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Simple Modules for Affine Vertex Algebras in the Minimal Nilpotent Orbit

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We explicitly construct, in terms of Gelfand–Tsetlin tableaux, a new family of simple positive energy representations for the simple affine vertex algebra $V_k(\mathfrak{sl}_{n+1})$ in the minimal nilpotent orbit of $\mathfrak{sl}_{n+1}$. These representations are quotients of induced modules over the affine Kac–Moody algebra $\widehat{\mathfrak{sl}}_{n+1}$ and include in particular all admissible simple highest weight modules and all simple modules induced from $\mathfrak{sl}_2$. Any such simple module in the minimal nilpotent orbit has bounded weight multiplicities.

1 Introduction

Relaxed highest weight modules for affine Kac–Moody algebras attract a considerable interest due to their connection to the representation theory of conformal vertex algebras and conformal field theories, cf. [16], [2], [3], [4], [5], [1], [10] and the references therein. In particular, the importance of *relaxed highest weight* admissible modules for conformal field theory was shown in [34], [11], [12], see also [35], [15]. Using the Zhu's functor the study of positive energy representations of simple affine vertex algebras reduces to the representations of the underlined finite-dimensional Lie algebra. This allows to construct new families of simple representations of admissible vertex

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algebras. This has been exploited in [9], where new families of simple modules were constructed for the affine vertex algebra of $\mathfrak{sl}_n$ and a complete classification of all simple relaxed highest weight representations with finite-dimensional weight spaces in the case $n = 3$ was obtained. Further on, this approach was developed in [29*] and [30] using Fernando–Mathieu classification of simple weight representations with finite weight multiplicities of finite-dimensional simple Lie algebras. These papers provided an algorithm for classifying all admissible relaxed highest weight modules starting from admissible highest weight modules. The latter modules were classified in [27] and [7]. However, explicit construction of admissible relaxed highest weight representations beyond their classification is rather difficult. Moreover, the algorithm is limited to the modules with finite weight multiplicities. We refer to the recent paper [20], where the localization and the Wakimoto functors were used to construct *relaxed Wakimoto modules* for the universal and simple affine vertex algebras with infinite weight multiplicities.

The current paper aims to give an explicit construction of admissible representations extending [9], where the Gelfand–Tsetlin tableau realization was used to describe admissible families in type A. Such realization provides a basis and explicit formulas for the action of the Lie algebra. In this paper we describe several new families of simple positive energy weight representations of the admissible affine vertex algebra $V_k(\mathfrak{sl}_n)$. These families include all simple highest weight modules in the minimal nilpotent orbit described in [6]. They also include simple modules in the minimal nilpotent orbit induced from an $\mathfrak{sl}_2$-subalgebra. All constructed modules have bounded weight multiplicities. Our approach is based on the theory of *relation* Gelfand–Tsetlin modules developed in [21], where explicit tableau basis was constructed for different classes of simple Gelfand–Tsetlin modules for $\mathfrak{gl}_n$.

Our 1st main result is Theorem 4.8, which provides a tableau realization for simple highest weight modules, namely we give necessary and sufficient conditions for such modules to be relation modules with respect to the standard Gelfand–Tsetlin subalgebra. In Section 5 we apply twisted localization to relation Gelfand–Tsetlin modules. We establish the following result (cf. Theorem 5.4).

Theorem 1.1. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, Γ the standard Gelfand–Tsetlin subalgebra (corresponding to the chain of embeddings starting from the upper left corner), $f = E_{21}$, and M a Γ-relation module with an injective action of f. Then the twisted localization $D_f^x M$ of M is a Γ-relation Gelfand–Tsetlin module for any $x \in \mathbb{C}$.

Using the theorem above we obtain our 2nd main result—Theorem 5.6, where we classify all simple Γ-relation Gelfand–Tsetlin modules induced from $\mathfrak{sl}_2$-subalgebra generated by E_{12} and E_{21}. Also, in Theorem 5.8, we obtain an explicit construction of a family of simple $\mathfrak{sl}_{n+1}$-modules, which are parabolically induced from cuspidal $\mathfrak{a}$-modules, where $\mathfrak{a} \simeq \mathfrak{sl}_m$ is part of the Γ-flag, $m = 2, \dots, n$. Further, localization of simple highest weight relation modules is considered in Proposition 5.12 for a multiplicative set generated by any number of commuting generators of the form E_{k1}.

Finally, in Section 6, we establish our 3rd main result—explicit description of simple admissible highest weight $\mathfrak{sl}_{n+1}$-modules and modules induced from $\mathfrak{sl}_2$ subalgebra in the unique minimal non-trivial nilpotent orbit $\mathbb{O}_{min}$ of $\mathfrak{sl}_{n+1}$. Let

$$k + n = \frac{p}{q} - 1, \quad p > n,\ q \geq 1 \text{ and } (p, q) = 1,$$

$a \in \{1, 2, \dots, q-1\}$ and $\lambda_a = \lambda - \frac{ap}{q}\varpi_1$, where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a weight of $\mathfrak{sl}_{n+1}$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$, for all $i = 1, \dots, n$, $\lambda_1 + \ldots + \lambda_n < p - n$. Then we have (cf. Theorem 6.3 and Theorem 6.12):

Theorem 1.2.

(i) If Γ is the standard Gelfand–Tsetlin subalgebra then the simple highest weight module $L(\lambda_a)$ is a bounded Γ-relation Gelfand–Tsetlin $\mathfrak{sl}_{n+1}$-module. Moreover, all simple admissible highest weight modules in the minimal nilpotent orbit are bounded Γ-relation Gelfand–Tsetlin modules.

(ii) Let π be a basis of the root system of $\mathfrak{g}$, $\mathfrak{b}(\pi)$ the corresponding Borel subalgebra of $\mathfrak{g}$, β a positive (with respect to π) root of $\mathfrak{g}$, and ρ_π the half-sum of positive (with respect to π) roots. Let $L_{\mathfrak{b}(\pi)}(\lambda)$ be an admissible simple $\mathfrak{b}(\pi)$-highest weight $\mathfrak{g}$-module in the minimal orbit, such that $\langle \lambda + \rho_\pi, \beta^\vee \rangle \notin \mathbb{Z}$, and $f = f_\beta$. Denote by $A_{\pi,\beta}$ the set of all $x \in \mathbb{C} \setminus \mathbb{Z}$ for which $x + \langle \lambda + \rho_\pi, \beta^\vee \rangle \notin \mathbb{Z}$. Then the twisted localization modules $D_f^x L_{\mathfrak{b}(\pi)}(\lambda)$, $x \in A_{\pi,\beta}$, exhaust all simple $\mathfrak{sl}_2$-induced admissible modules in the minimal orbit. Moreover, there exists a flag $\mathcal{F}$ such that $D_f^x L_{\mathfrak{b}(\pi)}(\lambda)$ is $\Gamma_{\mathcal{F}}$-relation Gelfand–Tsetlin $\mathfrak{g}$-module.

2 Preliminaries

2.1 Weight modules

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra, $\mathfrak{h}$ a fixed Cartan subalgebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and W the Weyl group of $\mathfrak{g}$. By Δ we denote

the root system of $\mathfrak{g}$ and by Δ_+ the set of positive roots with respect to a fixed basis π of Δ. Put $Q = \sum_{\alpha\in\Delta} \mathbb{Z}\alpha$ for the root lattice and $Q^\vee = \sum_{\alpha\in\Delta} \mathbb{Z}\alpha^\vee$ for the coroot lattice, where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. Let ρ_π to be the half sum of positive roots.

By $\mathfrak{b}(\pi)$ we denote the standard Borel subalgebra of $\mathfrak{g}$ corresponding to the set π. In addition to the standard Borel subalgebra of $\mathfrak{g}$ we also consider the standard parabolic subalgebras of $\mathfrak{g}$. For a subset Σ of π denote by Δ_Σ the root subsystem in $\mathfrak{h}^*$ generated by Σ. Then the standard parabolic subalgebra $\mathfrak{p}_\Sigma(\pi)$ of $\mathfrak{g}$ associated to π and Σ is defined as $\mathfrak{p}_\Sigma(\pi) = \mathfrak{l}_\Sigma \oplus \mathfrak{u}_\Sigma^+$ with nilradical $\mathfrak{u}_\Sigma^+ := \bigoplus_{\alpha\in\Delta_+\setminus\Delta_\Sigma} \mathfrak{g}_\alpha$, opposite nilradical $\mathfrak{u}_\Sigma^- := \bigoplus_{\alpha\in\Delta_+\setminus\Delta_\Sigma} \mathfrak{g}_{-\alpha}$, and Levi subalgebra $\mathfrak{l}_\Sigma$ defined by

$$\mathfrak{l}_\Sigma := \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_\Sigma} \mathfrak{g}_\alpha.$$

Moreover, we have the corresponding triangular decomposition $\mathfrak{g} = \mathfrak{u}_\Sigma^- \oplus \mathfrak{l}_\Sigma \oplus \mathfrak{u}_\Sigma^+$. Note that if $\Sigma = \emptyset$ then $\mathfrak{p}_\Sigma(\pi) = \mathfrak{b}(\pi)$, and if $\Sigma = \pi$ then $\mathfrak{p}_\Sigma(\pi) = \mathfrak{g}$.

Recall that a $\mathfrak{g}$-module (respectively $\mathfrak{l}_\Sigma$-module) M is called *weight* if $\mathfrak{h}$ is diagonalizable on M. Let V be a simple weight $\mathfrak{l}_\Sigma$-module. Set $\mathfrak{p} := \mathfrak{p}_\Sigma(\pi)$ and consider V as a $\mathfrak{p}$-module with trivial action of the nilradical $\mathfrak{u}_\Sigma^+$. The *generalized Verma* $\mathfrak{g}$-module $M_\mathfrak{p}^\mathfrak{g}(\Sigma, V)$ is the induced module

$$M_\mathfrak{p}^\mathfrak{g}(\Sigma, V) = \mathrm{Ind}_\mathfrak{p}^\mathfrak{g} V = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V.$$

The module $M_\mathfrak{p}^\mathfrak{g}(\Sigma, V)$ has a unique maximal submodule and a unique simple quotient $L_\mathfrak{p}^\mathfrak{g}(\Sigma, V)$. We write $M(\lambda)$ for the Verma module $M_{\mathfrak{b}(\pi)}^\mathfrak{g}(\emptyset, \mathbb{C}v_\lambda)$ and $L(\lambda)$ for $L_{\mathfrak{b}(\pi)}^\mathfrak{g}(\emptyset, \mathbb{C}v_\lambda)$ when it is clear which Borel subalgebra is meant.

Let M be a weight $\mathfrak{g}$-module. For $\lambda \in \mathfrak{h}^*$ the subspace M_λ of those $v \in V$ such that $hv = \lambda(h)v$ is the *weight subspace* of weight λ. The dimension of M_λ is the *multiplicity* of weight λ. We say that a weight module is *bounded* if all weight multiplicities are uniformly bounded. A weight $\mathfrak{g}$-module M is *torsion free* if for any nonzero weight subspace M_λ and any root α, a nonzero root vector $X \in \mathfrak{g}_\alpha$ defines an isomorphism between M_λ and $M_{\lambda+\alpha}$. A weight module M is *cuspidal* if M is finitely generated torsion free module with finite weight multiplicities.

Proposition 2.1. [31, Corollary 1.4] For a simple weight $\mathfrak{g}$-module M with finite weight multiplicities, the following assertions are equivalent:

(a) M is cuspidal;

(b) M is torsion free;

(c) The support of M is exactly one Q-coset.

A weight module satisfying Proposition 2.1, (c) is called *dense*.

2.2 Annihilators

Recall that $\lambda \in \mathfrak{h}^*$ is *dominant* if $\langle \lambda + \rho_\pi, \alpha^\vee \rangle \notin \mathbb{Z}_{<0}$, for all $\alpha \in \Delta_+$. Similarly λ is *antidominant* if $\langle \lambda + \rho_\pi, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Delta_+$. Also, λ is *regular* if $\langle \lambda + \rho_\pi, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta$. If $\mathcal{Z}$ is the center of $U(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$, then $\chi_\lambda : \mathcal{Z} \to \mathbb{C}$ stands for the central character of $L(\lambda)$.

For an associative algebra A and an A-module M, the *annihilator* of M in A is the ideal $\{a \in A \mid am = 0 \text{ for all} m \in M\}$ and will be denoted by $Ann_A(M)$.

Proposition 2.2. [13, 8.5.8] If $\lambda \in \mathfrak{h}^*$ is dominant, then the annihilator of $L(\lambda)$ is the unique maximal two-sided ideal of $U(\mathfrak{g})$ containing $U(\mathfrak{g}) \ker \chi_\lambda$.

If λ is regular dominant, then the correspondence $I \mapsto IM(\lambda)$ gives an order-preserving bijection between two-sided ideals of $U(\mathfrak{g})$ containing $U(\mathfrak{g}) \ker \chi_\lambda$ and submodules of $M(\lambda)$ [24, Corollaries 4.3 and 4.8]. As a consequence of Proposition 2.2 we immediately have the following well-known assertion.

Corollary 2.3. Let $\lambda, \mu \in \mathfrak{h}^*$ be dominant. If $\mu = w \cdot \lambda$ for some $w \in W$ then $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda) = \mathrm{Ann}_{U(\mathfrak{g})} L(\mu)$. Moreover, the converse holds if λ is regular.

2.3 Affine vertex algebras

Consider the non-twisted affine Kac–Moody algebra associated with $\mathfrak{g}$: $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K$, where K is a central element. Denote by $\widehat{\Delta}$ the set of roots of $\widehat{\mathfrak{g}}$. Then $\widehat{\Delta}^{re} = \{\alpha + m\delta \mid \alpha \in \Delta, m \in \mathbb{Z}\}$ is the set of real roots and $\widehat{\Delta}^{re}_+ = \{\alpha + m\delta \mid \alpha \in \Delta_+, m \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\alpha + m\delta \mid \alpha \in \Delta_+, m \in \mathbb{Z}_{\geq 1}\}$ is the set of positive real roots with respect to the basis $\pi \cup \{-\theta + \delta\}$, where θ is the maximal positive root of $\mathfrak{g}$ and δ is the minimal indivisible positive imaginary root. By $\widetilde{W} = W \ltimes Q^\vee$ we denote the extended affine Weyl group of $\widehat{\mathfrak{g}}$.

For $k \in \mathbb{C}$ denote by $V^k(\mathfrak{g})$ the *universal affine vertex algebra* associated with $\mathfrak{g}$ at level k ([18, 26]):

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}})_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}v_k,$$

where $\mathfrak{g}[t]v_k = 0$ and $Kv_k = kv_k$.

The *simple affine vertex algebra* associated with $\mathfrak{g}$ at level k is the unique simple graded quotient of $V^k(\mathfrak{g})$, which will be denoted by $V_k(\mathfrak{g})$. In the conformal case (i.e., $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$) there is a one-to-one correspondence between simple positive energy representations of $V_k(\mathfrak{g})$ and simple $A(V_k(\mathfrak{g}))$-modules [36], where

$$A(V_k(\mathfrak{g})) \simeq U(\mathfrak{g})/I_k$$

is the *Zhu's algebra* of $V_k(\mathfrak{g})$ and I_k is some two-sided ideal of $U(\mathfrak{g})$.

For $\lambda \in \mathfrak{h}^*$ and $k \in \mathbb{C}$ denote by $\widehat{L}_k(\lambda)$ the simple highest weight $\widehat{\mathfrak{g}}$-module with the highest weight $\widehat{\lambda} := \lambda + k\Lambda_0 \in \widehat{\mathfrak{h}}^*$, where $\Lambda_0(K) = 1$, and $\Lambda_0(h) = 0$ for $h \in \mathfrak{h}$. Following [27], the module $\widehat{L}_k(\lambda)$, and the highest weight λ, will be called *admissible* if

(1) $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \widehat{\Delta}^{re}_+$;
(2) $\mathbb{Q}\widehat{\Delta}(\lambda) = \mathbb{Q}\widehat{\Delta}^{re}$,

where $\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{re} \mid \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\}$. The level k of $\widehat{\mathfrak{g}}$ is *admissible* if the $\widehat{\mathfrak{g}}$-module $V_k(\mathfrak{g}) \cong \widehat{L}_k(0)$ is admissible.

