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NON - DETERMINISTIC DYNAMICAL SYSTEMS

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RESUMO: Neste trabalho procura-se extender a teoria dos sistemas dinâmicos aos espaços topológicos. A diferença essencial de outras extensões é que são preservadas as relações entre sistemas dinâmicos e equações diferenciais. Isso é possível graças à matemática não-determinista que oferece uma teoria de derivação em espaços topológicos. No final faz-se uma aplicação à teoria do movimento Browniano.

ABSTRACT: This paper extends the theory of dynamical systems to topological spaces. It differs from other extensions because it preserves the connections between dynamical systems and differential equations. This is possible because in non - deterministic mathematics we have a theory of derivatives in topological spaces. An application to Brownian motion is given at the end.

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§I

Fundamental Concepts

1. In this work we start investigacions on the possibilities of extending the theory of dynamical systems to topological spaces in the spirit of non-deterministic mathematics. The foundations of non-deterministic mathematics were laid in 1964-65 with subsequent work by myself and former students at McMaster University.A recent view of the subject can be seen at [1] even though it does not include the theory of measure and integration in Gauss spaces. We assume here a knowledge of [1], or at least of its introductory paragraphs.

The work is divided into 5 sections whose contents are: in section §I we give the fundamental concepts of n-groups and n-dynamical system, together with basic notions needed for the subsequent sections.

In section §II we prove several theorems showing the connection between the classical concept of dynamical system and the non-deterministic one.

In section §III we show how many concepts of the classical theory of dynamical systems in metric space can be extended to topological spaces using ideas in non-deterministic mathematics. Due to lack of space we only touch the main concepts.

In section §IV we show that under proper conditions an n-dynamical system can be regarded as defined by a family of solutions of certain differential equations and vice-versa, namely, under proper conditions a family of solutions of certain differential equation define an n-dynamical system.

We close the section with an example and remarks to possible applications to Brownian motions.

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Let us start now with some general considerations.

Historically, the notion of dynamical system grew out from the work of S. Lie and H. Poincaré, connected with the study of groups of transformations and differential equations in R^n and can be introduced as follows: let

$$F: R^n \times R \to R^n$$

be a continuously differentiable function, where R^n is the n-dimensional Euclidean space and R stand for the real numbers, subjected to the conditions:

$$(D_1)\ F(x,0) = x,\ \forall x \in R^n.$$

(D_2) If $t_1, t_2 \in R$, then

$$F(x,t_1+t_2) = F(F(x,t_1),t_2).$$

Then we say that F is a dynamical system in R^n and under the point of view of the theory of group of transformations it is nothing else but a one-parameter group of transformations in R^n. For basic knowledge in this area, besides the classical works of S. Lie [2] and H. Poincaré [3] the best references are L. Pontrjagin [4] and Nemytski and Stepanov [5].

One of the reasons for the beauty of the classical theory lies on the fact that every dynamical system, under suitable conditions can be defined by a system of differential equations

$$(1) \qquad \frac{dx_i}{dt} = X_i(x_1 \ldots x_n) \quad i = 1 \ldots n$$

whose solutions for each inital values $(x_1^o \ldots x_n^o)$ are differentiable curves in R^n described by the parameter t and as a result of unicity of solutions of (1), under proper conditions on X_i, $i = 1 \ldots n$, we have that the function

$$F(x,t) = f(x,t)$$

where $f(x,t)$ is a solution of (1) passing through $x \in R^n$, defines a one-parameter group of transformations of R^n, which is a dynamical system, when $-\infty < t < +\infty$. Conversely, if we start from a dynamical system $F(x,t)$ in R^n we have that the family of functions of t $f(x,t) = F(x,t)$, for $x \in R^N$, are solutions of a system like (1). These are classical results and a proof can be seen at [5].

Coming back to the definition of dynamical system in R^n, we realize that conditions (D_1) and (D_2) make sense in any metric space. Of course, we have to drop the assumption of differentiability of F because such a concept does not exist for metric spaces in general.

As a consequence there are no differential equations and the situation described above for the classical cases does not exist. Naturally, a substitute to cover some of the difficulties, due to the lack of differentiability, was sought in the notion of tube and transversal sections, as developed by Bebutov and others (see [5], p. 332). This situation is one of the tragical consequences of the impious idea, which proclaims that derivatives are linear maps! Certainly, when some linear structure is available the concept of derivative is very conveniently expressed in terms of linear maps, but it is a lack of feeling for the deepest meaning of the concept of derivative not to realize that this concept is much richer than the concept of linear maps. One of the main points of the program of non-deterministic mathematics is the construction of a theory of derivatives in topological spaces completely free from the yoke of linear maps. Now as such a theory already exists [1], one might be able to extend the theory of dynamical systems to general topological spaces with the possibility of re-establishing the harmony between such a concept and that of differential equations, in the line of the classical ideas. Of course, dynamical systems and differential equations have to be understood in the sense of non-deterministic mathematics.

