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THEORY

Creation of Neutral Fermions with Anomalous Magnetic Moment from the Vacuum by Magnetic Steps

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Abstract—We present recent results on neutral fermion pair production by magnetic field inhomogeneities as external backgrounds. Vacuum instability characteristics are calculated in the framework of QED with x-steps and specified to a magnetic step that allows solving the relativistic wave equation pertinent to this problem.

Keywords: quantum electrodynamics, Dirac—Pauli equation, nonperturbative effects

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1. INTRODUCTION

According to the usual interpretation, pair production by external fields is intimately associated with the possibility of such backgrounds producing work on virtual pairs of charged particles [1—15]. This raises the question as to whether inhomogeneous magnetic fields, which, contrary to homogeneous magnetic fields, produce work on particles with a magnetic moment, can actually create pairs from the vacuum. The answer to this question is affirmative, provided the particles are neutral and have an anomalous magnetic moment. This work presents our recent results on neutral fermion pair production by magnetic field inhomogeneities [16]. Our study is based on the quantization of fermion fields in terms of neutral particles/antiparticles, whose states have well-defined spin polarizations [17] and for which a nonperturbative formulation in QED [18, 19] can be used. To employ this formulation—which relies on the possibility of solving the Dirac—Pauli equation exactly—we consider an external magnetic field given by an analytic function; see Section 2 for its definition. We consider the four-dimensional Minkowski spacetime, parameterized by coordinates $X = (X^\mu,\ \mu = 0, i) = (t, \mathbf{r})$, $t = X^0$, $\mathbf{r} = X^i = (x, y, z)$, $i = 1, 2, 3$, metric tensor $\eta_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)$, and employ natural units ($\hbar = 1 = c$).

2. SOLUTIONS OF THE DIRAC-PAULI EQUATION WITH STEPLIKE MAGNETIC FIELDS

The motion of a relativistic spin 1/2 neutral particle with anomalous magnetic moment μ, mass m, in external electromagnetic fields is described by the Dirac—Pauli (DP) equation [20, 21]. Considering steplike magnetic fields[1)] (or magnetic steps) as

[1)]Time-independent magnetic fields oriented along a specific direction (say, z-direction), inhomogeneous along another direction (say, y-direction), $\mathbf{B}(\mathbf{r}) = (0, 0, B_z(y))$, homogeneous at remote distances, $B_z(\pm\infty) = \text{const.}$, and whose gradient is always positive $\partial_y B_z(y) \geq 0$ (thus, $B_z(+\infty) > B_z(-\infty)$).

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external backgrounds $\mathbf{B}(\mathbf{r}) = (0, 0, B_z(y))$, the DP equation in the Schrödinger form reads

$$i\partial_t \psi(X) = \hat{H}\psi(X), \quad \hat{H} = \gamma^0 \left(\gamma^3 \hat{p}_z + \Sigma_z \hat{\Pi}_z\right),$$
$$\hat{\Pi}_z = \Sigma_z (\boldsymbol{\gamma}\hat{\mathbf{p}}_\perp + m) - \mathbb{I}\mu B_z(y),$$
$$\Sigma_z = i\gamma^1\gamma^2. \tag{1}$$

$\psi(X)$ is a four spinor, $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ are Dirac matrices, $\mathbb{I}$ is the 4×4 identity matrix, and μ is the algebraic value of the magnetic moment (e.g., $\mu = -|\mu_N|$ for a neutron). The set of operators $\hat{p}_0, \hat{p}_x, \hat{p}_z, \hat{\Pi}_z$, and $\hat{R} = \hat{H}\hat{\Pi}_z^{-1}\left[\mathbb{I} + \left(\hat{p}_z\hat{\Pi}_z^{-1}\right)^2\right]^{-1/2}$ are compatible and are integrals of motion (i.e., they commute with the Hamiltonian operator (1)). In particular, DP spinors obey the eigenvalue equations:

$$\hat{p}_0\psi_n(X) = p_0\psi_n(X), \quad \hat{p}_x\psi_n(X) = p_x\psi_n(X),$$
$$\hat{p}_z\psi_n(X) = p_z\psi_n(X),$$
$$\hat{\Pi}_z\psi_n(X) = s\omega\psi_n(X), \quad \hat{R}\psi_n(X) = s\psi_n(X),$$
$$p_0^2 = \omega^2 + p_z^2, \quad s = \pm 1. \tag{2}$$

