag{5.6}$$

Now we are in a position to show that $\|\widehat{\mathrm{w}_1} \pm \widehat{\mathrm{w}_2}\| \le \|\hat{\mathrm{v}}_1 \pm \hat{\mathrm{v}}_2\|$. We will prove that $\|\widehat{\mathrm{w}_1} + \widehat{\mathrm{w}_2}\| \le \|\hat{\mathrm{v}}_1 + \hat{\mathrm{v}}_2\|$. The proof of the other inequality is analogous.

Assume then that

$$\|\widehat{w_1} + \widehat{w_2}\| > \|\widehat{v_1} + \widehat{v_2}\|, \tag{5.7}$$

and from this we will derive a contradiction.

Claim. $\{w', w_1 + w_2\}$ is linearly independent.

Suppose the contrary. We have that w', w_1 and w_2 are in the same plane, then by (5.2) and Proposition 2.3.2, we conclude that $\{w_1, w_2\}$ is linearly dependent. Since w_1 and w_2 are colinear, have the same norm and (5.7) holds, we deduce that $w_1 = w_2$. Therefore, since we are supposing that $\{w', w_1 + w_2\}$ is linearly dependent, $\{w', w_1\}$ is linearly dependent, which contradicts (5.2).

It follows immediately by the above claim that

$$\left\{ \frac{\widehat{w_1} + \widehat{w_2}}{\|\widehat{v_1} + \widehat{v_2}\|} + \widehat{w'}, \quad \frac{\widehat{w_1} + \widehat{w_2}}{\|\widehat{v_1} + \widehat{v_2}\|} - \widehat{w'} \right\} \tag{5.8}$$

is linearly independent. Put

$$a = \frac{1}{2} + \frac{\|\widehat{v_1} + \widehat{v_2}\|}{2\|\widehat{w_1} + \widehat{w_2}\|},$$

and

$$b = \frac{1}{2} - \frac{\|\widehat{v_1} + \widehat{v_2}\|}{2\|\widehat{w_1} + \widehat{w_2}\|}.$$

Observe that by (5.7) we have that $a, b > 0$. Moreover $a + b = 1$ and

$$(\widehat{\widehat{w_1} + \widehat{w_2}}) \pm \widehat{w'} = a\left(\frac{\widehat{w_1} + \widehat{w_2}}{\|\widehat{v_1} + \widehat{v_2}\|} \pm \widehat{w'} \right) - b\left(\frac{\widehat{w_1} + \widehat{w_2}}{\|\widehat{v_1} + \widehat{v_2}\|} \mp \widehat{w'} \right).$$

So, by (5.1), (5.8) and (5.6) we conclude that

$$\left\|(\widehat{\widehat{w_1} + \widehat{w_2}}) \pm \widehat{w'}\right\| < a \left\| \frac{\widehat{w_1} + \widehat{w_2}}{\|\widehat{v_1} + \widehat{v_2}\|} \pm \widehat{w'} \right\| + b \left\| \frac{\widehat{w_1} + \widehat{w_2}}{\|\widehat{v_1} + \widehat{v_2}\|} \mp \widehat{w'} \right\| \leq \lambda(X),$$

a contradiction with the definition of $\lambda(X)$. □

6 | IRREGULAR POINTS OF K AND S VIA TWO AUXILIARY FUNCTIONS

At this point in the paper it is convenient to introduce two functions $\Phi : K \to \mathcal{P}(S)$ and $\Psi : S \to \mathcal{P}(K)$ given by

$$\Phi(k) = \bigcup \{\Gamma_w(k, v) : v, w \in X, v \neq 0 \quad \text{and} \quad \|w\| = \|v\|/M\},$$

and

$$\Psi(s) = \bigcup \{\Gamma_v(s, w) : v, w \in X, w \neq 0 \quad \text{and} \quad \|v\| = \|w\|/M\}.$$

Proposition 6.1. *Let $k \in K$. Suppose that $\Phi(k)$ is not a singleton set. Then k is an irregular point.*