The investigation of the possibility of such a theory is the aim of this work.

2. We start with the introduction of the fundamental concepts relevant to this work and we recall that we assume knowledge of the basic notions of non-deterministic mathematics as can be seen at [1].

We assume that topological space means Hausdorff, i.e., T_2 space, unless explicitly remarked otherwise, even though the basic definitions and concepts would be valid in general topological spaces.

In non-deterministic mathematics we do not consider points of a set X as a fundamental concept in the sense that the definitions of n-function and its continuity and differentiability do not use points but rather open sets. This even allows these concepts to be generalized in the level of categories as can be seen at [6]. However, in this work we need, in many cases, to "localize" concepts and so a substitute for "points" become necessary. That is the reason for introducing the concept of germ, below. In this paper everytime we consider a pair $(X, \mathcal{V})$ we always understand that $\mathcal{V}$ is a family of collections of open sets, eventually open coverings, of the topological space X.

Definition I - A germ p in a pair $(X, \mathcal{V})$ is a set of open sets A_σ^p of $\mathcal{V}$, with $A_\sigma^p \in \sigma \in \mathcal{V}$ and only one for each $\sigma \in \mathcal{V}$, such that

$$\bigcap_{\sigma \in \mathcal{V}} A_\sigma^p \neq \phi .$$

<u>Notation</u>:

$$p = \{A_\sigma^p\}_{\sigma \in \mathcal{V}}$$

and

$$A_\sigma^p \in p ,$$

when we want to call attention that A_σ^p belongs to the germ p. Certainly, when no confusion is possible we simplify notations by writing A_σ or A instead of A_σ^p .

Definition I' - (a) Let $U \subset X$ be an open set. We say that a germ $p = \{A^p_\sigma\}_{\sigma \in \mathcal{V}}$ is in U if there is $\sigma \in \mathcal{V}$ such that for any $\tau \in \mathcal{V}$, $\tau > \sigma$ we have $A_\tau \subset U$.

(b) A germ $p = \{A^p_\sigma\}_{\sigma \in \mathcal{V}}$ is compatible with refinements if $\tau > \sigma$, $\tau, \sigma \in \mathcal{V}$, implies $A_\tau \subset A_\sigma$.

Let us discuss now a few notions connected with the concept of germ.

If $p = \{A_\sigma\}_{\sigma \in \mathcal{V}}$ is a germ in $(X, \mathcal{V})$ we call the nucleus of p the set.

$$\bigcap_{\sigma \in \mathcal{V}} A_\sigma \neq \phi ,$$

denoted by $\langle p \rangle$.

If the nucleus of p reduces to a single point we say that p is simple.

Given a germ $p = \{A^p_\sigma\}_{\sigma \in \mathcal{V}}$ of $(X, \mathcal{V})$ and a subset E of X we denote by $p \cap E \neq \phi$ the fact that $E \cap A^p_\sigma \neq \phi$ for all $\sigma \in \mathcal{V}$, i.e. ,

$$p \cap E \neq \phi \iff E \cap A^p \neq \phi, \forall_{\sigma \in \mathcal{V}}.$$

Clearly, if $E \cap \langle p \rangle \neq \phi$, this implies $p \cap E \neq \phi$, but not conversely.

Sometimes it is also convenient to index the sets of a germ p not by $\mathcal{V}$ but by an arbitrary set I for which there is a surjection $I \to \mathcal{V}$.

We use the notation Cov X, to denote the set of all open coverings of X and we say that a family $\mathcal{V}$ of collections of sets of X is cofinal in Cov X if for any covering $\alpha \in$ Cov X there is a collection $\sigma \in \mathcal{V}$ with $\sigma > \alpha$.

An n-function

$$f: (X, \mathcal{V}) \to (Y, \mathcal{V}')$$

is cofinal in Cov Y if $\{f_{\mathcal{V}}(\sigma)\}_{\sigma \in \mathcal{V}}$ is cofinal in Cov Y.

Proposition 1- If $\mathcal{V}$ is cofinal in Cov X, then any germ p in $(X, \mathcal{V})$ is simple.

Proof - Assume by contradiction, that p is a germ in $(X, \mathcal{V})$ such that $\langle p \rangle$ contains more than one point and let $x_1, x_2 \in \langle p \rangle$, $x_1 \neq x_2$. Consider the open covering Γ of X defined as follows:

$$\Gamma = \{X - \{x_1\}, X - \{x_2\}\}$$

As $\mathcal{V}$ is cofinal in Cov X, there is $\sigma \in \mathcal{V}$ with $\sigma > \Gamma$. But now any set $A \in \sigma$ which contains x_1 cannot contain x_2 and vice-versa and this implies that $\langle p \rangle$ cannot contain both x_1 and x_2 contrary to our assumptions. Notice that this proposition is true also in spaces where any point is closed, i.e., more general than T_2.