Solutions to the eigenvalue equations can be presented in the form $\psi_n(X) = \exp(-ip_0t + ip_xx + ip_zz)\psi_n(y)$, where

$$\psi_n(y)$$
$$= (\mathbb{I} + sR)\left[\hat{\pi}_z + \mathbb{I}(\mu B_z(y) + s\omega)\right]\varphi_{n,\chi}(y)\, v_\kappa^{(\chi)},$$
$$\hat{\pi}_z = \Sigma_z\left(\gamma^1 p_x + \gamma^2\hat{p}_y + m\right),$$
$$R = \frac{\gamma^0\Sigma_z + (sp_z/\omega)\gamma^0\gamma^3}{\sqrt{1 + p_z^2/\omega^2}}, \quad i\gamma^1 v_\kappa^{(\chi)} = \chi v_\kappa^{(\chi)},$$
$$\gamma^0\gamma^2 v_\kappa^{(\chi)} = \kappa v_\kappa^{(\chi)}, \quad \chi = \pm 1 = \kappa. \tag{3}$$

Here, $n = (p_x, p_z, \omega, s)$ labels the complete set of quantum numbers, $v_\kappa^{(\chi)}$ are constant and orthonormal spinors, $v_{\kappa'}^{(\chi')\dagger} v_\kappa^{(\chi)} = \delta_{\chi'\chi}\delta_{\kappa'\kappa}$, and $\varphi_{n,\chi}(y)$ a scalar function, solution to the second-order ordinary differential equation

$$\left\{-\frac{d^2}{dy^2} - [s\omega + \mu B_z(y)]^2 + \pi_x^2 + i\mu\chi B_z'(y)\right\}$$
$$\times \varphi_{n,\chi}(y) = 0, \quad \pi_x^2 = m^2 + p_x^2. \tag{4}$$

To quantize fermion fields in the framework of QED with x-steps [18, 19], one needs to classify DP spinors based on their asymptotic properties. At remote areas—where the field can be considered homogeneous and do not accelerate particles—the coefficient proportional to χ in (4) is absent. Therefore, solutions of Eq. (4) have well-defined "left" (L) ${}_\zeta\varphi_{n,\chi}(y)$ and "right" (R) ${}^\zeta\varphi_{n,\chi}(y)$ asymptotic forms

$${}_\zeta\varphi_{n,\chi}(y) = {}_\zeta\mathcal{N}\exp\left(i\zeta\left|p^{\mathrm{L}}\right|y\right),$$
$$\zeta = \mathrm{sgn}\left(p^{\mathrm{L}}\right), \quad y \to -\infty,$$
$${}^\zeta\varphi_{n,\chi}(y) = {}^\zeta\mathcal{N}\exp\left(i\zeta\left|p^{\mathrm{R}}\right|y\right),$$
$$\zeta = \mathrm{sgn}\left(p^{\mathrm{R}}\right), \quad y \to +\infty, \tag{5}$$

with real asymptotic momenta $\left|p^{\mathrm{L/R}}\right| = \sqrt{\left[s\pi_s(\mathrm{L/R})\right]^2 - \pi_x^2}$ provided $\left[\pi_s(\mathrm{L/R})\right]^2 > \pi_x^2$, $\pi_s(\mathrm{L/R}) = \omega - sU_{\mathrm{L/R}}$. The potential energy of a fermion in the field is $sU(y)$, $U(y) = -\mu B_z(y)$, and $U_{\mathrm{L}} = U(-\infty)$, $U_{\mathrm{R}} = U(+\infty)$ denote the corresponding asymptotic potential energies. For a fermion with negative magnetic moment $\mu = -|\mu|$, the difference $\mathbb{U} \equiv U_{\mathrm{R}} - U_{\mathrm{L}}$ is always positive.