Proof. Pick two different points $s, s' \in \Phi(k)$. So, there are v, v', w and $w' \in X$ such that

$$s \in \Gamma_w(k, v) \quad \text{and} \quad s' \in \Gamma_{w'}(k, v').$$

By Propositions 3.1.1 and 3.1.3 there exist z and $z' \in X$ satisfying

$$k \in \Gamma_z(s, w) \cap \Gamma_{z'}(s', w'),$$

hence k is irregular. □

Before we prove another property of the irregular elements of S we will need to establish a simple consequence of the well-known Brouwer's invariance of domain theorem, i.e. the statement that if U is an open subset of $\mathbf{R}^n$ and $f : U \to \mathbf{R}^n$ is an injective continuous map, then $f(U)$ is open and f is a homeomorphism between U and $f(U)$ [5], [19, p. 954]. Then, the proof of next proposition is a standard exercise that we leave to the reader.

Proposition 6.2. *Let $h : A \subset S_X \to S_X$ be a continuous injective map. If A has non-empty interior then $h(A)$ has non-empty interior.*

Proposition 6.3. *Let $k \in K$. Suppose that $\Phi(k)$ is not a singleton set. Then $\Phi(k)$ has only irregular elements.*

Proof. By Proposition 6.1 applied to $\Psi(s)$, it suffices to prove that for all $s \in \Phi(k)$, $\Psi(s)$ is not a singleton set. Assume by contradiction that $\Psi(s)$ is a singleton set for some $s \in \Phi(k)$. Since $s \in \Phi(k)$, there exist v and w $\in X$ such that $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v})$. By Propositions 3.1.1 and 3.1.3 there exists $\mathrm{z} \in X$ satisfying $\Gamma_{\mathrm{z}}(s, \mathrm{w}) = \{k\}$. Then $k \in \Psi(s)$ and therefore $\Psi(s) = \{k\}$. This allows us, by virtue of Propositions 3.1.1 and 3.1.3, to define a map $\phi : S_X \to S_X$ by

$$\{k\} = \Gamma_{\frac{\phi(\mathrm{w})}{M}}(s, \mathrm{w}), \quad \text{for all} \quad \mathrm{w} \in S_X.$$

Notice that ϕ is well defined and it has the following properties:

(a) ϕ is continuous and injective. Indeed, for each $\mathrm{w}_1, \mathrm{w}_2 \in S_X$, we have that

$$k \in \Gamma_{\frac{\phi(\mathrm{w}_1)}{M}}(s, \mathrm{w}_1) \cap \Gamma_{\frac{\phi(\mathrm{w}_2)}{M}}(s, \mathrm{w}_2),$$

and, since k is an irregular point, by Proposition 5.1,

$$\|\phi(\mathrm{w}_1) \pm \phi(\mathrm{w}_2)\| \leq \|\mathrm{w}_1 \pm \mathrm{w}_2\|.$$

The continuity of ϕ follows from this inequality with "−". On the other hand, assume that $\phi(\mathrm{w}_1) = \phi(\mathrm{w}_2)$ for some $\mathrm{w}_1, \mathrm{w}_2 \in X$. Then by the above inequality with "+" we conclude that $2 = \|\mathrm{w}_1 + \mathrm{w}_2\|$. Thus, by the characterization of the strictly convex spaces mentioned in (5.1), $\mathrm{w}_1 = \mathrm{w}_2$.

(b) According to Proposition 6.1 k is an irregular point. So, there exist $s' \in S$, $s' \neq s$ and $\mathrm{w}', \mathrm{z}' \in X$ such that $k \in \Gamma_{\mathrm{z}'}(s', \mathrm{w}')$. We claim that

$$Im\,\phi \subset \left\{ y \in X : \left\|\widehat{\mathrm{z}'} \pm y\right\| = \lambda(X) \right\}. \tag{6.1}$$

Indeed, for each $\mathrm{w} \in S_X$, we have that

$$k \in \Gamma_{\frac{\phi(\mathrm{w})}{M}}(s, \mathrm{w}) \cap \Gamma_{\mathrm{z}'}(s', \mathrm{w}'),$$

and then by Proposition 4.1.1 it follows that $\left\|\widehat{\mathrm{z}'} \pm \phi(\mathrm{w})\right\| = \lambda(X)$, and we are done.