3. Later on we shall need to introduce a topology in the set of germs of $(X, \mathcal{V})$ and so we study this question in this paragraph.

We denote the set of germs of $(X, \mathcal{V})$ by $[(X, \mathcal{V})]$ and we introduce a topology in this set by specifying its open sets.

This is done very easily as follows: Let A be any open set in X and $p \in [(X, \mathcal{V})]$, with $\langle p \rangle \subset A$, define:

$$V_A(p) = \{q \in [(X, \mathcal{V})] : \langle q \rangle \cap A \neq \phi\}.$$

A subset E of $[(X,\mathcal{V})]$ is defined to be open, if for any $p \in E$, there is A open in X that

$$V_A(p) \subset E .$$

(1) Let E,F be open in $[(X,\mathcal{V})]$ and if $E \cap F \neq \phi$ take any $p \in E \cap F$. Then there are A,B open in X with

$$V_A(p) \subset E \quad \text{and} \quad V_B(p) \subset F.$$

Now

$$V_{A\cap B}(p) \subset V_A(p) \cap V_B(p) \subset E \cap F$$

and the same applies for finitely many $E_1,\ldots,E_n$ open sets in $[(X,\mathcal{V})]$.

(2) Clearly if $\{E_i\}$, $i \in I$, is an arbitrary family open sets in $[(X,\mathcal{V})]$ also $\bigcup_{i\in I} E_i$ is an open set in $[(X,\mathcal{V})]$.

Therefore (1) and (2) shows that we have a topology in $[(X,\mathcal{V})]$ having all sets $V_A(p)$ as a base.

In particular if all germs are simple any set $V_A(p)$ is an open neighbourhood of p. This is not true in general if p is not simple.

One can also investigate the question: what separation axioms of X are preserved in $[(X,\mathcal{V})]$? But, we shall not study these questions now as they will not be used later on. In general, when all the germs are simple most properties of X are preserved in $[(X,\mathcal{V})]$.

The topology in $[(X,\mathcal{V})]$ defined above will be called <u>canonical topology of $[(X,\mathcal{V})]$</u>.

4. We introduce now the concept of n-semi-group and intu -

itively speaking, the definition imitates the usual one for semi-groups, namely a semi-group G is a set together with an operation which associates to every pair (x,y) of elements of G another element z in G, being associative and having a zero. For an n-semi-group G we define an operation associating to every pair (A,B) of open sets in G another open set C in G. Naturally, these open sets belong to prescribed coverings of G and several conditions are attached to the map

$$(A,B) \to C.$$

As a preliminary to the definition we say that a n-function

$$f: (X,\mathcal{V}) \to (Y,\mathcal{V}')$$

is germ preserving if for every germ $p = \{A_\sigma\}_{\sigma \in \mathcal{V}}$ in $(X,\mathcal{V})$, we have that

$$p' = \{A'_{\sigma'}\}_{\sigma' \in \mathcal{V}'}$$

with $A'_{\sigma'} = f_\sigma(A_\sigma)$, $\sigma' = f_{\mathcal{V}}(\sigma)$, is a germ in $(Y,\mathcal{V}')$.

As a consequence of this definition for a n-function to be germ preserving it is necessary that $f_{\mathcal{V}}$ be surjective, but clearly this is not sufficient as it is very easy to give examples of n-functions, even continuous, which are not germ preserving. We can also write p' as

$$p' = \{f_\sigma(A_\sigma)\}_{\sigma \in \mathcal{V}}$$

because

$$f_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}'$$

defines, in the case we are considering, a surjection. Another convenient notation for p' is

$$p' = f(p).$$

Definition II - A non-deterministic semi-group $\mathcal{G}$, or n-semi-group $\mathcal{G}$, is a pair $(X,\mathcal{V})$ together with a germ preserving n-function.

$$f: (X \times X, \mathcal{W}) \to (X, \mathcal{V})$$

satisfying the condition:

