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CHARACTERIZING A PARALLEL OPERATION THROUGH COMPENSATOR TRANSFORM

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Characterizing a parallel operation through compensator transform.

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Abstract Willing to work in reliability theory under dependence condition in a general set up we intend to characterize a parallel operation through compensator transform.

Keywords: Martingale methods in reliability theory, compensator process, parallel operation.

1.Introduction

In reliability theory, even in the case where the components in a coherent system are stochastically independent, parallel operations are very important. It is basic to define a parallel system and relevant to decompose any coherent system in a series-parallel (parallel-series) structure (Barlow and Proschan (1981)). The performance of a parallel system are always better than the performance of any coherent system. The parallel system reliability is an upper bound for any coherent system reliability. It is used in replacement models and to optimize system reliability through active redundancy (Boland et al. (1992), Kuo and Prasad (2000), Kuo et al. (2001), among others). Such simple problems became very complicate ones if the components are stochastically dependent. Recently, some problems in which the components are dependent but identically distributed was developed.

Bueno and Carmo (2007) investigate active redundancy allocation for a k-out-of-n:F system of dependent components. They characterize the parallel operation through compensator transform of a component and its respective spare which was dependent but identically distributed lifetimes. The conclusion is that there exists a probability space in which it is optimal to perform active (parallel) redundancy operation on the weakest component in a sense of components compensator ordering (Shaked and Shantikumar (1994)) which implies stochastic ordering.

Bueno (2005) defined, using the same compensator transformation, a component reliability importance through a parallel improvement under dependence condition.

Such approach is very recent and to characterize such compensator transform for dependent lifetimes which are not identically distributed remains an open problem. In what follows we attempt to solve it.

2. Parallel operation through compensator transform

We consider to observe two component lifetimes T and S, which are positive random variables defined in a complete probability space $(\Omega, \Im, P)$ through the family of sub σ-algebras $(\Im_t)_{t\geq 0}$ of $\Im$, where

$$\Im_t = \sigma\{1_{\{S>s\}}, 1_{\{T>s\}}, 0 \leq s \leq t\}$$

satisfies Dellacherie's conditions. In what follows we assume that S and T are totally inaccessible $\Im_t$-stopping time and that $P(S = T) = 0$, that is, the lifetimes can be dependent but simultaneous failures are ruled out.

The parallel operation of S and T is defined by the maximum between S and T and denoted by

$$S \vee T = \max\{S, T\}.$$

There is several interpretation and/or application for such operation which defined a parallel system of two components, which can be interpreted as a redundancy improvement of the component lifetime T by the lifetime S and under which we can define a component reliability importance measure through a parallel improvement under dependence conditions, among others.

If we denote the survival functions of S and T as $\overline{G}(t) = P(S > t)$ and $\overline{F}(t) = P(T > t)$ respectively and if T and S are independent lifetimes the $\Im_t$-compensator processes of $N_B(t) = 1_{\{S\leq t\}}$ and $N_A(t) = 1_{\{T\leq t\}}$ are its hazard functions given by $B(t) = -\ln P(S > t|\Im_t) = -\ln(\overline{G}(t \wedge S))$ and $A(t) = -\ln P(T > t|\Im_t) = -\ln(\overline{F}(t \wedge T))$. As S and T are totally inaccessible $\Im_t$-stopping time the compensator processes are continuous.

Under the independence hypothesis we can calculate

$$P(S \vee T > t|\Im_t) = \overline{F}(t) + \overline{G}(t) - \overline{F}(t)\overline{G}(t) =$$

$$e^{-[A(t)+B(t)]}\{e^{A(t)} + e^{B(t)} - 1\}$$

and therefore, in the set $\{t < S \vee T\}$ the $\Im_t$- compensator of $S \vee T$ is

$$-\ln[P(S \vee T > t|\Im_t)] = A(t) + B(t) - \ln[e^{A(t)} + e^{B(t)} - 1].$$

In the dependent case, we intend to define a compensator transform to characterize, through compensator processes, the parallel operation, preserving the above intuition. As this operation is symmetric on S and T, the idea is to combine compensator transformation in $A(t)$ and $B(t)$.

