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(Key words)

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FINITE FORMS OF DE FINETTI'S TYPE THEOREM FOR UNIVARIATE UNIFORM DISTRIBUTIONS

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ABSTRACT

Let $X_1, X_2, \ldots, X_N$ be a collection of random variables and define $X_{(1)} = \min\{X_i, 1 \leq i \leq N\}$, $X_{(N)} = \max\{X_i, 1 \leq i \leq N\}$. We show that if the conditional distribution of $(X_1, \ldots, X_N)$ given $(X_{(1)}, X_{(N)})$ is uniform then for $n < N$ the joint distribution of $(X_1, \ldots, X_n)$ is close to a mixture of parametric uniform distributions depending on two parameters. We also consider the one parameter case. The closeness here is in the sense of the total variation distance. The infinite versions of the theorem is also obtained from the finite form results.

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1 Introduction

For infinite sequences of exchangeable random variables, de Finetti's Theorem (1937) says that the process law can be represented as a mixture of product measures. Essentially the theorem is an existence result, not necessarily providing the functional form of the model in the representation, unless we specify additional hypotheses to exchangeability or the sample space structure is very simple, e.g., finite. For instance, when $\{X_n\}_{n\in\mathbb{N}}$ is an infinite sequence of 0-1 exchangeable random variables de Finetti's Theorem states that, for each n in $\mathbb{N}$ and $(x_1,\ldots,x_n)\in\{0,1\}^n$,

$$P(X_1=x_1,\ldots,X_n=x_n)=\int_{[0,1]}\theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}\,d\mu(\theta)\,,$$

where μ is the P-law of the random variable

$$\Theta=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n X_i\,.$$

Following the predictivist viewpoint, due to de Finetti, the theorem describes how a subjectivist statistician sets the parameters to the model (Wechsler, 1993). The Bernoulli model in the representation can be understood as a consequence of subjective evaluation of exchangeability. Dawid (1982) calls such model intersubjective. Moreover, the theorem establishes the connection between the probability and the relative frequency (Barlow, 1991). A particular value of the limiting random variable gives the propensity as in Dawid (1982).

The mixing measure μ express the initial opinion with respect to such propensities. When the result is extended to exchangeable process on a general finite states space then we have the multinomial product model (Diaconis and Freedman, 1981). Other models like Poisson, Normal, Exponential can be put under such framework. Freedman (1962, 1963), Kingman (1972), Smith (1981), Diaconis and Freedman (1990), Barlow and Mendel (1992), Barlow and Spizzichino (1993) and Wechsler (1993) among others characterize classes of exchangeable random variables whose law can be represented as mixture of parametric exponential family. In another direction, Ressel (1985), by using Harmonic Analysis on semigroups, presents the most general result. Unfortunately there is a disadvantage as we can quote Diaconis (1988): "the results often wind up in highly analytic form, e.g., as conditions on the characteristic function of the process instead of on observables". Still quoting Diaconis (1988) we have "It is a worthwhile project to try to systematically translate these results to condition on observables." We should point out that whenever the results can be translated to condition on observables, one gets an interpretation of the model and the parameters in the Bayesian approach language.

Finite sequences of exchangeable random variables cannot be represented necessarily as a mixture of i.i.d. processes. When such representation does not exist, finite version of de Finetti's type theorem has been established. The idea is to estimate the total variation distance between the subcollection of finite sequence law and the mixture of product measures law. Diaconis and Freedman (1980), for instance, gives such finite version for binary sequences. Finite versions of de Finetti's type theorem, associated with some models in the parametric exponential family are given in Diaconis and Freedman (1987). They show for example that the n-dimensional distribution of orthogonaly invariant distribution on $\mathbb{R}^n$ ($n < \mathbb{N}$) is approximately a mixture of Normals. Finite forms associated to finite sequences of random vectors which are invariant under several orthogonal transformation groups have been given in Diaconis, Eaton and Lauritzen (1992).

Results related to finite version of de Finetti's type Theorem are interesting for at least two reasons. First, the infinite version of the theorem can be obtained from the finite version in a natural fashion. Second, the results are related to random quantities which are, at least conceptually, observable. The model parameters in the n-dimensional distribution of sequences of length N ($n \leq N$) are defined in a constructive way (de Finetti, 1949). This means they are functions of observable quantities. Moreover, the usual type of construction done to specifying the class of of distributions for $X_1, \ldots, X_N$ is usefull to solve problems in Finite Population Inference following a completely predictivist insight (Piccinato, 1986). Under such insight $X_1, X_2, \ldots, X_N$ represent the quantities associated to a finite population with N elements. The probabilistic models are specified through the opinions which express the uncertainty on the populational quantities. For instance, if $T(X_1, X_2, \ldots, X_N)$ is a populational quantity (e.g., the total) then one way to assign a model to $X_1, X_2, \ldots, X_N$ is starting from specifying first the conditional distribution of $X_1, \ldots, X_N$ given $T(X_1, \ldots, X_N)$. A a priori distribution to $T(X_1, \ldots, X_N)$ completes the process. $T(X_1, X_2, \ldots, X_N)$ can be inferred from samples comming from the population. More details on this can be found in Iglesias(1993). Infinite extendebility assumptions are not needed here as they are in the usual Bayesian approach for superpopulations (Ericson, 1969).

The aim of this paper is to present finite versions of de Finetti's type theorem, in the sense of Diaconis and Freedman (1987), first, for sequences of exchangeable random variables assuming non-negative values whose law is summarized by the order statistic $X_{(N)} = \max\{X_i : 1 < i \leq N\}$. Second, for finite sequences taking values on $\mathbb{R}$ whose conditional distribution given $(X_{(1)}, X_{(N)})$, where $X_{(1)} = \min\{X_i : 1 \leq i \leq N\}$, is uniform on the interval defined by these two quantities. The type of model we construct has been used to find predictors of the maximum and the minimum quantities on a finite population through the insight mentioned above. Finite forms in this context are usefull on studying approximate solutions.

The definitions, results and proofs are given in Sections 2 and 3.

Notation: We denote by $\mathcal{B}_n$ the Borel σ-field on $\mathbb{R}^n$ and by $\|\cdot\|$ the total variation distance; i.e., if P and Q are two probability measures on $(\Omega, \mathcal{A})$ then

$$\|P - Q\| = 2 \sup_{A \in \mathcal{A}} |P(A) - Q(A)| .$$

Also $\mathcal{X}^N$ will denote an N-fold product of a set $\mathcal{X}$ and $\mathbb{Z}_+$ the nonnegative integers.