(G_1) $\mathcal{W}$ is the "diagonal" of $\mathcal{V} \times \mathcal{V}$, i.e., coverings of $X \times X$ of the form

$$\sigma \times \sigma = \{A \times B;\ A,B \in \sigma \in \mathcal{V}\}.$$

and

$$f_{\mathcal{W}}(\sigma \times \sigma) = \sigma .$$

(G_2) There is a unique germ e in $(X,\mathcal{V})$, called the identity of $(X,\mathcal{V})$, such that for any $\sigma \in \mathcal{V}$ and any $A,B \in \sigma$, $B \in e$, we have

$$f_{\sigma\times\sigma}(A \times B) = f_{\sigma\times\sigma}(B \times A) = A.$$

(G_3) For any $\sigma \in \mathcal{V}$ and $A,B,C \in \sigma$ we have the law:

$$f_{\sigma\times\sigma}[A \times f_{\sigma\times\sigma}(B \times C)] = f_{\sigma\times\sigma}[f_{\sigma\times\sigma}(A \times B) \times C] .$$

Notations:

(1) $p \cdot q = \{f_{\sigma\times\sigma}(A \times B)\}_{\sigma \in \mathcal{V}}$, $A \in p$, $B \in q$, which is a germ in $(X, \mathcal{V})$.

(2) If we need to be more precise we indicate a n-semi-group $\mathcal{G}$ as $\mathcal{G} = \{(X, \mathcal{V}), f\}$.

Definition II' - A non-deterministic group, abreviated n-group is a n-semi-group satisfying the condition ;

(G) For any germ p in $(X, \mathcal{V})$ there is a unique germ s in $(X, \mathcal{V})$ such that for any $\sigma \in \mathcal{V}$ and $A, B \in \sigma$, with $A \in p$ and $B \in s$ we have

$$f_{\sigma\times\sigma}(A \times B) = f_{\sigma\times\sigma}(B \times A) \in e;$$

the germ s is called the inverse of p and it is denoted by $s = p^{-1}$. Also B is written as A^{-1}.

Remark - In this work we only use the concept of n-semi-group because many properties of non-deterministic dynamical systems, to be defined ahead, do not depend on the existence of on inverse. Also the theory of n-groups is not yet developed so that we even do not have good examples of n-groups. For n-semi-group non trivial examples are easier to supply , as will be seen by the end of this section. However some of the results will be expressed in terms of n-groups.

Definition III - A n-semi-group

$$\mathcal{G} = \{(X, \mathcal{V}), f\}$$

where f is continuous is called a continuous n-semi-group.

Definition IV - A n-group

$\mathcal{G} = \{(X, \mathcal{V}), f\}$ is called a differentable n-group or a n-Lie group if X has a structure $\mathcal{F}$ of Gauss space together with a Gauss transformation $G : \mathcal{F} \longrightarrow \mathcal{F}$, which is the identity map and f is differentiable relative to $\mathcal{F}$.

Remarks (a) We observe that definition IV is a natural generalization of a Lie group, in the spirit of non-deterministic mathematics. In the classical case we need X to be analytic manifold because we cannot speak of differentiable functions for topological spaces in general. But in non-determrministic mathematics we do not need such restriction because the concept of differentiability can be defined for topological spaces having a Gauss structure and when $G: \mathcal{F} \longrightarrow \mathcal{F}$ is the identity and f is differentiable, in the sense of non-deterministic mathematics, we can use Definition IV.

(b) Analysing definition IV we notice further that the structure of n-semi-group defined in $(X, \mathcal{V})$ induces a structure of semi-group, in the usual sense, in $[(X, \mathcal{V})]$ given by

$$(p, q) \longrightarrow p \cdot q$$

as defined above and then we call $[(X, \mathcal{V})]$ the semi-group of germs of $(X, \mathcal{V})$. Naturally one might ask if any usual semi-group G is isomorphic with the semi-group of germs of some pair $(X, \mathcal{V})$ with a struc ture of n-semi-group. If this is true we can study perhaps the properties of G by looking to $\{(X,\mathcal{V}), f\}$. For instance, suppose f generates some usual continuous functions $\phi : X \times X \longrightarrow X$ which might also give a semi-group structure in X and then we ask about relations, in terms of semi-group, between G and X . Furthermore, if X has a Gauss structure $\mathcal{F}$, it is possible that D_f, the n-derivative of f, defines a function $\psi : X \times X \longrightarrow R$ (reals) which can be interpreted as a "derivative" of the product ϕ in X and connected in some way with a "derivative" of the product in G . This is a program we do not consider now and we restrict ouselves later on in studying some connection between n-semi-groups and usual semi-groups.

5. Now we are prepared to introduce the idea of non-deterministic dynamical system and we start with some preliminary notions. Our first step is the generalization of the concept of family of transformations in the sense of non-deterministic mathematics and roughly speaking we want to introduce a family of n-functions

$$g: (X,\mathcal{V}) \to (X,\mathcal{V})$$

which is a continuous family in some sense and each n-function of the family is a kind of generalized homeomorphism.

