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Robust Recursive Regulator for Systems Subject to Polytopic Uncertainties

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ABSTRACT We present a robust recursive framework for the regulation of discrete-time linear systems subject to polytopic uncertainties. Based on regularized least-squares with a penalty parameter, we formulate a convex optimization problem and weight the polytope vertices altogether. In this sense, the main contribution of this paper consists of a robust recursive framework for the computation of stabilizing feedback gains. The solution does not require numerical optimization packages and relies ultimately on a single penalty parameter which is easily tuned. Moreover, the gains are obtained recursively through algebraic expressions, as opposed to related works which employ linear matrix inequalities. Under observability and controllability conditions, we demonstrate convergence and stability of the closed-loop system in terms of an algebraic Riccati equation. We provide numerical and real-world examples to validate the proposed approach and for comparison with a robust H_∞ controller.

INDEX TERMS Discrete-time systems, least-squares, optimization problem, penalty function, Riccati equations, robust control, robust regulator.

I. INTRODUCTION

For decades, researches on robust control have focused on finding mathematical tools to guarantee stability of linear systems subject to uncertainties. As it is usually impractical to minutely describe the entire set of features in a physical process, the need for diminishing the negative effects of unmodelled characteristics becomes crucial. Numerous approaches have been proposed throughout the years to treat structured uncertainties in general. In fact, the robust control problem is still an open challenge and recent publications continue to investigate new approaches [1]–[6]. More specifically, we are concerned with the robust control problem regarding systems whose regions of uncertainty are polytopes.

Systems subject to polytopic uncertainties have been attracting increasing attention, with successful practical applications including data transmission protocols [7], power systems [8]–[10], vehicular technology [11], [12], and cyber security control [13]. Different methods contributed with significant advances in robust control theory for the treatment of structured uncertainties varying inside convex hulls. In this regard, recently published results include H_2 and H_∞ synthesis [14]–[16]; model predictive control [17]–[19]; and gain-scheduled control [20]–[22]. Additionally, a comprehensive collection of classic remarkable results can also be found in [23]–[28], and references therein. In these works, linear matrix inequalities (LMIs) are defined based on the vertices of the polytope and the solution yields an output or state feedback gain. Fundamentally, three generic optimization problems should be solved for this class of controllers: feasibility, generalized eigenvalue minimization, and minimization of a linear objective under LMI constraints. As the number of vertices increases, so does the number of inequality constraints to be fulfilled. In turn, the computational complexity of these methods may become excessive and lead to a performance decay in online applications. It is important, therefore, to design robust solutions with lower computational burden irrespective of the quantity of polytope vertices.

With this in mind, recursive solutions provide a valuable way of computing feedback controllers and filters for discrete-time linear systems subject to uncertainties without solving a set of parameter-dependent LMIs. In terms of norm-bounded uncertainties, recursiveness has been explored in theoretical results and in implementations on actual systems,

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for instance, in [29]–[34]. It is also useful to join different decision and control approaches as deep learning and autonomous control systems to increase, for instance, the effectiveness of Internet of Things [35], and autonomous navigation in urban environments [36].

In this paper, we address the robust recursive regulation of discrete-time linear systems subject to polytopic uncertainties. We formulate a min-max optimization problem with equality constraints defined upon the polytope vertices. With a penalty parameter easily tuned beforehand, we incorporate the constraints into the cost function to obtain an equivalent unconstrained problem. It is possible, then, to explicitly weight the vertices in a single equation, thus avoiding multiple linear matrix inequality constraints. The solution presents a symmetric matrix structure, which can be expressed as a set of algebraic equations suitable for stability analysis. In summary, the main contributions of this paper are the following:

1) We design a robust recursive framework to solve the regulation problem of discrete-time linear systems subject to polytopic uncertainties.
2) We add polytopic uncertainties to redefine the regularized least-squares problem. In turn, we yield a cost function that weights the entire set of vertices at once.
3) We attain a recursive solution that does not require numerical optimization packages and whose complexity is independent of the number of vertices.
4) We carry out simulations to analyze its robustness in comparison with a robust H_∞ controller, based on a trajectory tracking model of a commercial quadrotor.

The remainder of the paper is organized as follows. In Section II, we enunciate the optimization problem to be investigated. In Section III, we present some useful preliminary results about the robust regularized least-squares problem. In Section IV, we show the robust recursive solution for linear systems subject to polytopic uncertainties and discuss the conditions for convergence and stability. In Section V, we provide illustrative examples. Finally, in Section VI we conclude the paper.

A. NOTATION

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^n$ the set of n-dimensional vectors with elements in $\mathbb{R}$, and $\mathbb{R}^{n\times m}$ the set of $n \times m$ real matrices. The superscript T denotes transposition. I_n is the $n \times n$ identity matrix. The Kronecker product operator is denoted by $\otimes$, whereas $\mathbb{1}_n := [1 \cdots 1]^T \in \mathbb{R}^n$. Let P be a real symmetric matrix, then $P > 0$ ($P \geq 0$) means that P is positive (semi)definite. The weighted squared Euclidean norm of x is denoted by $\|x\|_P^2 = x^T Px$. Whenever convenient, we adopt the notation $X^T P(\bullet) = X^T PX$.

II. PROBLEM FORMULATION

Consider the realization of a discrete-time linear system:

$$x_{k+1} = (F_{0,k} + \delta F_k)x_k + (G_{0,k} + \delta G_k)u_k, \tag{1}$$

for $k = 0, \ldots, N-1$, where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input, $F_{0,k} \in \mathbb{R}^{n\times n}$ and $G_{0,k} \in \mathbb{R}^{n\times m}$ are known nominal system matrices, and $\{\delta F_k, \delta G_k\}$ are polytopic uncertainty matrices described by

$$\begin{bmatrix}\delta F_k & \delta G_k\end{bmatrix} = \sum_{i=1}^{V} \alpha_{i,k} \begin{bmatrix}F_{i,k} & G_{i,k}\end{bmatrix}, \tag{2}$$

with known vertices $F_{i,k} \in \mathbb{R}^{n\times n}$ and $G_{i,k} \in \mathbb{R}^{n\times m}$, and unknown coefficients $\alpha_k := \begin{bmatrix}\alpha_{1,k} & \ldots & \alpha_{V,k}\end{bmatrix}^T$ belonging to the unitary simplex

$$\Lambda_V = \left\{\alpha \in \mathbb{R}^V : \alpha_i \geq 0, \sum_{i=1}^{V} \alpha_i = 1\right\}. \tag{3}$$

Notice that, since $\alpha_{i,k}$ can randomly change during operation, matrices $\{\delta F_k, \delta G_k\}$ encompass both the time-invariant and time-varying cases of uncertainties.