2 Definitions and results

Let $X_1, X_2, \ldots, X_N$ be a finite sequence of exchangeable random variables taking values on $\mathcal{X} \subseteq \mathbb{R}$. Let $T : \mathcal{X}^N \to \mathcal{Y}$ be a mapping where $\mathcal{Y}$ is a subset of $\mathbb{R}^p$. The usual T's are functions of order statistics. Specifically, we consider the statistics $X_{(N)}$ and $(X_{(1)}, X_{(N)})$. The sequence we take has the property that the conditional distribution of $X_1, \ldots, X_N$ given $T = t$ is uniform over $T^{-1}(t)$. In this section Q_{Ntn} denotes the law of the first n coordinates $(1 \leq n \leq N)$ of a point uniformly distributed over $T^{-1}(t)$. We simply write Q_{Nt} for Q_{NtN}. Let us call $\mathcal{C}_N$ the class of exchangeable probability measures P on $\mathcal{X}^N$ such that $P = \int_{\mathcal{Y}} Q_{Nt}\, d\mu$, for some probability measure μ on $\mathcal{Y}$. Finite forms of de Finetti's type theorem are obtained for n-dimensional distributions of P in $\mathcal{C}_N$. The infinite version for the class of probability measures P on $\mathcal{X} \times \mathcal{X} \times \cdots$, whose N-dimensional distributions are in $\mathcal{C}_N$ for each N, follows from the finite form.

2.1 Discrete Uniform Distributions

Let $X_1, X_2, \ldots, X_N$ be an exchangeable random variables taking values on $\mathbb{Z}_+$. We assume that $(X_1, X_2, \ldots, X_N \mid X_{(N)} = t)$ is uniform over the set

$$\mathcal{X}_t^N = \{(x_1, \ldots, x_N) \in \mathbb{Z}_+^N : \max_{1 \leq i \leq N} \{x_i\} = t\}\ , \quad t \in \mathbb{Z}_+ \ .$$

If Q_{Nt} denotes such law then by simple counting we get

$$Q_{Nt}(x_1, \ldots, x_N) = \left[(t+1)^N - t^N\right]^{-1} I_{\{t\}}\left(\max_{1 \leq i \leq N} \{x_i\}\right) \ .$$

Therefore the class $\mathcal{C}_N$ consists of probability measures P, exchangeable on $\mathbb{Z}^N$, so that

$$P((x_1, \ldots, x_N)) = \int_{\mathbb{Z}_+} \left[(t+1)^N - t^N\right]^{-1} I_{\{t\}}\left(\max_{1 \leq i \leq N} \{x_i\}\right) d\mu(t)\ ,$$

with $(x_1, \ldots, x_N) \in \mathbb{Z}_+^N$.

The n-dimensional distribution P_n of P is given by

$$P_n((x_1,\ldots,x_n)) = \int_{\mathbb{Z}_+} Q_{Ntn}(x_1,\ldots,x_n)\,d\mu(t)$$

where

$$Q_{Ntn}(x_1,\ldots,x_n) = \sum_{(z_1,z_2,\ldots,z_{N-n})\in C} Q_{Nt}(x_1,\ldots,x_n,z_1,z_2,\ldots,z_{N-n})$$

with

$$C = \left\{(x_{n+1},\ldots,x_N) \in \mathbb{Z}_+^{N-n} : \max_{1\le i\le N}\{x_i\} = t\right\} .$$

Computing the summation we get,

$$Q_{Ntn}(x_1,\ldots,x_n) = \begin{cases} \dfrac{1}{(t+1)^n}\left\{\dfrac{1-\left(\frac{t}{t+1}\right)^{N-n}}{1-\left(\frac{t}{t+1}\right)^{N}}\right\} & \text{if } \max\limits_{1\le i\le N}\{x_i\} < t \\[2ex] \dfrac{1}{(t+1)^n}\left\{\dfrac{1}{1-\left(\frac{t}{t+1}\right)^{N}}\right\} & \text{if } \max\limits_{1\le i\le N}\{x_i\} = t\ . \end{cases}$$

The main result in the discrete case says that the distribution of the n first coordinates of a point uniformly distributed on $\mathcal{X}_t^N$ is close to the law of n independent random variables with uniform common distribution.

Theorem 1: *Let P_θ^n denote the law of n independent random variables uniformly distributed on $\{0,1,\ldots,\theta\}$ and Q_{Ntn} as before. For each $t \in \mathbb{N}$ and $1 \le n \le N$,*

$$\|Q_{Ntn} - P_t^n\| \le \frac{2n}{N} .$$

A finite form follows from the last theorem. Let $P_{\mu n} = \int P_\theta^n\, d\mu(\theta)$ for some probability measure μ on $\mathbb{Z}_+$. Note that $P_{\mu n}$ is also a n-dimensional distribution of some P in C_N.

Corollary 1: *If P_n and $P_{\mu n}$ are the previously defined probability measures then there exists a probability measure on $\mathbb{Z}_+$ so that for each $1 \le n \le N$,*

$$\|P_n - P_{\mu n}\| \le \frac{2n}{N} .$$

Proof: From the definition of P_n we have that

$$P_n = \int Q_{Nmn}\, d\mu_N(m)$$

where μ_N is the P-law of $X_{(N)}$. Hence, by setting $\mu = \mu_N$, the result follows from the convexity of the total variation distance. □

Remark 1: The above result is a finite form (see Ressel, 1985, Example 4).

Remark 2: If $P \in C_N$ then $P_n \in C_n$ for each $1 \le n < N$. Of course, if $X_1, X_2, \ldots, X_N$ are random variables with law P in C_N then

$$P(X_1 = x_1, \ldots, X_n = x_n \mid X_{(n)} = t_s) = \frac{P(X_1 = x_1, \ldots, X_n = x_n) I_{X_{(n)}^{-1}(t_s)}(x_1, \ldots, x_n)}{P(X_{(n)} = t_s)}$$

if $P(X_{(n)} = t_s) > 0$ and $t_s \in \mathbb{Z}_+$. Now,

$$\begin{aligned} P(X_{(n)} = t_s) &= \sum_{(z_1, z_2, \ldots, z_n) \in X_{(n)}^{-1}(t_s)} P(X_1 = z_1, \ldots, X_n = z_n) \\ &= \sum_{(z_1, z_2, \ldots, z_n) \in X_{(n)}^{-1}(t_s)} \int_{y} Q_{Ntn}(z_1, \ldots, z_n)\, d\mu_N(t)\,, \end{aligned}$$

where μ_N is the P-law of $X_{(N)}$. But

$$Q_{Ntn}(z_1, z_2, \ldots, z_n) = Q_{Ntn}(x_1, \ldots, x_n) \quad \text{if} \quad \max_{1 \le i \le n} \{x_i\} = t_s\,.$$

Therefore,

$$P(X_{(n)} = t_s) = |X_{(n)}^{-1}(t_s)| P(X_1 = x_1, \ldots, X_n = x_n)$$

where $|A|$ denotes the cardinality of A.