In this paper we are interested in type A only, hence from now on we fix $n \geq 2$ and assume that $\mathfrak{g}$ is of type A_n. In this case the description of admissible levels is as follows.

Proposition 2.4. [28, Proposition 1.2] Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. The number k is admissible if and only if

$$k + n = \frac{p}{q} - 1 \quad \text{with } p, q \in \mathbb{N},\ (p, q) = 1,\ p \geq n + 1.$$

2.4 Relaxed highest weight modules

Fix an admissible number k. Following [9], we say that a $\mathfrak{g}$-module M is *admissible* of level k if M is an $A(V_k(\mathfrak{g}))$-module. The corresponding simple quotient of the induced module for $\widehat{\mathfrak{g}}$ is a *relaxed highest weight module*. We have a one-to-one correspondence between the set of isomorphism classes of simple admissible $\mathfrak{g}$-modules of level k and the isomorphism classes of simple positive energy representations of $V_k(\mathfrak{g})$.

Denote by Pr_k the set of admissible weights λ such that there exists $y \in \widetilde{W}$ satisfying $\widehat{\Delta}(\lambda) = y(\widehat{\Delta}(k\Lambda_0))$, and $Pr_{k,\mathbb{Z}} := \{\lambda \in Pr_k \mid \lambda(K) = k,\ \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i = 1, \ldots, l\}$, where l is the rank of $\mathfrak{g}$. If k has the denominator q such that $(q, r^\vee) = 1$, then

(see [28])

$$Pr_k = \bigcup_{\substack{y\in\widetilde{W} \\ y(\widehat{\Delta}(k\Lambda_0)_+)\subset\widehat{\Delta}_+^{re}}} Pr_{k,y}, \quad \text{and} \quad Pr_{k,y} := y\cdot Pr_{k,\mathbb{Z}}.$$

Let $\overline{Pr_k} := \{\bar{\lambda} \mid \lambda \in Pr_k\} \subset \mathfrak{h}^*$, where $\bar{\lambda}$ is the projection of λ to $\mathfrak{h}^*$. The importance of this set comes from the following Arakawa result by Arakawa.

Theorem 2.5. [7, Main Theorem] Let k be admissible, $\lambda \in \mathfrak{h}^*$. Then $\widehat{L}_k(\lambda)$ is a module over $V_k(\mathfrak{g})$ if and only if $\lambda \in \overline{Pr_k}$.

For $\lambda \in \mathfrak{h}^*$ denote by J_λ the corresponding primitive ideal $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda)$. By Theorem 2.5, a simple $\mathfrak{g}$-module M is admissible of level k if and only if $\mathrm{Ann}_{U(\mathfrak{g})} M = J_\lambda$ for some $\lambda \in \overline{Pr_k}$. As a consequence of Proposition 2.2 and Corollary 2.3, we have immediately the following statement.

Proposition 2.6. [9, Proposition 2.4] For $\lambda \in \overline{Pr_k}$, the primitive ideal J_λ is the unique maximal two-sided ideal of $U(\mathfrak{g})$ containing $U(\mathfrak{g})\ker\chi_\lambda$. In particular, $J_\lambda = J_\mu$ for $\lambda, \mu \in \overline{Pr_k}$ if and only if there exists $w \in W$ such that $\mu = w\cdot\lambda$.

Remark 2.7. The following characterization of the Zhu's algebra of $V_k(\mathfrak{g})$ was shown in [8, Theorem 3.4]:

$$A(V_k(\mathfrak{g})) \simeq \prod_{\lambda\in[\overline{\mathrm{Pr}_k}]} \frac{U(\mathfrak{g})}{J_\lambda}.$$

For a two-sided ideal I of $U(\mathfrak{g})$, denote by $\mathrm{Var}(I)$ the *associated variety* of I (the zero locus of the associated graded ideal grI in $\mathfrak{g}^*$, with respect to the PBW filtration). If I is primitive then $\mathrm{Var}(I) = \overline{\mathbb{O}}$ for some nilpotent orbit $\mathbb{O}$ of $\mathfrak{g}$ [25, Theorem 3.10].

Definition 2.8. A simple $\mathfrak{g}$-module M is *in the orbit* $\mathbb{O}$ if $\mathrm{Var}(\mathrm{Ann}_{U(\mathfrak{g})} M) = \overline{\mathbb{O}}$.

We have the following:

Theorem 2.9. [6, Corollary 9.2 and Theorem 9.5] A simple $\mathfrak{g}$-module M in the orbit $\mathbb{O}$ is admissible of level k if and only if $\mathrm{Ann}_{U(\mathfrak{g})} M = J_\lambda$ for some $\lambda \in [\overline{Pr_k}^{\mathbb{O}}]$. Moreover, there

exists a nilpotent orbit $\mathbb{O}_q$ that depends only on q such that

$$\mathrm{Var}(I_k) = \overline{\mathbb{O}_q}.$$

Hence, $\mathrm{Var}(J_\lambda) \subset \mathrm{Var}(I_k) = \overline{\mathbb{O}_q}$ for any $\lambda \in \overline{Pr}_k$. Moreover, by Theorems 2.5 and 2.9, we have

$$\overline{Pr}_k = \bigsqcup_{\mathbb{O} \subset \overline{\mathbb{O}_q}} \overline{Pr}_k^{\mathbb{O}},$$

where $\mathbb{O}$ is a nilpotent orbit of $\mathfrak{g}$ and $\overline{Pr}_k^{\mathbb{O}} = \{\lambda \in \overline{Pr}_k \mid \mathrm{Var}(J_\lambda) = \overline{\mathbb{O}}\}$.

Set $[\overline{Pr}_k] = \overline{Pr}_k/\sim$, where $\lambda \sim \mu$ if and only if there exists $w \in W$ such that $\mu = w \cdot \lambda$. By Proposition 2.6, the ideal J_λ depends only on the class of $\lambda \in \overline{Pr}_k$ in $[\overline{Pr}_k]$. Therefore,

$$[\overline{Pr}_k] = \bigsqcup_{\mathbb{O} \subset \overline{\mathbb{O}_q}} [\overline{Pr}_k^{\mathbb{O}}],$$

where $[\overline{Pr}_k^{\mathbb{O}}]$ is the image of $\overline{Pr}_k^{\mathbb{O}}$ in $[\overline{Pr}_k]$.

3 Twisted Localization

In this section $F := \{f_1, \dots, f_r\}$ will denote any set of locally ad-nilpotent commuting elements of $U(\mathfrak{g})$, $\mathbf{x}$ any element of $\mathbb{C}^r$. A $\mathfrak{g}$-module M will be called *F-bijective* if f_i acts bijectively on M for each $i = 1, 2, \dots, r$. By $D_F U(\mathfrak{g})$ we denote the localization of $U(\mathfrak{g})$ relative to the multiplicative set $\langle F \rangle$ generated by F. Similarly, for a $\mathfrak{g}$-module M denote by $D_F M = D_F U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$ the *localization of M relative to* $\langle F \rangle$. We will consider $D_F M$ both as a $U(\mathfrak{g})$-module and as a $D_F U(\mathfrak{g})$-module.

Proposition 3.1. Let M be an F-bijective $\mathfrak{g}$-module, L, N submodules of M, such that L is not F-bijective, and M/L is simple. If L is isomorphic to a proper submodule of $D_F N$ then $D_F N \simeq M$.

Proof. As D_F is an exact functor, then $D_F N$ is a submodule of $D_F M \simeq M$. Since L is a proper submodule of $D_F N$, the statement follows from the maximality of L in M. ∎

For $\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{C}^r$ consider the automorphism $\Theta_F^{\mathbf{x}}$ of $D_F U(\mathfrak{g})$ such that

$$\Theta_F^{\mathbf{x}}(u) := \sum_{i_1,\dots,i_r \geq 0} \binom{x_1}{i_1} \dots \binom{x_r}{i_r} \mathrm{ad}(f_1)^{i_1} \dots \mathrm{ad}(f_r)^{i_r}(u) f_1^{-i_1} \dots f_r^{-i_r},$$

for $u \in D_F U(\mathfrak{g})$, where $\binom{x}{i} := x(x-1)...(x-i+1)/i!$ for $x \in \mathbb{C}$, $i \in \mathbb{Z}_{>0}$ and $\binom{x}{0} := 1$ ([31, Section 4]).

Definition 3.2. For a $D_F U(\mathfrak{g})$-module N, we will denote by $\Phi_F^{\mathbf{x}} N$ the $D_F U(\mathfrak{g})$-module on N twisted by $\Theta_F^{\mathbf{x}}$, where the new action is given by

$$u \cdot v^{\mathbf{x}} := (\Theta_F^{\mathbf{x}}(u) \cdot v)^{\mathbf{x}},$$

for $u \in D_F U(\mathfrak{g})$, $v \in N$. Here $v^{\mathbf{x}}$ stands for the element v considered as an element of $\Phi_F^{\mathbf{x}} N$.

If M is a $\mathfrak{g}$-module and $\mathbf{x} \in \mathbb{C}^r$, then $D_F^{\mathbf{x}} M := \Phi_F^{\mathbf{x}} D_F M$ is the *twisted localization of M relative to F and* $\mathbf{x}$.

Remark 3.3. Note that for $\mathbf{x} \in \mathbb{Z}^r$, we have $\Theta_F^{\mathbf{x}}(u) = \mathbf{f}^{\mathbf{x}} u \mathbf{f}^{-\mathbf{x}}$, where $\mathbf{f}^{\mathbf{x}} := f_1^{x_1} ... f_r^{x_r}$. Moreover, $D_F^{\mathbf{x}} M$ and $D_F M$ are isomorphic for any $\mathfrak{g}$-module M.

The following important property of the twisted localization is a consequence of [23, Lemma 2.8].

Proposition 3.4. Let M be a weight $\mathfrak{g}$-module with finite-dimensional weight spaces and assume that F is injective on M. Then

$$Ann_{U(\mathfrak{g})} M = Ann_{U(\mathfrak{g})} D_F M \subset Ann_{U(\mathfrak{g})} D_F^{\mathbf{x}} M,$$

for any $\mathbf{x} \in \mathbb{C}^r$. Moreover, $Ann_{U(\mathfrak{g})} M \subset Ann_{U(\mathfrak{g})} N$, for any subquotient N of $D_F^{\mathbf{x}} M$.

Applying Proposition 3.4 and Proposition 2.2 to simple highest weight modules with dominant highest weights we obtain the following statement.

Corollary 3.5. Let $\lambda \in \mathfrak{h}^*$ be dominant, $\mathbf{x} \in \mathbb{C}^r$, and F be injective on $L(\lambda)$. If $N \neq 0$ is a simple subquotient of $\mathcal{D}_F^{\mathbf{x}} L(\lambda)$ then

$$Ann_{U(\mathfrak{g})} L(\lambda) = Ann_{U(\mathfrak{g})} D_F^{\mathbf{x}} L(\lambda) = Ann_{U(\mathfrak{g})} N.$$

4 Relation Gelfand–Tsetlin Modules

4.1 Gelfand–Tsetlin modules

For any flag $\mathcal{F} : \mathfrak{gl}_1 \subset \cdots \subset \mathfrak{gl}_{n+1}$ we have an induced flag $U_1 \subset \cdots \subset U_{n+1}$ of the corresponding universal enveloping algebras. Let $\mathcal{Z}_m$ be the center of U_m. Following [14], we call the subalgebra $\Gamma_{\mathcal{F}}$ of $U := U_{n+1}$ generated by $\{\mathcal{Z}_m \mid m = 1, \ldots, n+1\}$ the *Gelfand–Tsetlin subalgebra* of U (with respect to $\mathcal{F}$). If the flag is given by left-upper corner inclusions, the corresponding Gelfand–Tsetlin subalgebra will be called the *standard Gelfand–Tsetlin subalgebra* and will be denoted by Γ_{st}. Consider a standard basis E_{ij}, $i, j = 1, \ldots n+1$ of $\mathfrak{gl}_{n+1}$. Then the standard flag $\mathcal{F}_{st}$ of $\mathfrak{gl}_{n+1}$ consists of $\mathfrak{gl}_k$, $k = 1, \ldots, n+1$, with $\mathfrak{gl}_s$ generated by E_{ij}, $i, j = 1, \ldots s$ for all s. If $w \in W$ and $\mathcal{F} = w\mathcal{F}_{st}$ then $\Gamma_{\mathcal{F}} = w\Gamma_{st}$ with a natural action of w.

Definition 4.1. A finitely generated U-module M is called a $\Gamma_{\mathcal{F}}$-*Gelfand–Tsetlin module* if M splits into a direct sum of $\Gamma_{\mathcal{F}}$-modules:

$$M = \bigoplus_{\mathsf{m} \in Specm \Gamma_{\mathcal{F}}} M(\mathsf{m}),$$

where

$$M(\mathsf{m}) = \{v \in M \mid \mathsf{m}^k v = 0 \text{ for some } k \geq 0\}.$$

Identifying m with the homomorphism $\chi : \Gamma_{\mathcal{F}} \to \mathbb{C}$ with $\mathrm{Ker}\chi = \mathsf{m}$, we will call m a *Gelfand–Tsetlin character* of M if $M(\mathsf{m}) \neq 0$. The dimension $\dim M(\mathsf{m})$ will be called the *Gelfand–Tsetlin multiplicity of* m.

Remark 4.2. Let $\tau : \mathfrak{gl}_{n+1} \to \mathfrak{sl}_{n+1}$ be a natural projection that extends to a homomorphism $\bar{\tau} : U(\mathfrak{gl}_{n+1}) \to U(\mathfrak{sl}_{n+1})$. If Γ is a Gelfand–Tsetlin subalgebra of $\mathfrak{gl}_{n+1}$ then $\bar{\tau}(\Gamma)$ of Γ is called a Gelfand–Tsetlin subalgebra of $\mathfrak{sl}_{n+1}$.

Unless otherwise is stated, from now on we will assume that $\Gamma_{\mathcal{F}} = \Gamma_{st}$, which we simply denote by Γ. We will refer to Γ-Gelfand–Tsetlin modules simply as Gelfand–Tsetlin modules.

4.2 Relation modules

A class of *relation* Gelfand–Tsetlin modules was constructed in [21]. These modules have properties similar to finite-dimensional modules [22] and to generic Gelfand–Tsetlin modules [14]. We recall their construction here since it will be used later on.

Set $\mathfrak{V} := \{(i,j) \mid 1 \leq j \leq i \leq n+1\}$, and $\mathcal{R} := \mathcal{R}^- \cup \mathcal{R}^0 \cup \mathcal{R}^+ \subset \mathfrak{V} \times \mathfrak{V}$, where

$$\mathcal{R}^+ := \{((i,j);(i-1,t)) \mid 2 \leq j \leq i \leq n+1,\ 1 \leq t \leq i-1\}$$
$$\mathcal{R}^- := \{((i,j);(i+1,s)) \mid 1 \leq j \leq i \leq n,\ 1 \leq s \leq i+1\}$$
$$\mathcal{R}^0 := \{((n+1,i);(n+1,j)) \mid 1 \leq i \neq j \leq n+1\}$$

Any subset $\mathcal{C} \subseteq \mathcal{R}$ will be called a *set of relations*. With any $\mathcal{C} \subseteq \mathcal{R}$ we associate a directed graph $G(\mathcal{C})$ with the set of vertices $\mathfrak{V}$, which has an arrow from vertex (i,j) to (r,s) if and only if $((i,j);(r,s)) \in \mathcal{C}$. For convenience, we will picture the set $\mathfrak{V}$ as a triangular tableau with $n+1$ rows, where the k-th row is $\{(k,1),\dots,(k,k)\}$, $k = 1,\dots,n+1$.