Firstly, we consider the compensator transform

$$A^*(t) = \int_0^t \frac{e^{B(s)} - 1}{e^{A(s)} + e^{B(s)} - 1} dA(s) = \int_0^t 1 - [\frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1}] dA(s) =$$

$$A(t) - \int_0^t \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} dA(s).$$

To prove the main Theorem of this note we are going to use the following Lemma:

Lemma 2.1 The following process

$$L_A(t) = (\frac{e^{B(T)} - 1}{e^{A(T)} + e^{B(T)} - 1})^{1_{\{T \leq t\}}} e^{\int_0^t \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} dA(s)}$$

is a nonnegative local $\Im_t$-martingale and $E[L_A(t)] = 1$.

Proof We consider the $\Im_t$-stopping time defined by

$$V_n = \inf\{t \geq 0 : A(t) \geq n \;\; or \;\; B(t) \geq n\}.$$

It is sufficient to prove that the process

$$L_A^n(t) = (\frac{e^{B(T)} - 1}{e^{A(T)} + e^{B(T)} - 1})^{1_{\{T \leq t \wedge V_n\}}} e^{\int_0^{t \wedge V_n} \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} dA(s)}$$

is a bounded $\Im_t$-martingale.

For any $\Im_t$-stopping time $V \leq V_n$ we can write

$$L_A^n(V) = 1 - \int_0^V e^{\int_0^s \frac{e^{A(u)}}{e^{A(u)} + e^{B(u)} - 1} dA(u)} \frac{e^{A(s)}}{e^{A(s)} + e^{B(s)} - 1} d(N(s) - A(s))$$

where $N(t) = 1_{\{T \leq t\}}$. The procedure is easy:

On the set $\{V < T\}$ we have

$$1+\int_0^V e^{\int_0^s \frac{e^{A(u)}}{e^{A(u)}+e^{B(u)}-1}dA(u)} \frac{e^{A(s)}}{e^{A(s)}+e^{B(s)}-1}dA(s) =$$

$$e^{\int_0^V \frac{e^{A(u)}}{e^{A(u)}+e^{B(u)}-1}dA(u)} = L_A^n(V).$$

Otherwise, on the set $\{V \geq T\}$

$$1-\int_0^V e^{\int_0^s \frac{e^{A(u)}}{e^{A(u)}+e^{B(u)}-1}dA(u)} \frac{e^{A(s)}}{e^{A(s)}+e^{B(s)}-1}d(N(s)-A(s)) =$$

$$e^{\int_0^T \frac{e^{A(u)}}{e^{A(u)}+e^{B(u)}-1}dA(u)}[1-\frac{e^{A(T)}}{e^{A(T)}+e^{B(T)}-1}] = L_A^n(V).$$

As the integrand $e^{\int_0^s \frac{e^{A(u)}}{e^{A(u)}+e^{B(u)}-1}dA(u)} \frac{e^{A(s)}}{e^{A(s)}+e^{B(s)}-1}$ is an $\Im_t$-predictable process and $N(s)-A(s)$ is an $\Im_t$-martingale, $L_A^n(t)$ is an $\Im_t$-martingale with $E[L_A^n(t)] = 1$.

Secondly, we consider the compensator transform

$$B^*(t) = \int_0^t \frac{e^{A(s)}-1}{e^{A(s)}+e^{B(s)}-1}dB(s) = B(t) - \int_0^t \frac{e^{B(s)}}{e^{A(s)}+e^{B(s)}-1}dB(s)$$

and with the same argument to prove Lemma 2.1 we can prove Lemma 2.2:

Lemma 2.2 The following process

$$L_B(t) = (\frac{e^{A(S)}-1}{e^{A(S)}+e^{B(S)}-1})^{1_{\{S\leq t\}}} e^{\int_0^t \frac{e^{B(s)}}{e^{A(s)}+e^{B(s)}-1}dB(s)}$$

is a nonnegative local $\Im_t$-martingale with $E[L_B(t)] = 1$.