Finally, by (a) and Proposition 6.2 it follows that $Im\,\phi$ has non-empty interior. Consequently, according to (6.1) the set

$$\left\{ y \in X : \left\|\widehat{\mathrm{z}'} \pm y\right\| = \lambda(X) \right\}$$

has non-empty interior. But this is a contradiction with Proposition 2.3.4. □

7 | $\Phi(k)$ HAS AT MOST 2 ELEMENTS, FOR ALL $k \in K$

Our main purpose in this section is to prepare the proof of Proposition 8.1, which asserts that $\Phi(k)$ is a singleton set for each $k \in K$. So, we will prove the following proposition.

Proposition 7.1. *The cardinality of $\Phi(k)$ is at most 2, for every $k \in K$.*

Proof. Suppose that there are three distinct elements $s_1, s_2, s_3 \in \Phi(k)$. For each $1 \leq i \leq 3$, by the definition of $\Phi(k)$, fix v_i and w_i such that $s_i \in \Gamma_{\mathrm{w}_i}(k, \mathrm{v}_i)$. By Proposition 3.1.2 we may assume that $\|\mathrm{v}_i\| = M$ and consequently $\|\mathrm{w}_i\| = 1$, for $1 \leq i \leq 3$.

Now, by using Proposition 3.1.1, fix $z_i \in X$, $\|z_i\| = 1/M$ such that $\Gamma_{z_i}(s_i, w_i) \neq \emptyset$, for $1 \le i \le 3$. So, by Proposition 3.1.3 we conclude that

$$\Gamma_{z_1}(s_1, w_1) = \Gamma_{z_2}(s_2, w_2) = \Gamma_{z_3}(s_3, w_3) = \{k\}. \tag{7.1}$$

Pick $g_1, g_2, g_3 \in C_0(S)$ such that

(a) $g_i(S) \subset [0, 1]$;

(b) $g_i(s_i) = 1$;

(c) The supports of g_1, g_2 and g_3 are mutually disjoint.

Then, define for each $1 \le i \le 3$, $h_i = g_i \cdot (w_i/2)$.
It's straightforward that for each $i \neq j$,

$$\omega\left(s_i, h_i + h_j, w_i\right) = \omega\left(s_j, h_i + h_j, w_j\right) = 1/2.$$

Thus, according to (7.1) and Proposition 4.1.2, we have that

$$T^{-1}\left(h_i + h_j\right)(k) = \frac{z_i + z_j}{2}, \quad \text{for all} \quad i \neq j.$$

So, by the linearity of T^{-1} and combining the equations above, we obtain that

$$T^{-1} h_i(k) = \frac{z_i}{2}, \quad \text{for all} \quad 1 \le i \le 3.$$

Therefore for each pair $\sigma, \theta \in \{-1, 1\}$,

$$\left\| \frac{z_1}{2} + \sigma \frac{z_2}{2} + \theta \frac{z_3}{2} \right\| = \|T^{-1}(h_1 + \sigma h_2 + \theta h_3)(k)\| \le M\|h_1 + \sigma h_2 + \theta h_3\| = \frac{M}{2},$$

where the last equality is a consequence of (c).
But since $\|z_1\| = \|z_2\| = \|z_3\| = 1/M$, it follows that

$$\|\hat{z}_1 + \sigma \hat{z}_2 + \theta \hat{z}_3\| \le M^2 = \lambda(X). \tag{7.2}$$

On the other hand, by (7.1) and Proposition 4.1.1 we see that

$$\|\hat{z}_i \pm \hat{z}_j\| = \lambda(X), \quad \text{for all} \quad i \neq j. \tag{7.3}$$

