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General Section

# L-functions of Carlitz modules, resultantal varieties and rooted binary trees - I ☆

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A R T I C L E I N F O

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A B S T R A C T

We continue study of some algebraic varieties (called resultantal varieties) started in a paper of A. Grishkov, D. Logachev "Resultantal varieties related to zeroes of L-functions of Carlitz modules". These varieties are related with the Sylvester matrix for the resultant of two polynomials, from one side, and with the L-functions of twisted Carlitz modules, from another side. Surprisingly, these varieties are described in terms of finite weighted rooted binary trees. We give a (conjecturally) complete description of them, we find parametrizations of their irreducible components and their invariants: degrees, multiplicities, Jordan forms, Galois actions. Proof of the fact that this description is really complete is a subject of future research. Maybe a generalization of these results will give us

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a solution of the problem of boundedness of the analytic rank of twists of Carlitz modules.



There exists a more detailed arxiv version [GLZ] of the present paper containing many examples and proofs. Its notations and numeration of propositions, formulas etc. are concordant with the ones of the present paper. For a list of notations of both [GLZ] and the present paper see [GLZ, p. 17].

## 0. Introduction

There is a famous

**Open problem 0.0.1.** (a) Are the ranks of elliptic curves over $\mathbb{Q}$ bounded, or not? If they are bounded, what is the upper bound of ranks; what is the maximal number $r$ such that there exists infinitely many elliptic curves over $\mathbb{Q}$ of rank $r$?

(b) The same questions for the ranks of all $\bar{\mathbb{Q}}/\mathbb{Q}$-twists of a fixed elliptic curve $E$ over $\mathbb{Q}$.

Earlier it was thought that the ranks are unbounded. There are examples of curves of rank $\geq 28$ and infinite series of curves of rank $\geq 19$. Nevertheless, a recent paper [PPVW] gives evidence that the ranks of all elliptic curves over $\mathbb{Q}$ are bounded, moreover, it gives evidence that there exists only finitely many elliptic curves over $\mathbb{Q}$ of rank $\geq 22$.

The purpose of the present paper is to continue the study (started in [GL16]) of an analog of this problem for the function field case. A function field case analog of an elliptic curve is a Drinfeld module of rank 2. They are particular cases of Anderson t-motives. Really, we shall consider a simpler example of Anderson t-motives — the Carlitz module $\mathfrak{C}$ (= the Drinfeld module of rank 1) over a finite field $\mathbb{F}_q$ (where $q$ is a power of a prime $p$), its tensor powers $\mathfrak{C}^n$ and their twists. The twists of $\mathfrak{C}^n$ are parameterized by polynomials $P \in \mathbb{F}_q[\theta]$ (here $\theta$ is an independent variable). We denote the degree of $P$ by $m$,

$$P = \sum_{\iota=0}^{m} a_\iota \theta^\iota \quad \text{where } a_\iota \in \mathbb{F}_q \tag{0.0.1c}$$

and we denote the corresponding twist of $\mathfrak{C}^n$ by $\mathfrak{C}^n_P$ (see 0.1.2 for details).

We consider the analytic rank of $\mathfrak{C}^n_P$. There are various versions of $L$-functions of an Anderson t-motive $M$, see [GLZ], 0.1.3B for a discussion, and respectively various versions of the notions of the analytic rank and of the statements of the Birch — Swinnerton-Dyer conjecture. We shall consider in the present paper the $L$-function $L(M, T)$ which is defined, for example, in [L], upper half of page 2603 (warning: $\tau$ of the present paper is $u$ of [L]), or in [Ga], Chapter 4, III, Definition 4.10, or in [G92], 3.2.15, or in [TW], formula (2.1); this $L$-function is called a naïve $L$-function in [B12]. Also, a very simple and explicit formula for $L$-function is given in [GL16], I.1 (caution: this formula of [GL16] is erroneous, therefore it should be treated only as a first approximation to the correct formula. Clearly these errors do not have any influence to the results of [GL16]). In 0.1.3A of the present paper we give a definition of a slightly weaker object, namely a $L$-function up to finitely many Euler factors.

For $M = \mathfrak{C}_P^n$ we have: $L(\mathfrak{C}_P^n, T)$ belongs to $(\mathbb{F}_q[t])[T]$, i.e. it should be considered as a polynomial in $T$ with coefficients in $\mathbb{F}_q[t]$ where $\mathbb{F}_q[t]$ is a subring of the Anderson ring $\mathbb{F}_q(\theta)\{t, \tau\}$ (see (0.1.0) below, case $\Phi = \mathbb{F}_q(\theta)$).

The analytic rank of $\mathfrak{C}_P^n$ is, by definition, the order of zero of $L(\mathfrak{C}_P^n, T)$ at $T = 1$ (recall that for other definitions of $L$-functions we shall get other definitions of the analytic rank, see 0.1.3B). Its study was started in [GL16]. It is denoted by $r_1(n, P)$ or simply by $r_1(P)$ if $n$ is fixed (subscript 1 because of the order of zero at $T = 1$).

Hence, this analytic rank of t-motives is an analog of the analytic rank of elliptic curves. For example, [L], Proposition 2.1, p. 2604 can be considered as an analog of the strong form of the Birch and Swinnerton-Dyer conjecture for $L(M, T)$ at $T = 1$ (here $M$ can be even a generalization of an Anderson t-motive like in [L]): "l'ordre d'annulation de $L(X, (\mathcal{E}, u), T)$ en $T = 1$ est égal à (dimension of a cohomology group) et la valeur spéciale $L^*(X, (\mathcal{E}, u), 1)$ est égale à (an expression) dans $F$".

**0.0.1d.** The results of [GL16] show that the behavior of $r_1(P)$, while $P$ varies, resembles the behavior of ranks of twists of a fixed elliptic curve.

First, let us recall this behavior. Let $E$ be an elliptic curve over $\mathbb{Q}$ defined by the equation $y^2 = x^3 + g_2x + g_3$, where $g_2, g_3 \in \mathbb{Q}$ (to simplify notations, we consider only this form of its minimal model). For a squarefree $d \in \mathbb{Z}$ let $E_d$ be its $d$-twist defined by the equation $dy^2 = x^3 + g_2x + g_3$. The parity of $E_d$ ( = the sign of its functional equation) depends on a residue of $d$ by some module (depending on the conductor of $E$ and on a chosen model), i.e. its behavior is regular. Moreover, the half of residues give even curves and the half of residues give odd curves. The parity of the rank of $E_d$ is equal to the parity of $E_d$. For almost all $d$ the rank of $E_d$ takes the minimal possible value, i.e. it is 0 for even curves and 1 for the odd ones. Rare jumps of the rank of $E_d$ occur, i.e. occasionally the rank of the even $E_d$ can be 2 (rarely), 4 (much more rarely) etc, and respectively the rank of the odd $E_d$ can be 3 (rarely), 5 (much more rarely) etc. There jumps of rank seem to be irregular, i.e. we do not know a formula for $d$ such that the rank of $E_d$ has a jump.

The behavior of $r_1(P)$ is the following. For most $P$ we have $r_1(P) = 0$. There exists a coset of a subgroup of the index $q^2$ in the group of twists (see [GLZ], Lemma 0.1.2.3) such that for $P$ belonging to this coset we have $r_1(P) \geq 1$ ([GL16], (4.4), (4.5)). Further, rare jumps of $r_1(P)$ occur.

An analog of the parity in characteristic $p$ could be the residue modulo $p$. Unfortunately, for the functional field case there is no information either on the parity of $r_1(P)$

or its residue modulo $p$, because for Anderson t-motives there is no functional equation for their $L$-series.

Therefore, we see that the behavior of $r_1(P)$, while $P$ varies, is as similar as possible to the behavior of ranks of twists of a fixed elliptic curve, taking into consideration the absence of a functional equation.

The function field analog of Open Problem 0.0.1 (b) is

**Open problem 0.0.2.** Problem of boundedness of ranks of twists of Carlitz modules: Are the analytic ranks of $\mathfrak{C}_P^n$ bounded? ($q$, $n$ are fixed, $m$, $P$ vary). If it is bounded, what is the maximal value of rank; what is the maximal value of rank that occurs for infinitely many $P$?

Contents of the present paper should be considered as an approach (second step) to a solution of 0.0.2 (the first step was [GL16]).

Let us describe what was made in [GL16]. Theorem 3.3 of [GL16] gives an explicit formula for $L(\mathfrak{C}_P^n, T)$: it is the characteristic polynomial of a matrix $\mathcal{M}(P, n, \bar{\kappa})_{fc}$ (roughly speaking; see 0.1.4 for $\bar{\kappa}$, Theorem 0.1.6 for the exact statement. The subscript $fc$ always means "finite characteristic") whose entries are polynomials in $a_0, \dots, a_m$ from (0.0.1c), and in $t$, with coefficients in $\mathbb{F}_p$. The matrix $\mathcal{M}(P, n, \bar{\kappa})_{fc}$ is a generalization of the Sylvester matrix of the resultant of two polynomials.

For the case $q = 3$, $n = 1$ (this is the simplest non-trivial case, because there are no non-trivial twists for $q = 2$, see 0.1.2.7) elementary considerations of the type "dimension of a generic variety is equal to the quantity of variables minus the quantity of equations" show that the maximal value of $r_1(P)$ should be 3. In reality, there are examples of $P$ such that $r_1(P) = 6$ ([GL16], Table 6.11). Hence, we do not know even a conjectural answer to 0.0.2.

**0.0.2a.** The coefficients $a_0, \dots, a_m$ of $P$ of (0.0.1c) form a point of $\mathbb{A}^{m+1}(\mathbb{F}_q)$ (the affine space of dimension $m+1$). Let us fix $i$ and consider all $P$ of degree $m$ such that $r_1(P) \geq i$. The coefficients of these $P$ form a subset of $\mathbb{A}^{m+1}(\mathbb{F}_q)$ denoted by $X_1(q, n, m, i)_{set;\ fc}$ (the subscript *set* indicates that we consider it as a set, without any additional structure).

Formula for $L(\mathfrak{C}_P^n, T)$ and the definition of the analytic rank show that there exists an affine algebraic variety denoted by $X_1(q, n, m, i)_{fc}$ which is the set of $\bar{\mathbb{F}}_q$-zeroes of some (non-homogeneous) polynomials $D_{**fc} \in \mathbb{F}_p[a_0, \dots, a_m]$. This variety has the property:

$$X_1(q, n, m, i)_{set;\ fc} = X_1(q, n, m, i)_{fc}(\mathbb{F}_q).$$

See [GL16], between (6.2) and (6.3) for the definition of $D_{**fc}$ (in [GL16] polynomials $D_{**fc}$ are denoted by $D_{**}$); two subscripts ** mean that $D_{**fc}$ depend on 2 integer parameters. See (0.1.13) for details.

Hence, we can state

**Open problem 0.0.3.** What are the dimensions of $X_1(q,n,m,i)_{fc}$? Let $q,\ n$ be fixed. Are there numbers $i$ such that for all $m$ the varieties $X_1(q,n,m,i)_{fc}$ are empty sets? (If yes then the answer to 0.0.2 is: "bounded", but not the converse: a non-empty variety can have no $\mathbb{F}_q$-points).

**Remark.** To state a question on dimensions, we have to consider varieties $X_1(q,n,m,i)_{fc}$, because there is no notion of dimension of $X_1(q,n,m,i)_{set;\ fc}$ which is only a subset of $\mathbb{A}^{m+1}(\mathbb{F}_q)$.

**0.0.4.** If $D_{**fc}$ were independent then the dimensions of $X_1(q,n,m,i)_{fc}$ would be so small that they would be empty for all sufficiently large $i$. For example, for $q=3$, $n=1$ in this case we would have $X_1(q,n,m,i)_{fc}=\emptyset\ \forall\ i\geq 4,\ \forall\ m$, and the maximal value of $r_1(P)$ would be 3, as it was mentioned above. Nevertheless, calculations of [GL16], Section 6 (especially 6.7–6.10) give evidence that for $q=3$, $n=1$, $i\leq 3$ the codimensions of $X_1(3,1,m,i)_{fc}$ are less than the quantity of equations $D_{**fc}$, i.e. apparently $D_{**fc}$ are dependent.

It turns out that it is much easier to consider the behavior (as $P$ varies) of $L(\mathfrak{C}^n_P,T)$ not at $T=1$ but at $T=\infty$. The deficiency of the order of pole of $L(\mathfrak{C}^n_P,T)$ at $T=\infty$ (see (0.1.11) for the exact definition) is called the analytic rank of $P$ at infinity, it is denoted by $r_\infty(P)$. Analogously to the case of $r_1(P)$, we denote the set of $(a_0,\dots,a_m)\in\mathbb{A}^{m+1}(\mathbb{F}_q)$ such that $r_\infty(P)\geq i$ by $X_\infty(q,n,m,i)_{set;\ fc}$. Again like the case of $r_1(P)$, there exists an algebraic variety $X_\infty(q,n,m,i)_{fc}$ which is the set of $\bar{\mathbb{F}}_q$-zeroes of some homogeneous polynomials $H_{**fc}\in\mathbb{F}_p[a_0,\dots,a_m]$ such that

$$X_\infty(q,n,m,i)_{set;\ fc}=X_\infty(q,n,m,i)_{fc}(\mathbb{F}_q).$$

It is easy to show that $D_{**fc}$ are linear combinations of $H_{**fc}$ with integer coefficients, so we can expect that study of $X_\infty(q,n,m,i)_{fc}$ can shed light to study of $X_1(q,n,m,i)_{fc}$ and hence to a solution of the Problem 0.0.3. Numerical data of [GL16], 6.16 and 6.11 show that there exists a correlation between high values of $r_1(P)$ and $r_\infty(P)$ (for example, polynomials $P_2,\ P_3,\ P_4$ from [GL16], Table 6.11 having high $r_1(P)$, also have high $r_\infty(P)$).

Since $H_{**fc}$ are homogeneous, we can consider a projective variety of their zeroes; it is denoted by $X_\infty(q,n,m,i)_{fc}$ as well.

Polynomials $H_{**fc}$ are highly dependent ([GL16], Theorem 8.6), i.e. $X_\infty(q,n,m,i)_{fc}$ are of low codimension.

**0.0.6.** Recall that the entries of $\mathcal{M}(P,n,\bar{\kappa})_{fc}$ are polynomials in $a_0,\dots,a_m,t$. The coefficients of these polynomials are images of some binomial coefficients (integer numbers) under the residue map $\mathbb{Z}\to\mathbb{F}_p$, see (0.1.4a). This means that there are matrices $\mathcal{M}(P,n,\bar{\kappa})\in\mathbb{Z}[a_0,\dots,a_m,t]$ whose entries are given by the same formulas (0.1.4a) but in $\mathbb{Z}$ (see (0.2.0)), i.e. $\mathcal{M}(P,n,\bar{\kappa})$ is a distinguished $\mathbb{F}_p\to\mathbb{Z}$-lift of $\mathcal{M}(P,n,\bar{\kappa})_{fc}$.

Polynomials $H_{**fc}$ are defined using the entries of $\mathcal{M}(P,n,\bar{\kappa})_{fc}$. The same formulas applied to the entries of $\mathcal{M}(P,n,\bar{\kappa})$ give us polynomials $H_{**}$. We see that $H_{**}$ can be considered as canonical $\mathbb{F}_p \to \mathbb{Z}$-lifts of $H_{**fc}$. Hence we can consider the varieties of zeroes of $H_{**}$ in $\mathbb{P}^m(\mathbb{C})$. They are denoted by $X_\infty(q,n,m,i)$. These varieties are defined over $\mathbb{Q}$, because coefficients of $H_{**}$ are in $\mathbb{Z}$. It turns out that for this case of characteristic 0 the simplest possible value of $q$ is 2: although for $q=2$ there are no twists of $\mathfrak{C}^n$ (see (0.1.2.7)), the theory of $X_\infty(2,n,m,i)$ — unlike the theory of $X_\infty(2,n,m,i)_{set;\ fc}$ over $\mathbb{F}_2$ — is non-trivial. Moreover, it turns out that for $q=2$ the varieties $X_\infty(2,n,m,i)$ (conjecturally) do not depend on $n$, see Conjecture 0.2.4, Remark 0.2.5 (1).

**0.0.7.** Further, we can consider formally polynomials $H_{**}$ for $n=0$: although $\mathfrak{C}^0$ does not exist, the polynomials $H_{**}$, and hence varieties $X_\infty(2,0,m,i)$ have meaning for $n=0$, and (again conjecturally) $X_\infty(2,0,m,i)$ coincides with $X_\infty(2,n,m,i)$ for all $n>0$. Finally, the case $n=0$ is especially simple, because for this case the set of $H_{**}$ depends on 1 parameter and not on 2 parameters that simplifies their study. See Section 0.2 for a more detailed discussion.

Now we can formulate the subject of the present paper: study of varieties $X_\infty(2,0,m,i)$ which are denoted simply by $X(m,i)$. The above arguments show their importance. We prove some theorems and state some conjectures on their irreducible components, their rational parameterizations, degrees, Jordan forms, Galois actions and multiplicities.

It is necessary to emphasize that formally for reading of the present paper it is not necessary to know what is a Carlitz module, $L$-functions etc. — definitions of $X(m,i)$ are elementary, and methods are purely combinatorial. The reader can start reading from Section 1; the notion of the Carlitz modules is used only in the Introduction, in order to show the importance of the subject.

**0.0.8.** It is interesting that even in this interpretation varieties $X(m,i)$ "remember" their origin. For example, one of the important constructions (Proposition 7.1) depends on Conjecture 0.2.4 which indicates that $X_\infty(2,0,m,i)=X_\infty(2,n,m,i)$ for all $n>0$. Recall that the varieties $X_\infty(2,n,m,i)$ come from the $L$-functions of $\mathfrak{C}^n_P$.

The subsequent sections of the Introduction contain a more detailed exposition of the theory of $L$-functions of twisted Carlitz modules (Section 0.1) and more detailed definitions of the above objects (Section 0.2). Further, we give the contents of the present paper (Section 0.3) and the possibilities of further research (Section 0.4).

## *0.1. L-functions of twisted n-th tensor powers of Carlitz modules*

Carlitz modules, their tensor powers and twists are particular cases of Anderson t-motives, so we recall their definition. See, for example, [GL20].

Let $q$ be a power of $p$ and $\mathbb{F}_q$ the finite field of order $q$. The field $\mathbb{F}_q(\theta)$ is its field of rational functions, it is the function field analog of $\mathbb{Q}$. The field of the Laurent series

$\mathbb{F}_q((1/\theta))$ is the function field analog of $\mathbb{R}$. By definition, $\mathbb{C}_\infty$ is the completion of the algebraic closure of $\mathbb{F}_q((1/\theta))$, it is the function field analog of $\mathbb{C}$.

**0.1.0.** Let $\Phi$ be a subfield of $\mathbb{C}_\infty$ (we shall consider only the cases $\Phi = \mathbb{F}_q(\theta)$ and $\Phi = \mathbb{C}_\infty$). The Anderson ring $\Phi\{t,\tau\}$ is the ring of non-commutative polynomials in two variables $t$, $\tau$ over $\Phi$ satisfying the following relations:

$$t\tau = \tau t;\ \forall a \in \Phi \text{ we have }\ ta = at;\ \tau a = a^q\tau$$

We need the following simple version of the definition of Anderson t-motives over $\Phi$ (throughout the whole paper $M$ will mean an Anderson t-motive):

**Definition 0.1.1.** An Anderson t-motive $M$ is a $\Phi\{t,\tau\}$-module such that

1. $M$ considered as a $\Phi[t]$-module is free of finite rank $r$;
2. $M$ considered as a $\Phi\{\tau\}$-module is free of finite rank $n$;
3. $\exists \varkappa > 0$ such that $(t-\theta)^\varkappa M/\tau M = 0$.

An isomorphism of t-motives is an isomorphism of modules.

Numbers $r$, resp. $n$ are called the (ordinary) rank of $M$, resp. the dimension of $M$. The ordinary rank of $M$ should not be confused with $r_1$ — the analytic rank of $M$ considered below.

The Carlitz module $\mathfrak{C}$ is an Anderson t-motive over $\mathbb{C}_\infty$ having $r = n = 1$. Let $\{e\} = \{e_1\}$ be the only element of a basis of $M$ over $\mathbb{C}_\infty\{\tau\}$. $\mathfrak{C}$ is given by the relation $te = \theta e + \tau e$. We have: $e$ is also the only element of a basis of $\mathfrak{C}$ over $\mathbb{C}_\infty[t]$, and the multiplication by $\tau$ is given by the formula

$$\tau e = (t-\theta)e \tag{0.1.1a}$$

The $n$-th tensor power of $\mathfrak{C}$, denoted by $\mathfrak{C}^n$, is the $n$-th tensor power of $\mathfrak{C}$ over the ring $\mathbb{C}_\infty[t]$. Hence, it has the ordinary rank $r = 1$. We denote the only element $e \otimes e \otimes e \otimes \ldots \otimes e$ of a basis of $\mathfrak{C}^n$ over $\mathbb{C}_\infty[t]$ by $e_n$. The action of $\tau$ on $e_n$ is given by the formula

$$\tau e_n = (t-\theta)^n e_n \tag{0.1.1b}$$

It is easy to check that the $\Phi\{t,\tau\}$-module defined by this formula satisfies 0.1.1.2-3, i.e. it is an Anderson t-motive of the ordinary rank $r = 1$ and dimension $n$.