Consequently,

$$P(X_1 = x_1, \ldots, X_n = x_n \mid X_{(n)} = t_s) = \frac{1}{|X_{(n)}^{-1}(t_s)|} I_{X_{(n)}^{-1}(t_s)}(x_1, \ldots, x_n)\,.$$

Notice that $P_\theta^N \in C_N$. Moreover, the probability measures in $\mathcal{P} = \{P_\theta^N : \theta \in \mathbb{Z}_+\}$ are the only product probability measures P in C_N as can be seen in the next result.

Proposition 1: *If $X_1, X_2, \ldots, X_N$ are independent and identically distributed random variables with law $P \in C_N$ then $X_1, X_2, \ldots, X_N$ are uniformly distributed.*

Proof: By the hypothesis

$$P(X_1 = x_1, X_2 = x_2 \mid X_{(2)} = t) = \frac{\prod_{i=1}^{2} P(X_1 = x_i) I_{X_{(2)}^{-1}(t)}(x_1, x_2)}{[P(X_1 \leq t)]^2 - [P(X_1 \leq t-1)]^2}$$

if $P(X_{(2)} = t) > 0$.

On the other hand, $P \in C_N$ implies that $P_2 \in C_2$. Hence,

$$\frac{\prod_{i=1}^{2} P(X_1 = x_i) I_{X_{(2)}^{-1}(t)}(x_1, x_2)}{[P(X_1 \leq t)]^2 - [P(X_1 \leq t-1)]^2} = \frac{I_{X_{(2)}^{-1}(t)}(x_1, x_2)}{(t+1)^2 - t^2} .$$

Taking $x_1 = x_2 = t$ in the above expression we get

$$\frac{[P(X_1 = t)]^2}{[P(X_1 \leq t)]^2 - [P(X_1 \leq t-1)]^2} = \frac{1}{2t+1} .$$

After some computation it results

$$\frac{P(X_1 = t)}{P(X_1 \leq t)} = \frac{1}{t+1} .$$

Evaluating the above equality at $t = 0, 1, 2, \ldots$ we see that

$$P(X_1 = x) = P(X_1 = 0) \quad \text{for each } x \in \mathbb{Z}_+ .$$

As P is a probability measure we conclude that there exists a $K \in \mathbb{Z}_+$ so that $P(X_1 > K) = 0$. Therefore,

$$P(X_1 = x) = \frac{1}{K+1} I_{\{0,1,\ldots,K\}}(x) \qquad \square$$

The infinite version of Corollary 1 characterizes a class of infinite exchangeable sequences. Each element of this class can be represented in a unique way as mixture of independent random variables with common uniform distribution. This is the result of the next theorem.

Theorem 2: *Let $X_1, X_2, \ldots$ be an infinite sequence of exchangeable random variables taking values in $\mathbb{Z}_+$ and P_n denote the law of $(X_1, X_2, \ldots, X_n)$. If for each $n \in \mathbb{N}$ have $P_n \in C_n$ then there is an unique probability measure on $\mathbb{Z}_+$ such that for each $(x_1, \ldots, x_n) \in \mathbb{Z}_+^n$*

$$P(X_1 = x_1, \ldots, X_n = x_n) = \int_{\mathbb{Z}_+} \frac{1}{(\theta+1)^n} I_{\{0,1,\ldots,\theta\}}\left(\max_{1\le i\le n}\{x_i\}\right) d\mu(\theta) .$$

Remark 3: The converse for the last theorem also holds. It is enough to see that the mapping $T(x_1, \ldots, x_n) = \max_{1\le i\le n}\{x_i\}$ is a summary statistic for the law P_θ^n of $(X_1, \ldots, X_n)$ given θ.

2.2 Continuous uniform distribution depending on one parameter

We now consider finite forms for the case of uniform distributions over intervals of type $(0,\theta)$ and $(-\theta,\theta)$, for $\theta > 0$. Similarly to the previous ones, the results are a consequence of the fact that the distribution of the first n-coordinates ($n < N$) of a point uniformly distributed on

i) $\mathcal{X}_t^N = \left\{(x_1, \ldots, x_N) \in \mathbb{R}_+^N : \max_{1\le i\le N}\{x_i\} = t\right\}, t > 0$

or on

ii) $H_t^N = \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N : \max_{1\le i\le N}\{|x_i|\} = t\right\}, t > 0$

is close to the law of n independent random variables with the common distribution being uniform on $(0,t)$ and $(-t,t)$, respectively.

The uniform distribuition on H_t^N and $\mathcal{X}_t^N$ are defined through the $N-1$ dimensional volume as it follows. Let

$$M_i\left(t(-1)^J\right) = \{(x_1, \ldots, x_N) \in H_t^N : x_i = t(-1)^J\}, \quad J = 0, 1, \ t > 0 .$$

Then $H_t^N = \bigcup_{J=0}^{1} \bigcup_{i=1}^{M} M_i\left(t(-1)^J\right)$.

Let $\varphi^i : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be defined as $\varphi^i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and λ to be the $N-1$-dimensional Lebesgue measure. For $B \in B_N$ define

$$\mu_i(B) = \lambda(\varphi^i(B \cap M_i(t))) + \lambda(\varphi^i(B \cap M_i(-t))) .$$

Definition 1: *The probability function $Q_{Nt} : B_N \to [0,1]$ given by*

$$Q_{Nt}(B) = \frac{\sum_{i=1}^N \mu_i(B)}{2N(2t)^{N-1}}$$

is called uniform probability measure on H_t^N.

We notice that the function $\psi : B_N \times \mathbb{R}_+ \to [0,1]$ defined as $\psi(B,t) = Q_{Nt}(B)$ is a transition function. The above probability measure is defined on the border of the N-dimensional hypercube centered at the origin. In the next proposition we redefine it in terms of random variables, similar to Eaton (1981) when defined the uniform distribution on a N-sphere.