In this setting, the main goal of this work is to obtain a sequence $\{u_k\}_{k=0}^{N-1}$ to regulate the states of system (1) subject to the polytopic uncertainties (2). To this end, we formulate the following finite horizon problem:

$$\min_{x_{k+1}, u_k} \max_{\delta F_k, \delta G_k} \left\{ \|x_N\|_{P_N}^2 + \sum_{j=0}^{N-1} (\|x_j\|_{Q_j}^2 + \|u_j\|_{R_j}^2) \right\}, \tag{4}$$

$$\text{subject to} \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} x_{k+1} = \begin{bmatrix} F_{0,k} + V\delta F_{1,k} \\ \vdots \\ F_{0,k} + V\delta F_{V,k} \end{bmatrix} x_k + \begin{bmatrix} G_{0,k} + V\delta G_{1,k} \\ \vdots \\ G_{0,k} + V\delta G_{V,k} \end{bmatrix} u_k, \tag{5}$$

for $k = N-1, \ldots, 0$, where $P_N > 0$, $Q_j > 0$, $R_j > 0$, $\delta F_{i,k} := \alpha_{i,k} F_{i,k}$, and $\delta G_{i,k} := \alpha_{i,k} G_{i,k}$, for $i = 1, \ldots, V$.

Based on Bellman's Principle of Optimality and on concepts of dynamic programming [37], we solve problem (4) recursively by separating it into N one-step quadratic optimization problems of the form

$$\min_{x_{k+1}, u_k} \max_{\delta F_k, \delta G_k} \left\{ J_k = \|x_{k+1}\|_{P_{k+1}}^2 + \|x_k\|_{Q_k}^2 + \|u_k\|_{R_k}^2 \right\}, \tag{6}$$

subject to (5), for $k = N-1, \ldots, 0$, with $P_{k+1} > 0$, $Q_k > 0$, and $R_k > 0$.

Remark 1: Observe that the vertices $\{F_{i,k}, G_{i,k}\}$, $i = 1, \ldots, V$, *are explicitly expressed in the constraints* (5), *instead of considering only their convex combination. Nevertheless, we can recover the system* (1) *by pre-multiplying both sides of* (5) *by* $\mathbb{1}_V^T \otimes I_n$, *producing*

$$Vx_{k+1} = \left(VF_{0,k} + V\sum_{i=1}^{V} \alpha_{i,k} F_{i,k}\right) x_k + \left(VG_{0,k} + V\sum_{i=1}^{V} \alpha_{i,k} G_{i,k}\right) u_k,$$

which corresponds to (1).

We incorporate the constraints (5) into the quadratic cost function by means of the penalty function method [38]. In this manner, we attain an unconstrained optimization problem whose cost function weights all polytope vertices in a unified way. Observe that we can reorganize the equality constraints in terms of $\begin{bmatrix} x_{k+1}^T & u_k^T \end{bmatrix}^T$ and express (5) as $g(x_{k+1}, u_k) = 0$, where

$$\begin{aligned} &g(x_{k+1}, u_k) \\ &= \left(\begin{bmatrix} I_n & -G_{0,k} \\ \vdots & \vdots \\ I_n & -G_{0,k} \end{bmatrix} + \begin{bmatrix} 0 & -V\delta G_{1,k} \\ \vdots & \vdots \\ 0 & -V\delta G_{V,k} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} \\ &\quad - \left(\begin{bmatrix} F_{0,k} \\ \vdots \\ F_{0,k} \end{bmatrix} + \begin{bmatrix} V\delta F_{1,k} \\ \vdots \\ V\delta F_{V,k} \end{bmatrix} \right) x_k. \end{aligned} \tag{7}$$

By defining $C(x_{k+1}, u_k) = g(x_{k+1}, u_k)^T \mu g(x_{k+1}, u_k)$, with penalty parameter $\mu > 0$, we add the constraints into the cost function J_k. After some algebra, we attain the following unconstrained optimization problem:

$$\min_{x_{k+1}, u_k} \max_{\delta F_k, \delta G_k} \mathcal{J}_k(x_{k+1}, u_k, \delta F_k, \delta G_k), \tag{8}$$

with the one-step cost function shown in (9), as shown at the bottom of the page, in which $\{P_{k+1},\ Q_k,\ R_k\}$ are weighting matrices related to x_{k+1}, x_k, and u_k, respectively. The minimization in x_{k+1} and u_k aims to provide robustness and stability of the closed-loop system, as we will examine in Section IV.

By solving (8)–(9), we attain the recursive framework for regulation of systems subject to polytopic uncertainties. In addition, observe that problem (8)–(9) is a special case of the robust regularized least-squares problem, which is covered in the sequel.

III. ROBUST REGULARIZED LEAST-SQUARES

We present in this section the robust regularized least-squares problem subject to polytopic uncertainties in the data. Consider the following optimization problem:

$$\min_{x} \max_{\delta A, \delta b} \{ f(x) = \|x\|_{\mathcal{Q}}^2 + \|(A_0 + \delta A)x - (b_0 + \delta b)\|_{\mathcal{W}}^2 \}, \tag{10}$$

where A_0 and b_0 are known entities, x is the minimization variable, $\mathcal{Q} > 0$ and $\mathcal{W} > 0$ are weighting matrices, and $\{\delta A, \delta b\}$ belong to a polyhedral domain described by

$$\Pi_V := \left\{ \begin{bmatrix} \delta A & \delta b \end{bmatrix} = M \sum_{i=1}^{V} \alpha_i \begin{bmatrix} A_i & b_i \end{bmatrix},\ \alpha \in \Lambda_V \right\}, \tag{11}$$

with constant matrix M of adequate dimensions, known vertices $\{A_i, b_i\}$, and unitary simplex Λ_V as defined in (3).

Although M seems to be directly replaceable by an identity matrix, its presence in the definition of the convex set Π_V provides additional flexibility to the framework outlined in this section. In particular, we will use such flexibility in our favor to carry out specific matrix mappings in Section IV and to benefit from the following result.