**0.1.2.** The Carlitz module $\mathfrak{C}_\Phi$ over any $\Phi$ (not necessarily $\Phi = \mathbb{C}_\infty$) is an Anderson t-motive over $\Phi$ defined by the same formula (0.1.1a), its $n$-th tensor power $\mathfrak{C}^n_\Phi$ is defined by (0.1.1b). A twist of $\mathfrak{C}^n_\Phi$ is an Anderson t-motive $M$ over $\Phi$ which is isomorphic to $\mathfrak{C}^n$ over $\mathbb{C}_\infty$ (twists of Carlitz modules are considered also in [GL16] (any $n$); [Ge] ($n = 1$)).

Below we shall consider only the case $\Phi = \mathbb{F}_q(\theta)$. It is easy to check (see [GLZ], 0.1.2 for details) that for this $\Phi$ the twists of $\mathfrak{C}^n_\Phi$ are parameterized by polynomials $P$ from (0.0.1c). The twist corresponding to such $P$ is denoted by $\mathfrak{C}^n_P$. It is an Anderson t-motive given by the formula

$$\tau e_n = P(t-\theta)^n e_n \tag{0.1.2.5}$$

($e_n$ is as earlier). The group of twists is $\mathbb{F}_q(\theta)^*/\mathbb{F}_q(\theta)^{*(q-1)}$. Two twists $\mathfrak{C}^n_{P_1}$, $\mathfrak{C}^n_{P_2}$ are isomorphic over $\mathbb{F}_q(\theta)$ iff images of $P_1$, $P_2$ in the group of twists coincide, i.e. iff $P_1/P_2 \in \mathbb{F}_q(\theta)^{*(q-1)}$.

**(0.1.2.7).** In particular, there are no non-trivial twists for $q = 2$.

It is possible to define a t-motive $M$ not over a field $\Phi$, but also over a ring. For example, if the above $P \in \mathbb{F}_q[\theta]$ then $\mathfrak{C}^n_P$ is over $\mathbb{F}_q[\theta]$. Moreover, [GLZ], (0.1.2.6) shows that if $P_1 = PQ^{q-1}$ then $\mathfrak{C}^n_P$, $\mathfrak{C}^n_{P_1}$ are not isomorphic over $\mathbb{F}_q[\theta]$, i.e. they are different $\mathbb{F}_q[\theta]$-forms of the same t-motive over $\mathbb{F}_q(\theta)$.

**0.1.3A. *L*-functions.** As it was mentioned above, we consider in the present paper the $L$-function $L(M,T)$, and we give here (for brevity) a definition of a weaker object, namely $L_{fmef}(M,T)$ — a $L$-function up to finitely many Euler factors.

**Remark 0.1.3A.0.** It is necessary to emphasize that $L(M,T)$ depends not on $M$ as a t-motive over $\mathbb{F}_q(\theta)$ (or an extension of $\mathbb{F}_q(\theta)$) but on its model over $\mathbb{F}_q[\theta]$. Namely, let $M_1$, $M_2$ be two t-motives over $\mathbb{F}_q[\theta]$ which are isomorphic as t-motives over $\mathbb{F}_q(\theta)$. In this case we have $L_{fmef}(M_1,T) = L_{fmef}(M_2,T)$ but not necessarily $L(M_1,T) = L(M_2,T)$.

As an illustration, we can consider $\mathfrak{C}^n_P$, $\mathfrak{C}^n_{P_1}$ for $P$ and $P_1 = PQ^{q-1}$, $Q \in \mathbb{F}_q[\theta]$, from [GL16], 5.6. The Theorem 0.1.6 (= [GL16], Theorem 3.3) gives us both $L(\mathfrak{C}^n_P,T)$, $L(\mathfrak{C}^n_{P_1},T)$, and the formula [GL16], (5.6.1) gives us a relation between them, i.e. it indicates explicitly which Euler factors enter in their ratio.[1]

The formula [GL16], (5.6.1) is given without proof. It can be proved either using a definition of $L(M,T)$, or comparing the values of determinants of $\mathfrak{M}(P,n,t)$, $\mathfrak{M}(P_1,n,t)$ (notations of [GL16]). If deg $Q = 1$ this comparison is immediate; for deg $Q > 1$ this can be a difficult problem.

The definition of $L_{fmef}(M,T)$ is the following. Let $M$ be an Anderson t-motive over $\mathbb{F}_q(\theta)$. Let $f_* = (f_1,\dots,f_r)^{tr}$ be a basis of $M$ over $\mathbb{F}_q(\theta)[t]$ (here $tr$ means transposition: we consider elements of a basis as a column vector). Let $Q \in M_{r\times r}(\mathbb{F}_q(\theta)[t])$ be the matrix of multiplication by $\tau$ in this basis. Let $\mathfrak{P}$ be an irreducible polynomial in $\mathbb{F}_q[\theta]$.

[1] $P$, $P_1$, $Q$, $\mathfrak{C}^n_*$ of [GL16], Section 5.6 are the same as in the present paper. The first line of [GL16], Section 5.6 contains a typographic error: $\mathfrak{C}^n_P$, $\mathfrak{C}^n_{P_1}$ are different $\mathbb{F}_q[\theta]$-models.

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$\mathfrak{P}$ is called good for $(M,\ f_*)$ if all coefficients of all entries of $Q$ are integer at $\mathfrak{P}$. For fixed $(M,\ f_*)$ almost all $\mathfrak{P}$ are good for them.

We need the following notation. For $a \in (\mathbb{F}_q[\theta]/\mathfrak{P})[t]$, $a = \sum c_i t^i$ where $c_i \in \mathbb{F}_q[\theta]/\mathfrak{P}$, we denote $a^{(k)} := \sum c_i^{q^k} t^i$, for a matrix $A = (a_{ij}) \in M_{r\times r}((\mathbb{F}_q[\theta]/\mathfrak{P})[t])$ $A^{(k)} := (a_{ij}^{(k)})$ and $A^{[k]} := A^{(k-1)} \cdot \dots \cdot A^{(1)} \cdot A$.

Let us fix $f_*$ (hence $Q$ is also fixed). For a good $\mathfrak{P}$ the local $\mathfrak{P}$-factor $L_\mathfrak{P}(M,T)$ is defined as follows. Let $d$ be the degree of $\mathfrak{P}$ and $\tilde{Q} \in M_{r\times r}((\mathbb{F}_q[\theta]/\mathfrak{P})[t])$ the reduction of $Q$ at $\mathfrak{P}$. We define:

$$L_\mathfrak{P}(M,T) := \det(I_r - \tilde{Q}^{[d]}T^d)^{-1} \in \mathbb{F}_q[t][[T^d]] \tag{0.1.3A.1}$$

(because obviously det $(I_r - \tilde{Q}^{[d]}T) \in \mathbb{F}_q[t,T]$) and the global $L$-function is, as usual, the product of local factors:

$$L_{fmef}(M,T) := \prod_{\mathfrak{P}} L_\mathfrak{P}(M,T) \in \mathbb{F}_q[t][[T]] \tag{0.1.3A.2}$$

This function is defined up to finitely many Euler factors (subscript $fmef$). It does not depend on $f_*$ (see [GLZ] for an elementary proof). Further, [GLZ], 0.1.3A contains more examples and information on $L(M,T)$.

**0.1.3B.** The above $L$-function $L(M,T)$ considered in the present paper is a specialization (or a version) of the Goss' $L$-function $L_G(M,s)$. More exactly, $L(M,T)$ contains strictly more information than the value of $L_G(M,0)$, i.e. it refines the value of $L_G(M,0)$.[2] The function $L_G(M,s)$ or its versions for $\tau$-sheaves and crystals are defined for example in [B05], Definition 15, [B02], Definition 2.19, the original definition is due to [G92], 3.4.2a.

**Remark 0.1.3B.0.** In most earlier cited papers $L_G(M,s)$ is considered as a function in variable $s$, case $s \le 0$ (for $s \in \mathbb{Z}$, $s > 0$ we have a different theory). In terms of $\mathfrak{C}^n$, this is the same as to consider $n$ as a variable: dilatation of $s$ to an integer $n$ corresponds to the tensor multiplication of $M$ by $\mathfrak{C}^n$: see for example [B05], Proposition 9, or [GL16], (7.1) for $M = \mathfrak{C}^n$. Unlike this approach, we consider $n$ fixed, and we consider $T$ as a variable.

[GLZ], 0.1.3B contains a definition of the Goss' $L$-function and its relations with $L(M,T)$.

Further, there exists a Taelman $L$-function (or, better, a special value, see below) $L_{TA}(M/R) \in 1 + T^{-1}\mathbb{F}_q[[T^{-1}]]$ where $M$ is a Drinfeld module, $R$ is a finite extension of $\mathbb{F}_q[\theta]$, see [Ta], p. 371, below Remark 2. It is related with $L_G(M,s)$ by the following formula: $L_{TA}(M/\mathbb{F}_q[\theta])$ is the value at $s = 0$ (this explains the above terminology "a

[2] The authors are grateful to an anonymous reviewer who indicated them this statement, as well as some other statements of Sections 0.1.3A, B of the present paper and of [GLZ].

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special value") of a version of $L_G(M,s)$ obtained by consideration of the dual of the Tate module of $M$ ([Ta], Remark 5).

Let us compare possibilities to apply various definitions of $L$-functions to the statements of Birch — Swinnerton-Dyer conjecture. The dual Goss $L$-function has a disadvantage that the analytic rank of all $M$ is 0 ((because $L_{TA}(M/R)$ — which is its value at $s=0$ — is non-zero). From another side, this value contains an algebraic and transcendental part, as it should be by the analogy with the number field case.

The analytic rank $r_1$ defined as the order of zero at $T=1$ of the $L$-function $L(M,T)$ behaves like the analytic rank of elliptic curves (except the parity property), see 0.0.1d. From another side, $L(M,T)$ are polynomials in $T$, i.e. they are "too simple objects". Particularly, the coefficient $L^{(r_1)}(M,1)$ (entering to the Birch — Swinnerton-Dyer conjecture; $L^{(r_1)}$ means the $r_1$-th derivative) belongs to $\mathbb{F}_q[t]$, i.e. it has no transcendental part.

Finally, we indicate that

(a) Some results on vanishing of $L(\mathfrak{C}^n,T)$ at $T=1$ were obtained in [Th], [L].
(b) The zeta function of [AT90] is a specialization of the above $\zeta_{\mathbb{F}_q[\theta]}$ (i.e. its domain is a subset of $S_\infty$).

**0.1.4.** An explicit formula for $L(\mathfrak{C}_P^n,T)$ (based on the general formula) is given in [GL16], Theorem 3.3. Let $P$ be from (0.0.1c). We denote $\bar\kappa=[\frac{m+n}{q-1}]$ (here $[x]$ means the integer part of $x$), and let $\mathcal{M}(P,n,\bar\kappa)_{fc}=\mathcal{M}(a_*,n,\bar\kappa)_{fc}$ be the matrix in $M_{\bar\kappa\times\bar\kappa}(\mathbb{F}_q[t])$ whose $(\mathfrak{i},\mathfrak{j})$-th entry is defined by the formula

$$\mathcal{M}(P,n,\bar\kappa)_{fc,\mathfrak{i},\mathfrak{j}}=\sum_{l=0}^{n}(-\bar1)^l\overline{\binom{n}{l}}a_{\mathfrak{j}q-\mathfrak{i}-l}\ t^{n-l} \tag{0.1.4a}$$

(here $a_*=0$ if $*\notin[0,\dots,m]$, and $\bar1$, $\overline{\binom{n}{l}}$ are images of 1, $\binom{n}{l}$ in $\mathbb{F}_p\subset\mathbb{F}_q$).

**Remark.** Later (for example, in the proofs of Theorem 5.6 and Lemma 5.14.1) we shall consider linear transformations defined by matrices $\mathcal{M}(P,n,\bar\kappa)_{fc}$ (and its characteristic 0 versions). Nevertheless, we do not know any natural interpretation of these linear transformations.

In particular, for $n=1$ we have $\mathcal{M}(P,1,\bar\kappa)_{fc,\mathfrak{i},\mathfrak{j}}=a_{\mathfrak{j}q-\mathfrak{i}}t-a_{\mathfrak{j}q-\mathfrak{i}-1}$ and

$$\mathcal{M}(P,1,\bar\kappa)_{fc}=\begin{pmatrix} a_{q-1}t-a_{q-2} & a_{2q-1}t-a_{2q-2} & \dots & a_{\bar\kappa q-1}t-a_{\bar\kappa q-2}\\ a_{q-2}t-a_{q-3} & a_{2q-2}t-a_{2q-3} & \dots & a_{\bar\kappa q-2}t-a_{\bar\kappa q-3}\\ a_{q-3}t-a_{q-4} & a_{2q-3}t-a_{2q-4} & \dots & a_{\bar\kappa q-3}t-a_{\bar\kappa q-4}\\ \dots & \dots & \dots & \dots\\ a_{q-\bar\kappa}t-a_{q-\bar\kappa-1} & a_{2q-\bar\kappa}t-a_{2q-\bar\kappa-1} & \dots & a_{\bar\kappa q-\bar\kappa}t-a_{\bar\kappa q-\bar\kappa-1}\end{pmatrix} \tag{0.1.5}$$

**Theorem 0.1.6** *([GL16, Theorem 3.3]).* $L(\mathfrak{C}_P^n, T) = \det(I_{\bar\kappa} - \mathcal{M}(P, n, \bar\kappa)_{fc} T)$ *(if* $\bar\kappa = 0$ *then* $L(\mathfrak{C}_P^n, T) = 1$*).*

**0.1.7 Remarks.** 1. Theorem 0.1.6 implies that $L(\mathfrak{C}_P^n, T) \in (\mathbb{F}_q[t])[T]$ is of degree $\leq \bar\kappa$ in $T$.

2. $\mathfrak{M}(P, n, k)$ of [GL16], (3.1), (3.2) is $\mathcal{M}(P, n, \bar\kappa)_{fc}^{tr}$. Authors apologize for this non-concordance of notations. Really, the transposition of these matrices is not important, because we consider their determinants.

4. Formula for $L(\mathfrak{C}_P^n, T)$ is concordant with the natural inclusion of the set of polynomials of degree $\leq m$ to the set of polynomials of degree $\leq m'$, where $m' > m$.

**0.1.8. Non-trivial part.** If $\frac{m+n}{q-1}$ is integer then the last column of $\mathcal{M}(P, n, \bar\kappa)_{fc}$ has only one non-zero element, namely its lower element is equal to $(-1)^n a_m$. Hence, for this case we denote $\kappa := \bar\kappa - 1 = \frac{m+n}{q-1} - 1$, we consider a $\kappa \times \kappa$-submatrix of $\mathcal{M}(P, n, \bar\kappa)_{fc}$ formed by elimination of its last row and last column. We denote this submatrix by $\mathcal{M}_{nt}(P, n, \kappa)_{fc} = \mathcal{M}_{nt}(a_*, n, \kappa)_{fc}$ and $L_{nt}(\mathfrak{C}_P^n, T) := \det(I_\kappa - \mathcal{M}_{nt}(P, n, \kappa)_{fc} T)$ (the subscript $nt$ means the non-trivial part). We have

$$L(\mathfrak{C}_P^n, T) = L_{nt}(\mathfrak{C}_P^n, T) \cdot (1 - (-1)^n a_m T)$$

**0.1.9. Analytic rank.** Let $(a_0, \dots, a_m) \in \mathbb{F}_q^{m+1}$ be fixed, and $P$ from (0.0.1c). Let $n \geq 1$ and $\mathfrak{c} \in \mathbb{F}_q[t]$ be fixed.

**Definition 0.1.10.** The analytic rank of $\mathfrak{C}_P^n$ at $\mathfrak{c}$ is the order of 0 of $L(\mathfrak{C}_P^n, T)$ (considered as a function in $T$) at $T = \mathfrak{c}$. It is denoted by $r_\mathfrak{c} = r_\mathfrak{c}(n, P) = r_\mathfrak{c}(n, a_*)$.

Since for all $P$ we have $L(\mathfrak{C}_P^n, T) = 1$ at $T = 0$, the simplest values of $\mathfrak{c}$ to study the analytic rank are $\mathfrak{c} = \text{const}$. According to [GL16], Corollary 2.4.4, the case of any constant $\mathfrak{c}$ can be easily reduced to the case $\mathfrak{c} = 1$, so $r_1$ is the most natural object to study.

Also, we can consider the order of pole of $L(\mathfrak{C}_P^n, T)$ at $T = \infty$ (i.e. the degree of $L(\mathfrak{C}_P^n, T)$ as a polynomial in $T$). It is more convenient to consider its deficiency:

**Definition 0.1.11.** The analytic rank of $\mathfrak{C}_P^n$ at $\infty$ is $\bar\kappa - \deg_T L(\mathfrak{C}_P^n, T)$. It is denoted by $r_\infty = r_\infty(n, P) = r_\infty(n, a_*)$.

**Remark 0.1.12.** $r_\infty$ is not invariant under the natural inclusion of the set of polynomials of degree $m$ to the set of polynomials of degree $m'$, where $m' > m$.

**0.1.13. Varieties of points of a given analytic rank, and polynomials defining them.** We recall definitions of (0.0.2a), (0.0.4). We fix $q, n$, $m > i \geq 0$, and $\mathfrak{c} = 1$ or $\mathfrak{c} = \infty$. We consider the set of points $(a_0, \dots, a_m) \in \mathbb{F}_q^{m+1}$ such that $r_\mathfrak{c}(n, a_*) \geq i$. It is obviously an algebraic set, i.e. the set of $\mathbb{F}_q$-zeroes of some polynomials in $a_0, \dots, a_m$. These polynomials come from coefficients of $\det(I_{\bar\kappa} - \mathcal{M}(P, n, \bar\kappa)_{fc} T)$ considered as polynomials in $t, T$. We denote these polynomials by $D_{**fc}$, resp. $H_{**fc}$ for $\mathfrak{c} = 1$, resp. $\mathfrak{c} = \infty$ (we do not give an exact definition of $D_{**fc}$, $H_{**fc}$ because we do not need it. See (0.2.1a) for the definition of $H_{**}$ for the case that we need). For fixed $q, n, m, i$ their set depends on 2 parameters, because they are coefficients of polynomials in $t$, $T$.

Hence, we can consider algebraic varieties denoted by $X_{\mathfrak{c}}(q,n,m,i)_{fc}$ — the sets of zeroes in $\bar{\mathbb{F}}_q^{m+1}$ of $D_{**fc}$, resp. $H_{**fc}$ for $\mathfrak{c}=1$, resp. $\mathfrak{c}=\infty$. We have: the set of points $(a_0,\dots,a_m)\in\mathbb{F}_q^{m+1}$ such that $r_{\mathfrak{c}}(n,a_*)\ge i$ is $X_{\mathfrak{c}}(q,n,m,i)_{fc}(\mathbb{F}_q)$ — the set of $\mathbb{F}_q$-points of $X_{\mathfrak{c}}(q,n,m,i)_{fc}$.

Polynomials $H_{**fc}$ are homogeneous, hence we can consider $X_\infty(q,n,m,i)_{fc}\subset\mathbb{P}^m(\bar{\mathbb{F}}_q)$ as projective varieties.

**Remark 0.1.14.** For the case of integer $\frac{m+n}{q-1}$ we can consider the non-trivial parts $r_{\mathfrak{c},nt}$ of $r_{\mathfrak{c}}$, corresponding to zeroes of $L_{nt}(\mathfrak{C}_P^n,T)$. The same consideration is applied to the varieties $X_{\mathfrak{c}}(q,n,m,i)_{fc}$: we can consider varieties $X_{\mathfrak{c},nt}(q,n,m,i)_{fc}$.