Proposition 2: *Let $X_1, X_2, \ldots, X_N$ be independent random variables with the common distribution uniform on $(-1,1)$. Set $M_N = \max\{|X_1|, \ldots, |X_N|\}$ and $Y_i = t\frac{X_i}{M_N}$, $i = 1, 2, \ldots, N$. Then the vector $Y^N = (Y_1, Y_2, \ldots, Y_N)$ is uniformly distributed on H_t^N.*

Proof: Let P be the law of $(X_1, \ldots, X_N)$ and Q be the P-law of Y^N. It is clear from the definition of Y^N that

$$Q(H_t^N) = P(Y^N \in H_t^N) = 1 .$$

For $B \in B_N$ we have

$$\begin{aligned} Q(B) &= P(Y^N \in B) = P\left(Y^N \in \bigcup_{J=0}^{1} \bigcup_{i=1}^{N} \left(B \cap M_i(t(-1)^J)\right)\right) \\ &= \sum_{J=0}^{1} \sum_{i=1}^{N} P\left(Y^N \in B \cap M_i(t(-1)^J)\right) . \end{aligned}$$

On the other hand,

$$
\begin{aligned}
& P\left(Y^N \in B \cap M_i(t)\right) \\
= \; & P\left(\frac{X_i}{M_N} = 1, t\left(\frac{X_1}{M_N}, \ldots, \frac{X_{i-1}}{M_N}, \frac{X_{i+1}}{M_N}, \ldots, \frac{X_N}{M_N}\right) \in \varphi^i(B \cap M_i(t))\right) \\
= \; & P\left(X_i > 0, t\left(\frac{X_1}{M_N}, \ldots, \frac{X_{i-1}}{M_N}, \frac{X_{i+1}}{M_N}, \ldots, \frac{X_N}{M_N}\right) \in \varphi^i(B \cap M_i(t))\right),
\end{aligned}
$$

once $(z_1, \ldots, z_{N-1}) \in \varphi^i(B \cap M_i(t))$ implies $|z_j| \leq t$ for $j = 1, \ldots, N-1$. Therefore,

$$
P(Y^N \in B \cap M_i(t)) = \int_0^1 P\left((X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_N) \in \frac{x}{t}\varphi^i(B \cap M_i(t))\right) \frac{1}{2} \, dx
$$

$$
= \int_0^1 \lambda\left(\frac{x}{t}\varphi^i(B \cap M_i(t))\right) \frac{1}{2^N} \, dx = \frac{\lambda(\varphi^i(B \cap M_i(t)))}{N t^{N-1} 2^N}.
$$

Analogously, we can show that

$$
P\left(Y^N \in B \cap M_i(-t)\right) = \frac{\lambda\left(\varphi^i(B \cap M_i(-t))\right)}{N t^{N-1} 2^N}
$$

concluding the proof. □

Returning to $\mathcal{X}_t^N$, the uniform distribution $\tilde{Q}_{Nt}$ over $\mathcal{X}_t^N$ is defined in a similar fashion as before through the N-dimensional volume. Note that $\tilde{Q}_{Nt}$ corresponds to the Q_{Nt} law of T when $T(x_1, \ldots, x_N) = (|x_1|, \ldots, |x_N|)$. Hence

$$
\tilde{Q}_{Nt}(B) = \frac{\sum_{i=1}^N \lambda(\varphi^i(B \cap M_i(t)))}{N t^{N-1}}, \quad B \in \mathcal{B}_N
$$

where $M_i(t) = \left\{(x_1, \ldots, x_N) \in \mathcal{X}_t^N : x_i = t\right\}$.

The distribution in terms of random variables is given next.

Proposition 3: *Let $X_1, X_2, \ldots, X_N$ be independent random variables with the common distribution being uniform on $(0,1)$. Set $Y_i = t\frac{X_i}{X_{(N)}}$, $i = 1, 2, \ldots, N$. Then the random vector $Y^N = (Y_1, Y_2, \ldots, Y_N)$ is uniformly distributed on $\mathcal{X}_t^N$.*

Proof: It follows from the definition of $\tilde{Q}_{Nt}$ and the Proposition 2. □

Theorem 3: *Let P_θ^n be the law of n independent random variables with the common distribution uniform on $(-\theta,\theta)$. Then, for $1 \le n \le N$,*

$$\|Q_{Ntn} - P_t^n\| \le 4\frac{n}{N}\ .$$

The same inequality holds when we replace Q_{Ntn} by $\tilde{Q}_{Ntn}$ and then P_t^n corresponds to the law of n-independent random variables with the common distribution being uniform over $(0,t)$. As in the discrete case, a de Finetti's type result follows from the last theorem. Now C_N is the class of probability measures P on $\mathbb{R}^N$ so that $P = \int_{\mathbb{R}_+} Q_{Nt}\, d\mu(t)$ for some probability measure μ on $\mathbb{R}_+$ and P_n is a n-dimensional distribution from $P \in C_N$.

Corollary 2: *Let P_θ^n be the law of n independent random variables with the common uniform distribution over $(-\theta,\theta)$ and let $P_{\mu n} = \int_{\mathbb{R}_+} P_\theta^n\, d\mu(\theta)$ for some probability measure μ on $\mathbb{R}_+$. Then, for $1 \le n \le N$,*

$$\|P_n - P_{\mu n}\| \le \frac{4n}{N}\ .$$

Proof: It suffices to choose μ as the P-law of $M_N = \max\{|X_1|,\dots,|X_N|\}$. □

The associated infinite version characterizes the family of exchangeable random variables which has a unique representation as a mixture of independent random variables, all of them uniformly distributed on $(-\theta,\theta)$.

Theorem 4: *Let $X_1, X_2, \dots$ be an infinite sequence of exchangeable random variables taking values on $\mathbb{R}_+$ and P_n denote the law of $X_1, X_2, \dots, X_n$. If for each $n \in \mathbb{N}$, $P_n \in C_n$ then there exists an unique probability measure μ on $\mathbb{R}_+$ so that for each $(x_1,\dots,x_n) \in \mathbb{R}^n$*

$$P(X_1 \le x_1,\dots,X_n \le x_n) = \int_{\mathbb{R}_+} \prod_{i=1}^n \left\{\frac{x_i+\theta}{2\theta} I_{(-\theta,\theta)}(x_i) + I_{[\theta,+\infty)}(x_i)\right\} d\mu(\theta)\ .$$

Remark 4: Note that if $X_1, X_2, \ldots, X_n$ are random variables with law P_θ^n then $P_\theta^n \in C_n$. In fact, $M_n = \max_{1 \le i \le n}\{|X_i|\}$ is independent of $T_n = \left(\frac{X_1}{M_n}, \ldots, \frac{X_n}{M_n}\right)$ since M_n is a complete and sufficient statistic for the family $\{P_\theta^n, \theta > 0\}$ and T_n is ancillar (see Proposition 2 and Basu (1958, 1959)). Therefore, for bounded f we have

$$E_\theta(f(X_1, \ldots, X_n)|M_n = t) = E_\theta(f(T_n M_n)|M_n = t) = E_\theta(f(T_n t)) .$$

The converse of Theorem 3 also follows from this result.

Remark 5: Corollary 2 and Theorem 3 remain true if we replace Q_{Nt} by $\tilde{Q}_{Nt}$ and P_θ^n by the law of n independent random variables with common distribution uniform on $(0, \theta)$.

Let us consider now the case where the common uniform distribution is symmetric around a value $\lambda \in \mathbb{R}$. When λ is known the characterization of the class of exchangeable random variables which has representation as a mixture or quasi-mixture of independent random variables follows, directly from the results of the same type. However, if λ is unknown, then the problem is to obtain the characterization when the n independent variables are uniform on (θ_1, θ_2) with both θ_1 and θ_2 unknown. We deal with such problem next.