Lemma 1: Consider the min-max optimization problem (10)–(11). The following statements are equivalent:

1) *For $\mathcal{Q} > 0$, there is a unique $\hat{x}$ such that*

$$\hat{x} = \arg \min_{x} \max_{\delta A, \delta b} f(x).$$

2) *$(\xi, \zeta, \gamma, x) = (\hat{\xi}, \hat{\zeta}, \hat{\gamma}, \hat{x})$ is the optimal solution to*

$$\begin{bmatrix} \mathcal{Q}^{-1} & 0 & 0 & I \\ 0 & \mathcal{W}(\hat{\lambda})^{-1} & 0 & A_0 \\ 0 & 0 & \hat{\lambda}^{-1} I & \hat{A}_V \\ I & A_0^T & \hat{A}_V^T & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \\ \gamma \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ b_0 \\ \hat{b}_V \\ 0 \end{bmatrix},$$

where $\hat{\lambda}$ is a Lagrange multiplier, given by

$$\hat{\lambda} := \arg \min_{\lambda > \|M^T \mathcal{W} M\|} \Gamma(\lambda), \tag{12}$$

with

$$\begin{aligned} \Gamma(\lambda) &:= \|x(\lambda)\|_{\mathcal{Q}}^2 + \lambda \|\hat{A}_V x(\lambda) - \hat{b}_V\|^2 + \|A_0\, x(\lambda) - b_0\|_{\mathcal{W}(\lambda)}^2, \\ \mathcal{Q}(\lambda) &:= \mathcal{Q} + \lambda \hat{A}_V^T \hat{A}_V, \\ \mathcal{W}(\lambda) &:= \mathcal{W} + \mathcal{W} M (\lambda I - M^T \mathcal{W} M)^{-1} M^T \mathcal{W}, \\ x(\lambda) &:= [\mathcal{Q}(\lambda) + A_0^T \mathcal{W}(\lambda) A_0]^{-1} [A_0^T \mathcal{W}(\lambda) b_0 + \lambda \hat{A}_V^T \hat{b}_V]. \end{aligned}$$

In addition, the unique solution $\hat{x}$ and the corresponding cost $f(\hat{x})$ are obtained by

$$\begin{bmatrix} \hat{x} \\ f(\hat{x}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & b_0 \\ 0 & \hat{b}_V \\ I & 0 \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}^{-1} & 0 & 0 & I \\ 0 & \mathcal{W}(\hat{\lambda})^{-1} & 0 & A_0 \\ 0 & 0 & \hat{\lambda}^{-1} I & \hat{A}_V \\ I & A_0^T & \hat{A}_V^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ b_0 \\ \hat{b}_V \\ 0 \end{bmatrix}, \tag{13}$$

where $\hat{A}_V = \begin{bmatrix} A_1^T & \cdots & A_V^T \end{bmatrix}^T$ and $\hat{b}_V = \begin{bmatrix} b_1^T & \cdots & b_V^T \end{bmatrix}^T$.

Proof: Note that we recast the original robust regularized least-squares of [31] from a polytopic perspective, with $\{\delta A, \delta b\} \in \Pi_V$. Then, the statements follow directly by association with the procedures outlined in [31]. □

Remark 2: The Lagrange multiplier $\hat{\lambda}$ is computed by solving the auxiliary optimization problem (12). Even though the search for $\hat{\lambda}$ can be performed without concerns about local minima, given that $\mathcal{W} > 0$, it demands extra computational time. For this reason, we adopt the approximation $\hat{\lambda} \approx \beta \|M^T \mathcal{W} M\|$, for some $\beta > 1$. This is reasonable, since $\Gamma(\lambda)$ reaches amplitudes close to its global minimum for values

$$\mathcal{J}_k = \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} P_{k+1} & 0 \\ 0 & R_k \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} + \left\{ \left(\begin{bmatrix} 0 & 0 \\ I_n & -G_{0,k} \\ \vdots & \vdots \\ I_n & -G_{0,k} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -V\delta G_{1,k} \\ \vdots & \vdots \\ 0 & -V\delta G_{V,k} \end{bmatrix} \right) \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} - \left(\begin{bmatrix} -I_n \\ F_{0,k} \\ \vdots \\ F_{0,k} \end{bmatrix} + \begin{bmatrix} 0 \\ V\delta F_{1,k} \\ \vdots \\ V\delta F_{V,k} \end{bmatrix} \right) x_k \right\}^T \begin{bmatrix} Q_k & 0 \\ 0 & \mu I_{nV} \end{bmatrix} \left\{ \bullet \right\}. \tag{9}$$

of λ in the neighborhood of the lower bound $\|M^T \mathcal{W} M\|$, as discussed in [39], [40].

The positiveness of matrices P_{k+1} and R_k is fundamental to guarantee the existence of a unique solution. We outline this aspect in the next remark.

Remark 3: Notice that (10)–(11) *can be rewritten as*

$$\min_{x} \max_{\|y\| \le \phi(x)} \left\{ \|x\|_{\mathcal{Q}}^2 + \mathcal{R}(x, y) \right\},$$

where $\mathcal{R}(x, y) = \|A_0 x - b_0 + My\|_{\mathcal{W}}^2$ *and* $\phi(x) = \|\hat{A}_V x - \hat{b}_V\|$. *The function* $\mathcal{R}(x, y)$ *is convex in* x *for any* y. *Therefore, the maximum*

$$\mathcal{C}(x) := \max_{\|y\| \le \phi(x)} \mathcal{R}(x, y)$$

is a convex function in x. *Furthermore, since* $\mathcal{Q} > 0$, $\|x\|_{\mathcal{Q}}^2$ *is strictly convex in* x, *such that the cost* $\|x\|_{\mathcal{Q}}^2 + \mathcal{C}(x)$ *is also strictly convex in* x. *Therefore, the solution* $\hat{x}$ *presented in Lemma 1 for problem* (10)–(11) *is indeed unique.*

In the following section we combine the penalty function method with the robust regularized least-squares to obtain a solution for the regulation problem (8). In fact, we show that the designed robust recursive regulator stabilizes the system (1) subject to polytopic uncertainties for $\mu > 0$.

IV. ROBUST REGULATOR FOR SYSTEMS SUBJECT TO POLYTOPIC UNCERTAINTIES

In this section, we provide the solution for the optimization problem (8)–(9), through which we compute the feedback gains K_k. The solution is expressed in a symmetric matrix arrangement and as a set of algebraic equations. Afterwards, under assumptions on controllability and observability, we carry out the convergence and stability analysis of the proposed method. Specifically, we show that, for $\mu > 0$, the closed-loop system (1) is stable when $u_k = K_k x_k$, despite the polytopic uncertainties.