**Remark 0.1.15.** [3] We have: only the sets $X_{\mathfrak{c}}(q,n,m,i)_{set,\ fc}=X_{\mathfrak{c}}(q,n,m,i)_{fc}(\mathbb{F}_q)$ are canonically defined (as sets of twists having $r_{\mathfrak{c}}\ge i$), but not $X_{\mathfrak{c}}(q,n,m,i)_{fc}$ as varieties. Really, by definition, varieties $X_{\mathfrak{c}}(q,n,m,i)_{fc}$ are sets of $\bar{\mathbb{F}}_q$-zeroes of polynomials $D_{**fc}$, resp. $H_{**fc}$ for $\mathfrak{c}=1$, resp. $\mathfrak{c}=\infty$. In its turn, $D_{**fc}$, $H_{**fc}$ come from the proof of [GL16], Theorem 3.3. In principle, it can happen that we get another formula for $L(\mathfrak{C}_P^n,T)$ giving other systems of polynomials $\mathfrak{D}_*$, $\mathfrak{H}_*$. The sets of $\mathbb{F}_q$-zeroes of $\mathfrak{D}_*$, resp. $\mathfrak{H}_*$ are the same sets $X_{\mathfrak{c}}(q,n,m,i)_{fc}(\mathbb{F}_q)$ for $\mathfrak{c}=1$, resp. $\mathfrak{c}=\infty$ (because they are sets of twists having $r_{\mathfrak{c}}\ge i$), but varieties of $\bar{\mathbb{F}}_q$-zeroes of $\mathfrak{D}_*$, $\mathfrak{H}_*$ can be another.

We neglect this phenomenon, because at the moment we do not know other natural systems of polynomials that define $X_{\mathfrak{c}}(q,n,m,i)_{fc}(\mathbb{F}_q)$. See [GL16], page 121 for a possible invariant definition of the dimension of $X_{\mathfrak{c}}(q,n,m,i)_{fc}(\mathbb{F}_q)$.

### *0.2. Subject of the present paper: $\mathfrak{c}=\infty$, $q=2$, $n=0$, field $=\mathbb{C}$*

As we mentioned above, study of behavior of $L(\mathfrak{C}_P^n,T)$ at $T=\infty$ is much simpler than at $T=1$, so in the present paper we shall consider only this problem. Further, we can consider $a_0,\dots,a_m$ as abstract elements, not necessarily as elements of $\mathbb{F}_q$. Particularly, they can belong to $\mathbb{C}$. For simplicity, we shall consider the case of (0.1.8): $\frac{m+n}{q-1}$ is integer, $\kappa:=\frac{m+n}{q-1}-1$, we consider the non-trivial part of the $L$-function, and we omit the subscript "$nt$". Hence, we define a matrix $\mathcal{M}(P,n,\kappa)\in M_{\kappa\times\kappa}(\mathbb{Z}[t])$ using the same formula (0.1.4a), but in characteristic 0. Namely, the $(\mathfrak{i},\mathfrak{j})$-th entry of $\mathcal{M}(P,n,\kappa)$ is defined as follows:

$$\mathcal{M}(P,n,\kappa)_{\mathfrak{i},\mathfrak{j}}:=\sum_{l=0}^{n}(-1)^l\binom{n}{l}a_{\mathfrak{j}q-\mathfrak{i}-l}\ t^{n-l} \tag{0.2.0}$$

[3] The authors are grateful to an anonymous reviewer who indicated them the subject of the present remark.

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We consider its characteristic polynomial:

$$Ch(\mathcal{M}(a_*, n, \kappa), T) := \det(I_\kappa - \mathcal{M}(P, n, \kappa)T) \in \mathbb{Z}[a_0, \ldots, a_m][t][T] \tag{0.2.1}$$

Varieties $X_\mathfrak{c}(q, n, m, i) \subset \mathbb{C}^{m+1}$ are defined like the above $X_\mathfrak{c}(q, n, m, i)_{fc}$ (recall that we omit the subscript $nt$). For $\mathfrak{c} = \infty$ we use the same notation for its projectivization, i.e. $X_\infty(q, n, m, i) \subset \mathbb{P}^m(\mathbb{C})$.

More exactly, for $\mathfrak{c} = \infty$ the polynomials

$$H_{**} = H_{i\mathfrak{j},qn}(m) = H_{i\mathfrak{j},qn}(a_0, \ldots, a_m) \in \mathbb{Z}[a_0, \ldots, a_m] \tag{0.2.1a}$$

are defined by the following formula (they are coefficients of $Ch(\mathcal{M}(a_*, n, \kappa), T)$ at $t^*T^*$):

$$Ch(\mathcal{M}(a_*, n, \kappa), T) = \sum_{\iota=0}^{\kappa} \sum_{\mathfrak{j}=0}^{n(\kappa-\iota)} H_{\iota\mathfrak{j},qn}(m) t^{\mathfrak{j}} T^{\kappa-\iota} \tag{0.2.2}$$

Hence, $X_\infty(q, n, m, i)$ is the set of zeroes of $H_{\iota\mathfrak{j},qn}(m)$ for all $\iota \in [0, \ldots, i-1]$ and all $\mathfrak{j} \in [0, \ldots, n(\kappa - \iota)]$. We shall need also the corresponding projective schemes ($\iota$ and $\mathfrak{j}$ run over the same set $\iota \in [0, \ldots, i-1]$, $\mathfrak{j} \in [0, \ldots, n(\kappa - \iota)]$):

$$X_{S,\infty}(q, n, m, i) := \text{Proj } (\mathbb{C}[a_0, \ldots, a_m]/ < H_{\iota\mathfrak{j},qn}(m) >) \tag{0.2.2a}$$

(here $< H_{\iota\mathfrak{j},qn}(m) >$ is the ideal generated by all $H_{\iota\mathfrak{j},qn}(m)$, and the subscript $S$ indicates that we consider a scheme instead of a set of points as earlier).

**Remark 0.2.3.** According (0.1.2.7), for $q = 2$ there are no twists. An exact relation between $L$-functions of $\mathfrak{C}_P$, $\mathfrak{C}_{PQ^{q-1}}$ (here $Q \in \mathbb{F}_q[\theta]$) is given in [GL16], (5.6.1) — these $L$-functions are equal up to finitely many Euler factors. Clearly this formula holds only in characteristic $p$, and in characteristic 0 we have a non-trivial theory for $q = 2$.

**0.2.3a.** As it was mentioned in (0.0.7), we can consider formally polynomials $H_{**}$ for $n = 0$. For this case the matrix $\mathcal{M}_{nt}(a_*, 0, \kappa)$ does not depend on $t$, and hence the polynomials $H_{**}$ depend on only one parameter, not on two parameters. For $q = 2$ we denote them by $D(m, i) := H_{i0,20}(m)$ (see also (1.2.1); recall that $D(m, i) \in \mathbb{C}[a_0, \ldots, a_m]$).

Now we can formulate the subject of the present paper: study of $X_\infty(q, n, m, i)$, $X_{S,\infty}(q, n, m, i)$ for the case $q = 2$, $n = 0$ (see (0.0.7)). Varieties $X_\infty(2, 0, m, i)$, resp. schemes $X_{S,\infty}(2, 0, m, i)$ are denoted simply by $X(m, i)$, resp. $X_S(m, i)$. Namely, we define (here $\iota = 0, \ldots, i-1$; see also Definition 1.3):

$$X(m, i) = \{(a_0 : \ldots : a_m) \in \mathbb{P}^m | \; D(m, \iota)(a_0, \ldots, a_m) = 0\} \quad \text{(set of points)}$$

$$X_S(m, i) := \text{Proj } (\mathbb{C}[a_0, \ldots, a_m]/ < D(m, \iota) >) \quad \text{(projective scheme)}$$

It turns out that for $q = 2$ the case of any $n$ is conjecturally the same as the case $n = 0$. Namely, we have

**Conjecture 0.2.4.** *For $q = 2$, $\forall i, \mathfrak{j}, m, n \ \exists \ \varkappa$ such that*

$$(H_{i\mathfrak{j},2n}(m))^{\varkappa} \in < D(m,0), \dots, D(m,i) >$$

*— the ideal generated by $D(m,0), \dots, D(m,i)$ (recall that polynomials $H_{i\mathfrak{j},2n}(m)$, $D(m,*)$ are over $\mathbb{C}$, not over $\mathbb{F}_q$).*

This conjecture holds for $i = 0$ and some other cases, see [GL16], (II.6) and [GLZ], A1 for numerical data. See also [GLZ], Proposition 10.5 for a simple proof for $i = 0$, $n = 1$, and [ELS] for two particular cases.

**Remark 0.2.5.** 1. Conjecture 0.2.4 implies that $X_\infty(2, n, m, i)$ (which is equal to Supp $X_{S,\infty}(2, n, m, i)$ — the support of $X_{S,\infty}(2, n, m, i)$ ) does not depend on $n$ and is equal to $X(m, i)$, although the schemes $X_{S,\infty}(2, n, m, i)$ are different. In particular, multiplicities of their irreducible components depend on $n$. See, for example, [GL16], Conjecture 9.7.8: both $X_\infty(2, 0, m, 3)$, $X_\infty(2, 1, m, 3)$ consist of 4 irreducible components (because they coincide), but multiplicities of these components in $X_{S,\infty}(2, 0, m, 3)$, $X_{S,\infty}(2, 1, m, 3)$ are different (see the last two lines of the table of [GL16], Conjecture 9.7.8).

2. Finding of analogs of Conjecture 0.2.4 for $q > 2$ is a subject of further research. [GL16], Theorem 8.6a indicates that such analogs exist.

3. Analogs of Conjecture 0.2.4 for $q > 2$ show that $H_{ij,qn}(m)$ are highly dependent. Since $H_{**fc}$ of (0.1.13) is the reduction of $H_{i\mathfrak{j},qn}(m)$ and $D_{**fc}$ of (0.1.13) are linear combinations of $H_{**fc}$ we can expect that $D_{**fc}$ are also dependent, and hence dimensions of $X_1(q, n, m, i)_{fc}$ maybe are higher than the naïve parameter count predicts.

### *0.3. Contents of the present paper*

We give an explicit description of irreducible components of $X(m, i)$, and their rational parameterizations, in combinatorial terms. Some results are conditional: they depend on conjectures based on computer calculations. Their proof is a subject of further research.

There are two constructions. Irreducible components of $X(m, i)$ are denoted by $C_{ij\mathfrak{k}}(m) \subset X(m, i)$, where $j \geq 1$ is an invariant of an irreducible component (i.e. $j$ is canonically defined by the component), and $\mathfrak{k}$ is a label of this component (i.e. assignment of $\mathfrak{k}$ is arbitrary), see 2.4, 2.5 for details. Conjecturally, they form series: $i$, $j$, $\mathfrak{k}$ are fixed and $m$ grows. The minimal possible value of $m$ is $i + j$, and the corresponding component is called the minimal component (see [GLZ], 2.4.1a).

The first construction (Theorem 5.6; Conjecture 5.10) describes the minimal components. Namely, let $T$ be a finite rooted binary tree[4] (see (4.1)) and $F$ a forest — a

[4] This $T$ has nothing common with the argument of the $L$-function.

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disjoint union of such trees. We consider a weight function $w$ — a function on the set of nodes of $F$ satisfying some properties ((4.4), (a), (b), (c) or (a′), (b′), (c′)). There are 3 groups $\mathrm{Aut}(F)$, $\mathfrak{G}(F)$ and the Galois group $\mathrm{Gal}(\mathbb{Q}(exp(2\pi\sqrt{-1}/2^d))/\mathbb{Q})$ (here $d$ is a sufficiently large number) acting on the set of pairs $(F, w)$ where $F$ is a forest and $w$ is a weight function on it.

The main results of the first construction are the following:

**Theorem 5.6.** *Any pair $(F, w)$ defines an irreducible component of $X(m, i)$. If two pairs $(F_1, w_1)$ and $(F_2, w_2)$ belong to the same orbit of $\mathfrak{G}(F) \rtimes \mathrm{Aut}(F)$ then they define the same component. The action of the Galois group on the set of $(F, w)$ is the Galois action on the corresponding components.*

**Conjecture 5.10.** *These irreducible components of $X(m, i)$ are the minimal ones. All minimal irreducible components of $X(m, i)$ are obtained by the above construction, and if two pairs $(F_1, w_1)$ and $(F_2, w_2)$ define the same component then they belong to the same orbit of $\mathfrak{G}(F) \rtimes \mathrm{Aut}(F)$.*

This conjecture is supported by computer calculations (see [GLZ], Tables A2.2, A3).

The second construction (Proposition 7.1; Conjecture 7.3) describes all elements of a series in terms of the minimal component of this series. Proposition 7.1 is conditional, it depends on Conjecture 0.2.4.

It turns out that Conjecture 0.2.4 and polynomials $H_{ij,21}$ for $q = 2$, $n = 1$ play an important role in the second construction, as well as in the construction of the odd lift, see [GLZ], Corollary 10.4. This is wonderful, because a priori polynomials $D(m, i)$ “do not know” that they come from the theory of Carlitz modules, and that there is some relation between them and $H_{ij,21}$.

### *0.4. Subjects of further research*

(1) It is desirable to continue computer calculations of [GL16], Section 6, for the case $q = 3$, $n = 1$. Are there $P \in \mathbb{F}_3[\theta]$ such that $r_1(P) > 6$? Are there more $P \in \mathbb{F}_3[\theta]$ such that $r_1(P) = 6$? What are the dimensions of $X_1(3, 1, m, i)_{fc}$ for $i = 1$ and 2 (for other $i$ the calculation can be too difficult), or, at least, what are their growths as $m \to \infty$?

(2) To prove all conjectures of the present paper. Particularly, we should either prove Conjecture 0.2.4, or find an independent proof of Proposition 7.1 and Conjecture 7.3. Also, we should prove that Theorem 5.6 gives an exhaustive description of irreducible components, i.e. prove Conjecture 5.10.

(3) To find more properties of varieties $X(m, i)$ and their irreducible components. For example, from one side, they are surjective images of powers of $\mathbb{P}^1$. From another side, there are some tautological sheaves on them (see [GLZ], 1.5). We should find these sheaves in terms of $O(n)$ on $\mathbb{P}^1$.

(4) Generalize the results of the present paper to other $q$ and $n$ (i.e. to the case $q$, $n$ arbitrary, $T = \infty$, characteristic 0), first to the case $q = 3$, $n = 1$ — the first non-trivial case of Problem 0.0.2. Most likely we shall have to use $q$-ary trees for their description, instead of binary trees. Here the situation is more complicated, because the statement of Conjecture 0.2.4 should be modified. Particularly, we should study varieties of zeroes of $H_{i\mathfrak{j}}$ and not only of $D(m, i)$.

(5) Solve Problems 0.0.2, 0.0.3. This is the case $q$, $n$ arbitrary, $T = 1$, characteristic $p$. We think that to pass from $T = \infty$ to $T = 1$ is the most complicated part of the work.

(6) Further, we can generalize the above results to the case of other Drinfeld modules (not necessarily the Carlitz modules).

### *0.5. Organization of the paper*

Section 1 contains an elementary self-contained definition of varieties $X(m, i)$. We formulate in Section 2 conjectures on degrees, multiplicities and Jordan forms of their irreducible components, and we describe their series $C_{ij\mathfrak{k}}(m)$. Section 4 contains definitions concerning the weighted rooted binary trees, forests and weights — objects that will be used later. Section 5 is the main part of the paper. It gives the first construction — a description of the minimal irreducible components of $X(m, i)$ in terms of weighted rooted binary forests (Theorem 5.6; Conjecture 5.10). Further, formulas for the degree (Section 5.11), intersection with the trace hyperplane (Section 5.12), multiplicity (Section 5.13), Jordan form (Section 5.14), fields of definition and Galois action on them (Section 5.15) are given. [GLZ], Section 6 contains explicit examples of constructions of Section 5 for some types of forests. Section 7 contains the second construction - the construction of series corresponding to a fixed minimal irreducible component. [GLZ], Sections 8 and 9 contain another constructions of irreducible components which are called respectively the even and odd lifts. [GLZ], Section 10 gives a relation between the odd lift and Conjecture 0.2.4. The appendix to [GLZ] contains some tables. [GLZ], Table A1 justifies Conjecture 0.2.4. [GLZ], Tables A2, A3 describe irreducible components of $X(m, i)$ for $i \le 6$ and for some other cases. They justify Conjectures 5.10, 7.3.

## 1. Definitions

Let $m$, $i$ satisfy $m > i \ge 1$. We give here explicit definitions of $\mathcal{M}$, $D(m, i)$ and $X(m, i)$ for the case $q = 2$, $n = 0$, at $\mathfrak{c} = \infty$. We shall consider the non-trivial part of $\mathcal{M}$ and $L$ (see 0.1.8), hence $\kappa = m - 1$.

Let $a_* = (a_0, \dots, a_m)$ be any objects, $a_\varkappa = 0$ for $\varkappa \notin \{0, \dots, m\}$. The matrix $\mathcal{M}(a_*, 0, m-1)$ is denoted by $\mathfrak{M}(m)(a_0, \dots, a_m)$, i.e. it is a $(m-1) \times (m-1)$-matrix whose $(\alpha, \beta)$-th entry is equal to $a_{2\beta-\alpha}$ (this is $\mathfrak{M}(P, m)$ of [GL16], Section 9). If it is clear what objects $(a_0, \dots, a_m)$ are kept in mind, we write $\mathfrak{M}(m)$ instead of $\mathfrak{M}(m)(a_0, \dots, a_m)$.

See [GLZ], Section 1 for examples and more details. In particular, [GLZ], 1.1 contains explicit form of $\mathfrak{M}(m)$ for $m = 6, 7$, relations of $\mathfrak{M}(m)$ with some other objects.

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Let $Ch(\mathfrak{M}(m))$ be the $(-1)^{m-1}\cdot$ characteristic polynomial of $\mathfrak{M}(m)$:

$$Ch(\mathfrak{M}(m)) = |\mathfrak{M}(m) - U \cdot I_{m-1}|$$

**Definition 1.2.** $D(m,i) \in \mathbb{Z}[a_0,\dots,a_m]$ are coefficients at $U^i$ of $Ch(\mathfrak{M}(m))$ considered as a polynomial in $U$:

$$Ch(\mathfrak{M}(m)) = D(m,0) + D(m,1)\,U + D(m,2)\,U^2 + \dots + D(m,m-2)\,U^{m-2} + (-U)^{m-1} \tag{1.2.1}$$

They are homogeneous polynomials of degree $m-1-i$. Hence, $D(m,i) = H_{i0,0}(m)$ where $H_{ij,n}(m)$ are from (0.2.2). Particularly, we have: $D(m,0) = |\mathfrak{M}(m)|$, $D(m,m-2) = (-1)^m tr(\mathfrak{M}(m)) = (-1)^m(a_1 + a_2 + \dots + a_{m-1})$ (the trace hyperplane). [GLZ], 1.2.2 gives us explicit values of polynomials $D(m,i)$ for $m=4$.

**Definition 1.3. A.** $X_S(m,i) \subset \mathbb{P}^m(\mathbb{C})$ (subscript $S$ means scheme) is a projective scheme of the first $i$ polynomials $D(m,i')$, $i' = 0,1,\dots,i-1$:

$$X_S(m,i) := \operatorname{Proj} \mathbb{C}[a_0,\dots,a_m]/ < D(m,0),\dots,D(m,i-1) > \tag{1.3.A.1}$$

where (as in (0.2.4)) $< D(m,0),\dots,D(m,i-1) >$ is the ideal generated by $D(m,0),\dots,D(m,i-1)$.

**B.** $X(m,i)$ is the support of $X_S(m,i)$, i.e. the set of $(a_0 : \dots : a_m) \in \mathbb{P}^m(\mathbb{C})$ such that $\forall\ i' = 0,\dots,i-1$ we have $D(m,i')(a_0,\dots,a_m) = 0$.

[GLZ], 1.5 contains definitions of some sheaves on $X_S(m,i)$, and [GLZ], 1.7 explains the meaning of the words “a variety is defined over a field” that we shall use later.

## 2. Conjectural description of irreducible components of $X(m,i)$

The below conjectures generalize and correct the ones of [GL16], (9.7). They come from explicit calculations (see [GLZ], Tables A2.2, A3) and results and conjectures of Sections 5, 7. See [GLZ], Section 2 for a much more detailed exposition of the present section, with many examples.

**Conjecture 2.1.** *Formula (1.3.A.1) represents $X_S(m,i)$ as a complete intersection. Hence all irreducible components of $X_S(m,i)$ have codimension $i$ in $\mathbb{P}^m$.*

Let $\mathfrak{z}_d := exp(2\pi\sqrt{-1}/2^d)$, i.e. $\mathfrak{z}_1 = -1$, $\mathfrak{z}_2 = \sqrt{-1}$. In the present paper, number $i$ never means $\sqrt{-1}$.