The normalization constants ${}_\zeta\mathcal{N}$, ${}^\zeta\mathcal{N}$ can be calculated with the aid of the inner product on the time-like hyperplane $y = $ const., namely $(\psi, \psi')_y = \int dtdxdz\psi^\dagger(X)\gamma^0\gamma^2\psi'(X)$. Assuming that all processes take place in a macroscopically large space-time box, of volume TV_y, $V_y = L_xL_z$, and imposing periodic boundary conditions at the boundaries, the inner product is y-independent and the DP spinors may be subjected to the normalization conditions

$$\left({}_{\zeta'}\psi_{n'}, {}_\zeta\psi_n\right)_y = \zeta\eta_{\mathrm{L}}\delta_{n'n}\delta_{\zeta'\zeta},$$
$$\left({}^{\zeta'}\psi_{n'}, {}^\zeta\psi_n\right)_y = \zeta\eta_{\mathrm{R}}\delta_{n'n}\delta_{\zeta'\zeta}, \tag{6}$$

where $\eta_{\mathrm{L/R}} = \mathrm{sgn}\left[\pi_s(\mathrm{L/R})\right]$. Under these conditions, "left" ${}_\zeta\psi_n(X)$ and "right" ${}^\zeta\psi_n(X)$ sets of DP spinors are orthogonal and complete, and it allows expanding one set in terms of another as follows

$${}^\zeta\psi_n(X) = \eta_{\mathrm{L}}\sum_{\zeta'=\pm}\zeta' g\left({}_{\zeta'}|{}^\zeta\right){}_{\zeta'}\psi_n(X),$$
$${}_\zeta\psi_n(X) = \eta_{\mathrm{R}}\sum_{\zeta'=\pm}\zeta' g\left({}^{\zeta'}|{}_\zeta\right){}^{\zeta'}\psi_n(X), \tag{7}$$

in which $\left({}_\zeta\psi_n, {}^{\zeta'}\psi_{n'}\right)_y = \delta_{nn'}g\left({}_\zeta|{}^{\zeta'}\right) = \delta_{nn'}g\left({}^{\zeta'}|{}_\zeta\right)^*$.

To explicitly calculate vacuum instability characteristics (due to neutral fermion pair production from the vacuum), we consider the external field

$$B_z(y) = \varrho B'\tanh(y/\varrho), \quad B' > 0, \quad \varrho > 0, \tag{8}$$

as it enjoys the properties above discussed and allows solving the DP equation exactly. Exact solutions of Eq. (4) with "left" and "right" asymptotic properties (5) have the form [16]

$${}_\zeta\varphi_{n,\chi}(y) = {}_\zeta\mathcal{N}\exp\left(i\zeta\left|p^{\mathrm{L}}\right|y\right)$$

$$\times \left[1+\exp\left(2y/\varrho\right)\right]^{-i\varrho\left(\zeta|p^{\mathrm{L}}|+|p^{\mathrm{R}}|\right)/2}\,{}_{\zeta}u\left(\xi\right),$$
$${}^{\zeta}\varphi_{n,\chi}\left(y\right)={}^{\zeta}\mathcal{N}\exp\left(i\zeta\left|p^{\mathrm{R}}\right|y\right)$$
$$\times\left[1+\exp\left(-2y/\varrho\right)\right]^{i\varrho\left(|p^{\mathrm{L}}|+\zeta|p^{\mathrm{R}}|\right)/2}\,{}^{\zeta}u\left(\xi\right), \quad (9)$$