Let $(X,\mathcal{V})$, $(Y,\mathcal{V}')$ be pairs and consider the n-function

$$f: (X \times Y,\mathcal{W}) \longrightarrow (X,\mathcal{V}) .$$

with $\mathcal{W}$ defined as follows: we assume that there is a bijection,

$$\Lambda: \mathcal{V} \to \mathcal{V}'$$

compatible with refinements of $\mathcal{V}$ and $\mathcal{V}'$, i.e.,

$$\sigma < \tau \iff \Lambda(\sigma) < \Lambda(\tau)$$

and then any $\sigma_{\mathcal{W}} \in \mathcal{W}$ is of the type

$$\sigma_{\mathcal{W}} = \sigma \times \sigma'$$

with $\sigma \in \mathcal{V}$ and $\sigma' = \Lambda(\sigma)\sigma \in \mathcal{V}'$.

Under these conditions we say that <u>f is Λ-regulated</u>. Consider now any germ p' in $(Y, \mathcal{V}')$ and define

$$f^{p'}_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}$$

by

$$f^{p'}_{\mathcal{V}}(\sigma) = f_{\mathcal{W}}(\sigma\times\sigma') = \sigma,$$

where $\sigma \in \mathcal{V}$ and $\sigma' = \Lambda(\sigma)$. Also for every $\sigma \in \mathcal{V}$ we define

$$f^{p'}_{\sigma} : \sigma \longrightarrow \sigma$$

by

$$f^{p'}_{\sigma}(A) = f_{\sigma\times\sigma'}(A\times A')$$

for every $A \in \sigma$ and $A' \in \sigma'$, $A' \in p'$. This defines a n-function

$$f^{p'} : (X, \mathcal{V}) \longrightarrow (X, \mathcal{V})$$

The family $\{f^{p'}\}$, $p' \in [(Y, \mathcal{V}')]$ is called a <u>family of</u>

n-functions generated by f and $(Y, \mathcal{V}')$ is called space of parameters. We also say that $f^{p'}$ is generated by f at p'. The name transformation will apply when both $f^{p'}_{\mathcal{V}}$ and $f^{p'}_{\sigma}$ are one-to-one and onto for all p' and all σ.

Definition V - A n-dynamical system on $(X, \mathcal{V})$ is a germ preserving n-function

$$f: (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

where

(D_1) P is the space of a continuous n-semi-group $(P, \mathcal{V}_P)$

(D_2) f is Λ-regulated with $\Lambda : \mathcal{V} \longrightarrow \mathcal{V}_P$

(D_3) The family of n-functions $\{f^{p_P}\}$ having $(P, \mathcal{V}_P)$ as space of parameters generated by f is a family of transformations such that the map

$$p_P \longrightarrow f^{p_P}$$

is a homomorphism of $[(P, \mathcal{V}_P)]$ onto the set of transformations $\{f^{p_P}\}$ with internal law of composition given by the composition of two transformations, namely

$$p_P \cdot q_P \longrightarrow f^{q_P} \circ f^{p_P} = f^{p_P \cdot q_P}$$

Notation - A n-dynamical system as just defined will be denoted by

$$\{(X, \mathcal{V}), f\}.$$

<u>Remarks</u> (a) If the space of parameter is a n-group each n-function f^{p_P} must have an inverse, precisely $f^{p_P^{-1}}$, because

$$p_P \cdot p_P^{-1} = \text{identity} \longrightarrow f^{p_P^{-1}} \circ f^{p_P} = \text{identity}.$$

This means that for every $p_P \in [(P, \mathcal{V}_P)]$

$$f^{p_P} : \mathcal{V} \longrightarrow \mathcal{V}$$

$$f^{p_P} : \sigma \longrightarrow \sigma, \text{ all } \sigma \in \mathcal{V},$$

are all one-to-one and onto. So in this case the fact that all f^p are transformations is a consequence of the space of parameter being a n-group and need not be assumed in (D_3).

As a consequence we call the family $\{f^{p_P}\}$ <u>the continuous n-semi-group of transformations of $(X, \mathcal{V})$ generated by</u> f which we denote by

$$\mathcal{E}[(X, \mathcal{V}), f].$$

(b) We say that the n-dynamical system

$$f : (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

is continuous if f is continuous. In the same way we say that it is regular, fully regular, etc., if the same is true for f and these concepts are defined in §II, 1.

We do not claim that $p_P \longrightarrow f^{p_P}$ is one-to-one, so that to different p_P might correspond the same f^{p_P}. The concept of essential n-dynamical system given below will clarify this situation.

(c) We write n-D.S. for n-dynamical system and very often we "drop the n" and just write D.S. and dynamical system.

Later on we introduce the generalization of many concepts classically attached to dynamical system, like, point of equilibrium, limit sets,etc., but right now we only define some fundamental notions connected with n-D.S.