**Conjecture 2.2.** *All irreducible components of $X(m,i)$ are defined over $\mathbb{Q}(\mathfrak{z}_{(i-2)/2})$ for even $i$, $(\mathbb{Q}(\mathfrak{z}_{(i-1)/2}) \cap \mathbb{R}) \cdot \mathbb{Q}(\mathfrak{z}_{(i-3)/2})$ for odd $i$, see 5.15 for a justification.*

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**2.3. Jordan form.** Let $Irr(X(m,i))$ be the set of irreducible components of $X(m,i)$, and let us denote by $P(i)$ the set of partitions of $i$. Let $(a_0 : ... : a_m)$ be a generic point of $X(m,i)$. The Jordan form of its matrix $\mathfrak{M}(m)$ has $i$ zeroes on its diagonal, hence it defines a partition of $i$. This partition depends only on an irreducible component of $X(m,i)$, hence we get a map $\pi_{m,i} : Irr(X(m,i)) \to P(i)$. Later the expression "The Jordan form of a matrix" sometimes will mean "The partition of the Jordan form of a matrix".

**Main conjecture 2.4.** *For a fixed $i$ the set $Irr(X(m,i))$ does not depend on $m$ for $m \geq 2i$. Moreover, there exists an abstract set $Irr(i)$ and for any $m \geq 2i$ there exists an isomorphism of sets*

$$\beta_m = \beta_{m,i} : Irr(i) \to Irr(X(m,i)) \tag{2.4.1}$$

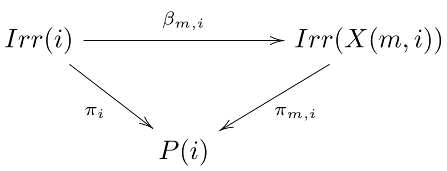
*such that the following holds:*

*For a fixed $C \in Irr(i)$ all $\beta_m(C)$ have the same Jordan form and the same multiplicity (denoted by $\mu(C)$), and their degrees are given by a simple formula:*

$$\deg \beta_m(C) = \mathfrak{d}(C)\binom{m-i}{j} \tag{2.4.2.1}$$

*where $\mathfrak{d}(C) \in \mathbb{Z}$ is a constant which does not depend on $m$, but only on $C$, and $j = j(C) \in \mathbb{Z}$, $1 \leq j \leq i$. See 7.4 for its justification.*

**2.4.3.** Particularly, there exists a map $\pi_i = \pi_{m,i} \circ \beta_{m,i} : Irr(i) \to P(i)$ which does not depend on $m$. Hence, we have a commutative diagram




**2.4.4.** The same holds for $m < 2i$, with the following exception: if $\deg \beta_m(C)$ given by (2.4.2.1) is equal to 0 ($\iff m < i + j(C)$) then the corresponding component $\beta_m(C)$ of $X(m,i)$ vanishes. If $\deg \beta_m(C) \neq 0$ then $\beta_m(C)$ has the same multiplicity and the same Jordan form as for the case $m \geq 2i$, and its degree is given by the same formula (2.4.2.1).

**2.5.** We use the following notations. The set of elements $C \in Irr(i)$ having $j(C) = j$ is denoted by $Irr(i,j)$, hence we have (here and below $\sqcup$ means the disjoint union):

$$Irr(i) = \bigsqcup_{j=1}^{i} Irr(i,j); \qquad Irr(X(m,i)) = \bigsqcup_{j=1}^{\min\,(i,m-i)} Irr(i,j)$$

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We order elements of $Irr(i,j)$ by some manner; the $\mathfrak{k}$-th element of $Irr(i,j)$ is denoted by $C_{ij\mathfrak{k}}$. The component $\beta_m(C_{ij\mathfrak{k}})$ of $X(m,i)$ is denoted by $C_{ij\mathfrak{k}}(m)$. The minimal possible value of $m$ such that $C_{ij\mathfrak{k}}(m) \neq \emptyset$ is $i+j$.

**Definition 2.4.1a.** The element $C_{ij\mathfrak{k}}(i+j) \in Irr(X(i+j,i))$ is called the minimal component (corresponding to $C_{ij\mathfrak{k}}$) and is denoted by $\bar{C}_{ij\mathfrak{k}}$.

**Conjecture 2.6.** *All irreducible components of $X(m,i)$ are rational varieties.*

This conjecture follows from Theorem 5.6, Conjectures 5.10, 7.3 (see [GLZ] for details).

**2.7. Formula for $\sum_{\mathfrak{k}\in Irr(i,j)} \mathfrak{d}(C_{ij\mathfrak{k}})\mu(C_{ij\mathfrak{k}})$.**

Most results of the present paper either are proved rigorously or have a serious justification. Exception is the formula (5.13.2) for $\mu(C)$: it is purely experimental. Its evidence comes not only from computer calculations, but also from the below formula (2.8). Let us deduce (2.8) from the above conjectures.

Since $X_S(m,i)$ is (conjecturally) a complete intersection and $\deg D(m,i') = m-i'-1$ we get that $\deg X_S(m,i) = (m-1)(m-2)\cdot \ldots \cdot (m-i)$. According (2.1) and (2.4.2.1), for any $m$ and $i$ the same number is equal to

$$\sum_{j=1}^{i} [\sum_{\mathfrak{k}\in Irr(i,j)} \mathfrak{d}(C_{ij\mathfrak{k}})\mu(C_{ij\mathfrak{k}})] \binom{m-i}{j}$$

Let $i$ be fixed. If constants $\Upsilon_{ij}$ $\forall m$ satisfy

$$\sum_{j=0}^{i} \Upsilon_{ij} \binom{m-i}{j} = (m-1)(m-2)\cdot \ldots \cdot (m-i)$$

then they are defined uniquely. A well-known combinatorial formula usually written as

$$\binom{\mathfrak{n}}{\mathfrak{l}} = \binom{\varkappa}{0}\binom{\mathfrak{n}-\varkappa}{\mathfrak{l}} + \binom{\varkappa}{1}\binom{\mathfrak{n}-\varkappa}{\mathfrak{l}-1} + \binom{\varkappa}{2}\binom{\mathfrak{n}-\varkappa}{\mathfrak{l}-2} + \ldots + \binom{\varkappa}{\mathfrak{l}}\binom{\mathfrak{n}-\varkappa}{0}$$

gives us immediately that Conjectures 2.1, 2.4 imply $\forall i,j$

$$\sum_{\mathfrak{k}\in Irr(i,j)} \mathfrak{d}(C_{ij\mathfrak{k}})\mu(C_{ij\mathfrak{k}}) = \binom{i-1}{i-j} \cdot i! \tag{2.8}$$

**Remark.** Numbers $\mathfrak{d}(C_{ij\mathfrak{k}})$, resp. $\mu(C_{ij\mathfrak{k}})$ are given by (5.11.11), resp. (5.13.2), (5.13.3), and the set $Irr(i,j)$ is defined by Conjecture 5.10. Recall that these formulas are conjectural. There is a problem: to prove (2.8) using (5.11.11), (5.13.2), (5.13.3) as definitions of $\mathfrak{d}(C_{ij\mathfrak{k}})$, $\mu(C_{ij\mathfrak{k}})$, and Conjecture 5.10 as definition of $Irr(i,j)$. We do not think that this is easy, especially because there is no uniform description of the sets of forests and weights entering in 5.10.

## 3. Intersection with the trace hyperplane[5]

We shall look in this section what happens if we pass from $X(m, m-2)$ to $X(m, m-1)$. Namely, $X(m, m-2)$ is a surface in $\mathbb{P}^m$ and $X(m, m-1)$ is its hyperplane section (recall that we assume truth of Conjecture 2.1). Really, we have

$$X(m, m-1) = X(m, m-2) \cap \text{ the trace hyperplane } H := \{a_1 + ... + a_{m-1} = 0\} \quad (3.1)$$

(because $D(m, m-2) = \pm(a_1 + a_2 + ... + a_{m-1})$).

We let $i = m-2$. We shall see that formula (3.1) and Conjectures 2.1, 2.4 imply some relations between the sets $Irr(i, j)$ and numbers $\mathfrak{d}(C_{ij\mathfrak{k}})$, $\mu(C_{ij\mathfrak{k}})$ for $j = 1, 2$. Hence, we assume in this section that these conjectures hold. We have (identifying elements of $Irr(*, *)$ with their $\beta_{*,*,*}$-images) $Irr(X(i+2, i)) = Irr(i, 1) \sqcup Irr(i, 2)$ and $Irr(X(i+2, i+1)) = Irr(i+1, 1)$. Formula (3.1) implies that we have a correspondence ( = binary relation = multi-valued function)

$$Y_i : Irr(i, 1) \sqcup Irr(i, 2) \to Irr(i+1, 1) \quad (3.2)$$

— an irreducible component $Z \subset X(i+2, i)$ goes to $Z \cap H$ which is a union of irreducible components of $X(i+2, i+1)$.

**Conjecture 3.3.** *The converse correspondence $Y_i^{-1} : Irr(i+1, 1) \to Irr(i, 1) \sqcup Irr(i, 2)$ is a function, i.e. any irreducible component of $X(m, m-1)$ is contained only in one irreducible component of $X(m, m-2)$, or, the same, no curve — intersection of irreducible components of $X(m, m-2)$ — is contained in the trace hyperplane $H$.*

See [GLZ] for a remark concerning these objects.

Intersection with a hyperplane preserves both degree and multiplicity. For $C_{i1\mathfrak{k}} \in Irr(i, 1)$ we have $\deg C_{i1\mathfrak{k}}(i+2) = 2\mathfrak{d}(C_{i1\mathfrak{k}})$, for $C_{i2\mathfrak{k}} \in Irr(i, 2)$ we have $\deg C_{i2\mathfrak{k}}(i+2) = \mathfrak{d}(C_{i2\mathfrak{k}})$, hence we get

**Proposition 3.4.** *If Conjectures 2.1, 2.4 hold then*

$$\text{For } C_{i1\mathfrak{k}} \in Irr(i, 1) \text{ we have } \sum_{\tilde{C} \in Y_i(C_{i1\mathfrak{k}})} \mathfrak{d}(\tilde{C}) = 2\mathfrak{d}(C_{i1\mathfrak{k}}) \quad (3.4.1)$$

$$\text{For } C_{i2\mathfrak{k}} \in Irr(i, 2) \text{ we have } \sum_{\tilde{C} \in Y_i(C_{i2\mathfrak{k}})} \mathfrak{d}(\tilde{C}) = \mathfrak{d}(C_{i2\mathfrak{k}}) \quad (3.4.2)$$

[5] Contents of the present section, as well as Subsection 4.6, are used only for results of Subsection 5.12, and hence can be skipped while the first reading.

$$\textit{For } C_{ij\mathfrak{k}} \in Irr(i,1) \sqcup Irr(i,2),\ \tilde{C} \in Y_i(C_{ij\mathfrak{k}}) \textit{ we have } \mu(\tilde{C}) = \mu(C_{ij\mathfrak{k}}). \tag{3.4.3}$$

$$(j = 1,\ 2)$$

□

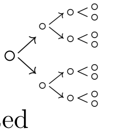
## 4. Rooted binary trees

Minimal irreducible components of $X(m,i)$ are defined in terms of rooted binary trees, their unions and weights on them.

**4.1.** A rooted tree is a tree having a fixed node (the root). The root defines an orientation on a tree (we shall consider the orientation as from the left to the right). A rooted tree is called binary if any node (including the root) has $\leq 2$ right neighbors. Later in the present paper the term "tree" always (without any exception) will mean a finite rooted binary tree. The quantity of nodes of a tree is denoted by $i$. See [GLZ], 4.1.1 for examples, pictures and some formulas.

**4.2a Definition.** The depth of a node is the distance between this node and the root (the length of any edge is 1; the depth of the root is 0), and the depth of a tree is the maximal depth of its nodes.

**4.2b Definition.** A simple tree is a tree without ramification nodes, its length (= depth + 1) is the quantity of nodes.

A complete tree of depth $k$ is a tree such that all its final nodes have depth $k$ and all its non-final nodes are ramification nodes. Example for $k = 3$: 

A forest $F$ is a disjoint unordered union of trees, it is denoted

$$F = \bigsqcup_{\alpha=1}^{j} T_\alpha \tag{4.2.1}$$

**Remark.** Later we associate to $F$ some minimal irreducible components of $X(i+j,i)$ where $j$ is from (4.2.1) and $i$ is the quantity of nodes of $F$.

Also, we can group isomorphic constituent trees as follows:

$$F = \bigsqcup_{\beta=1}^{\delta} \eta_\beta T_\beta \tag{4.2.2}$$

i.e. $F$ is a disjoint union of $\eta_1$ copies of $T_1$, $\eta_2$ copies of $T_2, \ldots, \eta_\delta$ copies of $T_\delta$, and $T_{\beta_1} \neq T_{\beta_2}$.

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**4.2.3.** By default, the quantity of nodes of a tree $T$ is denoted as above by $i$, the quantity of final nodes is denoted by $l$ (hence the quantity of ramification nodes is $l-1$), and the depth of $T$ is denoted by $d$. For a forest $F$ we use by default notations of (4.2.1) (i.e. $\alpha$ is the number (label) of a constituent tree in $F$, $j$ is the quantity of trees in $F$), numbers $i$, $l$, $d$ for $T_\alpha$ are denoted by $i_\alpha$, $l_\alpha$, $d_\alpha$ respectively. Numbers $i$, $l$, $d$ for $F$ mean the quantity of nodes, the quantity of final nodes and the depth respectively (the depth of a forest is, by definition, the maximal depth of its trees). Hence, we have $i=\sum_{\alpha=1}^{j} i_\alpha$, $l=\sum_{\alpha=1}^{j} l_\alpha$, $d=\max_{\alpha=1}^{j} d_\alpha$.

For a list of all forests having $i \le 6$ see the last column of [GLZ], Table A2.2, their $i$ is indicated in the first column of the table, their $j$ is given in the second column.

**4.3. Automorphisms of trees and forests.** Notation: Let $G$ be a group. By default, the semidirect product $G^n \rtimes S_n$ is with respect to the action of the symmetric group $S_n$ on $G^n$ by permutation of factors.

An automorphism of a tree (a forest) is a permutation of nodes preserving the root(s) and the edges. We denote the automorphism group of the complete tree of depth $k$ by $G_k$. It satisfies a recurrent formula $G_1=\mathbb{Z}/2\mathbb{Z}$, $G_k=G_{k-1}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$. We have $\#(G_k)=2^{2^k-1}$. Clearly for any tree $T$ we have $Aut(T)=G_{\varkappa_1}\times G_{\varkappa_2}\times \ldots \times G_{\varkappa_\gamma}$ for some $\varkappa_1,\varkappa_2,\ldots,\varkappa_\gamma$ and for a forest $F=\sqcup_{\beta=1}^{\delta}\eta_\beta T_\beta$ we have

$$Aut(F)=(Aut(T_1)^{\eta_1}\rtimes S_{\eta_1})\times(Aut(T_2)^{\eta_2}\rtimes S_{\eta_2})\times\ldots\times(Aut(T_\delta)^{\eta_\delta}\rtimes S_{\eta_\delta})$$

**4.4. Weighted trees.** A weight on a tree is a function $w$ on its nodes satisfying the below conditions. There are two equivalent descriptions of a weight: multiplicative and additive. Let $T$ be a tree. A multiplicative weight function

$$w_\mu:\ \{\text{nodes of } T\}\to <\mathfrak{z}_d> \subset \mathbb{C}^*$$

where $<\mathfrak{z}_d>$ (see lines below 2.1) is the multiplicative group generated by $\mathfrak{z}_d$, satisfies conditions:

(a) If $y$ is the left neighbor of $x$ then $w_\mu(y)=w_\mu(x)^2$;
(b) If $y$ is a ramification node, $x_1$ and $x_2$ are its right neighbors then

$$w_\mu(x_2)=-w_\mu(x_1)$$

(c) $w_\mu(\text{root})=1$.

The corresponding additive weight $w_a$ is defined as follows. Let $x$ be a node of depth $k$. We have: $w_a(x)\in\mathbb{Z}/2^k$ such that $w_\mu(x)=\mathfrak{z}_k^{w_a(x)}$. The above (a)-(c) become (a′) If $x$ is a right neighbor of $y$, where $y$ of depth $k$, then either $w_a(x) = w_a(y)$ or $w_a(x) = w_a(y) + 2^k$ (here $\forall\ \varkappa\ \mathbb{Z}/2^\varkappa$ is identified with $\{0, 1, \dots, 2^\varkappa - 1\}$);

(b′) If $y$ is a ramification node of depth $k$, $x_1$ and $x_2$ are its right neighbors then $w_a(x_2) \neq w_a(x_1)$, i.e. $w_a(x_2) = w_a(x_1) \pm 2^k$;

(c′) $w_a(\text{root}) = 0$.

A weight of a forest $F = \sqcup_{\alpha=1}^{j} T_\alpha$ is a set of weights $(w_1, \dots, w_j)$ of $T_1, \dots, T_j$. The set of weights of a tree $T$, resp. a forest $F$ is denoted by $W(T)$, resp. $W(F)$.

**4.4.1.** For any $T$ we have $\#(W(T)) = 2^{i-l}$ (see [GLZ] for a proof). Clearly $\#(W(F)) = 2^{i-l}$ as well ($i$, $l$ for $F$).

**4.5. Action of groups on $W(T)$, $W(F)$.** There are 3 group actions on $W(F)$. The first action is the obvious action of $Aut(F)$ on $W(F)$.

Let $F$ be a fixed forest. To define the second action, we let $\mathfrak{G}_\alpha = \mathfrak{G}(T_\alpha) := \mathbb{Z}/2^{d_\alpha}$ and $\mathfrak{G} = \mathfrak{G}(F) := \prod_{\alpha=1}^{j} \mathfrak{G}_\alpha = \prod_{\alpha=1}^{j} \mathbb{Z}/2^{d_\alpha}$. The second action is an action of the group $\mathfrak{G}$. It is defined as follows. Let $w_a$ be a weight written additively. Namely, if $x$ is a node of $T_\alpha \subset F$ of depth $k$ then $w_a(x) \in \mathbb{Z}/2^k$. Let $\mathfrak{g} = (\mathfrak{g}_1, \dots, \mathfrak{g}_j) \in \mathfrak{G}$ where $\mathfrak{g}_\alpha \in \mathfrak{G}_\alpha$. We define

$$(\mathfrak{g}(w_a))(x) := \bar{\mathfrak{g}}_\alpha + w_a(x) \tag{4.5.1}$$

where $\bar{\mathfrak{g}}_\alpha := \varepsilon_{d_\alpha,k}(\mathfrak{g}_\alpha)$ is the image of $\mathfrak{g}_\alpha$ in the epimorphism $\varepsilon_{d_\alpha,k} : \mathbb{Z}/2^{d_\alpha} \to \mathbb{Z}/2^k$.

**(4.5.2).** Formula (4.5.1) shows that this action of $\mathfrak{G}$ on $W(F)$ comes from the actions of $\mathfrak{G}_\alpha$ on $W(T_\alpha)$ defined by (4.5.1). See [GLZ], 4.5.3 for an example.

**(4.5.4).** We denote $\mathbb{Q}(d) := \mathbb{Q}(\mathfrak{z}_d)$, $\Gamma_d := \mathrm{Gal}(\mathbb{Q}(d)/\mathbb{Q}) = (\mathbb{Z}/2^d)^*$. The third (multiplicative) action is the Galois action of $\Gamma_d$ on values of $w_\mu(x)$. It is easy to see that in the above (additive) notations, it is defined by the formula

$$(\gamma(w_a))(x) := \bar{\gamma} \cdot w_a(x)$$

where $\gamma \in \Gamma_d = (\mathbb{Z}/2^d)^* \subset \mathbb{Z}/2^d$ and $\bar{\gamma} \in \mathbb{Z}/2^k$ is as above.

Clearly for one tree $T$ the action of $\mathrm{Aut}(T)$ commutes with the action of $\mathfrak{G}(T)$ on $W(T)$. For a forest $F$ such that some $\eta_\beta$ of (4.2.2) are $> 1$ the action of $\mathrm{Aut}(F)$ on $\mathfrak{G}(F)$ is not trivial (see [GLZ] for details). We denote the corresponding semidirect product by $\mathfrak{G}(F) \rtimes \mathrm{Aut}(F)$. This group acts on $W(F)$. Further, the actions of $\mathrm{Aut}(F)$ and of $\Gamma_d$ on $W(F)$ commute, and the actions of $\mathfrak{G}(F)$ and $\Gamma_d$ give an action on $W(F)$ of the group $\mathfrak{G}(F) \rtimes \Gamma_d$, the action of $\Gamma_d$ on $\mathfrak{G}(F)$ in this semidirect product is the natural one (namely, $\gamma \in \Gamma_d = (\mathbb{Z}/2^d)^*$ acts on $\mathfrak{g} = (\mathfrak{g}_1, \dots, \mathfrak{g}_j) \in \mathfrak{G}$ by the formula $\gamma(\mathfrak{g}) = (\bar{\gamma}\mathfrak{g}_1, \dots, \bar{\gamma}\mathfrak{g}_j)$ where for $\alpha$-th term of this expression we have $\bar{\gamma} \in (\mathbb{Z}/2^{d_\alpha})^*$ as above).