Before stating the main result of this section, let us introduce the following lemma on the invertibility of block matrices.

Lemma 2 [41]: Consider $\mathcal{T} \in \mathbb{R}^{p \times p}$ *and* $B \in \mathbb{R}^{p \times l}$. *Suppose* B *has rank* l *and* $\mathcal{T} > 0$. *Then, the matrix*

$$\begin{bmatrix} \mathcal{T} & B \\ B^T & 0 \end{bmatrix} > 0.$$

Lemma 2 will be important to ensure the nonsingularity of the proposed robust recursive approach. In sequence, we present the main result of this section.

Lemma 3: Consider the optimization problem (8)–(9), *for* $k = N-1, \ldots, 0$, *with known matrices* $P_N > 0$, $R_k > 0$, $Q_k > 0$, *and fixed* $\mu > 0$. *The robust solution and corresponding cost are recursively given by*

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1}, \hat{u}_k) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & \hat{x}_k^T \end{bmatrix} \begin{bmatrix} L_k \\ K_k \\ P_k \end{bmatrix} \hat{x}_k, \tag{14}$$

where

$$\begin{bmatrix} L_k \\ K_k \\ P_k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & -I_n & \hat{F}_{0,k}^T & \hat{F}_{V,k}^T & 0 & 0 \end{bmatrix}$$
$$\times \underbrace{\begin{bmatrix} P_{k+1}^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_k^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_k^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_{0,k} \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{G}_{V,k} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_{0,k}^T & -\hat{G}_{V,k}^T & 0 & 0 \end{bmatrix}^{-1}}_{\mathcal{M}_k} \begin{bmatrix} 0 \\ 0 \\ -I_n \\ \hat{F}_{0,k} \\ \hat{F}_{V,k} \\ 0 \\ 0 \end{bmatrix}, \tag{15}$$

with $\Phi := \mu^{-1}(1 - \beta^{-1}) I_{nV}$, $\Sigma := (\beta\mu)^{-1} I_{nV}$, $\beta > 1$, $\hat{F}_{0,k} := \mathbb{1}_V \otimes F_{0,k}$, $\hat{G}_{0,k} := \mathbb{1}_V \otimes G_{0,k}$, $\hat{I} := \mathbb{1}_V \otimes I_n$,

$$\hat{F}_{V,k} := V \begin{bmatrix} F_{1,k} \\ \vdots \\ F_{V,k} \end{bmatrix}, \text{ and } \hat{G}_{V,k} := V \begin{bmatrix} G_{1,k} \\ \vdots \\ G_{V,k} \end{bmatrix}.$$

Proof: Recall that $\delta F_{k,i} = \alpha_{i,k} F_{i,k}$ and $\delta G_{k,i} = \alpha_{i,k} G_{i,k}$, $i = 1, \ldots, V$, where $F_{i,k}$ and $G_{i,k}$ are the vertices of a polytope with coefficients $\alpha_k \in \Lambda_V$. The penalty parameter μ allows us to transform (4)–(5) into (8)–(9), which is a special case of the regularized least-squares problem. We make the following identifications with problem (10)–(11):

$$x \leftarrow \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}, \ \mathcal{Q} \leftarrow \begin{bmatrix} P_{k+1} & 0 \\ 0 & R_k \end{bmatrix}, \ \mathcal{W} \leftarrow \begin{bmatrix} Q_k & 0 \\ 0 & \mu I_{nV} \end{bmatrix},$$

$$A_0 \leftarrow \begin{bmatrix} 0 & 0 \\ I_n & -G_{0,k} \\ \vdots & \vdots \\ I_n & -G_{0,k} \end{bmatrix}, \ \delta A \leftarrow \begin{bmatrix} 0 & 0 \\ 0 & -V\delta G_{1,k} \\ \vdots & \vdots \\ 0 & -V\delta G_{V,k} \end{bmatrix}, \ M \leftarrow \begin{bmatrix} 0 \\ I_{nV} \end{bmatrix},$$

$$b_0 \leftarrow \begin{bmatrix} -I_n \\ F_{0,k} \\ \vdots \\ F_{0,k} \end{bmatrix} x_k, \ \delta b \leftarrow \begin{bmatrix} 0 \\ V\delta F_{1,k} \\ \vdots \\ V\delta F_{V,k} \end{bmatrix} x_k, \ f \leftarrow J_k,$$

$$A_i \leftarrow \begin{bmatrix} 0 & -VG_{i,k} \end{bmatrix}, \ b_i \leftarrow VF_{i,k} x_k, \ i = 1, \ldots, V.$$

From the above association, we derive the recursive solution based on Lemma 1, hence attaining matrices $\{L_k, K_k, P_k\}$ in (15). Observe that we approximate $\hat{\lambda}$ by $\hat{\lambda} = \beta\mu$, for some $\beta > 1$, to circumvent additional computational effort, as pointed out in Remark 2. In general, $\beta \in (1, 2]$ leads to appropriate results. Observe that, as $\mathcal{Q} > 0$, the convexity of the addressed optimization problem is ensured by Remark 3. Finally, the existence of the inverse of $\mathcal{M}_k$ in (15) is ensured by Lemma 2. □

With the procedure developed in Lemma 3, a single matrix equation is solved at each time step considering all vertices of the polytope at once. In turn, (14) computes the stabilizing control signal u_k and the future state x_{k+1} in terms of L_k. Notice also that matrix L_k expresses the equivalent closed-loop system when $u_k = K_k x_k$, with K_k given in (15).

In view of Lemma 1, item *(ii)*, it is possible to achieve explicit expressions for $\{L_k, K_k, P_k\}$ by further developing the matrix arrangement of Lemma 3. We then present our main result in the next theorem.