For $v \in \mathbb{C}^{\frac{(n+1)(n+2)}{2}}$ denote by $T(v)$ the image of v via the natural isomorphism between $\mathbb{C}^{\frac{(n+1)(n+2)}{2}}$ and $\mathbb{C}^{n+1} \times \cdots \times \mathbb{C}^1$. If $T(v) = (v^{(n+1)},\dots,v^{(1)})$ then we refer to $v^{(k)} = (v_{k1},\dots,v_{kk})$ as the k-th row of $T(v)$. Hence, we can picture $T(v)$ as a triangular tableau, a *Gelfand–Tsetlin tableau* of height $n+1$. Finally, by $\mathbb{Z}_0^{\frac{(n+1)(n+2)}{2}}$, we will denote the set of vectors v in $\mathbb{Z}^{\frac{(n+1)(n+2)}{2}}$ such that $v^{(n+1)} = \mathbf{0}$.

Definition 4.3. [21, Definition 4.2] Let $\mathcal{C}$ be a set of relations and $T(L)$ any Gelfand–Tsetlin tableau, where $L = (l_{ij}) \in \mathbb{C}^{\frac{(n+1)(n+2)}{2}}$.

(a) We say that $T(L)$ *satisfies* $\mathcal{C}$ if:
   (i) $l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0}$ for any $((i,j);(r,s)) \in \mathcal{C} \cap (\mathcal{R}^+ \cup \mathcal{R}^0)$;
   (ii) $l_{ij} - l_{rs} \in \mathbb{Z}_{>0}$ for any $((i,j);(r,s)) \in \mathcal{C} \cap \mathcal{R}^-$.

(b) We say that $T(L)$ is a *$\mathcal{C}$-realization* if $T(L)$ satisfies $\mathcal{C}$ and for any $1 \leq k \leq n$ we have $l_{ki} - l_{kj} \in \mathbb{Z}$ if only if (k,i) and (k,j) in the same connected component of the undirected graph associated with $G(\mathcal{C})$.

(c) We call $\mathcal{C}$ *noncritical* if for any $\mathcal{C}$-realization $T(L)$ one has $l_{ki} \neq l_{kj}$, $1 \leq k \leq n$, $i \neq j$, if (k,i) and (k,j) are in the same connected component of $G(\mathcal{C})$.

(d) Suppose that $T(L)$ satisfies $\mathcal{C}$. Then $\mathcal{B}_{\mathcal{C}}(T(L))$ denotes the set of all tableaux of the form $T(L+z)$, $z \in \mathbb{Z}_0^{\frac{(n+1)(n+2)}{2}}$ satisfying $\mathcal{C}$. Also, $V_{\mathcal{C}}(T(L))$ denotes the complex vector space spanned by $\mathcal{B}_{\mathcal{C}}(T(L))$.

Remark 4.4. Note that if $T(L)$ satisfies $\mathcal{C}$, and $\mathcal{C}$ is the maximal set of relations satisfied by $T(L)$, then $T(L)$ is a $\mathcal{C}$-realization.

Definition 4.5. [21, Definition 4.4] Let $\mathcal{C}$ be a set of relations. We call $\mathcal{C}$ *admissible* if for any $\mathcal{C}$-realization $T(L)$, $V_{\mathcal{C}}(T(L))$ has a structure of a $\mathfrak{sl}_{n+1}$-module, endowed with the following action of the generators of $\mathfrak{sl}_{n+1}$:

$$
\begin{aligned}
E_{k,k+1}(T(w)) &= -\sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k+1}(w_{ki}-w_{k+1,j})}{\prod_{j\neq i}^{k}(w_{ki}-w_{kj})}\right)T(w+\delta^{ki}),\\
E_{k+1,k}(T(w)) &= \sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k-1}(w_{ki}-w_{k-1,j})}{\prod_{j\neq i}^{k}(w_{ki}-w_{kj})}\right)T(w-\delta^{ki}),\\
H_k(T(w)) &= \left(2\sum_{i=1}^{k}w_{ki}-\sum_{i=1}^{k-1}w_{k-1,i}-\sum_{i=1}^{k+1}w_{k+1,i}-1\right)T(w).
\end{aligned}
\tag{1}
$$

A pair $\{(k,i),(k,j)\} \subset \mathfrak{V}$ is an *adjoining pair* for a graph G if $i<j$, there is a path in $G(\mathcal{C})$ from (k,i) to (k,j), and there is no path in $G(\mathcal{C})$ from (k,i) to (k,j) passing trough (k,t) with $i<t<j$.

Suppose that $\mathcal{C}$ is a noncritical set of relations whose associated graph $G=G(\mathcal{C})$ satisfies the following conditions:

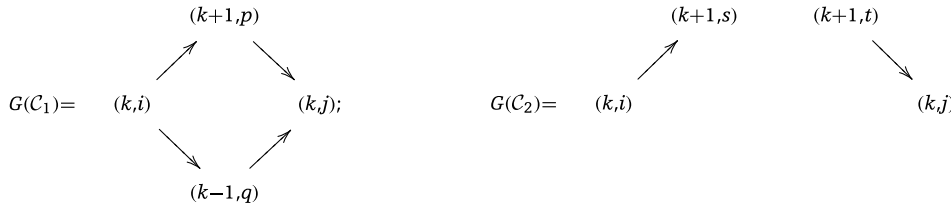
(i) G is reduced;

(ii) If there is a path in G connecting (k,i) and (k,j) with tail (k,i) and head (k,j), then $i<j$ (in particular G does not contain loops);

(iii) If the graph G contains an arrow connecting (k,i) and $(k+1,t)$ then $(k+1,s)$ and (k,j) with $i<j$, $s<t$ are not connected in G.

By [21, Theorem 4.33], the set $\mathcal{C}$ is admissible if and only if G is a union of connected graphs satisfying the following

$\diamond$-Condition: For every adjoining pair $\{(k,i),(k,j)\}$, $1 \leq k \leq n$, there exist p,q such that $\mathcal{C}_1 \subseteq \mathcal{C}$ or there exist $s < t$ such that $\mathcal{C}_2 \subseteq \mathcal{C}$, where the graphs associated to $\mathcal{C}_1$ and $\mathcal{C}_2$ are as follows:



From now on we will assume that $\mathcal{C}$ is an admissible set of relations and consider the $\mathfrak{sl}_{n+1}$-module $V_{\mathcal{C}}(T(L))$. The simplicity criteria for $V_{\mathcal{C}}(T(L))$ is as follows.

Theorem 4.6. [21, Theorem 5.6] The Gelfand–Tsetlin module $V_{\mathcal{C}}(T(L))$ is simple if and only if $\mathcal{C}$ is the maximal (admissible) set of relations satisfied by $T(L)$.

We will call $V_{\mathcal{C}}(T(L))$ a Γ-*relation Gelfand–Tsetlin module*. Similar construction can be made for any Gelfand–Tsetlin subalgebra $\Gamma_{\mathcal{F}} = w\Gamma_{st}, w \in W$. Applying w to the formulas (1) we obtain explicit action of $\mathfrak{sl}_{n+1}$ on $\Gamma_{\mathcal{F}}$-relation Gelfand–Tsetlin modules.

4.3 Classification of highest weight relation modules

The goal of this section is to describe all simple highest weight modules that can be realized as $V_{\mathcal{C}}(T(L))$ for some admissible set of relations $\mathcal{C}$. From now on we fix $\mathfrak{g} := \mathfrak{sl}_{n+1}$, the set of simple roots $\pi = \{\alpha_1, \ldots, \alpha_n\}$, and set $\alpha_{r,s} := \alpha_r + \ldots + \alpha_s$ for $1 \leq r \leq s \leq n$. We use elements E_{ij}, $i,j = 1,\ldots,n+1$, $i \neq j$, $H_k := E_{kk} - E_{k+1,k+1}$, $k = 1,\ldots,n$, as a basis of $\mathfrak{g}$.

From [21, Proposition 5.9], we have the following statement.

Lemma 4.7. If $\langle \lambda + \rho_\pi, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Delta_+ \setminus \{\alpha_{k,n} \mid k = 1,\ldots,n\}$, then the simple highest weight module $L(\lambda)$ is a Γ-relation module.

The following assertion classifies all simple highest weight Γ-relation modules.

Theorem 4.8. The simple highest weight module $L(\lambda)$ is a Γ-relation module if and only if one of the following conditions holds:

a) $\langle \lambda + \rho_\pi, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$, for all $\alpha \in \Delta_+ \setminus \{\alpha_{k,n} \mid k = 1,\ldots,n\}$.

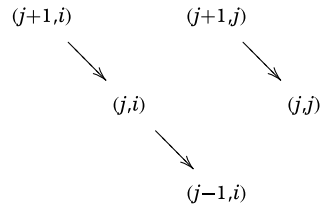
b) There exist unique i,j with $1 \leq i \leq j < n$ such that:

i) $\langle \lambda + \rho_\pi, \alpha_k^\vee \rangle \in \mathbb{Z}_{>0}$ for each $k > j$,

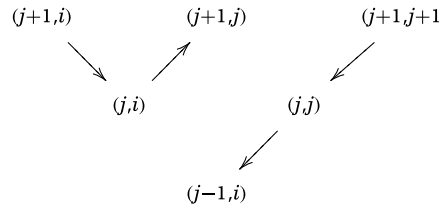
ii) $\langle \lambda + \rho_\pi, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Delta_+ \setminus \{\alpha_{i,k} \mid k \geq j\}$,

iii) $\left\langle \lambda + \rho_\pi, \alpha_{i,n}^\vee \right\rangle \in \mathbb{Z}_{\leq 0}$.

Proof. Suppose that $L(\lambda)$ is a Γ-relation Gelfand–Tsetlin module. Then $L(\lambda) \simeq V_{\mathcal{C}}(T(v))$ for some tableau $T(v)$. As $L(\lambda)$ is simple, we can assume without loss of generality that $T(v)$ is a highest weight vector of weight λ and $\mathcal{C}$ is the maximal set of relations satisfied by $T(v)$ (cf. Theorem 4.6). Then $E_{k,k+1}(T(v)) = 0$ for all $k = 1, \ldots, n$, which implies $v_{ij} = v_{kj}$ for all $1 \leq i,k < j \leq n+1$ (see (1)), and $\mathcal{C} \supset \{((i+1,j);(i,j)) \mid 1 \leq j \leq i \leq n\}$. Set $v_j := v_{n+1,j}$, and note that $v_i - v_{j+1} = \left\langle \lambda + \rho_\pi, \alpha_{i,j}^\vee \right\rangle$ for all $1 \leq i \leq j \leq n$. Now suppose that for some $\alpha \in \Delta_+ \setminus \{\alpha_{k,n} \mid k = 1, \ldots, n\}$ we have $\langle \lambda + \rho_\pi, \alpha^\vee \rangle \in \mathbb{Z}_{\leq 0}$. Hence, there exists a pair (r,s) with $1 \leq r \leq s < n$ such that $v_r - v_{s+1} = \langle \lambda + \rho_\pi, \alpha_{r,s}^\vee \rangle \in \mathbb{Z}_{<0}$. This shows that $I = \{(r,s) \mid v_s - v_r \in \mathbb{Z}_{>0} \text{ and } 1 \leq r < s < n\} \neq \emptyset$. Now choose $i = min\{r \mid (r,s) \in I\}$ and $j = min\{s \mid (i,s) \in I\}$. Then (j,i) and (j,j) form an adjoining pair and the associated graph looks as follows:



On the other hand, $\mathcal{C}$ is an admissible set of relations and, hence, by the $\diamond$-condition the associated graph must satisfy



Therefore, $v_j - v_{j+1} \in \mathbb{Z}_{>0}$ and $v_{j+1} - v_i \in \mathbb{Z}_{>0}$. Repeating the same argument, we conclude that $v_k - v_{k+1} \in \mathbb{Z}_{>0}$ for all $k = j, \ldots, n$ and $v_{n+1} - v_i \in \mathbb{Z}_{\geq 0}$. Consequently, we have $v_r - v_s \in \mathbb{Z}_{>0}$ for all $j \leq r < s \leq n+1$. On the other hand, from the choice of i, we have that $v_r - v_s \notin \mathbb{Z}_{\leq 0}$ for each $1 \leq r < i$ and $s > r$. From the definition of j we also have

$v_i - v_s \notin \mathbb{Z}_{\leq 0}$ for each $i < s < j$. Given that $v_s - v_i \in \mathbb{Z}_{\geq 0}$ for all $s \geq j$ and $v_i - v_r \notin \mathbb{Z}_{\leq 0}$ for all $i < r < j$, we obtain $v_s - v_r \notin \mathbb{Z}_{\leq 0}$ for all s, r such that $i < r < j \leq s \leq n+1$. Finally, $v_s - v_r \notin \mathbb{Z}_{\leq 0}$ for all s, r such that $i < r < s < j$. Indeed, assume that $v_s - v_r \in \mathbb{Z}_{<0}$ for some $i < r < s < j$. Then there exists $r < j' \leq s$ such that $v_k - v_{k+1} \in \mathbb{Z}_{>0}$ for all $k = j', \ldots, n$. In particular, $v_s - v_j \in \mathbb{Z}_{>0}$, which is a contradiction.

Conversely, suppose first that (a) holds. Then by Lemma 4.7 we have that $L(\lambda)$ is a Γ-relation module. Now assume (b) and let $v_s - v_{s+1} = \langle \lambda + \rho_\pi, \alpha_s^\vee \rangle$ for each $1 \leq s \leq n$ such that $\sum_{s=1}^{n+1} v_s = -\binom{n+1}{2}$. Then for some $1 \leq i \leq j < n$ the following conditions are satisfied:

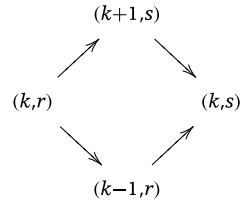
- $v_{n+1} - v_i \in \mathbb{Z}_{\geq 0}$,
- $v_r - v_s \in \mathbb{Z}_{>0}$ for all $j+1 \leq r < s \leq n+1$,
- $v_r - v_s \notin \mathbb{Z}_{\leq 0}$ for all $1 \leq r \leq j$, $r < s \leq n+1$ and $r, s \neq i$,
- $v_r - v_i \in \mathbb{Z}_{>0}$ for all $j+1 \leq r \leq n$,
- $v_r - v_i \notin \mathbb{Z}_{\leq 0}$ for all $1 \leq r \leq j$ and $r \neq i$.

Let $T(v)$ be a Gelfand–Tsetlin tableau with entries

$$v_{rs} = \begin{cases} v_s, & \text{if } 1 \leq s < i \text{ or } i \leq r \leq j, \\ v_{s+j-i+1}, & \text{if } i \leq s < r+i-j, \\ v_{s-r+j}, & \text{if } s \geq r+i-j, \end{cases}$$

for $1 \leq s \leq r \leq n+1$, and let $\mathcal{C}$ be the maximal set of relations satisfied by $T(v)$. To prove that $\mathcal{C}$ is admissible we consider the following cases.

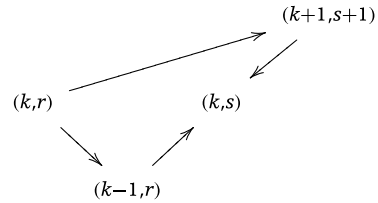
Case I: Suppose that (k, r) and (k, s) form an adjoining pair for some $1 \leq k < j$. Then we have an indecomposable subset $\mathcal{C}'$ of $\mathcal{C}$ with the associated graph



The same happens in the following cases:

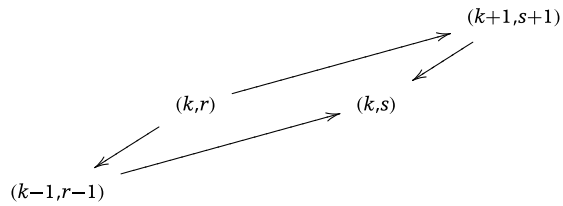
- $j \leq k \leq n$ and $1 < s < i$;
- $j < k \leq n$ and $1 \leq r < s = i$;
- $j+1 < k \leq n$ and $i < s < n+i-j$ where $r = s-1$.