where ${}_{-}u\left(\xi\right)=F\left(a,b;c;\xi\right)$, ${}_{+}u\left(\xi\right)=F(a+1-c, b+1-c;2-c;\xi)$, ${}^{-}u\left(\xi\right)=F(a,b;a+b+1-c; 1-\xi)$, and ${}^{+}u\left(\xi\right)=F(c-a,c-b;c+1-a-b; 1-\xi)$ are Hypergeometric functions [22], whose parameters are $a=(1-\chi)/2-i\varrho(\mathbb{U}+|p^{\mathrm{L}}|-|p^{\mathrm{R}}|)/2$, $b=(1+\chi)/2+i\varrho(\mathbb{U}+|p^{\mathrm{R}}|-|p^{\mathrm{L}}|)/2$, and $c=1-i\varrho|p^{\mathrm{L}}|$.

3. NEUTRAL FERMION PAIR CREATION FROM THE VACUUM

The condition that solutions have physical asymptotic momenta $p^{\mathrm{L/R}}$ introduces important limitations on the quantum numbers. For critical fields $\mathbb{U}>\mathbb{U}_{\mathrm{c}}=2m$, the manifold of quantum numbers divides into five subranges [18]. Particle creation takes place only in the so-called Klein zone Ω_3,

$$\Omega_3=\{n:U_{\mathrm{L}}+\pi_x\le s\omega\le U_{\mathrm{R}}-\pi_x,$$
$$\pi_{xz}\le\mathbb{U}/2\},\quad \pi_{xz}=\sqrt{\pi_x^2+p_z^2}. \quad (10)$$

The quantization is realized using exact solutions classified as particle/antiparticle and as incoming/outgoing waves. After a careful consideration of the inner product on t-constant hyperplane $(\psi_n,\psi'_{n'})=\int d\mathbf{r}\psi_n^{\dagger}(X)\psi'_{n'}(X)$, it follows that linearly independent sets of spinors are classified as [18]:

$$\text{in-solutions}: {}_{-}\psi_n\left(X\right), {}^{-}\psi_n\left(X\right),$$
$$\text{out-solutions}: {}_{+}\psi_n\left(X\right), {}^{+}\psi_n\left(X\right),\quad n\in\Omega_3. \quad (11)$$

Based on this classification, we may introduce "in" and "out" sets of operators

$$\text{in-set: } {}_{-}b_{n_3}\left(\text{in}\right), {}^{-}a_{n_3}\left(\text{in}\right),$$
$$\text{out-set: } {}_{+}b_{n_3}\left(\text{out}\right), {}^{+}a_{n_3}\left(\text{out}\right), \quad (12)$$

which, in turn, obey the anticommutation relations $\left[{}^{-}a_{n'_3}\left(\text{in}\right), {}^{-}a^{\dagger}_{n_3}\left(\text{in}\right)\right]_{+}=\left[{}_{-}b_{n'_3}\left(\text{in}\right), {}_{-}b^{\dagger}_{n_3}\left(\text{in}\right)\right]_{+}=\delta_{n'_3n_3}$, $\left[{}^{+}a_{n'_3}\left(\text{out}\right), {}^{+}a^{\dagger}_{n_3}\left(\text{out}\right)\right]_{+}=\left[{}_{+}b_{n'_3}\left(\text{out}\right), {}_{+}b^{\dagger}_{n_3}\left(\text{out}\right)\right]_{+}=\delta_{n'_3n_3}$, and annihilate the corresponding vacuum states

$${}_{-}b_{n_3}\left(\text{in}\right)|0,\text{in}\rangle={}^{-}a_{n_3}\left(\text{in}\right)|0,\text{in}\rangle=0,$$
$${}_{+}b_{n_3}\left(\text{out}\right)|0,\text{out}\rangle={}^{+}a_{n_3}\left(\text{out}\right)|0,\text{out}\rangle=0. \quad (13)$$