Theorem 1: Consider the optimization problem (8)–(9). *For* $k = N-1, \dots, 0$ *and fixed* $\bar{\mu} > 0$, *the solution given by* (14)–(15) *is equivalent to*

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{u}_k \\ J_k(\hat{x}_{k+1}, \hat{u}_k) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & \hat{x}_k^T \end{bmatrix} \begin{bmatrix} L_k \\ K_k \\ P_k \end{bmatrix} \hat{x}_k, \tag{16}$$

with

$$L_k = \bar{\mathcal{F}}_k - G_{0,k}\left(\bar{R}_k^{-1} + G_{0,k}^T P_{k+1} G_{0,k}\right)^{-1} G_{0,k}^T P_{k+1} \bar{\mathcal{F}}_k, \tag{17}$$

$$\begin{aligned} K_k &= -\bar{R}_k G_{0,k}^T \left(I + P_{k+1} G_{0,k} \bar{R}_k G_{0,k}^T\right)^{-1} P_{k+1} \bar{\mathcal{F}}_k \\ &\quad - R_k^{-1} \hat{G}_{V,k}^T (\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T)^{-1} \hat{F}_{V,k}, \end{aligned} \tag{18}$$

$$\begin{aligned} P_k &= \bar{Q}_k + \bar{\mathcal{F}}_k^T P_{k+1} \bar{\mathcal{F}}_k \\ &\quad - \bar{\mathcal{F}}_k^T P_{k+1} \bar{G}_k \left(I_m + \bar{G}_k^T P_{k+1} \bar{G}_k\right)^{-1} \bar{G}_k^T P_{k+1} \bar{\mathcal{F}}_k, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \bar{\mathcal{F}}_k &= F_{0,k} - G_{0,k} R_k^{-1} \hat{G}_{V,k}^T (\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T)^{-1} \hat{F}_{V,k}, \\ \bar{R}_k &= R_k^{-1}(I - \hat{G}_{V,k}^T (\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T)^{-1} \hat{G}_{V,k} R_k^{-1}), \\ \bar{Q}_k &= Q_k + \hat{F}_{V,k}^T (\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T)^{-1} \hat{F}_{V,k}, \\ \Sigma &= \bar{\mu}^{-1} I_{nV}, \quad \bar{G}_k = G_{0,k} \bar{R}^{1/2}. \end{aligned}$$

Proof: From Lemma 3, we see that (15) holds if, and only if, the system of simultaneous equations

$$\underbrace{\begin{bmatrix} P_{k+1}^{-1} & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & R_k^{-1} & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & Q_k^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & \hat{I} & -\hat{G}_{0,k} \\ 0 & 0 & 0 & 0 & \Sigma & 0 & -\hat{G}_{V,k} \\ I_n & 0 & 0 & \hat{I}^T & 0 & 0 & 0 \\ 0 & I_m & 0 & -\hat{G}_{0,k}^T & -\hat{G}_{V,k}^T & 0 & 0 \end{bmatrix}}_{\mathcal{M}_k} \begin{bmatrix} a \\ b \\ c \\ \bar{d} \\ e \\ L_k \\ K_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -I_n \\ \hat{F}_{0,k} \\ \hat{F}_{V,k} \\ 0 \\ 0 \end{bmatrix}, \tag{20}$$

where $\bar{d} = \mathbb{1}_V \otimes d$, $d \in \mathbb{R}^{n \times n}$, has a unique solution. Therefore, we take a large fixed $\bar{\mu} > 0$ and solve (20) for $(a, b, c, d, e, L_k, K_k)$ to obtain

$$\begin{aligned} L_k &= \bar{\mathcal{F}}_k - G_{0,k}\left(\bar{R}_k^{-1} + G_{0,k}^T P_{k+1} G_{0,k}\right)^{-1} G_{0,k}^T P_{k+1} \bar{\mathcal{F}}_k, \\ K_k &= -\bar{R}_k G_{0,k}^T \left(I + P_{k+1} G_{0,k} \bar{R}_k G_{0,k}^T\right)^{-1} P_{k+1} \bar{\mathcal{F}}_k \\ &\quad - R_k^{-1} \hat{G}_{V,k}^T (\Sigma + \hat{G}_{V,k} R_k^{-1} \hat{G}_{V,k}^T)^{-1} \hat{F}_{V,k}. \end{aligned}$$

Here, we omitted the straightforward algebraic manipulations carried out to determine (a, b, c, d, e) needed to ultimately generate L_k and K_k. Multiply both sides of (20) to the left by $\mathcal{M}_k^{-1}$ and substitute into (15) to yield

$$P_k = \begin{bmatrix} 0 & 0 & -I_n & \hat{F}_{0,k}^T & \hat{F}_{V,k}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ \bar{d} \\ e \\ L_k \\ K_k \end{bmatrix},$$

from which we obtain (19). It is worth noting that all inverses composing (17), (18), and (19) exist, since $\bar{R}_k > 0$ and $\bar{Q}_k > 0$. Finally, the introduction of $\bar{\mu} > 0$ provides a single parameter to be tuned with respect to the robustness of the solution. □

Remark 4: After presenting the set of equations in (20), *we can now explain better why the optimization problem* (8)–(9) *is solved over both variables* $\{u_k, x_{k+1}\}$. *With this selection of variables, we are able to provide, in a unified fashion, both stability and robustness to the control system by solving the following equations*:

$$\begin{aligned} \hat{I} L_k &= (\hat{F}_{0,k} + \hat{G}_{0,k} K_k) - \Phi \bar{d}, \\ \Sigma e &= (\hat{F}_{V,k} + \hat{G}_{V,k} K_k), \end{aligned}$$

which involve all polytope vertices of (1). *Recall that if* $\bar{\mu} \to \infty$, *we have that* $\Phi \to 0$ *and* $\Sigma \to 0$, *such that the convergences* $(\hat{F}_{0,k} + \hat{G}_{0,k} K_k) \to \hat{I} L_k$ *and* $(\hat{F}_{V,k} + \hat{G}_{V,k} K_k) \to 0$ *hold, so we obtain the optimal robust regulator. If it is not possible to tune* $\bar{\mu} \to \infty$, *we can adjust* $\bar{\mu}^{-1} \to \epsilon$ *in order to obtain a sub-optimal robust solution.*

Remark 5: To some extent, the robust recursive solution presented in Theorem 1 relates to Policy Iteration (PI) and Value Iteration (VI) algorithms [42] through Bellman's Principle of Optimality. However, there are fundamental differences among these methods. PI and VI usually relate to reinforcement learning algorithms, data-driven control and Markov decision processes. The proposed approach, in contrast, does not employ transition probabilities to compute the control actions and there is no learning procedure involved. Moreover, it provides an analytical solution for the regulation problem of linear systems subject to polytopic uncertainties.

The penalty parameter μ is closely related to the optimality of the solution. In fact, (16) (and (14)) converges to the optimal solution of the original constrained problem as $\mu \to \infty$. Nonetheless, although a finite positive μ yields a sub-optimal solution, the resulting feedback gain still stabilizes system (1). We elaborate on this aspect in the following subsection.