Case II: Suppose $j \leq k \leq n$, $1 \leq r < i$, and $s \geq k+i-j$. If (k,r) and (k,s) form an adjoining pair, then we have an indecomposable subset $\mathcal{C}'$ of $\mathcal{C}$ with the following associated graph:

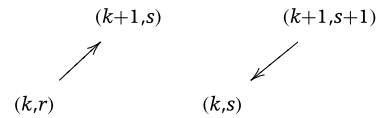


The graph is the same in the case $k = j$ and $r \geq i$.

Case III: Fix $j < k \leq n$ with $r \geq k+i-j$. If (k,r) and (k,s) form an adjoining pair, then the associated graph of the indecomposable subset $\mathcal{C}'$ of $\mathcal{C}$ is as follows:



Case IV: Finally, let $j < k \leq n$, $s = k+i-j$, and $r = s-1$. In this case (k,r) and (k,s) are an adjoining pair with the associated graph as follows:



The same happens in the case: $j < k \leq n$ and $r = k+i-j-1$ for each adjoining pair $\{(k,r),(k,s)\}$.

Therefore, $\mathcal{C}$ is admissible and $V_{\mathcal{C}}(T(v))$ is a simple module by Theorem 4.6. Hence, $V_{\mathcal{C}}(T(v)) = U(\mathfrak{g})T(v)$, $E_{k,k+1}(T(v)) = 0$, and $H_k(T(v)) = \langle \lambda, \alpha_k^\vee \rangle T(v)$ for all $k = 1,\dots,n$. ■

In particular, applying Theorem 4.8 and [31, Proposition 8.5], we get the following criterion for infinite-dimensional simple highest weight modules to be bounded Γ-relation modules.

Corollary 4.9. The simple highest weight module $L(\lambda)$ is a bounded infinite-dimensional Γ-relation module if and only if one of the following conditions holds:

(a) $\langle \lambda + \rho_\pi, \alpha_n^\vee \rangle \notin \mathbb{Z}_{>0}$, and $\langle \lambda + \rho_\pi, \alpha_k^\vee \rangle \in \mathbb{Z}_{>0}$ for all $k < n$;
(b) $\langle \lambda + \rho_\pi, \alpha_1^\vee \rangle \notin \mathbb{Z}$, and $\langle \lambda + \rho_\pi, \alpha_k^\vee \rangle \in \mathbb{Z}_{>0}$ for all $k > 1$;
(c) $\langle \lambda + \rho_\pi, \alpha_1^\vee \rangle \in \mathbb{Z}_{<0}$, $\langle \lambda + \rho_\pi, \alpha_{1n}^\vee \rangle \in \mathbb{Z}_{\leq 0}$, and $\langle \lambda + \rho_\pi, \alpha_k^\vee \rangle \in \mathbb{Z}_{>0}$ for all $k > 1$;
(d) There exists a unique $i \in \{2, \dots, n-1\}$ such that $\langle \lambda + \rho_\pi, \alpha_i^\vee \rangle \in \mathbb{Z}_{<0}$, $\left\langle \lambda + \rho_\pi, \alpha_{i-1,i}^\vee \right\rangle \in \mathbb{Z}_{>0}$, $\langle \lambda + \rho_\pi, \alpha_{in}^\vee \rangle \in \mathbb{Z}_{\leq 0}$, and $\langle \lambda + \rho_\pi, \alpha_k^\vee \rangle \in \mathbb{Z}_{>0}$ for all $k \neq i$;
(e) There exists a unique $i \neq n$ such that $\langle \lambda + \rho, \alpha_i^\vee \rangle \notin \mathbb{Z}$, $\left\langle \lambda + \rho_\pi, \alpha_{i+1}^\vee \right\rangle \notin \mathbb{Z}$, $\left\langle \lambda + \rho, \alpha_{i,i+1}^\vee \right\rangle \in \mathbb{Z}_{>0}$, and $\langle \lambda + \rho_\pi, \alpha_k^\vee \rangle \in \mathbb{Z}_{>0}$ for all $k \neq i, i+1$.

Remark 4.10. For $i = 1, 2, \dots, n$, the highest weight module $L(-\omega_i)$ is bounded but not a Γ-relation module.

Corollary 4.11. Let $\lambda \in \mathfrak{h}^*$. The following conditions are equivalent:

(a) The Verma module $M(\lambda)$ is a simple Γ-relation module;
(b) $\langle \lambda + \rho_\pi, \alpha^\vee \rangle \notin \mathbb{Z}$, for any $\alpha \in \Delta_+ \setminus \{\alpha_{k,n} \mid k = 1, \dots, n\}$, and $\left\langle \lambda + \rho_\pi, \alpha_{k,n}^\vee \right\rangle \notin \mathbb{Z}_{>0}$ for $k = 1, \dots, n$.

Remark 4.12. By Corollary 4.11, $M(\lambda)$ is a simple Γ-relation module if and only if $M(\lambda) \simeq V_{\mathcal{C}}(T(v))$, where $T(v)$ is a generic tableau and $\mathcal{C}$ is the maximal set of relations satisfied by $T(v)$. This generalizes [32, Proposition 1] (cf. [9, Corollary 3.7] and [21, Example 5.10]).

Since $L(\lambda)$ has finite-dimensional weight subspaces, then it is a $w\Gamma$-Gelfand–Tsetlin module for any $w \in W$. Denote by $\widetilde{W}$ the extension of the Weyl group of $\mathfrak{g}$ by the symmetries of the root system. Let $w \in \widetilde{W}$ and $\mathfrak{b} = w\mathfrak{b}_{st}$. Then, clearly, $L_{\mathfrak{b}}(w\lambda)$ is a $w\Gamma$-relation Gelfand–Tsetlin module if and only if $w\lambda$ satisfies the conditions of Corollary 4.9 for $w\alpha_k^\vee$, $k = 1, \dots, n$. In particular, we obtain the following:

Corollary 4.13. Let $\lambda \in \mathfrak{h}^*$ and $\left\langle \lambda + \rho_\pi, \beta_j^\vee \right\rangle \in \mathbb{Z}_{>0}$ for some simple roots β_j, $j = 1, \dots, t$. Consider the minimal $w \in W$ such that $\left\langle w(\lambda + \rho_\pi), \beta_j^\vee \right\rangle \in \mathbb{Z}_{\leq 0}$ for all $j = 1, \dots, t$, and set $\Gamma_{\mathcal{F}} = w\Gamma$. Then the simple highest weight module $L(\lambda)$ is a bounded infinite-dimensional $\Gamma_{\mathcal{F}}$-relation Gelfand–Tsetlin module if and only if λ satisfies one of the conditions of Corollary 4.9.

Proof. Since $L_{w\mathfrak{b}_{st}}(w\lambda) \simeq L(\lambda)$, it is sufficient to consider the weight $w\lambda$ and apply Corollary 4.9. ■

Corollary 4.13 shows that $L(\lambda)$ can have different realizations via Gelfand–Tsetlin tableaux as a relation module.

Lemma 4.14. Suppose λ satisfies conditions (a) or (e) of Corollary 4.9. Let $i > 1$ and set $w_i = s_{i-1}s_is_{i-2}s_{i-1}\cdots s_2s_3s_1s_2 \in W$, then $L_i = L_{w_i^{-1}b_{st}}(w_i^{-1}\lambda)$ is a Γ-relation Gelfand–Tsetlin module.

Proof. Let us consider $\{v_1, \dots, v_{n+1}\} \subseteq \mathbb{C}$ such that $v_1 - v_2 = \langle \lambda + \rho_\pi, \alpha_i^\vee \rangle$, $v_1 - v_{i+1} = -\langle \lambda + \rho_\pi, \alpha_{i-1}^\vee \rangle$, $v_2 - v_{i+2} = \langle \lambda + \rho_\pi, \alpha_{i+1}^\vee \rangle$, $v_{k+2} - v_{k+3} = \langle \lambda + \rho_\pi, \alpha_k^\vee \rangle$ for all $1 \le k \le i-2$, $v_k - v_{k+1} = \langle \lambda + \rho_\pi, \alpha_k^\vee \rangle$ for all $k \ge i+2$, and $\sum_{i=1}^{n+1} v_i = -\binom{n+1}{2}$.

Let $T(v)$ be the Gelfand–Tsetlin tableau with entries

$$v_{rs} = \begin{cases} v_1 + i - 1, & \text{if } r = s = 1, \\ v_1 + i + 1 - r, & \text{if } 1 < r \le i \text{ and } s = r-1, \\ v_1, & \text{if } r > i \text{ and } s = i, \\ v_2 + i + 1 - r, & \text{if } 1 < r \le i \text{ and } s = r, \\ v_2, & \text{if } r > i \text{ and } s = r, \\ v_3, & \text{if } r \ge 3 \text{ and } s = 1, \\ v_{s+2}, & \text{if } r \ge 4 \text{ and } 2 \le s < i, \\ v_s, & \text{if } r > s \ge i+1. \end{cases}$$

Consider the set of relations $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$, where

$$\mathcal{C}^+ := \{((r+1,s);(r,s)) \mid 1 \le s < r \le n \text{ or } r = s = 1\} \bigcup \{((r+1,r+1);(r,r)) \mid i+1 \le r \le n\}$$

$$\mathcal{C}^- := \{((r,r);(r+1,r+1)) \mid 1 < r \le i\} \bigcup \{((r,s);(r+1,s+1)) \mid 1 \le s < r \le n\}.$$

In this case, $\mathcal{C}$ is an admissible set of relations, $T(V)$ is a $\mathcal{C}$-realization, and $V_{\mathcal{C}}(T(v))$ is a simple module by Theorem 4.6. Moreover, $V_{\mathcal{C}}(T(v)) = U(\mathfrak{g})T(v)$, $E_{12}(T(v)) =$

$E_{i+1,1}(T(v)) = E_{2,i+2}(T(v)) = E_{k,k+1}(T(v)) = 0$ for all $k \notin \{i-1, i, i+1\}$, $H_1(T(v)) = \langle \lambda, \alpha_i^\vee \rangle T(v)$, $\sum_{k=1}^{i}(-H_k)(T(v)) = \left\langle \lambda, \alpha_{i-1}^\vee \right\rangle T(v)$, $\sum_{k=2}^{i+1} H_k(T(v)) = \left\langle \lambda, \alpha_{i+1}^\vee \right\rangle T(v)$, $H_{j+2}(T(v)) = \left\langle \lambda, \alpha_j^\vee \right\rangle T(v)$ for all $1 \leq j \leq i-2$, and $H_j(T(v)) = \left\langle \lambda, \alpha_j^\vee \right\rangle T(v)$ for all $j \geq i+2$. Hence, $V_{\mathcal{C}}(T(v)) \simeq L_i$. ■

5 Induced Relation Modules

In this section we fix $f = E_{21}$ and $\Gamma = \Gamma_{st}$.

5.1 Localization of relation modules

Lemma 5.1. Suppose that $T(v)$ is a $\mathcal{C}$-realization for some admissible set of relations $\mathcal{C}$.

(a) f is injective on $V_{\mathcal{C}}(T(v))$ if and only if $((1,1);(2,1)) \notin \mathcal{C}$ and $((1,1);(2,2)) \notin \mathcal{C}$.

(b) f is surjective on $V_{\mathcal{C}}(T(v))$ if and only if $((2,1);(1,1)) \notin \mathcal{C}$ and $((2,2);(1,1)) \notin \mathcal{C}$.

Proof. For any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ we have $f(T(w)) = T(w - \delta^{11}) \in \mathcal{B}_{\mathcal{C}}(T(v))$, since $((1,1);(2,1)) \notin \mathcal{C}$ and $((1,1);(2,2)) \notin \mathcal{C}$. Suppose that $0 \neq u = \sum_{i\in I} a_i T(w^i) \in V_{\mathcal{C}}(T(v))$. Then

$$f(u) = f\Big(\sum_{i\in I} a_i T(w^i)\Big) = \sum_{i\in I} a_i T(w^i - \delta^{11}) \neq 0,$$

as $a_i \neq 0$, $T(w^i - \delta^{11}) \in V_{\mathcal{C}}(T(v))$ and $T(w^i - \delta^{11}) \neq T(w^j - \delta^{11})$ for all $i \neq j \in I$. On the other hand, if $((1,1);(2,1)) \in \mathcal{C}$ or $((1,1);(2,2)) \in \mathcal{C}$, then there exists $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ such that $w_{11} - w_{21} = 1$ or $w_{11} - w_{22} = 1$, and hence $f(T(w)) = T(w - \delta^{11}) = 0$, as $T(w - \delta^{11}) \notin \mathcal{B}_{\mathcal{C}}(T(v))$. This shows item (a).

If $((2,1);(1,1)) \notin \mathcal{C}$ and $((2,2);(1,1)) \notin \mathcal{C}$ then for any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$, we have $T(w + \delta^{11}) \in \mathcal{B}_{\mathcal{C}}(T(v))$ and $f(T(w + \delta^{11})) = T(w)$. Hence, for any $u \in V_{\mathcal{C}}(T(v))$,

$$u = \sum_{i\in I} a_i T(w^i) = \sum_{i\in I} a_i f(T(w^i + \delta^{11})) = f\Big(\sum_{i\in I} a_i T(w^i + \delta^{11})\Big).$$

On the other hand, assume that $((2,1);(1,1)) \in \mathcal{C}$ or $((2,2);(1,1)) \in \mathcal{C}$ and f is surjective on $V_{\mathcal{C}}(T(v))$. Then there exists $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ with $w_{11} = w_{21}$ or $w_{11} = w_{22}$ and

$u \in V_{\mathcal{C}}(T(v))$, such that

$$f(u) = \sum_{i \in I} a_i T(w^i - \delta^{11}) = T(w).$$

Hence, $T(w) = T(w^i - \delta^{11})$ for some $i \in I$. As $T(w^i) \in \mathcal{B}_{\mathcal{C}}(T(v))$, we have $0 = w_{21} - w_{11} = w^i_{21} - w^i_{11} + 1 \in \mathbb{Z}_{>0}$ or $0 = w_{22} - w_{11} = w^i_{22} - w^i_{11} + 1 \in \mathbb{Z}_{>0}$. This implies (b). ■

Next we consider the twisted localization of $V_{\mathcal{C}}(T(v))$ with respect to f and its tableaux realization.

Lemma 5.2. Let $\mathcal{C}$ be an admissible maximal set of relations such that

$$\mathcal{C} \cap \{((1,1);(2,1)), ((1,1);(2,2))\} = \emptyset$$

and $T(v)$ a $\mathcal{C}$-realization. If $\mathcal{D} = \mathcal{C} \setminus \{((2,1);(1,1)), ((2,2);(1,1))\}$ then the localized module $D_f(V_{\mathcal{C}}(T(v)))$ is isomorphic to $V_{\mathcal{D}}(T(v))$.

Proof. Set $M = V_{\mathcal{C}}(T(v))$ and $N = V_{\mathcal{D}}(T(v))$. By Lemma 5.1, the action of f on M is injective but not bijective. Hence, M is a proper submodule of $D_f M$. On the other hand, consider the set of relations $\mathcal{C}_1 = \mathcal{D} \cup \{((1,1);(2,1))\}$. By Theorem 4.6, $V_{\mathcal{C}_1}(T(v + \delta^{11}))$ is a simple module isomorphic to N/M and hence, M is a maximal submodule of N. Finally, since f is bijective on N, by Proposition 3.1, we have $N \simeq D_f M$. ■

Under the assumptions of Lemma 5.2 the following is straightforward.