2.3 Continuous uniform distribution depending on two parameters

Consider the space

$$\mathcal{X}_{t_1,t_2}^N = \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N : \min_{1 \le i \le N}\{x_i\} = t_1 \quad \text{and} \quad \max_{1 \le i \le N}\{x_i\} = t_2\right\}$$

with t_1 and t_2 in $\mathbb{R}$, $t_1 < t_2$ and $N \ge 3$. Set

$$M_{ij}(t_1, t_2) = \left\{(x_1, \ldots, x_N) \in \mathcal{X}_{t_1,t_2}^N : x_i = t_1, x_j = t_2\right\} , \quad i \ne j$$

and $i, j \in \{1, 2, \ldots, N\}$. Then $\mathcal{X}_{t_1,t_2}^N = \bigcup_{\substack{i,j \in \{1,\ldots,N\} \\ i \ne j}} M_{ij}(t_1, t_2)$.

Let $\varphi^{ij} : \mathbb{R}^N \to \mathbb{R}^{N-2}$ to be defined by

$$\varphi^{ij}(x_1, x_2, \ldots, x_N) = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$$

and λ denote the $N-2$ dimensional Lebesgue measure. For B in $\mathcal{B}_N$ define

$$\mu_{ij}(B) = \lambda\left(\varphi^{ij}(B \cap M_{ij}(t_1, t_2))\right) .$$

Definition 2: The probability function $Q_{N(t_1,t_2)} : \mathcal{B}_N \to [0,1]$ defined by

$$Q_{N(t_1,t_2)}(B) = \sum_{i,j\in\{1,2,\ldots,N\}} \frac{\mu_{ij}(B)}{N(N-1)(t_2-t_1)^{N-2}}$$

is to be called *uniform probability distribution on* $\mathcal{X}^N_{t_1,t_2}$.

It is easy to see that $Q_{N(t_1,t_2)}$ is as transition function. In terms of random variables the uniform distribution can be seen as in the sequel.

Proposition 4: *Let $X_1, X_2, \ldots, X_N$ be independent random variables with the uniform $(0,1)$ common distribution. Set*

$$Y_i = (t_2 - t_1)\left\{\frac{X_i - X_{(1)}}{X_{(N)} - X_{(1)}}\right\} + t_1 , \quad \textit{with } t_1 < t_2 .$$

Then the random vector $Y^N = (Y_1, \ldots, Y_N)$ is uniformly distributed over $\mathcal{X}^N_{t_1,t_2}$.

Proof: Analogous to the one to Proposition 3. □

Let $P^n_{\theta_1,\theta_2}$ be the law of n iid random variables uniform on (θ_1, θ_2) and let $Q_{N(t_1,t_2)n}$ be the law of the n-first coordinates of a point uniformly distributed on $\mathcal{X}^N_{t_1,t_2}$.

Theorem 5: *For each $2 \leq n \leq N$,*

$$\|Q_{N(t_1,t_2)n} - P^n_{t_1,t_2}\| \leq \frac{2n(4N-n-3)}{N(N-1)} .$$

Let now C_N denote the class of probability measures P on $\mathbb{R}^N$ so that

$$P = \int Q_{N(t_1,t_2)}\, d\mu(t_1,t_2)$$

for some probability measure μ concentrated on $S = \{(x,y) \in \mathbb{R}^2 : x \leq y\}$ and P_n denote the n-dimensional distribution ($n < N$) from $P \in C_N$.

Corollary 3: *Set* $P_{\mu n} = \int P_{\theta_1,\theta_2}\, d\mu(\theta_1,\theta_2)$ *for some probability measure* μ *on* S. *Then, for* $2 \leq n \leq N$,

$$\|P_n - P_{\mu n}\| \leq \frac{2n(4N-n-3)}{N(N-1)}\,.$$

Proof: It suffices to choose μ as the P-law of $(X_{(1)}, X_{(N)})$. □

Theorem 6: *Let* $X_1, X_2, \ldots$ *be a sequence of infinite exchangeable random variables taking values on* $\mathbb{R}$ *and let* P_n *denote the law of* $X_1, X_2, \ldots, X_n$. *If for each* $n \in \mathbb{N}$ *we have* $P_n \in C_n$ *then there exists an unique probability measure* μ *on* S *so that for each* $n \in \mathbb{N}$ *and* $B \in B_n$,

$$P((X_1,\ldots,X_n) \in B) = \int_S P^n_{\theta_1,\theta_2}(B)\, d\mu(\theta_1,\theta_2)\,.$$

Remark 6: Note that if the sequence $(X_1, X_2, \ldots, X_n)$ has $P^n_{\theta_1,\theta_2}$ as its law then $P^n_{\theta_1,\theta_2}$ is in C_n. To see that we recall that $(X_{(1)}, X_{(n)})$ is a complete and sufficient statistic to the family $\{P^n_{\theta_1,\theta_2} : \theta_1 < \theta_2\}$ and $T_n = \left(\frac{X_1 - X_{(1)}}{X_{(n)} - X_{(1)}}, \ldots, \frac{X_n - X_{(1)}}{X_{(n)} - X_{(1)}}\right)$ is an ancillary statistic. The converse to Theorem 5 follows from this.

3 Proofs of the theorems

Proof of the Theorem 1: Let ν be the counting measure. Then

$$\|Q_{Ntn} - P^n_t\| = \int_{\{0,1,\ldots,t\}^n} \left|1 - \frac{Q_{Ntn}}{P^n_t}\right| P^n_t\, d\nu$$

$$= \int_{\{(x_1,\ldots,x_n)\in \mathbb{N}^n:\ \max_{1\le i\le n}\{x_i\}<t\}} \left\{ \frac{\left(\frac{t}{t+1}\right)^{N-n} - \left(\frac{t}{t+1}\right)^{N}}{1-\left(\frac{t}{t+1}\right)^{N}} \right\} \frac{1}{(t+1)^n}\, d\nu(x_1,\ldots,x_n)$$

$$+ \int_{\{(x_1,\ldots,x_n)\in \mathbb{N}^n:\ \max_{1\le i\le n}\{x_i\}=t\}} \left\{ \frac{\left(\frac{t}{t+1}\right)^{N}}{1-\left(\frac{t}{t+1}\right)^{N}} \right\} \frac{1}{(t+1)^n}\, d\nu(x_1,\ldots,x_n)$$

$$= 2\left\{ \frac{\left(\frac{t}{t+1}\right)^{N} - \left(\frac{t}{t+1}\right)^{N+n}}{1-\left(\frac{t}{t+1}\right)^{N}} \right\} .$$

Set $g(x) = \frac{x^N - x^{N+n}}{1-x^N}$, $0 \le x < 1$. We show here that $g(x) \le \frac{n}{N}$, any x in $[0,1)$. Note that

$$g(x) = \frac{x^N(1+x+\cdots+x^{n-1})}{1+x+x^2+\cdots+x^{N-1}} .$$

Also, $(1+x+\cdots+x^{n-1}) < n$ and $(1+x+x^2+\cdots+x^{N-1} > Nx^{N-1}$ so that

$$g(x) < \frac{x^N n}{Nx^{N-1}} < \frac{n}{N} ,$$

concluding the proof. □

It is not hard to see that $\lim_{x\to 1^-} g(x) = \frac{n}{N}$ and that $\frac{dg}{dx} \ge 0$ for $0 \le x < 1$, from which we can conclude that $\frac{2n}{N}$ is the best estimate for the total variation in this case.