Thus, we may quantize the DP field operator in the Klein zone as

$$\hat{\Psi}\left(X\right)=\sum_{n\in\Omega_3}\mathcal{M}_n^{-1/2}\Big[{}^{-}a_n\left(\text{in}\right){}^{-}\psi_n\left(X\right)$$
$$+{}_{-}b_n^{\dagger}\left(\text{in}\right){}_{-}\psi_n\left(X\right)\Big],$$
$$=\sum_{n\in\Omega_3}\mathcal{M}_n^{-1/2}\Big[{}^{+}a_n\left(\text{out}\right){}^{+}\psi_n\left(X\right)$$
$$+{}_{+}b_n^{\dagger}\left(\text{out}\right){}_{+}\psi_n\left(X\right)\Big], \quad (14)$$

where $\mathcal{M}_n=2\left|g\left({}_{+}|^{-}\right)\right|^2\mathcal{C}_n$ and $\mathcal{C}_n$ are some constants; see [16, 18, 19] for their definition. Using orthogonality relations between DP spinors, it is possible to derive linear canonical transformations between sets of operators. For example, one (out of four) of such thansformations have the form

$${}^{+}a_n\left(\text{out}\right)=-g\left({}_{-}|^{+}\right)^{-1}{}_{-}b_n^{\dagger}\left(\text{in}\right)$$
$$+g\left({}^{-}|_{+}\right)^{-1}g\left({}^{+}|_{+}\right){}^{-}a_n\left(\text{in}\right). \quad (15)$$

Using such a transformation, we may finally compute the differential mean numbers of "out" particles created from the "in" vacuum,

$$N_n^{\mathrm{cr}}=\left\langle 0,\text{in}\left|{}^{+}a_n^{\dagger}\left(\text{out}\right){}^{+}a_n\left(\text{out}\right)\right|\text{in},0\right\rangle$$
$$=\left|g\left({}_{-}|^{+}\right)\right|^{-2},\quad n\in\Omega_3, \quad (16)$$

the flux density of particles created with a given s

$$n_s^{\mathrm{cr}}=\frac{1}{V_yT}\sum_{n\in\Omega_3}N_n^{\mathrm{cr}}$$
$$=\frac{1}{(2\pi)^3}\int dp_z\int dp_x\int dp_0N_n^{\mathrm{cr}} \quad (17)$$

and the vacuum-vacuum transition probability

$$P_v=|\langle 0,\text{out}|0,\text{in}\rangle|^2$$
$$=\exp\left[\sum_{s=\pm1}\sum_{n\in\Omega_3}\ln\left(1-N_n^{\mathrm{cr}}\right)\right]. \quad (18)$$

It should be noted that $n_{+1}^{\mathrm{cr}}=n_{-1}^{\mathrm{cr}}$; thus, the total flux density of particles created is $n^{\mathrm{cr}}=n_{+1}^{\mathrm{cr}}+n_{-1}^{\mathrm{cr}}$. The exact differential mean number of pairs created from the vacuum by the magnetic step (8) has the form [16]

$$N_n^{\rm cr} = \frac{\sinh\left(\pi\varrho\left|p^{\rm R}\right|\right)\sinh\left(\pi\varrho\left|p^{\rm L}\right|\right)}{\sinh\left[\pi\varrho\left(\mathbb{U}+\left|p^{\rm L}\right|-\left|p^{\rm R}\right|\right)/2\right]\sinh\left[\pi\varrho\left(\mathbb{U}+\left|p^{\rm R}\right|-\left|p^{\rm L}\right|\right)/2\right]}. \quad (19)$$