It follows by Proposition 2.3.3 that $\{\hat{z}_1, \hat{z}_2, \hat{z}_3\}$ is linearly independent. Consequently $\{\hat{z}_1 + \hat{z}_2 + \hat{z}_3,\ \hat{z}_1 + \hat{z}_2 - \hat{z}_3\}$ is also linearly independent.
Hence by (7.3), the characterization of strictly convex spaces cited in (5.1) and (7.2), we obtain that

$$2\lambda(X) = 2\|\hat{z}_1 + \hat{z}_2\| < \|\hat{z}_1 + \hat{z}_2 + \hat{z}_3\| + \|\hat{z}_1 + \hat{z}_2 - \hat{z}_3\| \le 2\lambda(X),$$

this contradiction completes the proof of the proposition. □

8 | $\Phi(k)$ HAS ONLY ONE ELEMENT, FOR ALL $k \in K$

We are now in position to state the key proposition for proving Theorem 1.3.

Proposition 8.1. $\Phi(k)$ *is a singleton set for every* $k \in K$.

Proof. Assume that there exists $k \in K$ such that $\Phi(k)$ is not a singleton set. Then, by Proposition 7.1 there are $s_1, s_2 \in S$, with $s_1 \neq s_2$, such that $\Phi(k) = \{s_1, s_2\}$. For each $i \in \{1, 2\}$ put

$$V_i = \{v \in S_X \, : \, s_i \in \Gamma_w(k, v) \text{ for some } w\}.$$

It follows from the definition of $\Phi(k)$ that V_1 and V_2 are non-empty. Moreover, according to Proposition 3.1.1

$$S_X = V_1 \cup V_2.$$

Moreover, let us see that the sets V_1 and V_2 are closed. We will prove that V_1 is closed and the proof for V_2 is analogous. Pick $(\mathrm{v}_i)_{i\in I} \subset V_1$ converging to some $\mathrm{v} \in S_X$. We need to show that $\mathrm{v} \in V_1$. For each $i \in I$, fix w_i such that $\|\mathrm{w}_i\| = 1/M$ and $s_1 \in \Gamma_{\mathrm{w}_i}(k, \mathrm{v}_i)$, since $(\mathrm{w}_i)_{i\in I}$ is bounded, it admits a convergent subnet, so we may assume that $(\mathrm{w}_i)_{i\in I}$ converges to some w, with $\|\mathrm{w}\| = 1/M$. Given $f \in C_0(K, X)$, since for each $i \in I$, $s_1 \in \Gamma_{\mathrm{w}_i}(k, \mathrm{v}_i)$, we have

$$\|Tf(s_1) - \mathrm{w}_i\| \le M\omega(k, f, \mathrm{v}_i), \quad \text{for all } i \in I,$$

but then, since $\mathrm{v}_i \to \mathrm{v}$ and $\mathrm{w}_i \to \mathrm{w}$, we derive that

$$\|Tf(s_1) - \mathrm{w}\| \le M\omega(k, f, \mathrm{v}).$$

Therefore, $s_1 \in \Gamma_{\mathrm{v}}(k, \mathrm{w})$ and consequently $\mathrm{v} \in V_1$, as we wished.

Therefore, at least one of these sets has non-empty interior. So, we suppose without loss of generality that V_1 has non-empty interior.