Proof of Theorem 2: Let μ_N be the P-law of $X_{(N)}$. From the hypothesis and Corollary 1 we have for $n \le N$ and $N \in \mathbb{N}$ that

$$\|P_n - P_{\mu_N n}\| \le \frac{2n}{N} ,$$

where P_n is the law of $(X_1, X_2, \ldots, X_n)$ and

$$P_{\mu_N n}(x_1,\ldots,x_n) = \int \frac{1}{(\theta+1)^n} I_{\{0,1,\ldots,\theta\}}\left(\max_{1\le i\le n}\{x_i\}\right) d\mu_N(\theta) .$$

The proof follows from the above inequality if we show that the sequence $\{\mu_N\}_{N\in\mathbb{N}}$ is tight.

By the hypothesis

$$P(X_1 > k) = \int_{\mathbb{Z}_+} P(X_1 > k / X_{(N)} = t)\, d\mu_N(t)$$

$$= \int_{\mathbb{Z}_+} \left[1 - \sum_{x=0} Q_{Nt1}(x)\right] I_{\{k+1,\ldots\}}(t)\, d\mu_N(t)$$

$$\geq \int_{\mathbb{Z}_+} \left[1 - \left(\frac{k+1}{t+1}\right)\right] I_{\{k+1,\ldots\}}(t)\, d\mu_N(t) .$$

As the left-hand side goes to zero when k goes to ∞ we have that given $\eta > 0$ there exists $k_0 = k(\eta)$ such that

$$\eta \geq \int_{\mathbb{Z}_+} \left[1 - \left(\frac{k_0+1}{t+1}\right)\right] I_{\{k_0+1,\ldots\}}(t)\, d\mu_N(t) .$$

On the other hand, the function inside the integral increases to 1 as t goes to ∞. Therefore, given $\delta > 0$, there exists $M = M(\delta) > k_0$ so that if $t > M$

$$\int_{(M,+\infty)} (1-\delta)\, d\mu_N(t) \leq \eta ;$$

that is $\{\mu_N\}$ is tight. By choosing μ as one of the subsequential limit we have the first part of the proof.

For uniqueness, notice that

$$P(X_{(N)} \leq t) = \int_{\mathbb{Z}_+} [P_\theta(X_1 \leq t)]^N\, d\mu(\theta) = \int_{\{\theta \leq t\}} d\mu(\theta) + \int_{\{\theta > t\}} [P_\theta(X_1 \leq t)]^N\, d\mu(\theta) ,$$

where P_θ is uniformly distributed on $\{0, 1, \ldots, \theta\}$. Now $(P_\theta(X_1 \leq t))^N \to 0$ as $N \to \infty$ if $\theta > t$.

So,

$$\lim_{N\to\infty} P(X_{(N)} \leq t) = \mu(\{0, 1, \ldots, t\}) ,$$

concluding the proof. □

Remark 7: The uniqueness problem also appears in the finite case. In other words, if $X_1, X_2, \ldots, X_N$ are random variables taking values on $\mathbb{N}$ with law P in C_N then there is an unique probability measure μ over $\mathbb{Z}_+$ so that, for each $1 \leq n \leq N$,

$$P(X_1 = x_1, \ldots, X_n = x_n) = \int_{\mathbb{Z}_+} Q_{Ntn}(x_1, \ldots, x_n)\, d\mu(t) .$$

The representation is obvious since it is enough to choose μ as the P-law of $X_{(N)}$. For the uniqueness, it suffices to notice that, if $x \in \mathbb{N}$ then,

$$P(X_1 = x) - P(X_1 = x+1) = Q_{Nx1}(x)\mu(\{x\}) + (Q_{N(x+1)1}(x) - Q_{N(x+1)1}(x+1))\mu(\{x+1\}) ,$$

from what we can see that P_1, the law of X_1, determines μ.

Proof of Theorem 3: Taking into account that the total variation is invariant under a 1 to 1 transformation,

$$||Q_{Ntn} - P_t^n|| = ||Q_{N1n} - P_1^n|| .$$

Let $\tilde{Q}_1$ be the Q_{N1n}-law of $M_n = \max_{1 \leq i \leq n} \{|Y_i|\}$ and $\tilde{P}_1$ be the correspondent P_1^n-law.

Noticing that M_n is a sufficient statistic for both families $\{Q_{Ntn}, t > 0\}$ and $\{P_t^n, t > 0\}$, it is a property of the total variation distance (see Diaconis and Freedman, 1987) that

$$||Q_{Ntn} - P_1^n|| = ||\tilde{Q}_1 - \tilde{P}_1|| .$$

Using Proposition 3, after some computation, we get that $\tilde{Q}_1$ is given by the distribution function

$$F(\omega) = \begin{cases} 0 & \text{if } \omega \leq 0 \\ \left(\dfrac{N-n}{N}\right)\omega^n & \text{if } 0 < \omega < 1 \\ 1 & \text{if } \omega \geq 1 \end{cases}$$

and $\tilde{P}_1$ is given by

$$G(\omega) = \begin{cases} 0 & \text{if } \omega \leq 0 \\ \omega^n & \text{if } 0 < \omega < 1 \\ 1 & \text{if } \omega \geq 1 . \end{cases}$$

Hence, if $B \in \mathcal{B}_1$ then

$$\tilde{Q}_1(B) = \int_{B \cap (0,1)} \left(\frac{N-n}{N}\right) n\omega^{n-1}\, d\omega + \tilde{Q}_1(B \cap \{1\})$$

and

$$\tilde{P}_1(B) = \int_{B\cap(0,1)} n\omega^{n-1}\,d\omega\ .$$

Using that we obtain

$$\begin{aligned}
\|\tilde{Q}_1 - \tilde{P}_1\| &\leq 2\sup_{B\in\mathcal{B}_1}\left|\int_{B\cap(0,1)}\left(\frac{N-n}{N}\right)n\omega^{n-1}\,d\omega - \int_{B\cap(0,1)} n\omega^{n-1}\,d\omega\right| \\
&\quad +2\sup_{B\in\mathcal{B}_1}|\tilde{Q}_1(B\cap\{1\})| \leq 2\int_{(0,1)}\frac{n}{N}n\omega^{n-1}\,d\omega + \frac{2n}{N} \\
&= 4\frac{n}{N}\ .
\end{aligned}$$