Lemma 5.3. For any tableau $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ and any $x \in \mathbb{C}$ denote by $T(w)^x$ the image of $T(w)$ as an element of the twisted module $D^x_f(V_{\mathcal{C}}(T(v)))$. Then we have

$$E_{12} \cdot T(w)^x = -(w_{11} + x - w_{21})(w_{11} + x - w_{22}) T(w + \delta^{11})^x.$$

$$E_{k,k+1} \cdot T(w)^x = -\sum_{i=1}^{k} \left(\frac{\prod_{j=1}^{k+1} (w_{ki} - w_{k+1,j})}{\prod_{j \neq i}^{k} (w_{ki} - w_{kj})} \right) T(w + \delta^{ki})^x, \text{for all } k = 2, \ldots, n.$$

$$E_{21} \cdot T(w)^x = T(w - \delta^{11})^x.$$

$$E_{32} \cdot T(w)^x = \frac{w_{21} - (w_{11} + x)}{w_{21} - w_{22}} T(w - \delta^{21})^x + \frac{w_{22} - (w_{11} + x)}{w_{22} - w_{21}} T(w - \delta^{22})^x.$$

$$E_{k+1,k}\cdot T(w)^x=\sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k-1}(w_{ki}-w_{k-1,j})}{\prod_{j\neq i}^{k}(w_{ki}-w_{kj})}\right)T(w-\delta^{ki})^x,\ \textit{for all } k=3,\dots,n.$$

$$H_1\cdot T(w)^x=(2(w_{11}+x)-(w_{21}+w_{22})-1)T(w)^x.$$

$$H_2\cdot T(w)^x=\big(2(w_{21}+w_{22})-(w_{31}+w_{32}+w_{33})-(w_{11}+x)-1\big)\,T(w)^x.$$

$$H_k\cdot T(w)^x=\left(2\sum_{i=1}^{k}w_{ki}-\sum_{i=1}^{k+1}w_{k+1,i}-\sum_{i=1}^{k-1}w_{k-1,i}-1\right)T(w)^x,\ \textit{for all } k=3,\dots,n.$$

Theorem 5.4. Let M be a Γ-relation module with an injective action of f. Then $D_f^x M$ is a Γ-relation Gelfand–Tsetlin module.

Proof. By hypothesis, $M\simeq V_{\mathcal{C}}(T(v))$ for some $\mathcal{C}$-realization $T(v)$. By Lemma 5.2, the set $\{T(v+\ell\delta^{11})\mid \ell\in\mathbb{Z}\}$ is a basis of the localized module D_fM. Define the linear isomorphism ϕ from $D_f^x(V_{\mathcal{C}}(T(v)))$ to $V_{\mathcal{D}}(T(v+x\delta^{11}))$, such that $\phi(T(w)):=T(w+x\delta^{11})=T(w)^x\in\mathcal{B}_{\mathcal{D}}(T(v+x\delta^{11}))$ for any $T(w)\in\mathcal{B}_{\mathcal{C}}(T(v))$. Comparing the twisted action of $\mathfrak{g}$ from Lemma 5.3 with the Gelfand–Tsetlin formulas, we have

$$\phi(g\cdot T(w)^x)=g\phi(T(w)^x)$$

for any $g\in\mathfrak{g}$. Hence, ϕ is an isomorphism of modules, which completes the proof. ■

Corollary 5.5. Let M be a simple Γ-relation Gelfand–Tsetlin module.

(a) If f is bijective on M, then M is isomorphic to $D_f^x(N)$ for some simple Γ-relation Gelfand–Tsetlin module N with an injective action of f and $x\in\mathbb{C}\setminus\mathbb{Z}$.

(b) If f is surjective on M but not injective, then M is isomorphic to $D_f(N)/N$ for some simple Γ-relation Gelfand–Tsetlin module N with an injective action of f.

Proof. As M is a simple relation module, then $M\simeq V_{\mathcal{C}}(T(v))$ for some Gelfand–Tsetlin tableau and a maximal admissible set of relations $\mathcal{C}$ satisfied by $T(v)$.

Part (a): Suppose that f is bijective on M. By Lemma 5.1, we have $v_{11}-v_{21}\notin\mathbb{Z}$ and $v_{11}-v_{22}\notin\mathbb{Z}$. Without loss of generality, we assume that $v_{21}-v_{22}\notin\mathbb{Z}$. In this case, let $x=v_{11}-v_{21}$ and consider the Gelfand–Tsetlin tableau $T(v')=T(v+x\delta^{11})$. Then, $\mathcal{C}'=\mathcal{C}\cup\{((2,1);(1,1))\}$ is the maximal admissible set of relations satisfied by $T(v')$. Hence, by Theorem 4.6, $N=V_{\mathcal{C}'}(T(v'))$ is a simple Γ-relation Gelfand–Tsetlin module. By Lemma 5.1, f is injective on N but not surjective. Then $D_f^xN\simeq M$ by Theorem 5.4.

Part (b): Suppose that f is surjective on M, but not injective. By Lemma 5.1, we can assume that $v_{11}-v_{21}\in\mathbb{Z}_{>0}$ and $v_{11}-v_{22}\notin\mathbb{Z}$. Set again $x=v_{11}-v_{21}$ and $T(v')=T(v+x\delta^{11})$. Then $\mathcal{C}'=\Big(\mathcal{C}\setminus\{((1,1);(2,1))\}\Big)\cup\{((2,1);(1,1))\}$ is the maximal admissible set of relations satisfied by $T(v')$. Hence, $N=V_{\mathcal{C}'}(T(v'))$ is a simple relation Gelfand–Tsetlin module by Theorem 4.6. From Lemma 5.1, we have that f is injective on N and that the localized module $D_f(N)$ is isomorphic to $V_{\mathcal{C}\setminus\{((1,1);(2,1))\}}(T(v'))$ (cf. Lemma 5.2). Moreover, N is a maximal submodule of $D_f(N)$, and hence $D_f(N)/N\simeq V_{\mathcal{C}}(T(v'))$ is a simple module. We conclude that $D_f(N)/N\simeq V_{\mathcal{C}}(T(v))$. ■

5.2 $\mathfrak{sl}_2$-induced relation modules

Let $\mathfrak{p}\subset\mathfrak{g}$ be a parabolic subalgebra of $\mathfrak{g}$ with the Levi factor isomorphic to $\mathfrak{a}_\alpha=\mathfrak{sl}_2+\mathfrak{h}$ based on the root $\alpha=\alpha_1\in\pi$. Let $V=V(\gamma,\mu)$ be a simple cuspidal weight $\mathfrak{a}_\alpha$-module, where $\mu\in\mathfrak{h}^*$ is such that $V_\mu\neq 0$, and $\gamma\in\mathbb{C}$ is the eigenvalue of the Casimir element $c_\alpha=(E_{11}-E_{22}+1)^2+4E_{21}E_{12}$ of $\mathfrak{a}_\alpha$. Denote by $L(\gamma,\mu)=L_{\mathfrak{p}}^{\mathfrak{g}}(\{\alpha\},V)$ the unique simple quotient of the induced module $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\{\alpha\},V)$ with the trivial action of the radical of $\mathfrak{p}$ on V. Let Γ_α be any Gelfand–Tsetlin subalgebra corresponding to the flag containing $\mathfrak{a}_\alpha$ and $\mathfrak{h}$.

We have the following:

Theorem 5.6. Let $n>1$. The module $L(\gamma,\mu)$ is a Γ_α-relation Gelfand–Tsetlin $\mathfrak{g}$-module if and only if $L(\lambda)$ is a Γ_α-relation highest weight $\mathfrak{g}$-module, where $\langle\mu-\lambda,\alpha_1^\vee\rangle=2x$, $\langle\mu-\lambda,\alpha_2^\vee\rangle=-x$, $\langle\mu-\lambda,\alpha_i^\vee\rangle=0$ for each $i=3,\dots,n$, $\langle\lambda+\rho_\pi,\alpha_1^\vee\rangle\notin\mathbb{Z}_{\geq 0}$ and $(2x-\mu_1-1)^2=\gamma$. In this case, $L(\gamma,\mu)\simeq D_f^x(L(\lambda))$ and x satisfies the condition $x-\langle\mu+\rho_\pi,\alpha_1^\vee\rangle\notin\mathbb{Z}$. Moreover,

(a) $L(\gamma,\mu)$ is bounded if and only if $L(\lambda)$ is bounded;

(b) If λ is dominant then $s_1\cdot\lambda$ is dominant and $\gamma\neq m^2$ for all $m\in\mathbb{Z}$. In this case, $Ann_{U(\mathfrak{g})}L(\gamma,\mu)=Ann_{U(\mathfrak{g})}L(\lambda)=Ann_{U(\mathfrak{g})}L(s_1\cdot\lambda)$;

(c) If $\gamma=m^2$ for some $m\in\mathbb{Z}\setminus\{0\}$ then $L(\lambda)$ is bounded and the weight $w\cdot\lambda$ is integral, for all $w\in W$. In this case, $Ann_{U(\mathfrak{g})}L(\lambda)\subset Ann_{U(\mathfrak{g})}L(\gamma,\mu)\subset Ann_{U(\mathfrak{g})}L(w\cdot\lambda)$, if $w\cdot\lambda$ is dominant.

Proof. Since V is a simple dense $\mathfrak{sl}_2$-module, then $\gamma\neq(\mu_1-2k+1)^2$ for all $k\in\mathbb{Z}$, and hence $x\notin\mathbb{Z}$. On the other hand,$\gamma=m^2$ for some $m\in\mathbb{Z}$ if and only if $\langle\lambda+\rho_\pi,\alpha_1^\vee\rangle\in\mathbb{Z}$. Consider the following two cases:

Case 1: Let $\gamma \neq m^2$ for all $m \in \mathbb{Z}$. Assume that $L(\gamma, \mu)$ is a Γ_α-relation Gelfand–Tsetlin $\mathfrak{g}$-module. Then $L(\gamma, \mu) \simeq V_{\mathcal{C}}(T(v))$ for some set of relations $\mathcal{C}$ and a tableau $T(v) = T(v_{ij})$, such that

$$v_{ij} = \begin{cases} v_1 + x, & \text{if } i = 1 \\ v_1, & \text{if } i > j = 1 \\ v_j, & \text{if } j \geq 2, \end{cases}$$

with $v_1 - v_2 = \langle \mu + \rho_\pi, \alpha_1^\vee \rangle - 2x$, $v_2 - v_3 = \langle \mu + \rho_\pi, \alpha_2^\vee \rangle + x$, $v_j - v_{j+1} = \left\langle \mu + \rho_\pi, \alpha_j^\vee \right\rangle$ for each $3 \leq j \leq n$ and $\sum_{j=1}^{n+1} v_j = -n - 1$. On the other hand, f is bijective on $V_{\mathcal{C}}(T(v))$ if and only if $x - \langle \mu, \alpha_1^\vee \rangle \notin \mathbb{Z}$, in which case

$$\mathcal{C} \cap \{((1,1);(2,1)), ((1,1);(2,2)), ((2,1);(1,1)), ((2,2);(1,1))\} = \emptyset.$$

Consider the tableau $T(v') = T(v'_{ij})$ with entries $v'_{11} = v_{21}$ and $v'_{ij} = v_{ij}$ for $i \neq 1$. Then, $\mathcal{D} = \mathcal{C} \cup \{((2,1);(1,1))\}$ is the maximal set of relations satisfied by $T(v')$, since $\langle \lambda + \rho_\pi, \alpha_1^\vee \rangle \notin \mathbb{Z}$. Hence, the module $V_{\mathcal{D}}(T(v'))$ is simple by Theorem 4.6. Then $V_{\mathcal{D}}(T(v')) \simeq U(\mathfrak{g})T(v')$, $E_{k,k+1}(T(v')) = 0$, and $H_k(T(v)) = \lambda_k T(v)$ for all $k = 1, \ldots, n$, $\lambda_k = \langle \lambda, \alpha_k^\vee \rangle$. We get $V_{\mathcal{D}}(T(v')) \simeq L(\lambda)$. Moreover, $c_\alpha(T(v')) = (\lambda_1 + 1)^2 T(v') = \gamma T(v')$. Applying Theorem 5.4, we conclude that $D_f^x L(\lambda) \simeq L(\gamma, \mu)$.

Conversely, let $L(\lambda)$ be a Γ_α-relation Gelfand–Tsetlin module. Then $L(\lambda) \simeq V_{\mathcal{C}}(T(v))$, where $T(v) = T(v_{ij})$ is the Gelfand–Tsetlin tableau such that $v_{ij} = v_j$ with $v_j - v_{j+1} = \left\langle \lambda + \rho_\pi, \alpha_j^\vee \right\rangle$ for each $1 \leq j \leq n$, and $\sum_{j=1}^{n+1} v_j = -n - 1$. Note that $\mathcal{C}$ is the maximal set of relations satisfied by $T(v)$. Without loss of generality we assume that $((2,1);(1,1)) \in \mathcal{C}$ and $((1,1);(2,1)) \notin \mathcal{C}$. Hence, the localized module $D_f^x(L(\lambda))$ is isomorphic to $V_{\mathcal{D}}(T(v + x\delta^{11}))$, where $\mathcal{D} = \mathcal{C} \setminus \{((2,1);(1,1))\}$ by Theorem 5.4. Given that $x + v_{11} - v_{22} \notin \mathbb{Z}$, then $V_{\mathcal{D}}(T(v + x\delta^{11}))$ is a simple module. Further, the $\mathfrak{sl}_2$-module $\operatorname{span}_{\mathbb{C}}\{T(v+(x+\ell)\delta^{11}) \mid \ell \in \mathbb{Z}\}$ is isomorphic to V. By (1), we have $E_{k,k+1}(T(v+(x+\ell)\delta^{11})) = 0$ for all $k = 2, \ldots, n$ and $\ell \in \mathbb{Z}$. Hence, we get an epimorphism of $U(\mathfrak{g})$-modules

$$\phi : M_{\mathfrak{p}}^{\mathfrak{g}}(\{\alpha\}, V) \to V_{\mathcal{D}}(T(v + x\delta^{11})),$$

such that $\phi(u \otimes T(v)) = uT(v)$ for all $u \in U(\mathfrak{g})$. Therefore, $L(\gamma, \mu) \simeq V_{\mathcal{D}}(T(v + x\delta^{11}))$.

Case 2: Let $\gamma = m^2$ for some $m \in \mathbb{Z}$. In this case, $\langle \lambda + \rho_\pi, \alpha_1^\vee \rangle \in \mathbb{Z}_{<0}$ and the construction is similar. We leave the details out.

Note that if $\langle \lambda + \rho_\pi, \alpha_1^\vee \rangle = 0$ then $L(\gamma, \mu)$ is not a relation module. The statement (a) is clear from the construction, while (b) and (c) follow from Proposition 2.2, Corollary 2.3, and Corollary 3.5. ■

5.3 Family of induced relation modules

In this section, we give an explicit construction of a family of parabolically induced bounded Γ-relation modules.

Fix complex $\{u_i\}_{i=1,\dots,n+1}$ and $\{v_i\}_{i=1,\dots,n}$ satisfying conditions:

(a) $u_i - v_1 \notin \mathbb{Z}$ for any $1 \le i \le n$.

(b) $v_j - v_{j+1} \in \mathbb{Z}_{>0}$ for any $1 \le j < n$.

Let $T(v)$ be a Gelfand–Tsetlin tableau with entries

$$v_{ij} = \begin{cases} u_i, & \text{if } j = 1 \\ v_{j-1}, & \text{if } j \neq 1 \end{cases} \tag{2}$$

for $1 \le j \le i \le n+1$.