A. CONVERGENCE AND STABILITY ANALYSIS

To carry out the analysis, we assume invariant system parameters and allow coefficients α_k to be time-varying. Thus, we address the discrete-time realization

$$x_{k+1} = \left(F + \sum_{i=1}^{V} \alpha_{i,k} F_i\right) x_k + \left(G + \sum_{i=1}^{V} \alpha_{i,k} G_i\right) u_k,$$

with $\alpha_k \in \Lambda_V$, and rewrite P_k as

$$\begin{aligned} P_k &= \bar{Q} + \bar{\mathcal{F}}^T (P_{k+1} \\ &\quad - P_{k+1} \bar{G} \left(I_m + \bar{G}^T P_{k+1} \bar{G}\right)^{-1} \bar{G}^T P_{k+1}) \bar{\mathcal{F}}. \end{aligned} \tag{21}$$

The following theorem establishes the conditions for convergence and stability of the proposed solution.

Theorem 2: Assume the pair $\{\bar{Q}, \bar{\mathcal{F}}\}$ *is observable,* $\{\bar{\mathcal{F}}, \bar{G}\}$ *is controllable,* $\bar{\mu} > 0$ *and consider* (21) *with initial* *condition $P_0 > 0$. Then, P_{k+1} converges to the unique stabilizing solution $P > 0$ satisfying*

$$P = \bar{Q} + \bar{\mathcal{F}}^T \left(P - P\bar{G}\left(I_m + \bar{G}^T P\bar{G}\right)^{-1} \bar{G}^T P\right)\bar{\mathcal{F}}.$$

In addition, the corresponding closed-loop system matrix

$$L = \bar{\mathcal{F}} - G\left(\bar{R}^{-1} + G^T P_k G\right)^{-1} G^T P_k \bar{\mathcal{F}},$$

such that $x_{k+1} = Lx_k$, is Schur stable.

Proof: Notice that (21) resembles the classic algebraic Riccati equation, given by

$$P_k = Q + A^T\left(P_{k+1} - P_{k+1}B(R + B^T P_{k+1}B)^{-1}B^T P_{k+1}\right)A.$$

Based on this equivalence and on fundamental arguments presented in [37, Chapter 4], [43, Chapter 12], it follows that P_k converges to the unique stabilizing solution $P > 0$. Moreover, the stabilizing P is such that the eigenvalues of the closed-loop system matrix L are kept inside the open unit disc for any $\bar{\mu} > 0$. □

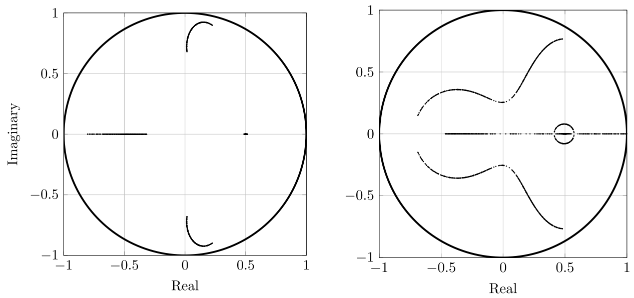
At this point, it is important to elaborate on how the penalty parameter is selected. As can be seen in Theorem 2, under plausible assumptions on controllability and observability, convergence and stability of the solution are guaranteed for any $\bar{\mu} > 0$. As $\bar{\mu}$ is kept fixed throughout operation, the choice of an adequate penalty can be performed offline without search algorithms. By this feature, the selection of $\bar{\mu}$ does not imply additional computational time, which is rather convenient from the application point of view.

Remark 6: Complexity is an important aspect to be discussed, as it directly relates to the number of stored variables. Observe that we compute the feedback gain K essentially based on the Riccati solution $P \in \mathbb{R}^{n\times n}$, since all other parameters are fixed. The complexity of the proposed solution is therefore independent of the number of polytope vertices. On the other hand, in LMI-based methods the feedback gains have the general form $\mathcal{K} = YX^{-1}$, where $Y \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{R}^{n\times n}$ are built upon a set of parameter-dependent Lyapunov functions. As such, the complexity of these approaches scales as the number of vertices increases.

The analysis in Remark 6 also implies that the proposed approach does not require numerical optimization packages to compute the feedback gain. In this sense, the robust recursive regulator has potential to benefit applications with limitations on memory space, software and hardware settings.

V. NUMERICAL EXAMPLES

We present two examples to illustrate the effectiveness of the proposed robust regulator. For comparison purposes, we adopt the robust controller presented in [26] and computed with the YALMIP Toolbox [44]. The experiments were carried out on a 2.50 GHz i5-3210M CPU with 8 GB of RAM.



(a) Robust regulator. (b) [26, Theorem 3].

FIGURE 1. Eigenvalues of the closed-loop system with $\rho = \bar{\rho}_{\text{ref [26]}}$: $\max\{\|\nu\|_{RLQR}\} = 0.937381$, $\max\{\|\nu\|_{\text{ref [26]}}\} = 0.999707$.

Example 1: Consider the discrete-time linear system adapted from [26]:

$$F_0 = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.03 \\ 0 & 0 & 1.0 & 0 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix},$$

with the parameterized vertices $(\rho F_i, G_i)$, $i = 1, 2$, with

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.8 & -0.5 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix},$$

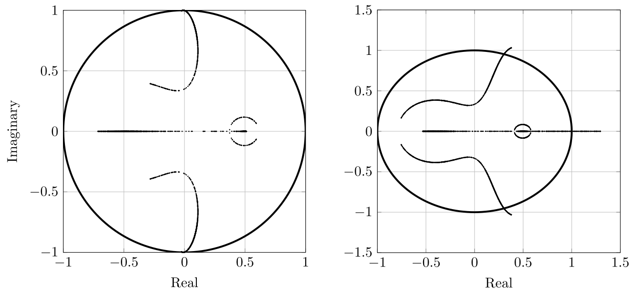
$$F_2 = -F_1, \quad G_2 = -G_1.$$

We search for the maximum values $\{\bar{\rho}_{RLQR}, \bar{\rho}_{\text{ref [26]}}\}$ such that the closed-loop system is stable with the robust regulator for any $\rho \le \bar{\rho}_{RLQR}$, and with the controller from [26, Theorem 3] for any $\rho \le \bar{\rho}_{\text{ref [26]}}$. Moreover, we select $\bar{\mu} = 1.2 \times 10^{15}$ and weights $P_0 = I_4$, $Q_k = I_4$ and $R_k = 1$ to compose the cost function (9).