□

Proof of Theorem 4: Let μ_N be the P-law of $M_N = \max_{1\leq i\leq N}\{|X_i|\}$. From the hypothesis and Propostion 2 we have

$$P(|X_1| > k) = \int_{(k,+\infty)} \tilde{Q}_{Nt1}((k,+\infty))\,d\mu_N(t)\ ,$$

where $\tilde{Q}_{Nt1}$ is the law of $\frac{tY_1}{Y_{(N)}}$ with $Y_{(N)} = \max_{1\leq i\leq N}\{Y_i\}$ and $Y_1, Y_2, \ldots, Y_n$ are iid random variables uniform on $(0,1)$.

After some calculus we have that $\tilde{Q}_{Nt1}$ has distribution function

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \dfrac{N-1}{N}\left(\dfrac{x}{t}\right) & \text{if } 0 < x < t \\ 1 & \text{if } x \geq t\ . \end{cases}$$

We now have

$$P(|X_1| > k) = \int_{(k,+\infty)}\left(1 - \left(\frac{N-1}{N}\right)\frac{k}{t}\right)d\mu_N(t) \geq \int_{(k,+\infty)}\left(1-\frac{k}{t}\right)d\mu_N(t)\ .$$

Using this we can conclude in a similar fashion as in the proof of Theorem 3 that $\{\mu_N\}$ is tight. Such fact and Corollary 2 concludes the proof. □

Proof of Theorem 5: Again, invariance of the total variation distance under a 1 to 1 transformations gives us

$$\|Q_{N(t_1,t_2)n} - P^n_{(t_1,t_2)}\| = \|Q_{N(0,1)n} - P^n_{(0,1)}\|\ .$$

$\tilde{Q}_1$ is the $Q_{N(0,1)n}$-law of $Z = \left(\min_{1\le i\le n}\{Y_i\}, \max_{1\le i\le n}\{Y_i\}\right)$ where $\{Y_i, 1 \le i \le n\}$ is as in the last proof. $\tilde{P}_1$ is the corresponding $P^n_{(0,1)}$-law. Using the fact that Z is a sufficient statistic for the family $\{P^n_{(\theta_1,\theta_2)} : \theta_1 < \theta_2, (\theta_1,\theta_2) \in \mathbb{R}^2\}$ we have that

$$\|Q_{N(0,1)n} - P^n_{(0,1)}\| = \|\tilde{Q}_1 - \tilde{P}_1\| .$$

The $P^n_{(0,1)}$-law of Z is defined by the density

$$g(z_1,z_2) = \begin{cases} n(n-1)(z_2-z_1)^{n-2} & \text{if } 0 < z_1 < z_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4 and laborious computation give the distribution function associated to $\tilde{Q}_1$,

$$F(z_1,z_2) = \begin{cases} 0 & \text{if } z_1 < 0 \text{ or } z_2 < 0 \\ \frac{n(N-n)}{N(N-1)} z_2^{n-1} & \text{if } z_1 = 0 \text{ and } 0 < z_2 < 1 \\ \frac{(N-n)(N-n-1)}{N(N-1)}\{z_2^n - (z_2-z_1)^n\} + \frac{n(N-n)}{N(N-1)} z_2^{n-1} & \text{if } 0 \le z_1 < z_2 < 1 \\ \frac{(N-n)(N-n-1)}{N(N-1)} z_2^n + \frac{n(N-n)}{N(N-1)} z_2^n & \text{if } 0 < z_2 \le z_1 < 1 \\ \frac{(N-n)(N-n-1)}{N(N-1)}\{1-(1-z_1)^n\} + \frac{n(N-n)}{N(N-1)}\{1-(1-z_1)^{n-1}\} + \frac{n}{N} & \text{if } z_2 = 1, 0 < z_1 < 1 \\ 1 & \text{if } z_1, z_2 \ge 1 \quad . \end{cases}$$

Define $B_1 = \{0\} \times \{1\}$, $B_2 = (0,1] \times \{1\}$, $B_3 = \{0\} \times [0,1)$, and $B_4 = \{(z_1,z_2) \in \mathbb{R}^2 : 0 < z_1 < z_2 < 1\}$, so that

$$\begin{aligned} \|\tilde{Q}_1 - \tilde{P}_1\| &= 2 \sup_{B\in\mathcal{B}_2} \left| \int_B d\tilde{Q}_1 - \int_B d\tilde{P}_1 \right| \\ &= 2 \sup_{B\in\mathcal{B}_2} \left| \sum_{i=1}^4 \int_{B\cap B_i} d\tilde{Q}_1 - \int_{B\cap B_4} n(n-1)(z_2-z_1)^{n-2}\, dz_1\, dz_2 \right| . \end{aligned}$$

Notice that

$$\int_{B\cap B_4} d\tilde{Q}_1 = \frac{(N-n)(N-n-1)}{N(N-1)} \int_{B\cap B_4} n(n-1)(z_2-z_1)^{n-2}\, dz_2\, dz_1$$

then

$$\begin{aligned}
\|\tilde{Q}_1 - \tilde{P}_1\| &= 2 \sup_{B \in \mathcal{B}_2} \left| \frac{2n}{N} \int_{B \cap B_4} n(n-1)(z_2 - z_1)^{n-2}\, dz_2\, dz_1 - \sum_{i=1}^{3} \int_{B \cap B_i} d\tilde{Q}_1 \right| \\
&= 2 \left\{ 2\frac{n}{N} + \sum_{i=1}^{3} \int_{B_i} d\tilde{Q}_1 \right\} .
\end{aligned}$$

Note that $\int_{B_1} d\tilde{Q}_1 = \frac{n(n-1)}{N(N-1)}$ and $\int_{B_2} d\tilde{Q}_1 = \int_{B_3} d\tilde{Q}_1 = \frac{n(N-n)}{N(N-1)}$, allowing us to write

$$\|\tilde{Q}_1 - \tilde{P}_1\| \leq \frac{2n(4N - n - 3)}{N(N-1)} ,$$

as desired. □

Proof of Theorem 6: As in earlier cases, it is enough to show that the sequence $\{\mu_N\}$ is tight, μ_N being the P-law of $(X_{(1)}, X_{(N)})$. From the hypothesis we have

$$P(|X_1| > K) = \int \left\{ 1 - Q_{N(t_1,t_2)1}([-K, K]) \right\} d\mu_N(t_1, t_2) .$$