To unveil important features about pair creation, it is worth discussing a situation where the field inhomogeneity evolves "gradually" along the y-axis, such that it stretches over a relatively wide region of the space. Such a configuration is achieved as long as the condition $\sqrt{\varrho\mathbb{U}/2} \gg \max\left(1, m/\sqrt{|\mu| B'}\right)$ is satisfied. For this configuration, the mean number of pairs created is approximately given by $N_n^{\rm cr} \approx \exp\left[-\pi\varrho\left(\mathbb{U}-\left|p^{\rm R}\right|-\left|p^{\rm L}\right|\right)\right]$. Performing summations over all quantum numbers, the total flux density of the pairs created $n^{\rm cr}$ and the vacuum-vacuum transition probability (18) have the form:

$$n^{\rm cr} \approx \frac{1}{2\pi^3}\varrho^2\left(|\mu| B'\right)^{5/2} e^{-\pi b'} I_{b'},$$
$$I_{b'} = \int_0^\infty \frac{du}{(u+1)^{5/2}} \ln\left(\frac{\sqrt{1+u}+\sqrt{1+2u}}{\sqrt{u}}\right) e^{-\pi b' u},$$
$$P_v = \exp\left(-\beta V_y T n^{\rm cr}\right),$$
$$\beta = \sum_{l=0}^{\infty} \frac{\epsilon_{l+1}}{(l+1)^{3/2}} e^{-l\pi b'},$$
$$\epsilon_l = \frac{I_{b'l}}{I_{b'}}, \quad b' = \frac{m^2}{|\mu| B'}. \quad (20)$$

If the field inhomogeneity is strong, $m^2/|\mu| B' \ll 1$, the total flux density of the pairs created is admits an approximate form $n^{\rm cr} = \varrho^2\left(\frac{\pi+\ln 2-1}{6\pi^3}\right)\left(|\mu| B'\right)^{5/2} \times e^{-\pi m^2/|\mu| B'}$. This approximation is quite similar to an earlier estimate, derived for the linearly-growing magnetic field [17].

The described mechanism raises the question about the critical magnetic field intensity, near which the phenomenon could be observed. It is possible to estimate such a field based on fermion's mass and anomalous magnetic moment. Since, within the present model, $\max\left(B_z(y)\right) = B_z(y) = \varrho B'$, the nontriviality of the Klein zone $\mathbb{U} > 2m$ yields the condition:

$$\mathbb{U} = 2|\mu|\varrho B' > 2m \to \varrho B'$$
$$= \max\left(B_z(y)\right) \equiv B_{\rm cr} > \frac{m}{|\mu|} \approx 1.73\times 10^8$$
$$\times\left(\frac{m}{1\ {\rm eV}}\right)\left(\frac{\mu_{\rm B}}{|\mu|}\right) {\rm G}. \quad (21)$$

Here, $\mu_{\rm B} = e/2m_e \approx 5.8\times 10^{-9}$ eV/G is Bohr's magneton [23]. For a neutron, $m_N \approx 939.6\times 10^6$ eV, $\mu_N \approx -1.042\times 10^{-3}\mu_{\rm B}$, the critical magnetic field (21) is $B_{\rm cr} \approx 1.6\times 10^{20}$ G. More optimistic values can be estimated for neutrino pair production. Considering a recent experimental limit to neutrinos effective magnetic moment $\mu_\nu \approx 2.9\times 10^{-11}\mu_{\rm B}$ [24] and mass $m_\nu \approx 10^{-2}$ eV [25], we find $B_{\rm cr} \approx 6\times 10^{16}$ G. This value lies within the range of values in astrophysical environments. For example, it was reported [26–28], that magnetic fields of the order of 10^{16}–10^{18} G could be generated during a supernova explosion or in the vicinity of magnetars. Moreover, ultra-intense magnetic fields (of order up to 10^{20} G) can be produced at the core of compact magnetars [29]. Thus, the mechanism here described can potentially explain additional sources of astrophysical neutrinos. Moreover, bearing in mind the possible existence of light sterile neutrinos, vacuum instability effects due to inhomogeneous magnetic fields may be relevant to studies on cosmological dark matter.

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CONFLICT OF INTEREST

The author declares that they have no conflicts of interest.

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