**4.6. Elimination of the root.**[6] Let $T$ be a tree and $r$ its root. There are two possibilities:

(a) $r$ is not a ramification node;
(b) $r$ is a ramification node.

In the case (a), we denote by $\bar{T}$ a tree obtained from $T$ by elimination of the root. We have a map $\tau : W(T) \to W(\bar{T})$ defined as follows. Let $\bar{r}$ be the right neighbor of $r$ (i.e. $\bar{r}$ is the root of $\bar{T}$), and $w \in W(T)$ written additively. We have $w(\bar{r}) = 0$ or $w(\bar{r}) = 1$, and for any $u \in \bar{T}$ we have $w(u) \equiv w(\bar{r}) \mod 2$. We let $\forall\ u \in \bar{T}$

$$\tau(w)(u) := \frac{w(u) - w(\bar{r})}{2}$$

Clearly $\forall\ w \in W(T)$, $\forall\ \mathfrak{g} \in \mathfrak{G}(T)\ \exists\ \bar{\mathfrak{g}} \in \mathfrak{G}(\bar{T})$ (depending on $w$) such that $\tau(\mathfrak{g}(w)) = \bar{\mathfrak{g}}(\tau(w))$. Warning: the map $\mathfrak{g} \mapsto \bar{\mathfrak{g}}$ (where $w$ is fixed) is not a homomorphism $\mathfrak{G}(T) \to \mathfrak{G}(\bar{T})$. For any $\bar{w} \in W(\bar{T})$ we have: $\tau^{-1}(\bar{w})$ consists of two elements which belong to one $\mathfrak{G}(T)$-orbit. Particularly, $\tau$ is an isomorphism on the set of $\mathfrak{G}$-orbits.

In the case (b), eliminating the root we get two trees. We denote them by $D_1(r)$, $D_2(r)$. Their disjoint union $D_1(r) \sqcup D_2(r)$ is a forest having $j = 2$. Let $\alpha = 1, 2$. Two maps $\tau_\alpha : W(T) \to W(D_\alpha(r))$ are defined by the similar manner. Let $r_\alpha$ be the root of $D_\alpha(r)$, and $u \in D_\alpha(r)$. We let

$$\tau_\alpha(w)(u) := \frac{w(u) - w(r_\alpha)}{2} \tag{4.6.1}$$

Maps $\tau_1$, $\tau_2$ define a map $\tau : W(T) \to W(D_1(r) \sqcup D_2(r))$. For $(w_1, w_2) \in W(D_1(r) \sqcup D_2(r))$ we have $\tau^{-1}(w_1, w_2)$ consists of 2 elements, because $\forall\ w \in W(T)$ we have $w(r_1) \neq w(r_2)$.

Clearly $\forall w \in W(T)$, $\forall\ \mathfrak{g} \in \mathfrak{G}(T)$, $\forall\ \alpha = 1, 2\ \exists\ \mathfrak{g}_\alpha \in \mathfrak{G}(D_\alpha(r))$ such that

$$\tau_\alpha(\mathfrak{g}(w)) = \mathfrak{g}_\alpha(\tau_\alpha(w)) \tag{4.6.2}$$

**(4.6.3).** The converse usually is not true: if $\mathfrak{g}_\alpha \in \mathfrak{G}(D_\alpha(r))$ $(\alpha = 1, 2)$ then it is few likely that $\exists\ \mathfrak{g} \in \mathfrak{G}(T)$ such that (4.6.2) holds for both $\alpha = 1, 2$. Moreover, for a given $(w_1, w_2) \in W(D_1(r) \sqcup D_2(r))$ two elements of $\tau^{-1}(w_1, w_2)$ usually do not belong to one $\mathrm{Aut}(T) \times \mathfrak{G}(T)$-orbit. See [GLZ] for an example.

## 5. The first construction: a weighted forest defines a minimal irreducible component

Let $F$ be a forest, and let $w \in W(F)$ be a weight of $F$. We let $m = i + j$ (see (4.2.3) for notations). We associate to the pair $(F, w)$ a map

[6] Contents of the present subsection are used only for results of Subsection 5.12.

$$\varphi = \varphi(F, w) : (\mathbb{P}^1)^j \to \mathbb{P}^m \tag{5.0}$$

Later we shall see that its image is contained in $X(m,i)$. Since $(\mathbb{P}^1)^j$ is irreducible, Conjecture 2.1 and the below Conjecture 5.8 imply that for any $(F,w)$ the image of $\varphi(F,w)$ is an irreducible component of $X(m,i)$. Some explicit examples are given in [GLZ], (5.5.2 A, B), (6.1), (6.2.1), (6.2.3), (6.5.1 — 6.5.3), (7.5.2), (7.5.3).

The definition of $\varphi(F,w)$ is given in (5.5.1), (5.5.D), numbers $\mathfrak{c}_*$ used in (5.5.1) are defined in (5.4.4). Nevertheless, it is more convenient to define $\varphi(F,w)$ not by the direct formulas (5.5.1), (5.4.4), but as a solution to some linear equations (5.2)-(5.3), see Section 5.1.1. This definition follows.

**5.0.1.** Let $\xi = \{(c_1 : c_1'), (c_2 : c_2'), \dots, (c_j : c_j')\} \in (\mathbb{P}^1)^j$. We denote $\varphi(\xi)$ by $(\lambda_0 : ... : \lambda_m) \in \mathbb{P}^m$. Numbers $\lambda_0, \dots, \lambda_m$ are polynomials in $c_1, c_1', c_2, c_2', \dots, c_j, c_j'$.

In order to simplify formulas (see [GLZ], 5.0.2, 5.0.3 for details), we can consider the case of the affine part of each $\mathbb{P}^1$ in $(\mathbb{P}^1)^j$, i.e. $\forall\ \alpha = 1, \dots, j$ we let $c_\alpha' = 1$.

We need a notation: for any $x$ we denote by $v(x)$ the following column vector of size $m-1 \times 1$:

$$v(x) := (1, x, x^2, x^3, \dots, x^{m-2})^{tr} \tag{5.1}$$

We use notations of (4.2.1), (4.2.3), and we use the multiplicative form $w_\mu$ of $w$.

**5.1.1.** The system of equations defining $\lambda_0, \dots, \lambda_m$ consists of:

**A.** One matrix equation for any final node $u$ of $F$. Let $k = k(u)$ be the depth of $u$ (see 4.2a), and $\alpha$ the number of the tree containing $u$: $u \in T_\alpha \subset F$. The corresponding equation ($\lambda_0, \dots, \lambda_m$ are unknowns, $c_1, \dots, c_j$ parameters) is the following (here 0 is the 0-matrix column of size $m-1 \times 1$):

$$\mathfrak{M}(m)(\lambda_0, \dots, \lambda_m) \cdot v(w_\mu(u) c_\alpha^{2^{d_\alpha - k}}) = 0 \tag{5.2}$$

**B.** One matrix equation for any non-final, non-ramification node $u$ of $F$. Let $k$ and $\alpha$ be as above, and $u'$ the right neighbor of $u$. The corresponding equation is the following: $\exists$ a scalar $\beta_u \in \mathbb{C}$ depending on $u$ such that the below holds:

$$\mathfrak{M}(m)(\lambda_0, \dots, \lambda_m) \cdot v(w_\mu(u) c_\alpha^{2^{d_\alpha - k}}) = \beta_u \cdot v(w_\mu(u') c_\alpha^{2^{d_\alpha - k - 1}}) \tag{5.3}$$

**Proposition 5.4.** *For any $(F,w)$, for any generic $(c_1, \dots, c_j) \in \mathbb{C}^j$ equations (5.2), (5.3) have 1 non-zero solution, up to a multiplication by a scalar, i.e. one solution in $\mathbb{P}^m$ (generic means that $(c_1, \dots, c_j)$ does not belong to a finite union of hypersurfaces).*

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**Proof.** For any number $\mathfrak{b}$ we define matrices

$$A_1(\mathfrak{b}) := \begin{pmatrix} 0 & 1 & 0 & \mathfrak{b} & 0 & \mathfrak{b}^2 & 0 & \mathfrak{b}^3 & \dots & \mathfrak{b}^{\frac{m}{2}-1} & 0 \\ 1 & 0 & \mathfrak{b} & 0 & \mathfrak{b}^2 & 0 & \mathfrak{b}^3 & 0 & \dots & 0 & \mathfrak{b}^{\frac{m}{2}} \end{pmatrix} \text{ for even } m,$$

$$A_1(\mathfrak{b}) := \begin{pmatrix} 0 & 1 & 0 & \mathfrak{b} & 0 & \mathfrak{b}^2 & 0 & \mathfrak{b}^3 & \dots & 0 & \mathfrak{b}^{\frac{m-1}{2}} \\ 1 & 0 & \mathfrak{b} & 0 & \mathfrak{b}^2 & 0 & \mathfrak{b}^3 & 0 & \dots & \mathfrak{b}^{\frac{m-1}{2}} & 0 \end{pmatrix} \text{ for odd } m$$

(for any $m$ it is a $2 \times (m+1)$-matrix) and

$$A_2(\mathfrak{b}) := \begin{pmatrix} 1 & -\mathfrak{b} & \mathfrak{b}^2 & -\mathfrak{b}^3 & \mathfrak{b}^4 & \dots & (-1)^m \mathfrak{b}^m \end{pmatrix} \text{ (a } 1 \times (m+1)\text{-matrix).}$$

Further, we denote a column matrix $(\lambda_0, \dots, \lambda_m)^{tr}$ by $\lambda_*$.

(5.2) is equivalent to

$$A_1(w_\mu(u) c_\alpha^{2^{d_\alpha - k}}) \cdot \lambda_* = 0 \tag{5.4.1}$$

and (5.3) is equivalent to

$$A_2(w_\mu(u') c_\alpha^{2^{d_\alpha - k - 1}}) \cdot \lambda_* = 0 \tag{5.4.2}$$

To prove these affirmations, it is sufficient to write down explicitly (5.2), (5.3), (5.4.1), (5.4.2), and we get a proof immediately. For example, (5.4.2) is equivalent to (5.3), because $w_\mu(u) c_\alpha^{2^{d_\alpha - k}} = (w_\mu(u') c_\alpha^{2^{d_\alpha - k - 1}})^2$; $\beta_u$ of (5.3) is equal to $\lambda_1 + \lambda_3 \mathfrak{b}^2 + \lambda_5 \mathfrak{b}^4 + \dots$ where $\mathfrak{b} = w_\mu(u') c_\alpha^{2^{d_\alpha - k - 1}}$.

**5.4.2a.** Recall that $F$ is a forest. Let us make a list of all non-ramification nodes of $F$: $u_1, \dots, u_{i-l+j}$ and fix this order $O$ (see (4.2.3) for $l$). We form a $m \times (m+1)$-matrix $A = A(F, w, c_*)$ as follows. $A$ is a block matrix having one block column and $i - l + j$ block lines. For $\varkappa = 1, \dots, i - l + j$ the $A_{\varkappa,1}$-th block entry of $A$ is a matrix $A_1(w_\mu(u_\varkappa) c_\alpha^{2^{d_\alpha - k_\varkappa}})$ (resp. $A_2(w_\mu(u'_\varkappa) c_\alpha^{2^{d_\alpha - k_\varkappa - 1}})$) if $u_\varkappa$ is a final (resp. non-final, non-ramification) node of $T_\alpha \subset F$, where as above $u'_\varkappa$ is the right neighbor of a non-final node $u_\varkappa$, and $k_\varkappa$ is the depth of $u_\varkappa$. So, $A$ is a disjoint union of matrices $A_1(w_\mu(u_\varkappa) c_\alpha^{2^{d_\alpha - k_\varkappa}})$, $A_2(w_\mu(u'_\varkappa) c_\alpha^{2^{d_\alpha - k_\varkappa - 1}})$ arranged vertically, according the order $O$, for all non-ramification nodes of $F$. See [GLZ], (6.1.1) for an example. (5.4.1), (5.4.2) are equivalent to

$$A \lambda_* = 0 \tag{5.4.3}$$

It is sufficient to show that for a generic $(c_1, \dots, c_j)$ the matrix $A$ is of the maximal rank $m$. Let $V_{\varkappa,\lambda}(x_1, \dots, x_\varkappa)$ be the Vandermonde matrix of size $\varkappa \times \lambda$ (its $(\mu, \nu)$-th entry is $x_\mu^{\nu - 1}$). Let $\mathcal{C}_2 = \mathcal{C}_2(\mathfrak{b})$ (an elementary correcting matrix) be $\begin{pmatrix} \sqrt{\mathfrak{b}} & 1 \\ -\sqrt{\mathfrak{b}} & 1 \end{pmatrix}$, so we have $\mathcal{C}_2(\mathfrak{b}) \cdot A_1(\mathfrak{b}) = V_{2,m+1}(\sqrt{\mathfrak{b}}, -\sqrt{\mathfrak{b}})$. Let $\mathcal{C}$ (a correcting matrix) be a block diagonal matrix consisting of $\mathcal{C}_2(w_\mu(u) c_\alpha^{2^{d_\alpha - k}})$ on block diagonal positions corresponding to final nodes

$u$ of $F$ and of 1-s on the diagonal positions corresponding to non-final, non-ramification nodes. All blocks are ordered by the order $O$.

**5.4.4.** Let us consider the sequence $u_1, \dots, u_{i-l+j}$ from 5.4.2a. Now we replace any final (resp. non-final, non-ramification) node $u$ by two numbers $\pm\sqrt{w_\mu(u)c_\alpha^{2^{d_\alpha-k}}}$ (resp. by one number $-w_\mu(u')c_\alpha^{2^{d_\alpha-k-1}}$, where $u'$ is the right neighbor). We denote the obtained sequence by $\mathfrak{c}_* = (\mathfrak{c}_1, \dots, \mathfrak{c}_m)$. See examples in [GLZ], 5.5.2.

In these notations we have

$$\mathcal{C} \cdot A(F, w, c_*) = V_{m,m+1}(\mathfrak{c}_*) \tag{5.4.5}$$

Hence, (5.4.3) is equivalent to

$$V_{m,m+1}(\mathfrak{c}_*) \cdot \lambda_* = 0 \tag{5.4.6}$$

For any $(x_1, \dots, x_m)$ the $\varkappa$-th maximal minor (the determinant of the matrix obtained by elimination of its $(m+1-\varkappa)$-th column, $\varkappa = 0, \dots, m$) of $V_{m,m+1}(x_1, \dots, x_m)$ is

$$\sigma_{m-\varkappa}(x_1, \dots, x_m) \cdot \prod_{m \geq \iota > \daleth \geq 1} (x_\iota - x_\daleth)$$

where $\sigma$ is the symmetric polynomial (we let $\sigma_0 = 1$).

**5.4.7.** For a generic $(c_1, \dots, c_j)$ the numbers $(\mathfrak{c}_1, \dots, \mathfrak{c}_m)$ are different (this is obvious; see [GLZ] for a proof). If they are different then $\lambda_* \in \mathbb{P}^m$ are unique. □

**5.5.** If $(\mathfrak{c}_1, \dots, \mathfrak{c}_m)$ are different then

$$\lambda_\varkappa = (-1)^{m-\varkappa}\sigma_{m-\varkappa}(\mathfrak{c}_*), \quad \varkappa = 0, \dots, m \tag{5.5.1}$$

Obviously $\sigma_{m-\varkappa}(\mathfrak{c}_*)$ are polynomials in $(c_1, \dots, c_j)$, we denote them by $\lambda_\varkappa(c_1, \dots, c_j)$. We can consider $\lambda_\varkappa(c_1, \dots, c_j)$ for any $c_1, \dots, c_j$, not necessarily $c_1, \dots, c_j$ satisfying the property that all $(\mathfrak{c}_1, \dots, \mathfrak{c}_m)$ are different.

**Remark 5.5.1.1.** (a) It is convenient (see (5.11.4)) to consider a polynomial $\mathfrak{P}(\mathfrak{Y}) = \mathfrak{P}_{F,w}(\mathfrak{Y}) := \prod_{\varkappa=1}^m (\mathfrak{Y} - \mathfrak{c}_\varkappa) = \sum_{\varkappa=0}^m \lambda_\varkappa \mathfrak{Y}^\varkappa$ where $\mathfrak{Y}$ is an abstract variable.[7] Here $\lambda_\varkappa = \lambda_\varkappa(c_1, \dots, c_j)$, i.e., strictly speaking, $\mathfrak{P}_{F,w}(\mathfrak{Y}) \in \mathbb{Q}(\mathfrak{z}_d)[c_1, \dots, c_j, \mathfrak{Y}]$.

(b) Polynomials $\lambda_\varkappa$ do not depend on the order $O$. Really, is we change $O$, then we get the corresponding change of the set $\mathfrak{c}_1, \dots, \mathfrak{c}_j$. Since $\lambda_\varkappa$ are symmetric polynomials in $\mathfrak{c}_1, \dots, \mathfrak{c}_j$, they do not depend on their order.

[7] This polynomial has nothing common with polynomials defining $X(m, i)$ and/or its irreducible components, as algebraic varieties. $\mathfrak{P}(\mathfrak{Y})$ will be used for calculation of the degrees of irreducible components of $X(m, i)$ in Section 5.11.

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**Definition 5.5.D.** For any $\xi = \{(c_1 : 1), (c_2 : 1), \dots, (c_j : 1)\} \in (\mathbb{P}^1)^j$ we define $\varphi(F, w)(\xi) \in \mathbb{P}^m(\mathbb{C})$ as $(\lambda_0(c_1, \dots, c_j) : \dots : \lambda_m(c_1, \dots, c_j))$.

We extend this definition to $(\mathbb{P}^1)^j$ from $(\mathbb{A}^1)^j$ by homogenization of polynomials $\lambda_\varkappa(c_1, \dots, c_j)$, see [GLZ], (5.0.3). See also [GLZ], 5.5.2 for examples.

**Theorem 5.6.** *$\forall$ $F, w$ we have: im $\varphi(F, w) \subset X(m, i)$.*

**Proof.** We fix a generic $c_* = (c_1, \dots, c_j) \in (\mathbb{A}^1)^j$, and let $\lambda_*$ be defined by (5.5.1) = (5.5.D). First, we need.

**Lemma 5.6.1.** *Let $u$ be any ramification node, $u'_1$, $u'_2$ its right neighbors, $\alpha, k$ as in (5.1.1.A), $c_\alpha \neq 0$. Then $\exists$ scalars $\beta_{1,u}$, $\beta_{2,u} \in \mathbb{C}$ such that*

$$\mathfrak{M}(m)(\lambda_0, \dots, \lambda_m) \cdot v(w_\mu(u) c_\alpha^{2^{d_\alpha - k}}) = \beta_{1,u} \cdot v(w_\mu(u'_1) c_\alpha^{2^{d_\alpha - k - 1}}) + \beta_{2,u} \cdot v(w_\mu(u'_2) c_\alpha^{2^{d_\alpha - k - 1}}) \tag{5.6.2}$$

**Proof.** We let $\mathfrak{b} = w_\mu(u'_1) c_\alpha^{2^{d_\alpha - k - 1}}$, hence $w_\mu(u'_2) c_\alpha^{2^{d_\alpha - k - 1}} = -\mathfrak{b}$ and $w_\mu(u) c_\alpha^{2^{d_\alpha - k}} = \mathfrak{b}^2$. We have: if numbers $y_1, y_2$ are roots to the system

$$\begin{aligned} \lambda_1 + \lambda_3 \mathfrak{b}^2 + \lambda_5 \mathfrak{b}^4 + \dots &= y_1 + y_2 \\ \lambda_0 + \lambda_2 \mathfrak{b}^2 + \lambda_4 \mathfrak{b}^4 + \dots &= \mathfrak{b} y_1 - \mathfrak{b} y_2 \end{aligned} \tag{5.6.3}$$

then values $\beta_{1,u} = y_1$, $\beta_{2,u} = y_2$ satisfy (5.6.2). System (5.6.3) has determinant $-2\mathfrak{b} \neq 0$. □

According (5.4.7), for generic $c_*$ numbers $w_\mu(u) c_\alpha^{2^{d_\alpha - k}}$ are different; it is easy to see that this is true for ramification nodes $u$ as well. We consider these $c_*$. For any node $u$ of $F$ we denote the column vector $v(w_\mu(u) c_\alpha^{2^{d_\alpha - k}})$ by $\chi_u$. They are linearly independent, there are $i$ such vectors (because $i$ is the quantity of nodes in the forest; we have one column vector for a node. The matrix formed by these column vectors is the transpose of a Vandermonde matrix. If all numbers $w_\mu(u) c_\alpha^{2^{d_\alpha - k}}$ are different then the rank of this matrix is $i$ (the quantity of columns; this is because the determinant of a square Vandermonde matrix with different second-column entries is non-zero), hence these vectors are linearly independent).