On the other hand, from Proposition 4, and after some computation, we have the distribution associated to $Q_{N(t_1,t_2)1}$ is

$$F_{t_1,t_2}(x) = \begin{cases} 0 & \text{if } x < t, \\ \frac{N-2}{N} \left(\frac{x - t_1}{t_2 - t_1} \right) & \text{if } t_1 \leq x \leq t_2 \\ 1 & \text{if } x \geq t_2 \ . \end{cases}$$

A probability property states that given $\eta > 0$ there exists a K_0 so that $K > K_0$ implies $P(|X_1| > K) < \eta$. Therefore by taking $K_1 > K_0 > 0$ we have

$$\eta \geq P(|X_1| > K_1) = \int_{-\infty < t_1 < t_2 < +\infty} \left\{ 1 - P(|X_1| \leq K_1 \mid X_{(1)} = t_1, X_{(N)} = t_2) \right\} d\mu_N(t_1, t_2) .$$

If we define

$$\begin{aligned}
A_1(\eta) &= \{(t_1, t_2) \in \mathbb{R}^2 : t_2 < -K_1 \quad \text{or} \quad t_1 > K_1, t_1 < t_2\} \\
A_2(\eta) &= \{(t_1, t_2) \in \mathbb{R}^2 : t_1 = -K_1 \quad \text{or} \quad t_2 = -K_1, t_1 < t_2\}
\end{aligned}$$

$$A_3(\eta) = \{(t_1,t_2) \in \mathbb{R}^2 : t_1 < -K_1 < t_2 \le K_1\}$$

$$A_4(\eta) = \{(t_1,t_2) \in \mathbb{R}^2 : -K_1 \le t_1 < K_1 < t_2\}$$

and

$$A_5(\eta) = \{(t_1,t_2) \in \mathbb{R}^2 : t_1 < -K_1, t_2 > K_1, t_1 < t_2\}$$

then

$$\begin{aligned} P(|X_1| > K_1) &= \int_{A_1(\eta)} d\mu_N(t_1,t_2) + \int_{A_2(\eta)} \left(1 - \frac{1}{N}\right) d\mu_N(t_1,t_2) \\ &\quad + \int_{A_3(\eta)} \left\{\frac{N-2}{N}\left(\frac{-K_1-t_1}{t_2-t_1}\right) + \frac{1}{N}\right\} d\mu_N(t_1,t_2) \\ &\quad + \int_{A_4(\eta)} \left\{1 - \frac{N-2}{N}\left(\frac{K_1-t_1}{t_2-t_1}\right) - \frac{1}{N}\right\} d\mu_N(t_1,t_2) \\ &\quad + \int_{A_5(\eta)} \left\{1 - \frac{N-2}{N}\left(\frac{-2K_1}{t_2-t_1}\right)\right\} d\mu_N(t_1,t_2) \\ &\ge \mu_N(A_1(\eta)) + \frac{1}{2}\mu_N(A_2(\eta)) + \frac{N-2}{N}\int_{A_3(\eta)} \left(\frac{-K_1-t_1}{t_2-t_1}\right) d\mu_N(t_1,t_2) \\ &\quad + \int_{A_4(\eta)} \left\{1 - \left(\frac{K_1-t_1}{t_2-t_1}\right) - \frac{1}{N}\right\} d\mu_N(t_1,t_2) \\ &\quad + \int_{A_5(\eta)} \left\{1 - \frac{2K_1}{t_2-t_1}\right\} d\mu_N(t_1,t_2)\ , \end{aligned}$$

if $N > 2$.

Setting $g(t_1,t_2) = \frac{-K_1-t_1}{t_2-t_1}$ for $t_1 < -K_1 < t_2 \le K_1$ we have that $g(t_1,t_2) \to 1$ as $t_1 \to -\infty$. Hence, given t_2 and $0 < \eta_3 < 1$ there exists $L_1 > 0$ so that if $t_1 < -L_1$ then $g(t_1,t_2) > 1 - \eta_3$.

If we define

$$A_3(\eta,\eta_3) = \left\{(t_1,t_2) \in \mathbb{R}^2 : t_1 < \frac{-K_1 - (1-\eta_3)t_2}{\eta_3}\ ,\ \ -K_1 < t_2 \le K_1, t_1 < t_2\right\}$$

then

$$\frac{N-2}{N}\int_{A_3(\eta,\eta_3)} g(t_1,t_2)\, d\mu_N(t_1,t_2) \ge \frac{1}{2}(1-\eta_3)\mu_N(A_3(\eta,\eta_3))\ ,$$

if $N > 4$.

If we now let $h(t_1,t_2) = \frac{t_2-K_1}{t_2-t_1}$ for $-K_1 \le t_1 < K_1 < t_2$, then for each $t_1 \in [-K_1, K_1)$, $h(t_1,t_2) \to 1$ as $t_2 \to +\infty$. Therefore, if $0 < \eta_4 < \frac{1}{2}$ then, there exists $L_2 > 0$ so that $h(t_1,t_2) > 1-\eta_4$ whenever $t_2 > L_2$.

Defining

$$A_4(\eta,\eta_4) = \left\{(t_1,t_2) \in \mathbb{R}^2 : t_2 > \frac{K_1-(1-\eta_4)t_1}{\eta_4}, \quad -K_1 \le t_1 < K_1, t_1 \le t_2\right\}$$

we have

$$\int_{A_4(\eta)} \left\{1 - \left(\frac{K_1-t_1}{t_2-t_1}\right) - \frac{1}{N}\right\} d\mu(t_1,t_2) \ge \left(\frac{1}{2} - \eta_4\right) \mu_N(A_4(\eta,\eta_4)) ,$$

if $N > 2$.

For $0 < \eta_5 < 1$, defining

$$A_5(\eta,\eta_5) = \left\{(t_1,t_2) \in \mathbb{R}^2 : t_2 - t_1 > \frac{2K_1}{\eta_5}, \quad t_1 < -K_1, t_2 > K_1\right\}$$

it follows that

$$\int_{A_5(\eta)} \left(1 - \frac{2K_1}{t_2-t_1}\right) d\mu_N(t_1,t_2) \ge (1-\eta_5)\, \mu_N(A_5(\eta,\eta_5)) .$$

Therefore, by setting

$$\varepsilon = 2\eta\left(2 + \frac{1}{1-\eta_3} + \frac{1}{1-2\eta_4} + \frac{1}{1-\eta_5}\right) \quad \text{and} \quad A(\varepsilon) = A_1(\eta) \cup A_2(\eta) \cup \bigcup_{i=3}^{5} A_i(\eta,\eta_i)$$

we get

$$\mu_N(A(\varepsilon)) \le \varepsilon , \quad \text{if } N > 4 .$$

□

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