If we give a germ p in $(X,\mathcal{V})$ we can define also as explained before a n-function

$$f^p: (P,\mathcal{V}_P) \longrightarrow (X,\mathcal{V})$$

which will be called, <u>motions</u> of the given dynamical system <u>passing through p</u>. We say that a <u>germ q of $(X,\mathcal{V})$ belongs to the image of f^p</u>, using the notation $q \in f^p$, if there is a germ q_P in $(P,\mathcal{V}_P)$ such that

$$f^p(q_P) = q.$$

If the motion f^p is continuous, then when e_P, the identity in $(P,\mathcal{V}_P)$, is compatible with refinements the same holds for p.

<u>Proposition 2</u> - Given a n-dynamical system

$$f: (X\times P,\mathcal{W}) \longrightarrow (X,\mathcal{V})$$

with f cofinal in Cov X, then for every germ p in $(X,\mathcal{V})$ or p_P in $(P,\mathcal{V}_P)$, both f^p and f^{p_P} are also cofinal in Cov X.

Proof - Let Γ be an arbitrary open covering of X. Then there is $\sigma_{\mathcal{W}} \in \mathcal{W}$ with

$$f(\sigma_{\mathcal{W}}) > \Gamma$$

By definition of f^p we have that, with $\sigma_{\mathcal{W}} = \sigma\times\sigma_P$, $\sigma_P = \Lambda(\sigma)$,

$$f^p_{\mathcal{V}_P}(\sigma_P) = f_{\mathcal{W}}(\sigma_{\mathcal{W}}) > \Gamma$$

and the same for f^{p_P},

$$f^{p_P}_{\mathcal{V}}(\sigma) = f_{\mathcal{W}}(\sigma_{\mathcal{W}}) > \Gamma$$

what proves the proposition.

We finish this paragraph by introducing a few other concepts which, even though not necessarily needed in this paper, might be used in further works and this justifies their introduction here, whose main goal is to lay down foundations for further developments.

Proposition 3 - Let

$$f\colon (X\times X,\mathcal{W}) \longrightarrow (X,\mathcal{V})$$

be a continuous n-semi-group. Then f is also a n-D.S. with $(X,\mathcal{V})$ as space of parameters.

<u>Proof</u> - From definition V we see that (D_1) is given and (D_2) is obvious by taking $A: \sigma \longrightarrow \sigma$ as the identity. As for (D_3) we proceed as follows: for any $\sigma \in \mathcal{V}$ and any $A \in \sigma$ we have

$$f_\sigma^{p\cdot q}(A) = f_{\sigma\times\sigma}[A \times f_{\sigma\times\sigma}(B\times C)]$$

where

$$B, C \in \sigma \; , \; B \in p, \; C \in q. \quad \text{Also}$$

$$f_\sigma^q[f_\sigma^p(A)] = f_{\sigma\times\sigma}[f^p(A) \times C] = f_{\sigma\times\sigma}[f_{\sigma\times\sigma}(A\times B) \times C]$$

$$= f_{\sigma\times\sigma}[A \times f_{\sigma\times\sigma}(B\times C)] ,$$

what shows that

$$f^{p\cdot q} = f^q \circ f^p$$

proving (D_3) and also the proposition.

<u>Definition VI</u> - A n-dynamical system

$$f: (X\times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

is essential if:

(E_1) For any open set $U \subset P$ the set $U(P)$ of the germs of P in U is not empty;

(E_2) there is an open covering Γ of P, such that for any $U \in \Gamma$ the function

$$p_P \longrightarrow f^{p_P}$$

for $p_P \in U(P)$ is one-to-one.

This definition is just the generalization of the idea, very popular for Lie groups, that each point of the space of parameters has a neighborhood homeomorphic with an open set of the group. We have not yet studied the question : if any n-D.S. has this kind of parametrization or not, even though this seems to be an interesting question.

<u>Definition VII</u> - A morphism of n-dynamical systems is a pair (h, k) of germ preserving n-functions, such that the diagram

$$\begin{array}{ccc} (X \times P, \mathcal{W}) & \xrightarrow{\ f\ } & (X, \mathcal{V}) \\ \downarrow h & & \downarrow k \\ (X' \times P', \mathcal{W}') & \xrightarrow{\ f'\ } & (X', \mathcal{V}') \end{array}$$

commutes. We denote this by

$$\mathcal{H} = (h, k) : f \longrightarrow f'$$

We observe that we have a category whose objects are n-D.S. and the morphisms are as defined above. The isomorphisms of this category are called <u>equivalences of n-dynamical systems</u>.