We order vectors $\chi_u$ according the value of $k(u)$ (see (5.1.1.A) for the definition of $k(u)$): if $k(u_1) < k(u_2)$ then the vector $\chi_{u_1}$ is preceding to $\chi_{u_2}$; if $k(u_1) = k(u_2)$ then the ordering of $\chi_{u_1}, \chi_{u_2}$ is arbitrary. Let $\psi_1, \dots, \psi_{m-i-1}$ be other vectors such that $\{\chi_u, \psi_*\}$ form a basis of $\mathbb{C}^{m-1}$.

We consider vectors $\{\chi_u, \psi_*\}$ as subdivided in two blocks: $\{\chi_1, \dots, \chi_i\}$ is the first block denoted by $\chi_*$, and $\psi_1, \dots, \psi_{m-i-1}$ is the second block denoted by $\psi_*$. Let us consider any linear operator on $\mathbb{C}^{m-1}$. This block subdivision of the basis defines a $2 \times 2$ block structure on the matrix of this linear operator in this basis.

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Let this linear operator be the linear operator $\mathfrak{M}$ on $\mathbb{C}^{m-1}$ defined by the matrix $\mathfrak{M}(m)(\lambda_0, \dots, \lambda_m)$ in the initial basis of $\mathbb{C}^{m-1}$. Hence, the matrix of $\mathfrak{M}$ in the basis $\{\chi_*, \psi_*\}$ is a $2 \times 2$ block matrix. Formulas (5.2), (5.3), (5.6.2) show that this matrix is block triangular, namely its $(2,1)$-block is 0. This is because the action of $\mathfrak{M}$ on any $\chi_u$ is a linear combination of other $\chi_*$ (formulas (5.2), (5.3), (5.6.2)) and not of $\psi_*$.

**5.6.4.** Further, the $(1,1)$-block of this matrix (corresponding to the action of $\mathfrak{M}$ on $\{\chi_*\}$) is strictly lower triangular (i.e. its diagonal entries are 0).

{This is because the image of any $\chi_u$ under the action of $\mathfrak{M}$ is a linear combination of some $\chi_{\tilde{u}}$ where:

For a non-final, non-ramification node $u$ we have: $\tilde{u}$ is $u'$ — the right neighbor of $u$ (see (5.3));

For a ramification node $u$ we have: $\tilde{u}$ consists of $u'_1$, $u'_2$ — the right neighbors of $u$ (see (5.6.2));

$\mathfrak{M}(\chi_u) = 0$ for a final node $u$, see (5.2). Since $k(u'),\ k(u'_1),\ k(u'_2) > k(u)$ we have that $\chi_{u'},\ \chi_{u'_1}, \chi_{u'_2}$ are to the right from the vector $\chi_u$ in the ordering of the basis $\{\chi_*, \psi_*\}$. This implies strict lower-triangularity.}

Finally, for any $2 \times 2$ block matrix such that its $(2,1)$-block is 0 and its (1,1)-block of size $i \times i$ is strictly lower triangular we have: $i$ lower coefficients of its characteristic polynomial (in degrees $0, \dots, i-1$) are 0. This implies the statement of the theorem for a generic $c_*$. Since $X(m,i)$ is closed, we get the desired. □

**Proposition 5.7.** *$\forall\ g \in Aut(F)$, $\mathfrak{g} \in \mathfrak{G}$ we have: if $w_2 = g(\mathfrak{g}(w_1))$ then* im $\varphi(F, w_1) =$ im $\varphi(F, w_2)$ *(as subsets of $X(m,i) \subset \mathbb{P}^m(\mathbb{C})$).*

**Proof.** It is sufficient to prove this proposition independently for the cases $w_2 = g(w_1)$, $w_2 = \mathfrak{g}(w_1)$. Let us consider the case $w_2 = g(w_1)$. In this case we have: for fixed $c_1, \dots, c_j$ the set of $\mathfrak{c}_1, \dots, \mathfrak{c}_m$ of $\{F, g(w_1), c_1, \dots, c_j\}$ is a permutation (induced by an automorphism $g$) of the set of $\mathfrak{c}_1, \dots, \mathfrak{c}_m$ of $\{F, w_1, c_1, \dots, c_j\}$, hence $\forall\ \varkappa$ the polynomials $\lambda_\varkappa(c_1, \dots, c_j)$ for $w_1$ and $w_2$ coincide.

Let $w_2 = \mathfrak{g}(w_1)$, where $\mathfrak{g} = (\mathfrak{g}_1, \dots, \mathfrak{g}_j) \in \mathfrak{G}$ from (4.5). Let $u$ be a non-final, non-ramification node, resp. a final node of $F$ and $\mathfrak{c}(u, w, c_1, \dots, c_j)$ be an element, resp. two elements of the set $\mathfrak{c}_1, \dots, \mathfrak{c}_m$ corresponding to the node $u$ and the weight $w$ according (5.4.4). Definition (5.4.4) and the multiplicative form of (4.5.1) (formula for the action of $\mathfrak{g}$ on weights) imply immediately that for the above $w_1,\ w_2,\ \mathfrak{g}$ we have

$$\mathfrak{c}(u, w_2, c_1, \dots, c_j) = \mathfrak{c}(u, w_1, \mathfrak{z}_{d_1}^{\mathfrak{g}_1} \cdot c_1, \dots, \mathfrak{z}_{d_j}^{\mathfrak{g}_j} \cdot c_j)$$

hence three maps: an isomorphism $\{c_\alpha \mapsto \mathfrak{z}_{d_\alpha}^{\mathfrak{g}_\alpha} \cdot c_\alpha\}$, and $\varphi(F, w_1)$, and $\varphi(F, w_2)$, form a commutative triangle. This means that im $\varphi(F, w_1) =$ im $\varphi(F, w_2)$. □

**Conjecture 5.8.** *$\forall\ F, w$ dim im $\varphi(F, w) = m - i$.*

**Remark 5.8.1.** The above conjecture is not surprising, because $j = m - i$, the source of $\varphi$ is $(\mathbb{P}^1)^j$ and $\varphi$ "cannot have a fibre of dimension $> 0$". This can be easily checked for any fixed $F$, $w$ (see, for example, [GLZ], Example 6.2.3), but a proof for all $F$, $w$ is too complicated.

**Corollary 5.9.** *Conjectures 2.1, 5.8 imply that* $\forall$ $F, w$ *we have: im* $\varphi(F, w)$ *is an irreducible component of* $X(m, i)$. $\square$

For fixed $i, j$ we denote by $\mathfrak{F}_{ij}$ the set of pairs $(F, w)$ for all $F$ whose $i, j$ are from (4.2.3), i.e. $F$ consists of $j$ trees and has $i$ nodes. We have:

**5.9.1.** Let us assume truth of Conjectures 2.1, 5.8. Under this assumption, Corollary 5.9 implies that there exists a map $\mathfrak{F}_{ij} \to Irr(X(m, i))$: $(F, w) \mapsto$ im $\varphi(F, w)$. We denote it by $\varphi_{ij}$.

**5.9.2.** Let $\bar{\mathfrak{F}}_{ij}$ be the quotient set of $\mathfrak{F}_{ij}$ by the equivalence relation of Proposition 5.7, i.e. $(F, w_1)$ is equivalent $(F, w_2)$ iff $\exists$ $g \in Aut(F)$, $\mathfrak{g} \in \mathfrak{G}$ such that $w_2 = g(\mathfrak{g}(w_1))$. The natural projection $\mathfrak{F}_{ij} \to \bar{\mathfrak{F}}_{ij}$ will be denoted by $\mathfrak{p}_{ij}$. Proposition 5.7 implies (here and below we assume the truth of Conjectures 2.1, 5.8) that $\varphi_{ij}$ factors throw $\bar{\mathfrak{F}}_{ij}$.

Let us recall some definitions of Conjecture 2.4 (see [GLZ] for details). There exist (abstract) sets $Irr(i, j)$ and injective maps [GLZ], (2.4.0)

$$\beta_{m,i,j} : Irr(i, j) \hookrightarrow Irr(X(m, i))$$

For $m = i+j$ the elements $\beta_{m,i,j}(Irr(i, j))$ are called the minimal irreducible components of $X(m, i)$, see [GLZ], 2.4.1a. We identify $Irr(i, j)$ with a subset of $Irr(X(m, i))$ via the inclusion $\beta_{m,i,j}$.

**Conjecture 5.10.** *We assume truth of Conjectures 2.1, 2.4, 5.8. In notations of (2.4), (2.5),* im $\varphi_{ij} = Irr(i, j) \subset Irr(X(m, i))$. *Hence, there exists a map* $\bar{\varphi}_{ij}$ *making the following diagram commutative:*

$$\begin{array}{ccc} \mathfrak{F}_{ij} & \xrightarrow{\varphi_{ij}} & Irr(X(m,i)) \\ \downarrow{\scriptstyle \mathfrak{p}_{ij}} & & \uparrow{\scriptstyle \beta_{m,i,j}} \\ \bar{\mathfrak{F}}_{ij} & \xrightarrow{\bar{\varphi}_{ij}} & Irr(i,j) \end{array} \tag{5.10.1}$$

*Moreover, we conjecture that* $\bar{\varphi}_{ij} : \bar{\mathfrak{F}}_{ij} \to Irr(i, j)$ *is an isomorphism.*

We see that Conjecture 5.10 describes the set $Irr(i, j)$ in combinatorial terms: it is the quotient set of $\mathfrak{F}_{ij}$ (a set of pairs {forest, weight}) by the equivalence relation of

Proposition 5.7 (see 5.9.2). Since for any $m$ the set $Irr(X(m,i))$ is a union of some $Irr(i,j)$, we get that Conjectures 5.10, 2.4 give a complete description of $Irr(X(m,i))$ in combinatorial terms.

**Remark 5.10.2.** (1) Particularly, $\forall\ F, w$ we have: im $\varphi(F,w)$ is a minimal component of $X(m,i)$ (because its dimension is $j$ which is $m-i$).

(2) Clearly Conjecture 5.10 has meaning if Conjecture 2.4 is true (if not then the sets $Irr(i,j)$ have no meaning).

(3) The origin of Conjecture 5.10 is purely experimental. We proved (Theorem 5.6) that im $\varphi(F,w) \subset X(m,i)$. Subsequent conjectures indicate that $\varphi(F,w)$ gives us irreducible components of $X(m,i)$. Computer calculations ([GLZ], Table A2.2) show that $\varphi(F,w)$ gives us:

(a) Only minimal irreducible components of $X(m,i)$ (i.e. not non-minimal components);
(b) All minimal irreducible components of $X(m,i)$;
(c) Any minimal irreducible component of $X(m,i)$ is obtained only once (up to the action of $Aut(F)$, $\mathfrak{G}$ on $\mathfrak{F}_{ij}$, see (5.7)).

This is exactly the contents of Conjecture 5.10.

(4) To prove Conjecture 5.10 we need to show that all generalized eigenvectors of $\mathfrak{M}(m)(\lambda_0,\dots,\lambda_m)$ with generalized eigenvalues 0 (here $(\lambda_0,\dots,\lambda_m)$ belongs to a minimal component of $X(m,i)$ ) have the form described in (5.1), for $x=$ one of $\mathfrak{c}_\alpha$. Also, we can use (2.8): namely, we must prove that for fixed $i,\ j$, for all possible $F$ having the given $i,\ j$, and for all their $w$, for the corresponding $\mathfrak{d}(C_{ij\mathfrak{k}})$, $\mu(C_{ij\mathfrak{k}})$ we have: (2.8) holds. This will mean that there is no other components. Taking into consideration that apparently there is no good description of $F$ and $w$ (see below), it will not be easy to realize this idea.

**5.11. Finding of the degree of** im $\varphi(F,w)$. First, we consider the case of one tree: $F=T$, hence $m=i+1$. We have one parameter $c=c_1$, and deg im $\varphi(T,w)$ is less than or equal to $\sum_{\alpha=1}^m \deg \mathfrak{c}_\alpha$ as a polynomial in $c$.

**Proposition 5.11.1.** $\sum_{\alpha=1}^m \deg \mathfrak{c}_\alpha = 2^d$.

**Proof.** The contribution of a node of depth $k$ to $\sum_{\alpha=1}^m \deg \mathfrak{c}_\alpha$ is:

$2^{d-k}$ $\left(= \deg \left(\sqrt{w_\mu(u)c^{2^{d-k}}}\right) \cdot \left(-\sqrt{w_\mu(u)c^{2^{d-k}}}\right)\right)$ for a final node;

$2^{d-k-1}$ $(= \deg\ w_\mu(u')c^{2^{d-k-1}})$ for a non-ramification non-final node;

0 for a ramification node.

The sum of these numbers over all nodes of $T$ is $2^d$, this is easily proved by induction by the quantity of branches of $T$. See [GLZ] for details. □

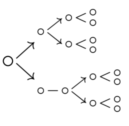
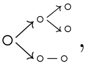
For further applications we shall need the exact value of $\lambda_0$ for any forest $F$.

**Lemma 5.11.1.1.** $\lambda_0 = (-1)^j \prod_{\alpha=1}^{j} c_\alpha^{2^{d_\alpha}}$.

**Proof.** The above proposition means that $\lambda_0 = \rho \prod_{\alpha=1}^{j} c_\alpha^{2^{d_\alpha}}$ where a constant $\rho = \rho(F, w)$ is defined as follows: $\rho = (-1)^l \prod_u w(u) \prod_{u'} w(u')$ where the first product runs over all final nodes of $F$, and the second product runs over all nodes $u'$ which are the right neighbors of non-final non-ramified nodes of $F$. This follows immediately from (5.4.4), (5.5.1). Further, for any tree $T$ and any its weight $w$ we have $\rho(T, w) = -1$. Really, this is obvious for a simple tree. If $T$ is any tree, $T'$ a tree obtained from $T$ by cutting a final branch, $w'$ the restriction of $w$ from $T$ to $T'$ then it is easy to check that $\rho(T, w) = \rho(T', w')$, hence we get the desired by induction. The proof for a forest follows immediately. See [GLZ] for details. □

Let us evaluate the degree of the covering $\varphi(T, w): P^1 \to \text{im } \varphi(T, w)$. We need the following

**Definition 5.11.2.** Let $T'$ be a tree. A tree $T$ obtained from $T'$ by replacing some final nodes of the maximal depth of $T'$ by complete trees (of some depths, probably different for different nodes) is called an extension of $T'$, and $T'$ is called a contraction of $T$. Number $\gamma := d(T) - d(T')$ is called the depth of the contraction.

**Example 5.11.3.** $T$ :  , $T'$ :  , $\gamma = 2$.

**Proposition 5.11.4.** *Let $T$ have a contraction $T'$ of depth $\gamma$. Then $\lambda_*$ from (5.5.1) are polynomials in $c^{2^\gamma}$.*

**Proof.** Non-ramified nodes of $T$ either have depth $< d - \gamma$, or — for some $\gamma'$ satisfying $0 \le \gamma' \le \gamma$ — are final nodes of complete subtrees of depth $\gamma'$ that replace some final nodes of depth $d - \gamma$ of $T'$. If $u$ is a non-ramified non-final node of $T$ of depth $< d - \gamma$ then the factor $\mathfrak{Y} - w(u)c^{2^{d-k-1}}$ of $\mathfrak{P}(\mathfrak{Y})$ (Remark 5.5.3) corresponding to $u$ contains $c$ in a power which is a multiple of $2^\gamma$. If $u$ is a final node of $T$ of depth $< d - \gamma$ then the same holds for the factor $(\mathfrak{Y} - \sqrt{w(u)c^{2^{d-k}}})(\mathfrak{Y} + \sqrt{w(u)c^{2^{d-k}}})$ of $\mathfrak{P}(\mathfrak{Y})$ corresponding to $u$. Let now $u'$ be a final node of depth $d - \gamma$ of $T'$ which is replaced by a complete tree $CT_{u'}$ of depth $\gamma'$ in $T$. We have ($u$ runs over the set of final nodes of $CT_{u'}$)

$$\prod_u (\mathfrak{Y} - \sqrt{w(u)c^{2^{\gamma-\gamma'}}}\ )\ (\mathfrak{Y} + \sqrt{w(u)c^{2^{\gamma-\gamma'}}}\ ) = \mathfrak{Y}^{2^{\gamma'+1}} - w(u')c^{2^\gamma}$$

We see that $c$ enters in all factors of $\mathfrak{P}(\mathfrak{Y})$ with a degree which is a multiple of $2^\gamma$, hence the proposition. See [GLZ] for details. □

**Corollary 5.11.5.** *If $T$ has a contraction of depth $\gamma$ then the degree of the covering $\varphi$ is $\ge 2^\gamma$, and hence deg im $\varphi(T, w)$ is $\le 2^{d-\gamma}$.*

**5.11.5a.** Let $T$ be fixed. Its contraction $T'$ such that the depth of $T'$ is the minimal possible (and hence the depth of contraction is the maximal possible) is called the minimal contraction of $T$.

**Conjecture 5.11.6.** *The degree of the covering $\varphi$ is $2^\gamma$, and hence deg im $\varphi(T, w)$ is $2^{d-\gamma}$, where $\gamma$ is the depth of the minimal contraction of $T$.*

Like for Remark 5.8.1, this is also "obvious" and can be easily checked for any fixed $T$, $w$.

**Corollary 5.11.7.** *Irreducible components of degree 1 (having $j = 1$) correspond to complete trees.*

See [GLZ], (6.4), (11.3), A7 for explicit examples.

We denote the (conjectural) degree of im $\varphi(T, w)$ by $\delta = 2^{d-\gamma}$. Let $\chi : P^1 \to P^\delta$ be the Veronese map (symmetric product). There exists a linear rational map $\omega : P^\delta \to P^{i+1}$ such that $\varphi(T, w) = \omega \circ \chi$.

**5.11.8.** Let us consider the case of arbitrary $F$. The above notations will bear an index $\alpha$ for a tree $T_\alpha$. Particularly, we have maps $\omega_\alpha \circ \chi_\alpha : P^1 \to P^{i_\alpha+1}$ whose image is $\varphi(T_\alpha, w_\alpha)$. We consider (see [GLZ] for details)

(a) their set-theoretic product

$\mathfrak{p} := (\omega_1 \circ \chi_1) \times (\omega_2 \circ \chi_2) \times \dots (\omega_j \circ \times \chi_j) : P^1 \times P^1 \times \dots \times P^1 \to P^{i_1+1} \times P^{i_2+1} \times \dots \times P^{i_j+1}$,

(b) the Segre map

$\psi : P^{i_1+1} \times P^{i_2+1} \times \dots \times P^{i_j+1} \to P^N$ $(N + 1 = \prod_\alpha (i_\alpha + 2)$ ) and

(c) We can choose a linear rational map $\omega : P^N \to P^m$ such that the map $\omega \circ \psi \circ \mathfrak{p} : P^1 \times P^1 \times \dots \times P^1 \to P^m$ is $\varphi(F, w)$ of (5.0).

Let us calculate deg im $\varphi(F, w)$. For any Segre map

$$s : P^{n_1} \times P^{n_2} \times \dots \times P^{n_j} \to P^N$$

we denote by $L_\alpha$ the class in $CH(P^{n_1} \times \dots \times P^{n_j})$ of $P^{n_1} \times \dots \times P^{n_{\alpha-1}} \times H_\alpha \times P^{n_{\alpha+1}} \dots \times P^{n_j})$, where $H_\alpha$ is a hyperplane in $P^{n_\alpha}$. Let $L$ be the class of the hyperplane of $P^N$. The Chow ring of $P^{n_1} \times P^{n_2} \times \dots \times P^{n_j}$ is $\mathbb{Z}$-generated by $L_1^{c_1} \cdot L_2^{c_2} \cdot \dots \cdot L_j^{c_j}$, where $c_\alpha \leq n_\alpha$. Let $CH(s)_*$ be the direct image map of Chow groups. We have:

$$CH(s)_*(L_1^{c_1} \cdot L_2^{c_2} \cdot \dots \cdot L_j^{c_j}) = \frac{(\sum_{\alpha=1}^{j} (n_\alpha - c_\alpha))!}{\prod_{\alpha=1}^{j} (n_\alpha - c_\alpha)!} \quad L^{N-(\sum_{\alpha=1}^{j} (n_\alpha - c_\alpha))} \tag{5.11.9}$$

For our situation we have $n_\alpha = i_\alpha + 1$, $c_\alpha = i_\alpha$, i.e. all $n_\alpha - c_\alpha$ are 1, hence the multinomial coefficient of (5.11.9) is $j!$, hence the degree of the map $\omega \circ \psi \circ \mathfrak{p}$ is $\prod_{\alpha=1}^{j} \delta_\alpha \cdot j!$

Isomorphisms between $(T_\alpha, w_\alpha)$ and $(T_{\alpha'}, w_{\alpha'})$ imply that $\omega \circ \psi \circ \mathfrak{p}$ is a finitely-sheeted covering, hence deg im $\varphi(F, w)$ is a divisor of $\prod_{\alpha=1}^{j} \delta_\alpha \cdot j!$ Namely, let us consider an equivalence relation on the set $T_1, \dots, T_j$: two elements $T_\alpha$, $T_{\alpha'}$ are equivalent iff there exists an isomorphism $\iota : T_\alpha \to T_{\alpha'}$ and $\mathfrak{g} \in \mathfrak{G}_\alpha$ such that $w_\alpha = \mathfrak{g}(w_{\alpha'} \circ \iota)$ (it is easy to check that this is really an equivalence relation).