We define $\phi : V_1 \to S_X$ as follows. Given $\mathrm{v} \in V_1$, pick $\mathrm{w} \in X$ such that $s_1 \in \Gamma_{\mathrm{w}}(k, \mathrm{v})$. Then, by Propositions 3.1.1 and 3.1.3, fix $\mathrm{z} \in X$ such that $\Gamma_{\mathrm{z}}(s_1, \mathrm{w}) = \{k\}$ and put $\phi(\mathrm{v}) = M^2\mathrm{z}$. Observe that ϕ has the following properties:

(a) ϕ is continuous and injective. Indeed, given $\mathrm{v}_1, \mathrm{v}_2 \in S_X$, let $\mathrm{w}_1, \mathrm{w}_2 \in X$ be as in the definition of ϕ. Then

$$s_1 \in \Gamma_{\mathrm{w}_1}(k, \mathrm{v}_1) \cap \Gamma_{\mathrm{w}_2}(k, \mathrm{v}_2).$$

By Proposition 6.3 we know that s_1 is an irregular point. Thus, by using Proposition 5.1, we get that

$$\|\widehat{\mathrm{w}_1} \pm \widehat{\mathrm{w}_2}\| \le \|\mathrm{v}_1 \pm \mathrm{v}_2\|. \tag{8.1}$$

Moreover, we have that

$$\Gamma_{\frac{\phi(\mathrm{v}_1)}{M^2}}(s_1, \mathrm{w}_1) \cap \Gamma_{\frac{\phi(\mathrm{v}_2)}{M^2}}(s_1, \mathrm{w}_2) = \{k\}.$$

On the other hand, according to Proposition 6.1 we know that k is also an irregular point. Hence by Proposition 5.1 and (8.1) we see that

$$\|\phi(\mathrm{v}_1) \pm \phi(\mathrm{v}_2)\| \le \|\widehat{\mathrm{w}_1} \pm \widehat{\mathrm{w}_2}\| \le \|\mathrm{v}_1 \pm \mathrm{v}_2\|.$$

The continuity of ϕ follows from this inequality with "−" and the injectivity of ϕ follows from the inequality with "+" and the strict convexity of X.

(b) There exists some $\mathrm{z} \in S_X$ such that

$$\phi(V_1) \subset \{y \in X : \|y \pm \mathrm{z}\| = \lambda(X)\}. \tag{8.2}$$

Indeed, since V_2 is non-empty, there are v' and $\mathrm{w}' \in X$ such that $s_2 \in \Gamma_{\mathrm{w}'}(k, \mathrm{v}')$. By Propositions 3.1.1 and 3.1.3, there exists $\mathrm{z}' \in X$ such that $\Gamma_{\mathrm{z}'}(s_2, \mathrm{w}') = \{k\}$.
Given $\mathrm{v} \in V_1$, pick $\mathrm{w} \in X$ such that $\Gamma_{\frac{\phi(\mathrm{v})}{M^2}}(s_1, \mathrm{w}) = \{k\}$. Then

$$k \in \Gamma_{\frac{\phi(\mathrm{v})}{M^2}}(s_1, \mathrm{w}) \cap \Gamma_{\mathrm{z}'}(s_2, \mathrm{w}'),$$

and by Proposition 4.1.1, we have that

$$\left\|\phi(\mathrm{v}) \pm \widehat{\mathrm{z}'}\right\| = \lambda(X).$$

So $\mathrm{z} = \widehat{\mathrm{z}'}$ satisfies the stated property.

Now, since V_1 has non-empty interior and (a) holds, it follows by Proposition 6.2 that $\phi(V_1)$ has non-empty interior. But then, by (8.2), it follows that

$$\{y \in X : \|y \pm z\| = \lambda(X)\}$$

has non-empty interior, a contradiction with Proposition 2.3.4. □

9 | COMPLETING THE PROOF OF THE MAIN THEOREM

The great work to prove Theorem 1.3 of this paper has already been made in the preceding sections. Here, we finish the proof of this theorem by showing that K and S are homeomorphic. Thanks to Proposition 8.1 we can define two functions $\varphi : K \to S$ and $\psi : S \to K$ by

$$\Phi(k) = \{\varphi(k)\} \quad \text{and} \quad \Psi(s) = \{\psi(s)\}.$$

Therefore, it suffices to prove that the functions $\varphi : K \to S$ and $\psi : S \to K$ are continuous and $\psi = \varphi^{-1}$. The proof of these facts is now direct and follows step by step that of [16, Proposition 6.1].

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