Consider the set of relations $\mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^-$, where

$$\mathcal{Q}^+ := \{((i+1,j);(i,j)) \mid 2 \le j \le i \le n\}$$

$$\mathcal{Q}^- := \{((i,j);(i+1,j+1)) \mid 2 \le j \le i \le n\}.$$

Lemma 5.7. Let $T(v)$ be a Gelfand–Tsetlin tableau (2). Then, $V_{\mathcal{Q}}(T(v))$ is a bounded dense $\mathfrak{g}$-module. Moreover, $V_{\mathcal{Q}}(T(v))$ is simple if and only if $u_i - u_{i+1} \notin \mathbb{Z}$ for all $1 \le i \le n$.

Proof. Follows from [33, Lemmas 3.1, 3.2, and 3.3]. ■

If $\mathcal{C}$ is any admissible set of relations containing $\mathcal{Q}$ and $T(v)$ is a $\mathcal{C}$-realization, then $V_{\mathcal{C}}(T(v))$ is a submodule of $V_{\mathcal{Q}}(T(v))$ and hence a bounded module. In particular, $V_{\mathcal{C}}(T(v))$ has finite length [17].

For $m \in \{2 \dots, n\}$ consider the tableau $T^m(v)$ as in (2) satisfying the conditions

$$u_i = u_m \text{ for } i = m+1, \dots, n+1, \text{ and } u_i - u_{i+1} \notin \mathbb{Z} \text{ for } i = 1, 2, \dots, m-1.$$

Let $\mathcal{C}^m = \mathcal{Q} \cup \{((i+1,1);(i,1)) \mid m \le i \le n\}$. Then, $T^m(v)$ is a $\mathcal{C}^m$-realization and the relation module $V_{\mathcal{C}^m}(T^m(v))$ is simple.

Consider the subset $\Sigma_m = \{\alpha_1, \dots, \alpha_{m-1}\} \subset \pi$ of simple roots and the corresponding parabolic subalgebra $\mathfrak{p}_m = \mathfrak{p}_{\Sigma_m} \subset \mathfrak{g}$. Then, $\mathfrak{p}_m$ has the Levi subalgebra isomorphic to $\mathfrak{sl}_m + \mathfrak{h}$. We have the following:

Theorem 5.8. For $m \in \{2, \dots, n\}$, the module $V_{\mathcal{C}^m}(T^m(v))$ is isomorphic to $L^{\mathfrak{g}}_{\mathfrak{p}_m}(\Sigma_m, V)$ for some simple cuspidal $\mathfrak{sl}_m$-module V.

Proof. By construction, $\mathcal{C}^m$ is the maximal set of admissible relations satisfied by $T^m(v)$. Hence, $V_{\mathcal{C}^m}(T^m(v))$ is a simple Gelfand–Tsetlin $\mathfrak{g}$-module by Theorem 4.6. Let $\mathcal{D}^m = \mathcal{D}^+ \cup \mathcal{D}^-$, where

$$\mathcal{D}^+ = \{((i+1,j);(i,j)) \mid 2 \le j \le i \le m-1\}$$
$$\mathcal{D}^- = \{((i,j);(i+1,j+1)) \mid 2 \le j \le i \le m-1\}.$$

Then $V = V_{\mathcal{D}^m}(T^m(v)) = \operatorname{span}_{\mathbb{C}} \mathcal{B}_{\mathcal{D}^m}(T^m(v))$ is a simple cuspidal $\mathfrak{sl}_m$-module by Lemma 5.7, where $\mathcal{B}_{\mathcal{D}^m}(T^m(v))$ denotes the set of $\mathcal{D}^m$-realizations of the form $T^m(v+z)$, $z \in \mathbb{Z}_0^{\frac{m(m+1)}{2}}$. It follows from (1) that V is a $\mathfrak{p}_m$-module with the trivial action of $\mathfrak{u}^+_{\Sigma_m}$. Hence, we have a homomorphism

$$\phi : M^{\mathfrak{g}}_{\mathfrak{p}}(\Sigma, V) \to V_{\mathcal{C}^m}(T^m(v))$$

of $U(\mathfrak{g})$-modules, such that $\phi(u \otimes T^m(v)) = uT^m(v)$ for all $u \in U(\mathfrak{g})$. Since $V_{\mathcal{C}^m}(T^m(v))$ is a simple $\mathfrak{g}$-module, then ϕ is surjective, and $L^{\mathfrak{g}}_{\mathfrak{p}}(\Sigma, V) \simeq V_{\mathcal{C}^m}(T^m(v))$. ∎

5.4 Localization of highest weight modules with respect to E_{m1}

For any $2 < m \le n+1$ and any $k \le m-1$ fix $i_k \in \{1, \dots, k\}$. Associated with the set $\{i_1, \dots, i_{m-1}\}$ define

$$\varepsilon(i_1, \dots, i_{m-1}) := -\delta^{1i_1} - \delta^{2i_2} - \delta^{3i_3} - \dots - \delta^{m-1,i_{m-1}} \in \mathbb{Z}_0^{\frac{(n+1)(n+2)}{2}}.$$

Suppose that $T(v)$ is a $\mathcal{C}$-realization for some admissible set of relations $\mathcal{C}$. For each $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ and any $1 \le i_k \le k \le m-1$ define

$$a(w, i_1, \dots, i_{m-1}) := \begin{cases} 0, & \text{if } T(w + \varepsilon(i_1, \dots, i_{m-1})) \notin \mathcal{B}_{\mathcal{C}}(T(v)) \\ \prod_{s=2}^{m-1} \frac{\prod_{t \ne i_{s-1}}^{s-1} (w_{si_s} - w_{s-1,t})}{\prod_{t \ne i_s}^{s} (w_{si_s} - w_{st})}, & \text{if } T(w + \varepsilon(i_1, \dots, i_{m-1})) \in \mathcal{B}_{\mathcal{C}}(T(v)). \end{cases}$$

One can easily check by direct computation the following analog of [19,Proposition 3.13].

Proposition 5.9. Let $\mathcal{C}$ be an admissible set of relations and $T(v)$ a $\mathcal{C}$-realization. If $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ then

$$E_{m1}(T(w)) = \sum_{\substack{k=1,\dots,m-1 \\ (i_1,\dots,i_{m-1})\in\{1,\dots,k\}^{m-1}}} a(w,i_1,\dots,i_{m-1})\ T(w+\varepsilon(i_1,\dots,i_{m-1})), \tag{3}$$

for $m \in \{3,\dots,n+1\}$.

Theorem 5.10. Let $T(v)$ be the Gelfand–Tsetlin tableau satisfying (2), $\mathcal{C}$ an admissible set of relations containing $\mathcal{Q}$, for which $T(v)$ is a $\mathcal{C}$-realization. Then

(a) E_{m1} is injective on $V_{\mathcal{C}}(T(v))$ if and only if $((m-1,1);(m,1)) \notin \mathcal{C}$;

(b) E_{m1} is surjective on $V_{\mathcal{C}}(T(v))$ if and only if $((m,1);(m-1,1)) \notin \mathcal{C}$.

Proof.

(a) For any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ we have

$$T(w+\varepsilon(1,\dots,1)) = T(w-\delta^{11}-\delta^{21}-\delta^{31}-\dots-\delta^{m-1,1}) \in \mathcal{B}_{\mathcal{C}}(T(v)).$$

In fact, $(w_{i+1,1}-1)-(w_{i1}-1) = w_{i+1,1}-w_{i1}$ for all $i=1,\dots,m-2$, $w_{m1}-(w_{m-1,1}-1) = w_{m1}-w_{m-1,1}+1 \notin \mathbb{Z}_{\leq 1}$, and $(w_{i1}-1)-w_{i+1,2} = w_{i1}-w_{i+1,2}-1 \notin \mathbb{Z}$ for all $i=1,\dots,m-1$. On the other hand, for any $s \in \{3,\dots m-1\}$ given that $w_{s1}-w_{s-1,2} \notin \mathbb{Z}$ and $w_{s-1,2}-w_{s-1,t} \in \mathbb{Z}$ for all $t \in \{2,\dots s-1\}$, we have $w_{s1}-w_{s-1,t} \notin \mathbb{Z}$ for all $t \in \{2,\dots s-1\}$. Hence, $a(w,1,\dots,1) \neq 0$. Using the fact that $T(w+\varepsilon(1,\dots,1)) \neq T(w+\varepsilon(i_1,\dots,i_{m-1}))$ for all $(i_1,\dots,i_{m-1}) \neq (1,\dots,1)$, we obtain (cf. formula (3))

$$E_{m1}(T(w)) = a(w,1,\dots,1)\ T(w+\varepsilon(1,\dots,1)) +$$

$$+\sum_{k=2}^{m-1} \sum_{(i_1,\dots,i_{m-1})\neq(1,\dots,1)} a(w,i_1,\dots,i_{m-1})\ T(w+\varepsilon(i_1,\dots,i_{m-1})) \neq 0.$$

Now, suppose that $0 \neq u = \sum_{i\in I} c_i T(w^i) \in V_{\mathcal{C}}(T(v))$ with $c_i \neq 0$ for all $i \in I$. Then

$$E_{m1}(u) = \sum_{\substack{i\in I,\ k=1,\dots,m-1 \\ (i_1,\dots,i_{m-1})\in\{1,\dots,k\}^{m-1}}} c_i\ a(w^i, i_1, \dots, i_{m-1})\ T(w^i - \varepsilon(i_1, \dots, i_{m-1})) \neq 0.$$

In fact, for any $i \in I$, we have $c_i a(w^i, 1, \dots, 1)\ T(w^i + \varepsilon(1, \dots, 1)) \neq 0$ and $T(w^i + \varepsilon(1, \dots, 1)) \neq T(w^i + \varepsilon(i_1, \dots, i_{m-1}))$ for all $(i_1, \dots, i_{m-1}) \neq (1, \dots, 1)$. Also, as $T(w^i) \neq T(w^j)$, we obtain $T(w^i + \varepsilon(1, \dots, 1)) \neq T(w^j + \varepsilon(1, \dots, 1))$ for all $i \neq j \in I$. Finally, suppose that for any $i \in I$, there exists $j \in I$ such that $j \neq i$ and $T(w^i + \varepsilon(1, \dots, 1)) = T(w^j + \varepsilon(i_1, \dots, i_{m-1}))$ for some $(i_1, \dots, i_{m-1}) \neq (1, \dots, 1)$. Then there exists $s \in \{2, \dots m-1\}$ with $w^i_{s1} - 1 = w^j_{s1}$, and hence

$$\sum_{j\in I} w^j_{s1} = \sum_{i\in I}(w^i_{s1} - 1) = \sum_{i\in I} w^i_{s1} - \#I = \sum_{j\in I} w^j_{s1} - \#I,$$

which is a contradiction. Thus, there exists $i \in I$ such that $T(w^i + \varepsilon(1, \dots, 1)) \neq T(w^j + \varepsilon(i_1, \dots, i_{m-1}))$ for all $j \neq i$ and for all $(i_1, \dots, i_{m-1}) \neq (1, \dots, 1)$.

Conversely, let $((m-1, 1); (m, 1)) \in \mathcal{C}$. By the hypothesis,

$$\{((m-1, j); (m, j+1)) \mid j = 2, \dots, m-1\} \subset \mathcal{C}$$

and hence there exists $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ such that $w_{m-1,1} - w_{m1} = w_{m-1,j} - w_{m,j+1} = 1$ for each $j = 2, \dots m-1$. Therefore, $T(w+\varepsilon(i_1, \dots, i_{m-1})) \notin \mathcal{B}_{\mathcal{C}}(T(v))$ for any $1 \leq i_k \leq k \leq m-1$ and $E_{m1}(T(w)) = 0$.

(b) First, note that for any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$, we have

$$T(w') := T(w - \varepsilon(1, \dots, 1)) \in \mathcal{B}_{\mathcal{C}}(T(v)),$$

since $((m, 1); (m-1, 1)) \notin \mathcal{C}$.

Using (3) we obtain

$$\begin{aligned} E_{m1}(T(w')) =& b(w, 1, \dots, 1)\ T(w) \\ &+ \sum_{\substack{k=2,\dots,m-1 \\ (i_2,\dots,i_{m-1})\neq(1,\dots,1)}} b(w, i_2, \dots, i_{m-1})\ T(w + \varepsilon'(i_2, \dots, i_{m-1})), \end{aligned}$$

with $\varepsilon'(i_2,\dots,i_{m-1}) := \varepsilon(i_1,\dots,i_{m-1}) - \varepsilon(1,\dots,1)$, and
$b(w,i_2,\dots,i_{m-1}) :=$

$$= \begin{cases} 0, & \text{if } T(w+\varepsilon'(i_2,\dots,i_{m-1})) \notin \mathcal{B}_{\mathcal{C}}(T(v)) \\ \prod_{s=2}^{m-1} \frac{\prod_{t\neq i_{s-1}}^{s-1}(w_{si_s}-w_{s-1,t+\delta_{1i_s}-\delta_{1t})}}{\prod_{t\neq i_s}^{s}(w_{si_s}-w_{st+\delta_{1i_s}-\delta_{1t})}}, & \text{if } T(w+\varepsilon'(i_2,\dots,i_{m-1})) \in \mathcal{B}_{\mathcal{C}}(T(v)). \end{cases}$$

In particular, for any $s \in \{3,\dots m-1\}$, we have $w_{s1} - w_{s-1,t} \notin \mathbb{Z}$ for any $t \in \{2,\dots s-1\}$, and hence

$$b(w,1,\dots,1) = \prod_{s=2}^{m-1} \frac{\prod_{t=2}^{s-1}(w_{s1}-w_{s-1,t})}{\prod_{t=2}^{s}(w_{s1}-w_{st}+1)} \neq 0.$$

As $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$, we have $w_{ij} - w_{i+1,j+1} \in \{1,2,\dots,d_{ij}\}$, where $d_{ij} := v_{j-1} - v_{n+j-i} - n + i \in \mathbb{Z}_{>0}$. We consider the following cases:

Case I: $w_{22} - w_{33} \in \{1,2,\dots,d_{22}\}$ and $w_{ij} - w_{i+1,j+1} = 1$ for all $3 \leq i \leq n$ and $2 \leq j \leq i$. In this case, $T(w+\varepsilon'(i_2,\dots,i_{m-1})) \notin \mathcal{B}_{\mathcal{C}}(T(v))$ for all $(i_2,\dots,i_{m-1})$ such that $i_k > 1$ for some $k = 3,\dots,m-1$, since $(w_{ij}-1) - w_{i+1,j+1} = 0$ for all $3 \leq i \leq n$ and $2 \leq j \leq i$. Therefore,

$$E_{m1}(T(w')) = b(w,1,1,\dots,1)T(w) + b(w,2,1,\dots,1)T(w+\delta^{21}-\delta^{22}).$$

If $((1,1);(2,1)) \in \mathcal{C}$ or $((3,1);(2,1)) \in \mathcal{C}$, then $T(w+\delta^{21}-\delta^{22}) \notin \mathcal{B}_{\mathcal{C}}(T(v))$, when $w_{11} - w_{21} = 1$ or $w_{31} - w_{21} = 0$. Hence, $b(w,2,1,\dots,1) = 0$, and $E_{m1}(T(w')) = b(w,1,1,\dots,1)T(w)$.

Suppose now that $((1,1);(2,1)) \notin \mathcal{C}$ and $((3,1);(2,1)) \notin \mathcal{C}$ and consider the following cases:

(i) $w_{22} - w_{33} = 1$.
In this case $T(w+\delta^{21}-\delta^{22}) \notin \mathcal{B}_{\mathcal{C}}(T(v))$ implying $E_{m1}(T(w')) = b(w,1,1,\dots,1)T(w)$.