<u>A zero automorphism of $(P, \mathcal{V}_P)$</u> is a n-function

$$\chi : (P, \mathcal{V}_P) \longrightarrow (P, \mathcal{V}_P)$$

such that

$$\chi_{\mathcal{V}_P} : \mathcal{V}_P \longrightarrow \mathcal{V}_P$$

is the identity and for any $A_{\sigma_P} \in \sigma_P$ we have

$$\chi_{\sigma_P}(A_{\sigma_P}) = A^{\circ}_{\sigma_P} \in e$$

where e is the identity of $(P, \mathcal{V}_P)$. Clearly, this definition is also good for n-group $(X, \mathcal{V})$.

5. We discuss now several examples and make several remarks corcerning the motions introduced above.

Example 1 - Consider the set of non-negative real numbers R^+ with the canonical family of coverings $\mathcal{V}$, namely $\mathcal{V}$ is made up of open coverings $\sigma_1, \sigma_2 \dots \sigma_n \dots$ defined as follows: σ_n is the set of all intervals

$$I^n_i = \{\frac{i - 1}{2^n} < x < \frac{i + i}{2^n}\} \qquad i > 1$$

plus

$$I^n_\circ = \{0 \leq x < 1\} .$$

To introduce in $(R^+, \mathcal{V})$ a structure of n-semi-group we define

$$f: (R^+ \times R^+, \mathcal{W}) \longrightarrow (R^+, \mathcal{V})$$

as follows: $\mathcal{W}$ is the family of all coverings $\sigma^n_{\mathcal{W}} = \sigma_n \times \sigma_n$ and

$$f_{\mathcal{W}}: \mathcal{W} \longrightarrow \mathcal{V}$$

is defined as

$$f_{\mathcal{W}}: (\sigma^n_{\mathcal{W}}) = \sigma^n$$

Now for any $\sigma^n_{\mathcal{W}} \in \mathcal{W}$ we define

$$f_{\sigma^n_{\mathcal{W}}} : \sigma^n_{\mathcal{W}} \longrightarrow \sigma^n$$

by

a) $f_{\sigma^n_{\mathcal{W}}} (A \times B) = A$, $A, B \in \sigma^n$

if $A \times B$ has a larger area below the diagonal Δ of $R^+ \times R^+$.

b) $f_{\sigma^n_{\mathcal{W}}} (A \times B) = B$

if $A \times B$ has a larger area above the diagonal Δ.

c) When Δ divides $A \times B$ in equal areas then clearly $A=B$ and so apply either a) or b) above.

The germ $e = \{I^n_\circ\}$, $n \geq 1$ is the identity and we leave for the reader to check all conditions of definition II. The structure of n-semi-group so defined in $(R^+, \mathcal{V})$ satisfies also the commutative law.

For later use we want to discuss an important property of $(R^+, \mathcal{V})$. Suppose we have a sequence of number $\{t_i\}$, $i \geq 1$, in R^+ with

$$\lim_{i \to \infty} t_i = + \infty$$

and consider in σ_n a sequence of intervals A_i, $i \geq 1$, with $t_i \in A_i$. Then we say that the sequence $\{A_i\}$ tends to $+ \infty$, writing

$$A_i \longrightarrow + \infty .$$

Now if we take any $B \in \sigma_n$ and if we denote with a dot the multiplication in $(R^+, \mathcal{V})$ defined above we have that

$$A_i \cdot B \longrightarrow + \infty$$

as well. This property will be used in §III.

Example 2 - Let T be the set of real numbers with the discrete topology and let $\mathcal{V}$ be the family containing the unique covering of R whose open sets are points. Define

$$f\colon (R \times R, \mathcal{W}) \longrightarrow (R, \mathcal{V})$$

as follows

$$f_{\mathcal{W}}(\sigma \times \sigma) = \sigma$$

and for each pair $A, B \in \sigma$ define

$$f_{\sigma \times \sigma}(A \times B) = A + B$$

where $+$ is the usual addition in R because A and B are single points.

In this case we have in $(R,\mathcal{V})$ really a structure of n-group which is essentially the same as the usual additive group in R . This is a trivial example, but so far we do not know other examples very different from the one just given. More precisely if we take $\mathcal{V}$ as the canonical family of coverings in R defined in similar way as for example 1, there is no continuous n-group structure in $(R,\mathcal{V})$ having as identity the germ $C = \{\overset{\circ}{A}_n\}$, $n \geq 1$, with

$$\overset{\circ}{A}_n = \{\frac{-1}{2^n} < x < \frac{-1}{2^n}\}$$

and such that the inverse of $A_n \in \sigma_n$ is its symmetric relative to the origin.