Let $f_1, \dots, f_{j'}$ be quantities of elements in these equivalence classes. We get that the degree of the covering $\omega \circ \psi \circ \mathfrak{p} : (P^1)^j \to \text{im } \omega \circ \psi \circ \mathfrak{p}$ is $\geq f_1! \cdot \ldots \cdot f_{j'}!$

**Conjecture 5.11.10.** *This degree is equal to $f_1! \cdot \ldots \cdot f_{j'}!$, hence (taking into consideration Conjecture 5.11.6) we get:*

$$\text{deg im } \varphi(F, w) = \frac{2^{\sum_{\alpha=1}^{j} d_\alpha} \cdot j!}{2^{\sum_{\alpha=1}^{j} \gamma_\alpha} \cdot f_1! \cdot \ldots \cdot f_{j'}!} \tag{5.11.11}$$

All entries of [GLZ], Table A2.2 satisfy this conjecture. See [GLZ], (6.1.4), (6.2.2), (6.3.3), (6.4.1) for explicit examples.

**5.12.** Recall that in Section 3 we considered relations between $X(m, m-2)$ and its hyperplane section $X(m, m-1)$, see formula (3.1). Now we show that the correspondence $Y_i$ of (3.2) has a natural interpretation in terms of weighted forests. Roughly speaking, the inverse map $Y_i^{-1}$ corresponds to elimination of the root, see (4.6). This fact will be used for a deduction of the multiplication formula (5.13.3) from (5.13.2).

Let us give details. We denote epimorphisms $\bar{\varphi}_{ij} \circ \mathfrak{p}_{ij} : \mathfrak{F}_{ij} \to Irr(i, j)$ (see (5.10.1)) by $\tilde{\varphi}_{ij}$. Further, we define a map $\mathfrak{Z}_i : \mathfrak{F}_{i+1,1} \to \mathfrak{F}_{i1} \sqcup \mathfrak{F}_{i2}$ as follows. Let $r$ be the root of $T$. We use notations of (4.6).

(a) $r$ is a non-ramification node. We have $\mathfrak{Z}_i(T, w) = (\bar{T}, \tau(w)) \in \mathfrak{F}_{i1}$.
(b) $r$ is a ramification node. We have $\mathfrak{Z}_i(T, w) = (D_1(r) \sqcup D_2(r), \tau(w)) \in \mathfrak{F}_{i2}$.

**Proposition 5.12.1.** *Conjectures 0.2.4, 3.3, 5.10 imply that the diagram*

$$\begin{array}{ccccccc} \mathfrak{Z}_i : & \mathfrak{F}_{i+1,1} & \to & \mathfrak{F}_{i1} & \sqcup & \mathfrak{F}_{i2} \\ & \tilde{\varphi}_{i+1,1} \downarrow & & \tilde{\varphi}_{i1} \downarrow & & \tilde{\varphi}_{i2} \downarrow \\ Y_i^{-1} : & Irr(i+1, 1) & \to & Irr(i, 1) & \sqcup & Irr(i, 2) \end{array}$$

*is commutative.*

See [GLZ] for the proof. □

**Remark.** Without assuming truth of Conjecture 3.3, we can understand commutativity of the above diagram as follows: for any $(T, w) \in \mathfrak{F}_{i+1,1}$ we have: $\tilde{\varphi}_{i+1,1}(T, w) \in Y_i(\tilde{\varphi}_{i\alpha}(\mathfrak{Z}_i(T, w)))$, where $\alpha = 1$ if $\mathfrak{Z}_i(T, w) \in \mathfrak{F}_{i1}$, and $\alpha = 2$ if $\mathfrak{Z}_i(T, w) \in \mathfrak{F}_{i2}$.

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**Remark 5.12.2.** Conjecture 5.10 gives evidence that the converse correspondence $Y_i^{-1}$ : $Irr(i+1,1) \to Irr(i,1) \sqcup Irr(i,2)$ is a function (considered as a particular case of a correspondence). Really, according Conjecture 5.10, a fiber of $\tilde{\varphi}_{i+1,1}$ is a $\mathrm{Aut}(T) \times \mathfrak{G}$-orbit. Description of $\mathfrak{Z}_i$ shows that $\mathfrak{Z}_i$ of this orbit is contained in an orbit of the analogous group for $(\bar{T}, \tau(w))$ or $(D_1(r) \sqcup D_2(r), \tau(w))$. Hence, its $\tilde{\varphi}_{i1}$ or $\tilde{\varphi}_{i2}$-image is one point.

This is not a proof of the fact that Conjecture 5.10 implies Conjecture 3.3, because Proposition 5.12 depends on Conjecture 3.3. To get rid of a vicious circle, we need more work; this is a subject of further research.

**Remark 5.12.3.** (4.6.3) shows why $Y_i$ is not 1–1, i.e. $Y_i^{-1}$ is not injective. We have $\mathfrak{G}(F) = \mathfrak{G}(D_1(r)) \times \mathfrak{G}(D_2(r))$. $(\mathfrak{g}_1, \mathfrak{g}_2)$ of (4.6.3) belongs to $\mathfrak{G}(F)$, and $\tilde{\varphi}_{i2}(F, \tau(w)) = \tilde{\varphi}_{i2}(F, (\mathfrak{g}_1, \mathfrak{g}_2)(\tau(w)))$. The fiber $\tau^{-1}(\tau w)$ consists of two elements, and 4 elements $\tau^{-1}(\tau w)$, $\tau^{-1}((\mathfrak{g}_1, \mathfrak{g}_2)(\tau w))$ usually are not $\mathfrak{G}(T)$-equivalent.

Example for $i = 5$, see [GLZ], Table A2.2: $Y_5^{-1}(C_{618a}) = Y_5^{-1}(C_{618b}) = C_{522}$. This is the only example for $i \le 5$ of non-injectivity of $Y_i^{-1}$. We have $C_{618a}, C_{618b}$ are defined over $\mathbb{Q}[\sqrt{-1}]$, they are $\mathbb{Q}$-conjugate (i.e. $C_{618b} = \sigma(C_{618a})$ where $\sigma$ is the complex conjugation. This follows easily from the results of Section 5.15).

See [GLZ] for one more remark.

**5.13. Multiplicity formula.** Let $u$ be a ramification node of $T$. All its descendants form two trees denoted by $D_1(u)$, $D_2(u)$. We denote by $q_\alpha(u)$ the quantity of nodes of $D_\alpha(u)$ ($\alpha = 1, 2$).

**Conjecture 5.13.1.** *The multiplicity* $\mu(\mathrm{im}\ \varphi(F, w)) = \mu(F)$ *of* $\mathrm{im}\ \varphi(F, w)$ *depends only of* $F$*. It is given by the formulas:*

$$\mu(F) = \frac{i!}{i_1! \cdot i_2! \cdot \ldots \cdot i_j!} \prod_{\alpha=1}^{j} \mu(T_\alpha) \tag{5.13.2}$$

$$\mu(T) = \prod_u \binom{q_1(u) + q_2(u)}{q_1(u)} \tag{5.13.3}$$

*where* $u$ *runs over the set of all ramification nodes of* $T$ *(the empty product is 1).*

At it was mentioned above, we have no theoretical justification of (5.13.2), it is purely experimental, based on computer calculations, see [GLZ], Table A2.2. Explicit examples are given in [GLZ], (5.13.4), (6.1.4), (6.2.2), (6.3.3), (6.4.2), (6.4.3).

**Deduction of** (5.13.3) **from** (5.13.2) **and Conjectures 0.2.4, 5.10.** The below text is only an idea of this deduction. It should be considered as an argument supporting Conjecture 5.13.1.

We use induction: let (5.13.3) hold for all trees having $\le i$ nodes. Let $T$ be a tree having $i + 1$ nodes such that its root $r$ is a ramification node, and $w \in W(T)$. We have $\tilde{\varphi}_{i+1,1}(T,w) \in Irr(i+1,1)$. We consider $Y_i^{-1}(\tilde{\varphi}_{i+1,1}(T,w)) \in Irr(i,2)$. By 5.12.1, $Y_i^{-1}(\tilde{\varphi}_{i+1,1}(T,w)) = \tilde{\varphi}_{i2}(D_1(r) \sqcup D_2(r), \tau(w))$. By induction supposition and by (5.13.2),

$$\mu(\text{im } \varphi(D_1(r) \sqcup D_2(r), \tau(w))) = \frac{i!}{\#(D_1(r))!\#(D_2(r))!} \cdot \mu(\text{im } \varphi(D_1(r), \tau_1(w))) \cdot$$
$$\cdot \mu(\text{im } \varphi(D_2(r), \tau_2(w)))$$

Because of (3.4.3), we deduce that (5.13.3) holds for im $\varphi(T,w)$.

For the case of $T$ having $i+1$ nodes such that its root $r$ is a non-ramification node, (5.12.1) and (3.4.3) show immediately that truth of (5.13.3) for $\bar{T}$ implies truth of (5.13.3) for $T$. □

**5.14. Jordan form.**[8] We need more definitions. Let $T$ be a tree and $u$ its node. We denote by $R_u$ the subtree of $T$ formed by $u$ and all its descendants; $u$ is the root of $R_u$. The depth of $R_u$ is called the height of $u$, it is denoted by $h(u)$. Finally, like in the proof of Theorem 5.6, we can consider a matrix $\mathfrak{M}(m)(\lambda_0, \dots, \lambda_m)$ as an operator $\mathfrak{M}$ on $\mathbb{C}^{m-1}$.

**Lemma 5.14.1.** *Conjecture 5.10 implies: for any $F, w$, for generic $c_1, \dots, c_j$, for $(\lambda_0, \dots, \lambda_m) = \varphi(F,w)(c_1, \dots, c_j)$, for any $k$ we have dim* $\text{Ker}\, \mathfrak{M}^k$ *= the quantity of nodes $u$ of $F$ such that $h(u) \le k-1$.*

**Proof.** Let $u$ be a node such that $h(u) < k$. The proof of Theorem 5.6 shows that $\mathfrak{M}^k(\chi_u) = 0$. For generic $c_1, \dots, c_j$ vectors $\chi_u$ are linearly independent, hence for generic $c_1, \dots, c_j$ we have dim Ker $\mathfrak{M}^k \ge$ the quantity of nodes $u$ of $F$ such that $h(u) \le k-1$. Let us prove that we have an equality. Numbers $\beta_u$ from (5.3), $\beta_{1,u}$, $\beta_{2,u}$ from (5.6.2) are polynomials in $c_1, \dots, c_j$ and hence are $\ne 0$ for generic $c_1, \dots, c_j$. This means that if $u$ is a node of height $k$ then $\mathfrak{M}^k(\chi_u) \ne 0$. Hence, dim (Ker $\mathfrak{M}^k / \text{Ker}\, \mathfrak{M}^{k-1}$) $\ge$ {the quantity of nodes $u$ of $F$ such that $h(u) = k-1$}. This means that if for some $k_0$ dim Ker $\mathfrak{M}^{k_0} / \text{Ker}\, \mathfrak{M}^{k_0-1} >$ {the quantity of nodes $u$ of $F$ such that $h(u) = k-1$} then dim Ker $\mathfrak{M}^{d+1} = i'$ for some $i' > i$, and hence im $\varphi(F,w) \subset X(m,i')$. This contradicts to Conjecture 5.10. □

Let $M$ be any 0-Jordan matrix (i.e. a union of Jordan blocks of eigenvalue 0) of size $n$. It defines a partition of $n$ denoted by $\sigma$. There is a relation between the rank of powers of $M$ and the conjugate of $\sigma$:

**5.14.2.** If $n = p_1 + ... + p_k$ is the conjugate of $\sigma$ then the rank of $M^\gamma$ is $n - p_1 - ... - p_\gamma$, i.e. dim Ker $M^\gamma$ is $p_1 + ... + p_\gamma$ (here and below, by default, for any partition we assume $p_1 \ge ... \ge p_k$).

[8] See [GLZ] for a much more detailed exposition of the present section, with many examples.

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We give below explicit constructions of a partition corresponding to a forest, and conversely, forests corresponding to a partition. These constructions are combinatorial, see [GLZ] for their proofs (based on Lemma 5.14.1 and 5.14.2).

**5.14.3.** We shall need a notion of a partition of a partition. Let $\sigma : \{i = d_1 + ... + d_l\}$ be a partition of $i$. A partition of $\sigma$ is a representation of the set $\{1, \dots, l\}$ as a disjoint union: $\{1, \dots, l\} = \cup_{\alpha=1}^{j} Z_\alpha$. We denote $Z_\alpha = \{z_{\alpha 1}, \dots, z_{\alpha, l_\alpha}\}$, we let $i_\alpha := d_{z_{\alpha 1}} + ... + d_{z_{\alpha, l_\alpha}}$. Hence, we get partitions (here maybe $i_1 \ge ... \ge i_j$ does not hold):

$$\bar{\sigma} : \{i = i_1 + ... + i_j\}; \tag{5.14.3.1}$$

$$\sigma_\alpha : \{i_\alpha = d_{z_{\alpha 1}} + ... + d_{z_{\alpha, l_\alpha}}\} \tag{5.14.3.2}$$

We do not distinguish different unions $\{1, \dots, l\} = \bigcup_{\alpha=1}^{j} Z_\alpha$ if they give us isomorphic (up to order) $\bar{\sigma}$, $\sigma_\alpha$. For example, if $\sigma : \{i = d_1 + d_2 + d_2\}$ then unions $\{1, 2, 3\} = \{1, 2\} \cup \{3\}$ and $\{1, 2, 3\} = \{1, 3\} \cup \{2\}$ give us the same partition of $\sigma$.

Two extreme partitions corresponding to the cases $j = 1$ and $j = l$ are called the upper and lower trivial partitions respectively.

(5.14.3.3) We need also the opposite construction. If partitions $\sigma_\alpha$ from (5.14.3.2) are given then $\sigma$ is called their union.

**5.14.4. From a forest to a partition.** Here we describe $\pi_{m,i}(\text{im } \varphi(F, w))$ — the partition of the Jordan form of $\text{im } \varphi(F, w)$. Lemma 5.14.1 shows that it does not depend on $w$. We denote it by $\mathfrak{s}(F)$. First, we consider a case $j = 1$, i.e. $F = T$ is a tree. The corresponding partition $\sigma = \mathfrak{s}(T)$ is described as follows.

**Definition 5.14.4.0.** A ramification node $u$ is called a final ramification node if one or two of trees $D_1(u)$, $D_2(u)$ is (are) a simple tree. These simple trees are called the final branches. See [GLZ], 5.11.1a for examples.

We denote $T_{(1)} := T$. First, let $h_1$ be the minimal length of the final branches of $T_{(1)}$. Further, let $a_1$ be the quantity of the final ramification nodes having one or two final branches of length $h_1$. We write the end of $\sigma$ as follows:

$$i = ... + ... + ... \quad ... + \underbrace{h_1 + h_1 + ... + h_1}_{a_1 \text{ times}}.$$

Now, we cut off $h_1$ nodes from the end of all final branches of $T_{(1)}$. Particularly, if $u$ is a final ramification node such that both $D_1(u)$, $D_2(u)$ are simple trees of length $h_1$ then after this cutting $u$ will become a final node. If $u$ is a final ramification node such that only one of $D_1(u)$, $D_2(u)$ is a simple tree of length $h_1$ then after this cutting $u$ will become a simple (non-ramification, non-final) node.

We denote the obtained tree by $T_{(2)}$, and we repeat the procedure: we denote by $h_2$ the minimal length of the final branches of $T_{(2)}$, we denote by $a_2$ the quantity of the final ramification nodes of $T_{(2)}$ having one or two final branches of length $h_2$. We write the end of $\sigma$ as follows:

$$i = ... + ... + ... \quad ... + \underbrace{(h_1 + h_2) + (h_1 + h_2) + ... + (h_1 + h_2)}_{a_2 \text{ times}} + \underbrace{h_1 + h_1 + ... + h_1}_{a_1 \text{ times}}.$$

Now, we cut off $h_2$ nodes from the end of all final branches of $T_{(2)}$, etc. The value of $a_{max}$ is always 1. We shall have $\mathfrak{s}(T) = \{i = c_1 + ... + c_l\}$, where $c_1 = d - 1 = h_1 + h_2 + ... + h_{max}$, $c_l = h_1$ and $l =$ the quantity of final nodes.

An illustration of this process for the tree [GLZ], (4.1.1) is given in [GLZ].

The proof of truth of the above construction is straightforward. Therefore, we have

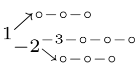
**Proposition 5.14.4.1.** *Let $F$ be forest, $w$ its weight,* im $\varphi(F, w)$ *its irreducible component belonging to $Irr(i, j)$, and $\{i = d_1 + ... + d_l\}$ its partition. We have: the present $l$ is $l$ of* (4.2.1), (4.2.3), *and the present $d_1$ is $d + 1$ where $d$ is from* (4.2.3). □

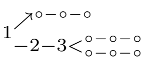
**5.14.5. From a partition to forests.** Let us consider the inverse process, i.e. we describe the set of forests $\mathfrak{s}^{-1}(\sigma)$ where

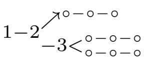
$$\sigma : \{i = \underbrace{c_1 + ... + c_1}_{a_1 \text{ times}} + \underbrace{c_2 + ... + c_2}_{a_2 \text{ times}} + ... + \underbrace{c_k + ... + c_k}_{a_k \text{ times}}\} \tag{5.14.5.1}$$

is a partition of $i$, where $c_1 > c_2 > ... > c_k$, and we let $c_{k+1} = 0$. First, we describe the subset $\mathfrak{s}^{-1}(\sigma)_1$ of trees in $\mathfrak{s}^{-1}(\sigma)$. We must have $a_1 = 1$: if $a_1 \neq 1$ then $\mathfrak{s}^{-1}(\sigma)_1 = \emptyset$.

Here numbering of $a_*$ and of $T_{(*)}$ is opposite with respect to the Section 5.14.4, because the process runs in the opposite direction. The tree $T_{(1)}$ is a simple tree of length $c_1 - c_2$. Now, we must choose $a_2$ nodes of $T_{(1)}$ (this can be made by $\binom{c_1 - c_2}{a_2}$ ways; if $c_1 - c_2 < a_2$ then $\mathfrak{s}^{-1}(\sigma)_1 = \emptyset$). To get $T_{(2)}$, we join $a_2$ simple trees of length $c_2 - c_3$ to chosen nodes of $T_{(1)}$, and one complementary simple tree of length $c_2 - c_3$ to the final node of $T_{(1)}$. Particularly, the final node of $T_{(1)}$ becomes a ramification node of $T_{(2)}$ iff it belongs to the set of chosen nodes. The below picture gives examples for $T_{(2)}$ for the case $c_1 - c_2 = 3,\ a_2 = 2,\ c_2 - c_3 = 3$ (nodes of $T_{(1)}$ are denoted by numbers 1 (root node), 2, 3 (final node)).


(nodes 1, 2 of $T_{(1)}$ are chosen);


(nodes 1, 3 of $T_{(1)}$ are chosen);


(nodes 2, 3 of $T_{(1)}$ are chosen).

To get $T_{(3)}$, we must choose $a_3$ nodes of $T_{(2)}$ which are not ramification nodes. $T_{(2)}$ has $c_1 - c_2$ nodes of $T_{(1)}$ and $(a_2 + 1)(c_2 - c_3)$ nodes joined while we extended $T_{(1)}$ to $T_{(2)}$. Also, $T_{(2)}$ has $a_2$ ramification nodes. Therefore, a choice can be made by

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$$\binom{(c_1-c_3)+a_2(c_2-c_3-1)}{a_3}$$

ways; if $(c_1-c_3)+a_2(c_2-c_3-1)<a_3$ then $\mathfrak{s}^{-1}(\sigma)_1=\emptyset$. To get $T_{(3)}$, we join $a_3$ simple trees of length $c_3-c_4$ to chosen nodes of $T_{(2)}$, and one complementary simple tree of length $c_2-c_3$ to each of the $a_2+1$ final nodes of $T_{(2)}$. Like above, a final node of $T_{(2)}$ becomes a ramification node of $T_{(3)}$ iff it belongs to the set of chosen nodes.