Observe that the same expression for the $\Im_t$-compensator of $1_{\{T\vee S\leq t\}}$ in the independent case is obtained in the dependent case through the transformation:

$$A^*(t) + B^*(t) = A(t) + B(t) - \int_0^t \frac{e^{A(s)}dA(s) + e^{B(s)}dB(s)}{e^{A(s)}+e^{B(s)}-1} =$$

$$A(t) + B(t) - \ln[e^{A(t)} + e^{B(t)} - 1].$$

We propose the compensator transforms:

$$A^*(t) = \int_{)}^{t} \alpha(s)dA(s), \quad \alpha(s) = \frac{e^{B(s)} - 1}{e^{A(s)} + e^{B(s)} - 1},$$

and

$$B^*(t) = \int_0^t \beta(s)dB(s), \quad \beta(s) = \frac{e^{A(s)} - 1}{e^{A(s)} + e^{B(s)} - 1}.$$

Now, we can write the main Theorem:

Theorem 2.3 The following process

$$L(t) = L_A(t)L_B(t) = (\alpha(T))^{1\{T\leq t\}}(\beta(S))^{1\{S\leq t\}}[e^{A(t)} + e^{B(t)} - 1]$$

is a nonnegative local $\Im_t$-martingale and $E[L(t)] = 1$.

Proof. Using Lemma 2.1, Lemma 2.2 and the Stieltjes differentiation rule we have

$$L_A(t)L_B(t) - 1 = \int_0^t L_A(s-)dL_b(s) + \int_0^t L_B(s-)dL_A(s) + \sum_{s\leq t} \Delta L_A(s)\Delta L_B(s).$$

As by assumption, $A(t)$ and $B(t)$ are continuous and $P(S = T) = 0$ we have $\sum_{s\leq t} \Delta L_A(s) \Delta L_B(s) = 0$ and therefore $L_A(t)L_B(t)$ in an local $\Im_t$-martingale with $E[L_A(t)L_B(t)] = 1$ and the theorem is proved.

Therefore, we are looking for a probability measure Q, such that, under Q, $C^*(t) = A^*(t) + B^*(t)$ becomes the $\Im_t$-compensator of $1_{\{S\vee T\leq t\}}$ with respect to this modified probability measure.

Under certain conditions, it is possible to find Q. Indeed, assume that the process $L(t)$ is uniformly integrable. Then it follows from well known results on point process martingales (Bremaud (1981)) that the desired measure Q is given by the Radon Nikodyn derivative $\frac{dQ}{dP} = L(\infty)$.

Remarks

i) In the case where T and S are identically distributed, we have $A(t) = B(t)$ and the compensator transform is given by

$$A^*(t) = 2\int_0^t \frac{e^{A(s)}-1}{2e^{A(s)}-1}dA(s) = \int_0^t \frac{2-2e^{-A(s)}}{2-e^{-A(s)}}dA(s).$$

which is used in Bueno and Carmo (2007) to define active redundancy operation when the component and the spare are dependent but identically distributed.

ii) The random variable $L(\infty)$ is given by

$$L(\infty) = (\frac{e^{B(T)}-1}{e^{A(T)}+e^{B(T)}-1})(\frac{e^{A(S)}-1}{e^{A(S)}+e^{B(S)}-1})[e^{A(T)}+e^{B(S)}-1] =$$

$$\frac{(e^{B(S\wedge T)}-1)(e^{A(S\wedge T)}-1)}{e^{A(S\wedge T)}+e^{B(S\wedge T)}-1}$$

where $S \wedge T = \min\{S, T\}$.

iii) Based in Bueno (2005) we can define the reliability importance of a component with lifetime T for the system reliability, with lifetime τ under a parallel improvement with the lifetime S as

$$cov(\tau, L(\infty)) = cov(\tau, \frac{(e^{B(S\wedge T)}-1)(e^{A(S\wedge T)}-1)}{e^{A(S\wedge T)}+e^{B(S\wedge T)}-1})$$

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