By carrying out an iterative search, we obtain the maxima $\bar{\rho}_{RLQR} = 1.9130$ and $\bar{\rho}_{\text{ref [26]}} = 1.0511$. In Figs. 1 and 2, we show the eigenvalues ν of the closed-loop system considering $\{\delta F_k, \delta G_k\}$ with vertices $(\bar{\rho}_{\text{ref [26]}} F_i, G_i)$ and $(\bar{\rho}_{RLQR} F_i, G_i)$, respectively. Observe that, in the limit $\rho = \bar{\rho}_{\text{ref [26]}}$, both methods are able to stabilize the system. Nonetheless, with $\rho = \bar{\rho}_{RLQR}$, only the robust regulator remains effective. Thus, it provides a wider region of stability.

Furthermore, by varying μ it is possible to verify how it affects the region of stability through the parameter $\bar{\rho}_{RLQR}$. We present this result in Table 1, along with the maximum norms of closed-loop eigenvalues ν_μ for each case. It can be seen that, even for small values of penalty μ, the proposed robust regulator outperforms [26, Theorem 3] by yielding $\bar{\rho}_{RLQR} > \bar{\rho}_{\text{ref [26]}}$. As μ increases, $\bar{\rho}_{RLQR}$ converges to 1.9130.

Finally, we examine the computational effort to calculate the feedback gains for both controllers. The average time spent to compute the feedback gain via Theorem 1 was 1.7 ms, while [26, Theorem 3] demanded 149.7 ms.



(a) Robust regulator. (b) [26, Theorem 3].

FIGURE 2. Eigenvalues of the closed-loop system with $\rho = \bar{\rho}_{RLQR}$: $\max\{\|\nu\|_{RLQR}\} = 0.999980$, $\max\{\|\nu\|_{\mathrm{ref}\,[26]}\} = 1.296115$.

TABLE 1. Effects of μ over $\bar{\rho}_{RLQR}$ and maximum norms of closed-loop eigenvalues with the robust recursive regulator.

μ	$\bar{\rho}_{RLQR}$	$\|\nu_\mu\|$
1	1.20730	0.999966
10	1.75800	0.999970
10^5	1.91290	0.999973
10^{10}	1.91300	0.999980
10^{12}	1.91300	0.999980

This result indicates that the proposed robust regulator is also suitable for online implementation.

Example 2: The following 4-DOF system is based on [45] and describes a trajectory tracking model of a commercial quadrotor:

$$F_0 = \begin{bmatrix} \Xi_0 & 0 \\ 0.01\, I_4 & I_4 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 0.01\, I_4 \\ 0 \end{bmatrix},$$

with

$$\Xi_0 = \begin{bmatrix} 0.9985 & 0.0003 & 0 & 0 \\ 0.0003 & 0.9970 & 0 & 0 \\ 0 & 0 & 0.9755 & 0 \\ 0 & 0 & 0 & 0.9893 \end{bmatrix},$$

and $x_k = \begin{bmatrix} e_{\mathrm{v}_k}^T & e_{\mathrm{p}_k}^T \end{bmatrix}^T \in \mathbb{R}^8$, with $e_{\mathrm{v}_k} = \begin{bmatrix} e_{\mathrm{vx}} & e_{\mathrm{vy}} & e_{\mathrm{vz}} & e_{\mathrm{v}_\psi} \end{bmatrix}^T$ and $e_{\mathrm{p}_k} = \begin{bmatrix} e_{\mathrm{x}} & e_{\mathrm{y}} & e_{\mathrm{z}} & e_\psi \end{bmatrix}^T$, in which $\{e_{\mathrm{vx}}, e_{\mathrm{x}}\}$, $\{e_{\mathrm{vy}}, e_{\mathrm{y}}\}$, and $\{e_{\mathrm{vz}}, e_{\mathrm{z}}\}$ are velocity and position errors along the global x, y and z axes, respectively, and $\{e_{\mathrm{v}_\psi}, e_\psi\}$ are angular velocity and orientation errors, in that order. In addition, the commands are given by $v_{drone} = u_{ref} - u$, where u_{ref} is the reference control input and u is the signal computed according to the adopted control law.

Our goal is to track an 8-shaped reference trajectory starting at the origin of the global coordinate frame. Uncertainties $\{\delta F_k, \delta G_k\}$ belong to a polytopic domain and represent variations introduced by unmodeled dynamics, nonlinearities, and discretization errors. Then, we consider two polytope vertices

$$F_i = \begin{bmatrix} 10^{-2}\Xi_i & 0 \\ 0 & 0 \end{bmatrix}, \quad G_i = \begin{bmatrix} 10^{-3}\Upsilon_i \\ 0 \end{bmatrix}, \quad i = 1, 2,$$

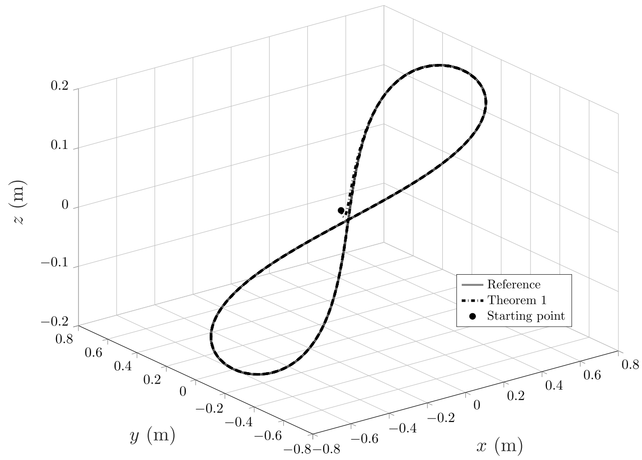


FIGURE 3. Resulting trajectory of the quadrotor in the global coordinate frame.

where

$$\Xi_1 = -\begin{bmatrix} 1.37 & 5.99 & 0 & 0 \\ 4.64 & 6.82 & 0 & 0 \\ 0 & 0 & 1.29 & 0 \\ 0 & 0 & 0 & 3.12 \end{bmatrix}, \quad \Upsilon_1 = -\begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 1.1 & 0.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 2.0 \end{bmatrix},$$

$$\Xi_2 = \begin{bmatrix} 0.80 & 2.66 & 0 & 0 \\ 3.10 & 6.21 & 0 & 0 \\ 0 & 0 & 1.04 & 0 \\ 0 & 0 & 0 & 5.13 \end{bmatrix}, \quad \Upsilon_2 = \begin{bmatrix} 6.5 & 7.4 & 0 & 0 \\ 0.8 & 4.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 1.9 \end{bmatrix}.$$