(ii) $w_{22} - w_{33} = i \in \{2,3,4,\dots,d_{22}\}$.
In this case $T(w+\delta^{21}-\delta^{22}) \in \mathcal{B}_{\mathcal{C}}(T(v))$ as $(w_{22}-1)-w_{33} = w_{22}-w_{33}-1 = i-1$ and $w_{32}-(w_{22}-1) = w_{32}-w_{22}+1 \in \mathbb{Z}_{>0}$. By the induction on i, we have $T(w+\delta^{21}-\delta^{22}) = E_{m1}(T(w_1))$, where $T(w_1)$ is a tableau in $V_{\mathcal{C}}(T(v))$. So, $E_{m1}(T(w')) = b(w,1,1,\dots,1)T(w) + b(w,2,1,\dots,1)T(w+\delta^{21}-\delta^{22})$. Consequently, $T(w) = E_{m1}(T(w_0))$ for some $T(w_0) \in V_{\mathcal{C}}(T(v))$.

Case II: $w_{22} - w_{33} \in \{1, 2, \dots, d_{22}\}$, $w_{32} - w_{43} = j \in \{2, 3, 4, \dots, d_{32}\}$ and $w_{ij} - w_{i+1,j+1} = w_{33} - w_{44} = 1$ for all $4 \leq i \leq n$ and $2 \leq j \leq i$.

In this case, $T(w + \varepsilon'(i_2, \dots, i_{m-1})) \notin \mathcal{B}_{\mathcal{C}}(T(v))$ for all $(i_2, \dots, i_{m-1})$ such that $i_3 = 3$ or $i_k > 1$ for any $k = 4, \dots, m-1$, as $(w_{ij} - 1) - w_{i+1,j+1} = (w_{33} - 1) - w_{44} = 0$ for all $4 \leq i \leq n$ and $2 \leq j \leq i$.

(i) If $w_{22} - w_{33} = 1$ then $T(w + \delta^{21} - \delta^{22})$ and $T(w + \delta^{21} - \delta^{22} + \delta^{31} - \delta^{32})$ do not belong $\mathcal{B}_{\mathcal{C}}(T(v))$. On the other hand, $T(w + \delta^{31} - \delta^{32}) \in \mathcal{B}_{\mathcal{C}}(T(v))$, as $w_{42} - (w_{32} - 1) \in \mathbb{Z}_{>0}$, $(w_{32} - 1) - w_{43} = j - 1 \geq 2$ and $(w_{32} - 1) - w_{22} = w_{43} - w_{44} + j - 3 \in \mathbb{Z}_{>0}$. Following the case I-(i), we conclude that there exists $T(w_1) \in V_{\mathcal{C}}(T(v))$ such that $T(w + \delta^{31} - \delta^{32}) = E_{m1}(T(w_1))$. Therefore, $E_{m1}(T(w'))) = b(w, 1, 1, \dots, 1)T(w) + b(w, 1, 2, \dots, 1)E_{m1}(T(w_1))$.

(ii) Assume $w_{22} - w_{33} = i \in \{2, 3, \dots, d_{22}\}$. We immediately get that $T(w + \delta^{21} - \delta^{22})$ and $T(w + \delta^{21} - \delta^{22} + \delta^{31} - \delta^{32})$ belong to $\mathcal{B}_{\mathcal{C}}(T(v))$. On the other hand, $T(w + \delta^{31} - \delta^{32}) \in \mathcal{B}_{\mathcal{C}}(T(v))$ if and only if $w_{43} - w_{44} \geq i - j + 2$, since $w_{42} - (w_{32} - 1) \in \mathbb{Z}_{>0}$ and $(w_{32} - 1) - w_{22} = w_{43} - w_{44} + j - i - 2$. Then, following the cases I and II-(i), there exists $T(w_k) \in V_{\mathcal{C}}(T(v))$ for each $k = 1, 2, 3$ such that $T(w + \delta^{21} - \delta^{22}) = E_{m1}\left(T(w_1)\right)$, $T(w + \delta^{21} - \delta^{22} + \delta^{31} - \delta^{32}) = E_{m1}\left(T(w_2)\right)$ and $T(w + \delta^{31} - \delta^{32}) = E_{m1}\left(T(w_3)\right)$ (if $w_{43} - w_{44} < i - j + 2$, then $T(w_3) = 0$).

Hence,

$$\begin{aligned} E_{m1}(T(w')) =& b(w, 1, 1, \dots, 1)T(w) + b(w, 2, 1, \dots, 1)E_{m1}(T(w_1)) \\ &+ b(w, 2, 2, \dots, 1)E_{m1}(T(w_2)) + b(w, 1, 2, \dots, 1)E_{m1}(T(w_3)). \end{aligned}$$

Repeating the process, after finitely many steps, we obtain that for any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ there exists $T(w_0) \in V_{\mathcal{C}}(T(v))$ such that $T(w) = E_{m1}(T(w_0))$, which implies the surjectivity of E_{m1}.

Conversely, assume that E_{m1} is surjective on $V_{\mathcal{C}}(T(v))$ but $((m, 1); (m-1, 1)) \in \mathcal{C}$. Choose $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ such that $w_{m-1,1} = w_{m1}$. As $((m, j); (m-1, j)) \in \mathcal{C}$ for all $j = 2, \dots, m-1$, we can assume without loss of generality that $w_{m-1,j} = w_{m,j}$ for all $j = 2, \dots, m-1$. On the other hand, there exists $u \in V_{\mathcal{C}}(T(v))$ such that

$$E_{m1}(u) = \sum_{\substack{i \in I,\ k=1,\dots,m-1 \\ (i_1,\dots,i_{m-1}) \in \{1,\dots,k\}^{m-1}}} c_i\, a(w^i, i_1, \dots, i_{m-1})\ T(w^i - \varepsilon(i_1, \dots, i_{m-1})) = T(w).$$

Hence, $T(w) = T(w^i - \varepsilon(i_1, \dots, i_{m-1}))$ for some $i \in I$ and $1 \leq i_k \leq k \leq m-1$. Then, there exists $j \in \{1, \dots, m-1\}$ such that $0 = w_{mj} - w_{m-1,j} = w^i_{mj} - w^i_{m-1,j} + 1 > 0$. Therefore, $((m,1);(m-1,1)) \notin \mathcal{C}$. ∎

Remark 5.11. Note that in Theorem 5.10 the set of relations $\mathcal{C}$ does not need to be neither indecomposable nor maximal set satisfied by $T(v)$. This is the case, for example, when $u_{i+1} - u_i \in \mathbb{Z}_{\geq 0}$ for any $1 \leq i < n$ and $\mathcal{C} = \{((i+1,1);(i,1)) \mid 1 \leq i \leq n\}$ or $u_{i+1} - u_i \notin \mathbb{Z}$ for any $1 \leq i < n$ and $\mathcal{C} = \emptyset$.

Let $F := \{E_{m_i 1} \mid i = 1, \dots, k\}$ such that, $m_i \in \{2, \dots, n+1\}$ for each $i = 1, \dots, k$. We have

Proposition 5.12. Let $\lambda \in \mathfrak{h}^*$.

(a) If λ satisfies conditions (b), (c) or (e) for $i = 1$ of Corollary 4.9, then $D_F L(\lambda)$ is a bounded relation Gelfand–Tsetlin module.

(b) If $M = L(\lambda)$ and $F_m = \{E_{m1}\}$ for some $m \in \{2, \dots, n+1\}$ such that E_{m1} acts injectivity on M, then $D_{F_m}(M)/M$ is simple.

Proof. Suppose that λ satisfies the condition (b) of Corollary 4.9 (the proof of other cases is similar). Then $L(\lambda) \simeq V_{\mathcal{C}}(T(v))$ where $T(v)$ is the Gelfand–Tsetlin tableau (2) such that $u_i = u_1$ for all $i = 2, 3, \dots, n+1$ and $\mathcal{C} = \mathcal{Q} \cup \{((i+1,1);(i,1)) \mid 1 \leq i \leq n\}$. Set $M = V_{\mathcal{C}}(T(v))$ and $N = V_{\mathcal{D}}(T(v))$, where $\mathcal{D} = \mathcal{C} \setminus \{((m_i,1);(m_i-1,1)) \mid i = 1, \dots, k\}$. Then $M \subset D_F M \subset N$ by Theorem 5.10.

Suppose first that $k = 1$. Then $F = F_m = \{E_{m1}\}$. Since E_{m1} is injective but not bijective on M, we conclude by Theorem 5.10 that M is a proper submodule of $D_{F_m} M$. On the other hand, consider the set of relations $\mathcal{D}_m = \mathcal{D} \cup \{((m-1,1);(m,1))\}$. Then $V_{\mathcal{D}_m}(T(v + \delta^{m-1,1}))$ is a simple module by Theorem 4.6, and $V_{\mathcal{D}_m}(T(v + \delta^{m-1,1})) \simeq N/M$. Hence, M is a maximal submodule of N. Finally, given that E_{m1} acts bijectively on N, we get that $N \simeq D_{F_m} M$ by Proposition 3.1. This completes the proof in the case $k = 1$.

Now, suppose that statement holds for all subsets of F with $k-1$ elements. Define $F_i := F \setminus \{E_{m_i 1}\}$ and $\mathcal{D}_i := \mathcal{D} \cup \{((m_i,1);(m_i-1,1))\}$ for any $i = 1, \dots, k$. Then $D_{F_i} M \simeq V_{\mathcal{D}_i}(T(v))$. Set $L = V_{\mathcal{D}_1}(T(v)) + \cdots + V_{\mathcal{D}_k}(T(v))$. As $D_{F_i} M \subset D_{\{E_{m_i 1}\}} D_{F_i} M \simeq D_F M$, we have $L \subset D_F M$. Since L is not F-bijective, then L is a proper submodule of $D_F M$. On the other hand, let $T(w) = T(v + i_1 \delta^{m_1 - 1,1} + (i_1 - 1)\delta^{m_2 - 1,1} + \cdots + \delta^{m_{i_1} - 1,1} + \cdots + (k - i_s + 1)\delta^{m_{i_s+1} - 1,1} + \cdots + \delta^{m_k - 1,1})$. Consider $\mathcal{A} = \mathcal{D} \cup \{((m_i - 1,1);(m_i,1)) \mid i = 1, \dots, k\}$,

where $\{m_i \mid i = 1, 2, \dots k\} = \{m_1, \dots, m_{i_1}\} \cup \dots \cup \{m_{i_s}, \dots, m_k\}$ is a disjoint union of sets with consecutive elements. Then $\mathcal{A}$ is a maximal set of relations satisfied by $T(w)$ and $V_{\mathcal{A}}(T(w))$ is a simple module by Theorem 4.6. As $N/L \simeq V_{\mathcal{A}}(T(w))$, we conclude that L is a maximal submodule of N and $D_F M \simeq N$ by Proposition 3.1. ■

6 Simple Modules in the Minimal Nilpotent Orbit

Recall that $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

6.1 Minimal nilpotent orbit

Let k an admissible number for $\widehat{\mathfrak{g}}$ with denominator $q \in \mathbb{N}$. In this section, we discuss explicit construction of simple admissible highest weight and $\mathfrak{sl}_2$-induced $\mathfrak{g}$-modules in the minimal nilpotent orbit $\mathbb{O}_{min}$. The orbit $\mathbb{O}_{min}$ is the unique minimal non-trivial nilpotent orbit of $\mathfrak{g}$ with $\dim \mathbb{O}_{min} = 2n$. We have the following description of $[\overline{Pr}_k^{\mathbb{O}_{min}}]$:

Proposition 6.1. [9, Proposition 2.10] Then

$$[\overline{Pr}_k^{\mathbb{O}_{min}}] = \bigsqcup_{a=1}^{q-1} \{[\bar{\lambda} - \frac{ap}{q}\varpi_1] \mid \lambda \in \widehat{P}_+^{p-n-1}\},$$

where $\widehat{P}_+^{p-n-1}$ is the set of level $p-n-1$ integral dominant weights of $\widehat{\mathfrak{g}}$ and ϖ_1 is the 1st fundamental weight.

To describe simple admissible $\mathfrak{g}$-modules in the minimal orbit $\mathbb{O}_{min}$ of level k we need to find those simple $\mathfrak{g}$-modules V for which $\mathrm{Ann}_{U(\mathfrak{g})} V = J_\lambda$, for each $\lambda \in [\overline{Pr}_k^{\mathbb{O}_{min}}]$. We start with the highest weight modules.

6.2 Explicit realization of highest weight modules

Let

$$k + n = \frac{p}{q} - 1, \quad p > n,\ q \geq 1 \text{ and } (p, q) = 1.$$

By Proposition 6.1, an element of $[\overline{Pr}_k^{\mathbb{O}_{min}}]$ has the form

$$\bar{\lambda} - \frac{ap}{q}\varpi_1 = \left(\lambda_1 - \frac{ap}{q}, \lambda_2, \lambda_3, \dots, \lambda_{n-1}, \lambda_n\right),$$

where $\lambda_i \in \mathbb{Z}_{\geq 0}$, for all $i = 1, \ldots, n$ are such that $\lambda_1 + \ldots + \lambda_n < p - n$ and $a \in \{1, 2, \ldots, q-1\}$.

Example 6.2. Let $\mathfrak{g} = \mathfrak{sl}_3$. As $\Lambda_1 = (\lambda_1 - \frac{ap}{q}, \lambda_2)$ is regular dominant, by Corollary 2.3, the simple admissible highest weight modules in the minimal orbit are $L(\Lambda_i)$, $i = 1, 2, 3$, where:

- $\Lambda_2 = s_1 \cdot \Lambda_1 = (\frac{ap}{q} - \lambda_1 - 2, \lambda_1 + \lambda_2 - \frac{ap}{q} + 1)$;
- $\Lambda_3 = s_2 s_1 \cdot \Lambda_1 = (\lambda_2, \frac{ap}{q} - \lambda_1 - \lambda_2 - 3)$.

These modules have weight multiplicities bounded by $\lambda_2 + 1$.

Applying Corollary 4.9, (b), we get the following:

Theorem 6.3. Any simple admissible highest weight module in the minimal nilpotent orbit is a bounded Γ-relation Gelfand–Tsetlin module.

Example 6.4. Let $\mathfrak{g} = \mathfrak{sl}_4$ and $\Lambda_1 = (\lambda_1 - \frac{ap}{q}, \lambda_2, \lambda_3)$. Then the simple admissible highest weight modules in the minimal orbit are $L(\Lambda_i)$, $i = 1, 2, 3, 4$, where:

- $\Lambda_2 = s_1 \cdot \Lambda_1 = (\frac{ap}{q} - \lambda_1 - 2, \lambda_1 - \frac{ap}{q} + \lambda_2 + 1, \lambda_3)$;
- $\Lambda_3 = s_2 s_1 \cdot \Lambda_1 = (\lambda_2, \frac{ap}{q} - \lambda_1 - \lambda_2 - 3, \lambda_1 + \lambda_2 + \lambda_3 - \frac{ap}{q} + 2)$;
- $\Lambda_4 = s_3 s_2 s_1 \cdot \Lambda_1 = (\lambda_2, \lambda_3, \frac{ap}{q} - \lambda_1 - \lambda_2 - \lambda_3 - 4)$.

These modules are bounded, for example, the weight multiplicities of $L(\Lambda_1)$ are bounded by $\frac{1}{2}(\lambda_2 + 1)(\lambda_3 + 1)(\lambda_2 + \lambda_3 + 2)$.

Let $F := \{E_{m_i 1} \mid i = 1, \ldots, k\}$ such that $m_i \in \{2, \ldots, n+1\}$ for each $i = 1, \ldots, k$. From Corollary 3.5 and Proposition 5.12, we immediately obtain the following:

Corollary 6.5. Let $n \geq 2$. All simple subquotients of $D_F L(\lambda')$ are admissible bounded Γ-relation Gelfand–Tsetlin $\mathfrak{g}$-modules in the minimal orbit.

Remark 6.6.

(i) Corollary 6.5 for $\mathfrak{sl}(3)$ was shown in [9, Theorem 5.6].

(ii) All simple modules in Corollary 6.5 are highest weight modules (with respect to some Borel subalgebra) with bounded weight multiplicities.