Continuing the process, we get that $\forall\ \gamma\ T_{(\gamma)}$ has

$$(c_1-c_{\gamma+1})a_1+(c_2-c_{\gamma+1})a_2+...+(c_\gamma-c_{\gamma+1})a_\gamma \text{ nodes},$$

$a_1+a_2+...+a_\gamma$ final nodes and $a_2+a_3+...+a_\gamma$ ramification nodes. We let $T=T_{(k)}$. We see that $\#(\mathfrak{s}^{-1}(\sigma)_1)$ is $\le$

$$\binom{c_1-c_2}{a_2}\binom{(c_1-c_3)+a_2(c_2-c_3-1)}{a_3}\binom{(c_1-c_4)+a_2(c_2-c_4-1)+a_3(c_3-c_4-1)}{a_4}\cdot$$
$$\dots\binom{(c_1-c_k)+a_2(c_2-c_k-1)+a_3(c_3-c_k-1)+...+a_{k-1}(c_{k-1}-c_k-1)}{a_k}$$

It can be less, because there are symmetries — see [GLZ], 5.14.5.2–5: it contains an example of the above construction for the partitions $\sigma_n:\{\binom{n+1}{2}=n+...+2+1\}$ for small $n$, as well as some other information.

**5.14.5.6.** Now we consider the general case. Let $\sigma$ be a partition. We consider all partitions $\{\sigma_\alpha\}$ of $\sigma$ (see (5.14.3.2)), and we consider $F=\cup T_\alpha$ (a disjoint union) where $T_\alpha\in\mathfrak{s}^{-1}(\sigma_\alpha)_1$. This $F$ belongs to $\mathfrak{s}^{-1}(\sigma)$, and all elements of $\mathfrak{s}^{-1}(\sigma)$ are obtained by this manner.

**Example 5.14.5.7.** The only $\sigma$ such that $\#(\mathfrak{s}^{-1}(\sigma))=1$ are partitions $\sigma_e(i,d):=\ \{i=d+d+...+d\}$ of equal parts. Really, $\mathfrak{s}^{-1}(\sigma)_1=\emptyset$ if $a_1$ of (5.14.5.1) is $\neq 1$. This means that the only partition of $\sigma_e(i,d)$ giving a forest is the lower trivial partition $\{d=d\},\ \{d=d\},\ \dots,\{d=d\}$. The above construction shows that $\mathfrak{s}^{-1}(\{d=d\})$ consists of one element — the simple tree of length $d$.

Obviously for other partitions $\sigma$ we have $\#(\mathfrak{s}^{-1}(\sigma))>1$. Really, if $\sigma:\{i=d_1+...+d_l\}$ is a partition and $d_k>d_{k+1}$ then $\sigma$ has at least two partitions: the lower trivial and a partition such that $Z_1=\{k,k+1\}$ and other $Z_\alpha$ consist of one element. We have $\mathfrak{s}^{-1}(\sigma_1)\neq\emptyset$, hence $\mathfrak{s}^{-1}(\sigma)$ contains at least two elements.

**5.15. Irreducible components, field(s) of definition and Galois orbits.**

Let $F$ be a fixed forest. We have $W(F)\subset\mathfrak{F}_{ij}$. We denote the set $\tilde\varphi_{ij}(W(F))$ (it is a subset of $Irr(i,j)$; see (5.12) for a definition of $\tilde\varphi_{ij}$) by $C(F)$. According (4.5), groups $\mathfrak{G}(F)\rtimes\mathrm{Aut}(F)$ and $\Gamma_d$ act on $W(F)$. Conjecture 5.10 affirms that the set of orbits of the action of $\mathfrak{G}(F)\rtimes\mathrm{Aut}(F)$ on $W(F)$ is isomorphic to $C(F)$. We shall give here some propositions concerning the action of $\Gamma_d$ on $C(F)$. Recall that an action of a group $G$ on a set $S$ is called a free action if $\forall\ g \in G,\ s \in S$ we have $\{g(s) = s\} \Rightarrow \{g = 1\}$, i.e. every orbit has $\#G$ elements.

Apparently, for a general case there is no simple description of Galois action on $C(F)$ (see [GLZ], Example 5.15.4), hence we give only some particular results.

First, let us consider the case $F = T$ a tree such that $\mathrm{Aut}(T) = 1$. We choose a longest simple subtree $B$ (a simple tree whose root coincides with the root of $T$, and whose final node is a final node of $T$) of $T$. We denote by $k$ the depth of the forest $T - B$.

**Proposition 5.15.1.** *Conjecture 5.10 implies that for $T$ such that $\mathrm{Aut}(T) = 1$, for all $w \in W(T)$ we have: $\mathrm{im}\ \varphi(T, w)$ is defined over $\mathbb{Q}(k+1)$ (see (4.5.4) for the notation). The action of $\Gamma_{k+1}$ on $C(T)$ is free.*

**Proof.** For any $w \in W(T)$ there exists the only element $\mathfrak{g} \in \mathfrak{G}(T)$ such that $\mathfrak{g}(w)_a$ is 0 on all nodes of $B$, i.e. the action of $\mathfrak{G}(T)$ on $W(T)$ is free. Hence, $\#C(T) = 2^{i-d-l}$. These $w \in W(T)$ (such that $w_a(u) = 0$ for all $u \in B$) form a set of representatives $R$ of orbits of the $\mathfrak{G}(T)$-action. For $w \in R$ we have:

numbers $w_\mu(u)$, $u \in T$, belong to $\mathbb{Q}(k+1)$;

$\Gamma_{k+1}$ acts on $R$, and this action is free — this is obvious.

Hence, $\Gamma_{k+1}$ acts freely on $C(T)$. □

An example of such $T$ is given in [GLZ] (6.3), see the tree (6.3.1) and the Remark 6.3.1a.

Let us consider now the case $F = T$, $\mathrm{Aut}(T)$ arbitrary. Let $T'$ be the minimal contraction of $T$ (see (5.11.5a)) such that the depth of $T'$ is $d' := d - \beta$, and $i'$, $l'$ the quantities of nodes, resp. final nodes of $T'$. Let $\varphi : W(T) \to W(T')$ be a map of forgetting the values of $w$ on nodes of $T - T'$. We have also natural surjections $\mathfrak{G}(T) \to \mathfrak{G}(T')$, $\mathrm{Aut}(T) \to \mathrm{Aut}(T')$. They define a map of the set of orbits $\bar{\varphi} : (\mathfrak{G}(T) \times \mathrm{Aut}(T))\backslash W(T) \to (\mathfrak{G}(T') \times \mathrm{Aut}(T'))\backslash W(T')$.

**Proposition 5.15.2.** *$\bar{\varphi}$ is an isomorphism; $\mathfrak{G}(T') \times \mathrm{Aut}(T')$ acts freely on $W(T')$. Hence, Conjecture 5.10 implies: the quantity of elements of the set $\tilde{\varphi}(T, W(T))$ is $2^{i'-d'-l'}/\#\,\mathrm{Aut}(T')$.*

**Proof.** First, we prove a lemma:

**Lemma 5.15.2.1.** *Let $T = T'$ be a non-contractible tree, $g \in \mathrm{Aut}(T)$, $\mathfrak{g} \in \mathfrak{G}(T)$, $w \in W(T)$. If $g(w) = \mathfrak{g}(w)$ then $g = 1$, $\mathfrak{g} = 0$ (i.e. $\mathfrak{G}(T) \times \mathrm{Aut}(T)$ acts freely on $W(T)$).*

**Proof.** Let us fix a final node $u$ of $T$ of the maximal depth $d$, and let $v = v(u, g)$ be the rightmost common predecessor of $u$, $g(u)$, i.e. a ramification node such that $u \in D_1(v)$, $g(u) \in D_2(v)$ (such $v$ exists and unique). Let $\gamma = \gamma(u, g)$ be the quantity of ramification

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nodes in the (only) way joining $v$ and $u$, including $v$ itself. We have $g^{2^\gamma}(u) = u$. This is proved by induction by $\gamma$. Really, $g$ interchanges $D_1(v)$ and $D_2(v)$, hence $g^2$ stabilizes both $D_1(v)$, $D_2(v)$. This means that $g^2(u) \in D_1(v)$, $\gamma(u, g^2) < \gamma(u, g)$, and the induction argument holds.

Replacing $w$ by $\tilde{\mathfrak{g}}(w)$ for some $\tilde{\mathfrak{g}} \in \mathfrak{G}$ we can assume that $w_a(u) = 0$, hence $w_a(v) = 0$. Let the depth of $v$ be $d - k$, obviously $k \geq \gamma$. We have $w_a(g(u)) \equiv 2^{d-k} \mod 2^{d-k+1}$ (because $w_a(v_2') = 2^{d-k}$ where $v_1' \in D_1(v)$, $v_2' \in D_2(v)$ are right neighbors of $v$).

We consider $\mathfrak{g}$ as an element in $\mathbb{Z}/2^d$. We have: $g(w) = \mathfrak{g}(w)$ implies $\mathfrak{g} = w_a(g(u))$. $g^{2^\gamma}(u) = u$ implies $2^\gamma \mathfrak{g} = 0$. This equality and $k \geq \gamma$, $\mathfrak{g} \equiv 2^{d-k} \mod 2^{d-k+1}$ imply $k = \gamma$, all nodes $g^\delta(u)$ for $0 \leq \delta < 2^\gamma$ are different and form a complete subtree of $T$ of node $v$ and depth $k$.

Let $\bar{u}$ be another final node of $T$ of the maximal depth $d$. Numbers $\bar{k} = \bar{\gamma}$ for $\bar{u}$ are the same, because $\mathfrak{g}$ is the same. If $k \neq 0$ we get a contradiction to the condition that $T$ is a non-contractible. □

So, if $T$ is non-contractible, then the group $\mathrm{Aut}(T) \times \mathfrak{G}(T)$ acts freely on $W(T)$, hence for this case (5.15.2) is proved. Let us consider the general case. We have a trivial

**Lemma 5.15.2.2.** *Let $T$ be a complete tree. In this case, the group $\mathrm{Aut}(T)$ acts simply transitively on $W(T)$ (see [GLZ] for a proof).* □

We denote by $\mathrm{Aut}(T/T')$ the kernel of the natural map $\mathrm{Aut}(T) \to \mathrm{Aut}(T')$. Lemma 5.15.2.2 implies that if $\varphi(w_1) = \varphi(w_2)$ then $\exists\ g \in \mathrm{Aut}(T/T')$ such that $w_2 = g(w_1)$. Using Lemma 5.15.2.1, we get the proposition. □

To consider the case of a forest, we need one trivial lemma more. Namely, let $G$ be any group acting on any set $\mathfrak{S}$, and $\eta$ a number. The group $G^\eta \rtimes S_\eta$ acts naturally on $\mathfrak{S}^\eta$ ($G^\eta$ acts on $\mathfrak{S}^\eta$ coordinatewise, and $S_\eta$ interchange coordinates in $\mathfrak{S}^\eta$).

**Lemma 5.15.3.** *In the above notations, let $\mathfrak{O}$ be the set of orbits of the action of $G$ on $\mathfrak{S}$. Then the set of orbits of the action of $G^\eta \rtimes S_\eta$ on $\mathfrak{S}^\eta$ is $S^\eta(\mathfrak{O})$ (the symmetric product) (see [GLZ] for a proof).* □

First, we apply this lemma to the case $F = T \sqcup T \sqcup ... \sqcup T$ — a disjoint union of $\eta$ copies of $T$. (5.15.2), (5.15.3) give us for this case a description of the set of orbits of the action of $\mathfrak{G}(F) \rtimes \mathrm{Aut}(F)$ on $W(F)$ and hence, by Conjecture 5.10, of the set $C(F)$.

For a general case $F = \sqcup_{\beta=1}^{\delta} \eta_\beta T_\beta$ where $T_\beta$ are different (see (4.2.2)), we have that the set of orbits of the action of $\mathfrak{G}(F) \rtimes \mathrm{Aut}(F)$ on $W(F)$ is the product of the corresponding sets for $F_\beta := \eta_\beta T_\beta$. This gives a complete description of $C(F)$.

There is no simple description of the Galois orbits, i.e. the action of $\Gamma_{k+1}$ on $C(F)$. See [GLZ], 5.15.4 for an example showing that Galois orbits of elements of $C(T)$ can be different for different $w \in W(T)$.

See [GLZ], Section 6 for examples on the above theory.

## 7. The second construction: from $\bar{C}_{ij*}$ to $C_{ij*}(m)$

We use notations of (2.5). The second construction shows how — starting from the minimal irreducible component $\bar{C}_{ij*}$ — we can get all components of its series.

For given $m > 0$, $\vartheta > 0$ we define a map $\nu = \nu_{m,\vartheta} : \mathbb{P}^m \times \mathbb{P}^\vartheta \to \mathbb{P}^{m+\vartheta}$ like a product of polynomials. Namely, let

$$((\lambda_0 : \lambda_1 : ... : \lambda_m); (b_0 : ... : b_\vartheta)) \in \mathbb{P}^m \times \mathbb{P}^\vartheta.$$

We associate $(\lambda_0 : \lambda_1 : ... : \lambda_m)$ with $P_1 := \sum_{i=0}^m \lambda_i x^i$ and $(b_0 : ... : b_\vartheta)$ with $P_2 := \sum_i b_i x^i$, then $\nu(\lambda_*; b_*)$ is associated with $P_1P_2$. Explicitly, the coordinates $(a_0 : ... : a_{m+\vartheta}) \in \mathbb{P}^{m+\vartheta}$ of $\nu(\lambda_*; b_*)$ are defined as follows: for $s \in [0, \dots, m+\vartheta]$ we let

$$a_s = \sum_{\gamma \in \mathbb{Z}} \lambda_\gamma b_{s-\gamma}$$

where $\lambda_* = 0$, resp. $b_* = 0$, if $* \notin [0, \dots, m]$, resp. $[0, \dots, \vartheta]$.

**Proposition 7.1.** *If Conjecture 0.2.4 is true*[9] *for $n = 1$ then the following holds:* $\forall\ m, i, \vartheta$

$$\nu(X(m, i) \times \mathbb{P}^\vartheta) \subset X(m + \vartheta, i) \tag{7.1.1}$$

**Proof.** We can represent the above $P_2$ as a product of linear polynomials: $P_2 = \prod_{\varkappa=1}^{\vartheta} P_{2\varkappa}$ where $P_{2\varkappa} = b_{0\varkappa} + b_{1\varkappa}x$. Now, we define inductively numbers $\lambda_{\varkappa\gamma}$ as follows:

$$\lambda_{0\gamma} = \lambda_\gamma, \quad \gamma = 0, \dots, m$$

$$\nu_{m+\varkappa,1}((\lambda_{\varkappa 0} : ... : \lambda_{\varkappa,m+\varkappa}); (b_{0,\varkappa+1} : b_{1,\varkappa+1})) = (\lambda_{\varkappa+1,0} : ... : \lambda_{\varkappa+1,m+\varkappa+1})$$

We have $\nu_{m,\theta}((\lambda_0 : \lambda_1 : ... : \lambda_m); (b_0 : ... : b_\vartheta)) = (\lambda_{\vartheta 0} : ... : \lambda_{\vartheta,m+\vartheta})$. This implies that if (7.1.1) holds for $\vartheta = 1$ then it holds for any $\vartheta$. Hence, we shall prove the proposition for $\vartheta = 1$.

Let us consider a point $(\lambda_0 : ... : \lambda_m) \in X(m, i)$, a point $(b_0 : b_1) \in \mathbb{P}^1$, and let $\nu((\lambda_0 : ... : \lambda_m), (b_0 : b_1)) = (a_0 : ... : a_{m+1}) \in \mathbb{P}^{m+1}$.

We have: the matrix $\mathfrak{M}(m + 1)(a_0, \dots, a_{m+1})$ becomes $\mathcal{M}_{nt}(\lambda_0, \dots, \lambda_m;\ 1, m)$ (see (0.1.8) for the notation) after the substitution $b_0 \to t$, $b_1 \to -1$, hence

$$D(m + 1, i)(a_0 : ... : a_{m+1}) = \sum_{\mathrm{j}=0}^{m-i} (-1)^{m-i-\mathrm{j}} H_{i\mathrm{j},21}(\lambda_0, \dots, \lambda_m)\ b_0^{\mathrm{j}} b_1^{m-i-\mathrm{j}} \tag{7.1.2}$$

This implies the proposition. □

[9] The reader can think that Conjecture 0.2.4 is in characteristic 2. Really, it is over $\mathbb{C}$, see Section 0.2.

**Remark 7.2.** The idea used in the above proof of (7.1) also can be used for a proof of the following (7.2.1):

**Proposition 7.2.1.** *If Conjecture 0.2.4 holds for $n = 1$ for all $m$, then it holds for all $n$.*

**Proof.** The matrix $\mathfrak{M}(m+\vartheta)(a_0, \dots, a_{m+\vartheta})$ becomes $\mathcal{M}_{nt}(\lambda_0, \dots, \lambda_m; \vartheta; m+\vartheta-1)$ after the substitution $b_\mathfrak{j} \to (-1)^\mathfrak{j}\binom{\vartheta}{\mathfrak{j}}t^{\vartheta-\mathfrak{j}}$, hence

$$D((m+\vartheta), i)(a_0, \dots, a_{m+\vartheta}) = \sum_{\mathfrak{J}} \tilde{H}_{\mathfrak{J},\vartheta}(\lambda_0, \dots, \lambda_m)b^{\mathfrak{J}}$$

where $\mathfrak{J} = (\mathfrak{j}_0, \dots, \mathfrak{j}_\vartheta)$ is a multiindex satisfying $\sum_{\varkappa=0}^{\vartheta} \mathfrak{j}_\varkappa = m+\vartheta-1-i$, $b^{\mathfrak{J}} := \prod_{\varkappa=0}^{\vartheta} b_\varkappa^{\mathfrak{j}_\varkappa}$ and $\tilde{H}_{\mathfrak{J},\vartheta}$ are some polynomials. This implies that $H_{i\mathfrak{j},2\vartheta}$ are linear combinations of the corresponding $\tilde{H}_{\mathfrak{J},\vartheta}$.

If Conjecture 0.2.4 holds for $n = 1$ for all $m$, then (7.1.1) holds, and hence some powers of $\tilde{H}_{\mathfrak{J},\vartheta}$ belong to $< D(m,0), \dots, D(m, i-1) >$. Therefore, the same holds for $H_{i\mathfrak{j},2\vartheta}$. □

Let us fix $i$, $j$ and an element $C_{ij\mathfrak{k}} \in Irr(i,j)$. We consider $\bar{C}_{ij\mathfrak{k}} \subset X(i+j, i) \subset \mathbb{P}^{i+j}$ from (2.5), and let $m \geq i+j$ be arbitrary. We let $\vartheta := m - (i+j)$.

**Conjecture 7.3.** $C_{ij\mathfrak{k}}(m) = \nu_{i+j,\vartheta}(\bar{C}_{ij\mathfrak{k}} \times \mathbb{P}^\vartheta)$.

This conjecture should be understood as follows. According Proposition 7.1, we get that $\nu_{i+j,\vartheta}(\bar{C}_{ij\mathfrak{k}} \times \mathbb{P}^\vartheta)$ is (conjecturally) an irreducible component of $X(m,i)$. We conjecture that if we denote this irreducible component of $X(m,i)$ by $C_{ij\mathfrak{k}}(m)$ then all affirmations of (2.4) hold.

For example, we have the following

**7.4. Justification of** (2.4.2.1)**.** We have $\dim \bar{C}_{ij\mathfrak{k}} = j$, $\dim \nu_{i+j,\vartheta}(\bar{C}_{ij\mathfrak{k}} \times \mathbb{P}^\vartheta) = j+\vartheta = m-i$, hence

$$\begin{aligned}&\deg \nu_{i+j,\vartheta}(\bar{C}_{ij\mathfrak{k}} \times \mathbb{P}^\vartheta) = \\ &\quad = \deg \bar{C}_{ij\mathfrak{k}} \cdot \text{ (the degree of the Segre embedding } \mathbb{P}^j \times \mathbb{P}^\vartheta \to \mathbb{P}^{j\cdot\vartheta+j+\vartheta}\text{ )}\end{aligned}$$

Since the degree of the Segre embedding is $\binom{m-i}{j}$ we get (2.4.2.1).

Proof (or, at least, a justification) of the fact that the multiplicity and the Jordan form of $\nu(\bar{C}_{ij\mathfrak{k}} \times \mathbb{P}^\vartheta)$ coincide with the ones of $\bar{C}_{ij\mathfrak{k}}$, as well as other affirmations of Conjectures 2.4, is a subject of future research. A simple particular case is treated in [GLZ], (11.3.4).

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