For the quadratic cost function (9), we choose a penalty parameter $\bar{\mu} = 10^{10}$ and weighting matrices $P_0 = 10^{12} I_8$, $R_k = I_4$ and $Q_k = \mathrm{diag}\{0.5 \cdot 10^{10} I_4,\ 10^{10} I_4\}$. Therefore, the feedback gain K provided by Theorem 1 converges to

$$K = \begin{bmatrix} -60.7374 & 23.1583 & 0 & 0 \\ 22.3468 & -48.7937 & 0 & 0 \\ 0 & 0 & -99.4591 & 0 \\ 0 & 0 & 0 & -74.5335 \\ -82.7444 & 33.4338 & 0 & 0 \\ 31.64482 & -64.1943 & 0 & 0 \\ 0 & 0 & -140.2319 & 0 \\ 0 & 0 & 0 & -102.0202 \end{bmatrix}^T.$$

Additionally, we use the conditions from [26, Theorem 3] to obtain the feedback gain

$$K_{\mathrm{ref}\,[26]} = \begin{bmatrix} -72.3857 & 2.6812 & 0 & 0 \\ 25.2426 & -78.7751 & 0 & 0 \\ 0 & 0 & -96.8010 & 0 \\ 0 & 0 & 0 & -97.2112 \\ -41.1403 & 1.7914 & 0 & 0 \\ 13.8486 & -43.3649 & 0 & 0 \\ 0 & 0 & -56.3373 & 0 \\ 0 & 0 & 0 & -54.8025 \end{bmatrix}^T.$$

A total of 1000 Monte Carlo experiments, each with time horizon $N = 3000$ (30 seconds flights), are performed. The initial condition is the same for both controllers, being $x_0 = \begin{bmatrix} 0.40 & 0.33 & -0.64 & 0.01 & 0 & 0 & 0.01 & 0 \end{bmatrix}^T$. The resulting motion of the quadrotor along with the reference trajectory in the global coordinate frame are presented in Fig. 3. In Figs. 4 and 5 we show the norms of tracking errors and control inputs averaged over all experiments, respectively. The average norms and standard deviations of velocity and

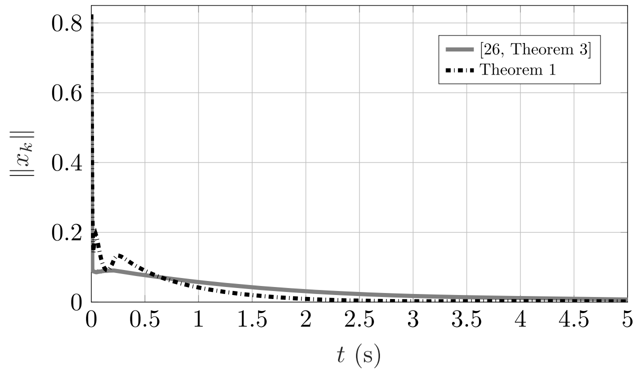


FIGURE 4. Averaged norms of tracking errors obtained with Theorem 1 and [26, Theorem 3].

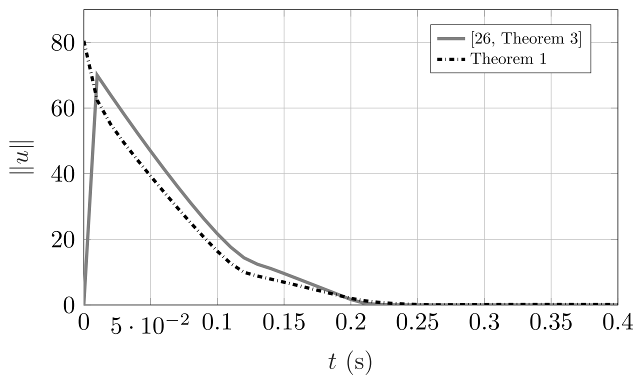


FIGURE 5. Averaged norms of control inputs computed with Theorem 1 and [26, Theorem 3].

TABLE 2. Average trajectory tracking errors and standard deviations.

Controller	$\|e_v\|_{\mathcal{L}_2}$	σ_v	$\|e_p\|_{\mathcal{L}_2}$	σ_p	$\|u\|_{\mathcal{L}_2}$
Theorem 1	1.8996	0.0347	0.3366	0.0060	152.8521
[26, Theorem 3]	1.8503	0.0338	0.4185	0.0073	155.3303

position tracking errors, $\|e_v\|_{\mathcal{L}_2}$, σ_v, $\|e_p\|_{\mathcal{L}_2}$, and σ_p, respectively, and the average control input norm, $\|u\|_{\mathcal{L}_2}$, are summarized in Table 2.

To perform an additional experiment, we assume parameterized vertices $(\rho F_i, G_i)$. We search for $\{\bar{\rho}_{RLQR}, \ \bar{\rho}_{\text{ref [26]}}\}$ such that the robust regulator and the controller borrowed from [26, Theorem 3] stabilize the closed-loop system for any $\rho \leq \bar{\rho}_{RLQR}$ and for any $\rho \leq \bar{\rho}_{\text{ref [26]}}$, respectively. Hence, we find $\bar{\rho}_{RLQR} = 11.5002$ with the robust regulator, and $\bar{\rho}_{\text{ref [26]}} = 11.4967$ with [26, Theorem 3].

VI. CONCLUSION

We presented a robust recursive framework for regulation of discrete-time linear systems subject to polytopic uncertainties. We formulated an optimization problem by combining robust regularized least-squares and a penalty parameter which is straightforwardly tuned beforehand. It was possible, therefore, to comprise all polytope vertices at once whilst ensuring the existence of a unique global minimum. The solution has a structure similar to the recursive Riccati equation and we take advantage of this feature to establish conditions for convergence and stability. With illustrative examples, we verify the effectiveness of the robust regulator in comparison to a robust controller borrowed from the specialized literature. The proposed solution guaranteed stability for wider polytopic regions of uncertainty and required lower computational effort to compute feedback gains. As such, the results also reveal the potential of application in real platforms.

For future works, the robust regulation problem for discrete-time Markov jump linear systems subject to polytopic uncertainties shall be addressed.

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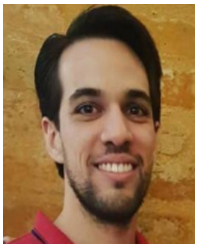

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