We have from Corollary 4.13

Corollary 6.7. Let $\mathfrak{b} = \mathfrak{b}(\pi)$, $\beta_j \in \pi$, $j = 1, \dots, t$ with $\left\langle \lambda + \rho_\pi, \beta_j^\vee \right\rangle \in \mathbb{Z}_{\geq 0}$ for all j, and let $w \in W$ be such that $\left\langle w(\lambda + \rho_\pi), \beta_j^\vee \right\rangle \in \mathbb{Z}_{<0}$ for all $j = 1, \dots, t$. Then $L(\lambda) \simeq L_{\mathfrak{b}(w\pi)}(w\lambda)$ is a $\Gamma_{\mathcal{F}}$-relation Gelfand–Tsetlin module, where $\Gamma_{\mathcal{F}} = w\Gamma$.

We also have the following result.

Corollary 6.8. Let $\lambda \in \mathfrak{h}^*$, π a basis of the root system, $L(\lambda) = L_{\mathfrak{b}(\pi)}(\lambda)$ an admissible highest weight module in the minimal orbit (with respect to the Borel subalgebra $\mathfrak{b}(\pi)$) and $\beta \in \pi$ is such that $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}_{\geq 0}$. Then

(a) The module $L_{\mathfrak{b}(s_\beta \pi)}(\lambda + \beta)$ is an admissible $s_\beta\Gamma$-relation Gelfand–Tsetlin module in the minimal orbit.

(b) Let $i > 1$, $\beta = \beta_i$ the 1st simple root of π such that $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}_{\geq 0}$ and $w = s_{i-1}s_i s_{i-2} s_{i-1} \cdots s_2 s_1 s_3 s_1 s_2 \in W$. Then $L_{\mathfrak{b}(w^{-1}\pi)}(w^{-1}\lambda)$ is a Γ-relation Gelfand–Tsetlin module in the minimal orbit.

(c) For any $x \in \mathbb{C}$, $D^x_{f_\beta} L(\lambda)$ is a $w\Gamma$-relation Gelfand–Tsetlin module in the minimal orbit.

Proof. Let $\Lambda = \bar{\lambda} - \frac{ap}{q}\varpi_1$, where $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$ with non-negative integers λ_i for all $i = 1, \dots, n$. Then $\lambda = s_t \dots s_1 \cdot \Lambda$ for some $t \leq n$. If $t = n$ then λ has the last component non-integral in which case $\beta = \alpha_n$. If $t < n$ then λ has exactly two non-integral components in the places t and $t+1$ and $\beta = \alpha_t$ or $\beta = \alpha_{t-1}$. Hence, $L(s_\beta \cdot \lambda)$ is a simple admissible highest weight module in the minimal orbit. Note that $s_\beta \cdot \lambda = s_\beta(\lambda + \beta)$ and consider $L_{\mathfrak{b}(s_\beta \pi)}(\lambda + \beta)$. This is the highest weight module with respect to the Borel subalgebra $\mathfrak{b}(s_\beta \pi)$ and the corresponding highest weight (with respect to $\mathfrak{b}(s_\beta \pi)$) is $\lambda + \beta$. Therefore, $L_{\mathfrak{b}(s_\beta \pi)}(\lambda + \beta)$ is an admissible module in the minimal orbit, and it is a Γ'-relation Gelfand–Tsetlin module where Γ' is the standard Gelfand–Tsetlin subalgebra of $s_\beta \pi$, that is $\Gamma' = s_\beta \Gamma$. This shows (a).

Since $L(\lambda)$ is a module in the nilpotent orbit, then $L_{w^{-1}\mathfrak{b}}(w^{-1}\lambda)$ is also an admissible module in the nilpotent orbit. Hence, $L_{\mathfrak{b}(w^{-1}\pi)}(w^{-1}\lambda)$ is a $w^{-1}\Gamma$-relation Gelfand–Tsetlin module. The statement (b) follows from Lemma 4.14. Now, twisting $L_{\mathfrak{b}(w^{-1}\pi)}(w^{-1}\lambda)$ by w, that is applying w to the corresponding Gelfand–Tsetlin formulas, we obtain that $L(\lambda)$ is a $w\Gamma$-relation Gelfand–Tsetlin module. Note that $w \in W$ is the element of minimal length such that β is the 1st simple root of $w\pi$. Hence, (c) follows from Theorem 5.4. ■

6.3 Classification of $\mathfrak{sl}_2$-induced modules

Let $k = \frac{p}{q} - n - 1$ be an admissible number for $\widehat{\mathfrak{g}}$. In this section, we complete the construction of all simple admissible $\mathfrak{g}$-modules in the minimal orbit, which are the quotients of modules induced from parabolic subalgebras with the Levi factor isomorphic to $\mathfrak{sl}_2 + \mathfrak{h}$. All such modules are $\Gamma_{\mathcal{F}}$-relation modules for some $\mathcal{F}$.

Let M be an admissible $\mathfrak{g}$-module of level k. Consider a parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_\Sigma = \mathfrak{l}_\Sigma + \mathfrak{n}_\Sigma$ of $\mathfrak{g}$, where Σ consists of one simple root β. Denote by $M^{\mathfrak{n}}$ the subspace of all $\mathfrak{n}_\Sigma$-invariants, which is an $\mathfrak{l}_\Sigma$-module. Suppose $M = L^{\mathfrak{g}}_{\mathfrak{p}}(\Sigma, N)$ for some simple weight $\mathfrak{l}_\Sigma$-module N with $\mathfrak{n}_\Sigma N = 0$. Then $M^{\mathfrak{n}} \simeq N$ is admissible $\mathfrak{l}_\Sigma$-module of level $k_\beta = \frac{2}{(\beta,\beta)}(k+n+1) - 2$ by [9, Theorem 2.12].

The following is straightforward.

Proposition 6.9. Let $\mathfrak{a} = \mathfrak{sl}_2$, $V = V(\gamma, \mu)$ a simple dense weight $\mathfrak{a}$-module, $\gamma, \mu \in \mathbb{C}$. Then V is admissible of level k in the minimal orbit if and only if $\mu = \lambda - \dfrac{ap}{q} + 2x$ and $\gamma = \left(\lambda - \dfrac{ap}{q} + 1\right)^2$, where $\lambda \in \{0, 1, \dots, p-2\}$, $a \in \{1, \dots, q-1\}$, $x \in \mathbb{C} \setminus \mathbb{Z}$ and $x - \dfrac{ap}{q} \notin \mathbb{Z}$.

Remark 6.10. In the proposition above we have $V = V_{\mathcal{C}}(T(v))$, where $\mathcal{C} = \emptyset$, $T(v) = T(v_{ij})$ is the Gelfand–Tsetlin tableau with height 2, such that $v_{11} = \dfrac{\mu}{2}$, $v_{21} = \dfrac{1}{2}\left(\lambda - \dfrac{ap}{q}\right)$, and $v_{22} = \dfrac{1}{2}\left(\dfrac{ap}{q} - \lambda - 2\right)$ (cf. Theorem 5.6). The set $\mathcal{B}_{\mathcal{C}}(T(v)) = \{T(v + \ell\delta^{11}) \mid \ell \in \mathbb{Z}\}$ is a basis of V.

Applying Proposition 6.9, Theorem 2.9, and Theorem 5.6, we obtain the following statement.

Corollary 6.11. Let $n > 1$, $\gamma \in \mathbb{C}$, $\mu = (\mu_1, \dots, \mu_n) \in \mathfrak{h}^*$, such that $V \simeq V(\gamma, \mu_1)$ is a simple dense weight $\mathfrak{sl}_2$-module and $\mu_i = \langle \mu, \alpha_i^\vee \rangle$, $i = 1, \dots, n$. Let $L(\gamma, \mu) = L^{\mathfrak{g}}_{\mathfrak{p}}(\{\alpha\}, V)$. Then, $L(\gamma, \mu)$ is admissible of level k in the minimal orbit if and only if $\mu_1 = \lambda_1 - \dfrac{ap}{q} + 2x$, $\mu_2 = \lambda_2 - x$, $\mu_j = \lambda_j$ for all $j = 3, \dots, n$ and $\gamma = \left(\lambda_1 - \dfrac{ap}{q} + 1\right)^2$ with $\{\lambda_i\}_{i=1,\dots,n} \subset \mathbb{Z}_{\geq 0}$ such that $\lambda_1 + \ldots + \lambda_n < p - n$, $a \in \{1, 2, \dots, q-1\}$, $x \in \mathbb{C} \setminus \mathbb{Z}$ and $x - \dfrac{ap}{q} \notin \mathbb{Z}$.

Theorem 6.12. Let π be a basis of the root system of $\mathfrak{g}$, β a positive root of $\mathfrak{g}$ (with respect to π). Let $L_{\mathfrak{b}(\pi)}(\lambda)$ be an admissible simple $\mathfrak{b}(\pi)$-highest weight $\mathfrak{g}$-module in the

minimal orbit, such that $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}$, and $f = f_\beta$. Denote by $A_{\pi,\beta}$ the set of all $x \in \mathbb{C} \setminus \mathbb{Z}$ such that $x + \langle \lambda + \rho_\pi, \beta^\vee \rangle \notin \mathbb{Z}$.

(a) The $\mathfrak{g}$-module $D_f^x L_{\mathfrak{b}(\pi)}(\lambda)$ is admissible in the minimal orbit for any $x \in A_{\pi,\beta}$;

(b) Modules $D_f^x L_{\mathfrak{b}(\pi)}(\lambda)$, where π runs the sets of simple roots of $\mathfrak{g}$, β runs positive roots with respect to π, $x \in A_{\pi,\beta}$, $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}$, and $L_{\mathfrak{b}(\pi)}(\lambda)$ is admissible module in the minimal orbit, exhaust all simple $\mathfrak{sl}_2$-induced admissible modules in the minimal orbit. All such modules have bounded weight multiplicities;

(c) There exists a flag $\mathcal{F}$ such that $D_f^x L_{\mathfrak{b}(\pi)}(\lambda)$ is $\Gamma_{\mathcal{F}}$-relation Gelfand–Tsetlin $\mathfrak{g}$-module.

Proof. Let $\pi = \{\alpha_1, \dots, \alpha_n\}$. First we prove (a). Let $\beta = \alpha_r + \ldots + \alpha_t$ for some consecutive simple roots α_j, $j = r, \dots, t$. Since λ is admissible in the minimal orbit then we have two possibilities: either there exists only one simple root i such that $\langle \lambda, \alpha_i^\vee \rangle \notin \mathbb{Z}$, or there are only two such roots which are consecutive. Assume the 1st case. Then either $i = r = 1$ or $i = t = n$. Without loss of generality we may assume that $i = 1$ (if $i = n$ then apply symmetry of the root system), and hence $\left\langle \lambda + \rho_\pi, \alpha_j^\vee \right\rangle \in \mathbb{Z}_{>0}$, $j = 2, \dots, t$. Take the following basis of the root system

$$\pi' = \{\beta, -\alpha_t, \dots, -\alpha_2, \alpha_2 + \ldots + \alpha_{t+1}, \alpha_{t+2}, \dots, \alpha_n\}.$$

Let $w \in W$ be such that $w\pi = \pi'$. Then w satisfies Corollary 6.7, and $L(\lambda) \simeq L_{\mathfrak{b}(w\pi)}(w\lambda)$ is a $w\Gamma$-relation Gelfand–Tsetlin module. Then the statement follows from Theorem 5.4.

Consider now the 2nd case. Assume that $\langle \lambda, \alpha_i^\vee \rangle \notin \mathbb{Z}$ for $i = k, k+1$ for some k. If $\beta = \alpha_k$ then the statement follows from Corollary 6.8, (c). If $\beta = \alpha_{k+1}$ then apply the symmetry of the Dynkin diagram and Corollary 6.8, (c). Let $\beta = \alpha_r + \ldots + \alpha_k$ for some $1 \leq r \leq k-1$. Take the minimal $w \in W$ such that $w\pi$ contains β and $-\alpha_r, \dots, -\alpha_{k-1}$ (such w clearly exists). Then $L(\lambda) \simeq L_{\mathfrak{b}(w\pi)}(w\lambda)$. Hence, the problem reduces to the case $\beta = \alpha_k$, which was argued above. If $\beta = \alpha_{k+1} + \ldots + \alpha_t$ for some $k+1 \leq t \leq n$, then the statement follows from the symmetry of the Dynkin diagram. Now (b) and (c) follow from (a) and Theorem 5.6. ■

We note that Theorem 6.12 was initially proved for $\mathfrak{g} = \mathfrak{sl}(3)$ in [9, Theorem 5.6].

Let β be a root of $\mathfrak{g}$ and π be a basis of the root system containing β as the 1st root (such π always exists by the conjugation by the Weyl group). Let $\mathfrak{p} = \mathfrak{a}_\beta \oplus \mathfrak{n}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}(\pi)$ with the Levi factor $\mathfrak{a}_\beta \simeq \mathfrak{sl}_2 + \mathfrak{h}$ based on the

root β. Let $V = V(\gamma_\beta, \mu)$, $\gamma_\beta \in \mathbb{C}$, $\mu = \mu_1\varpi_1 + \ldots + \mu_n\varpi_n \in \mathfrak{h}^*$, where γ_β is the eigenvalue of the Casimir element of $\mathfrak{a}_\beta$ and $\left\langle \varpi_i, \beta_j^\vee \right\rangle = \delta_{ij}$, $i = 1, \ldots, n$.

Corollary 6.13. The module $L_{\pi,\beta}(\gamma_\beta, \mu) := L_{\mathfrak{p}}^{\mathfrak{g}}(V)$ is admissible in the minimal orbit if and only if $\langle \mu - \lambda, \beta_1^\vee \rangle = 2x$, $\langle \mu - \lambda, \beta_2^\vee \rangle = -x$, $\langle \mu - \lambda, \beta_i^\vee \rangle = 0$ for each $i = 3, \ldots, n$, and $\gamma_\beta = \langle \lambda + \rho_{\mathfrak{b}}, \beta_1^\vee \rangle^2$, with $\langle \lambda + \rho_\pi, \beta_i^\vee \rangle \in \mathbb{Z}_{>0}$ for all $i \in \{2, \ldots, n\}$, $x \in \mathbb{C} \setminus \mathbb{Z}$, $x + \langle \lambda + \rho_\pi, \beta_1^\vee \rangle \notin \mathbb{Z}$ and $\left\langle \lambda + \frac{ap}{q}\varpi_1 + \rho_\pi, \beta_1^\vee \right\rangle \in \mathbb{Z}_{>0}$, $\left\langle \lambda + \frac{ap}{q}\varpi_1 + \rho_\pi, \beta_{1,n}^\vee \right\rangle < p$ for some $a \in \{1, 2, \ldots, q-1\}$.

Remark 6.14. Theorem 6.12 provides an algorithm how to list all simple $\mathfrak{sl}_2$-induced admissible modules in the minimal orbit:

- Consider all possible Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$;
- For each Borel subalgebra $\mathfrak{b}(\pi)$ describe $\lambda \in \mathfrak{h}^*$ for which $L_{\mathfrak{b}}(\lambda)$ is admissible using the Arakawa's classification for the standard Borel and applying the Weyl group;
- Choose any positive (with respect to π) root β such that $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}$, and define $D_f^x L_{\mathfrak{b}(\pi)}(\lambda)$ for any $x \in \mathbb{C} \setminus \mathbb{Z}$ such that $x + \langle \lambda + \rho_\pi, \beta^\vee \rangle \notin \mathbb{Z}$, where $f = f_\beta$.

Obtained modules exhaust all simple $\mathfrak{sl}_2$-induced admissible modules in the minimal orbit. Moreover, the proof of Theorem 6.12 explains how to define the flag $\mathcal{F}$ for which $L_{\pi,\beta}(\gamma_\beta, \mu)$ is a $\Gamma_{\mathcal{F}}$-relation Gelfand–Tsetlin $\mathfrak{g}$-module, and hence to obtain explicit tableaux realization for all such modules.

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