To see that consider σ_1 and $\sigma_2 > \sigma_1$ and let

$$A_1 = \{0 < x < 1\} \in \sigma_1$$

$$\overset{\circ}{A}_1 = \{-\frac{1}{2} < x < \frac{1}{2}\} \in \sigma_1$$

$$\overset{\circ}{A}_2 = \{-\frac{1}{4} < x < \frac{1}{4}\} \in \sigma_2$$

$$B_2 = \{0 < x < \frac{1}{2}\} \in \sigma_2$$

Then, does not matter how we define the multiplication in $(R,\mathcal{V})$ we have

$$A_1 \cdot A_1^\circ = A_1$$

$$B_2 \cdot B_2^{-1} = A_2^\circ .$$

But then the multiplication is not continuous because

$$A_2^\circ \not\subset A_1$$

Certainly one would object that we could define the inverse of A_n in a different manner. Maybe it is possible to do it but the situation above shows very clear what kind of difficulties we encounter to define continuous n-group on the real line. Of course, if we drop the continuity things are much easier.

Remark - As shown by the examples above we have a feeling that may be a more convenient definition of n-group could be thought. For instance one might relax the condition

$$f_{\mathcal{W}} (\sigma \times \sigma) = \sigma$$

by putting

$$f_{\mathcal{W}} (\sigma \times \sigma) = \tau \in \mathcal{V}$$

where the choice of τ depends on σ. However we do not investigate those question now and we postpone this question for future publications.

§ II

Relations between n-D.S. and usual dynamical systems

1. In his Ph.D. thesis [7], A.V. Jansen studies the connections between n-functions and usual functions in general and we intend to apply his results to the case of n-dynamical systems, as they are defined by a n-function

$$f : (X\times P, \mathcal{W}) \longrightarrow (X, \mathcal{V}).$$

Unfortunately, as Jansen's thesis has not been published, other than internally at Mc Master University, we start by recalling some of his results needed here. Some of these are also referred at [1].

Consider the continuous n-function

$$f : (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$$

and the usual continuous function

$$\phi : X \longrightarrow Y.$$

We say that f generate ϕ if for any $x \in X$ and any neighborhood V of $\phi(x)$, there is $\sigma \in \mathcal{V}$ and $A \in \sigma$, such that:

(i) $x \in A$,

(ii) $\phi(x) \in \overline{f_\sigma(A)}$,

(iii) $f_\sigma(A) \subset V$.

A n-function

$$f : (X,\mathcal{V}) \longrightarrow (Y,\mathcal{V}')$$

is

(I) Pointwise cofinal if for any $\Gamma \in$ Cov Y and any $x \in X$ there is $\sigma \in \mathcal{V}$ and $A \in \sigma$, with $x \in A$ and $f_\sigma(A) \subset D$, for some $D \in \Gamma$.

(II) Cofinal in Cov Y, as seen before in §I, if $\{f(\sigma)\}_{\sigma\in\mathcal{V}}$ is cofinal in Cov Y, in the sense that for any covering Γ of Y there is $\sigma \in \mathcal{V}$ such that $f(\sigma) > \Gamma$.

(III) regular , if $A,B, \in \sigma \in \mathcal{V}$ with $A \cap B \neq \phi \Longrightarrow f_\sigma(A) \cap f_\sigma(B) \neq \phi$.

(IV) fully regular, if $A \in \sigma$, $B \in \tau$, with $\sigma,\tau \in \mathcal{V}$, $A\cap B \neq \phi \Longrightarrow f_\sigma(A) \cap f_\tau(B) \neq \phi$.

Remark: Given a n- D.S.

$$f: (X\times P,\mathcal{W}) \longrightarrow (X,\mathcal{V})$$

its motions f^p, $p \in [(X,\mathcal{V})]$ are not necessarily continuous, unless p is compatible with refinements. However they are always fully regular if $\mathcal{W}$ is cofinal in Cov (X×P), as easily seen. This remark is very important to have in mind in the following.

Proposition 1 (A.V.Jansen) - Let a n-function

$$f : (X,\mathcal{V}) \longrightarrow (X,\mathcal{V}')$$

be continuous, pontwise cofinal, with $\mathcal{V}$ cofinal in Cov X and X,Y regular T_1. Then there is a unique continuous function $\phi: X \longrightarrow Y$ generated by f.

<u>Proof</u>: It will be too long to reproduce here and it can be seen at [7] and we content ourselves in showing how ϕ is defined, what is going to be important to have in mind later on. We start by proving that for any $x \in X$ there is exactly one point $y \in Y$ such that

$$y \in \overline{f_\sigma(A)}$$

for any $\sigma \in \mathcal{V}$ and any $A \in \sigma$ such that $x \in A$. After that we define

$$\phi(x) = y$$

and proceed to prove its continuity and unity.

As a corollary to this proposition we also have the result: if

$$f : (X,\mathcal{V}) \longrightarrow (X,\mathcal{V})$$

is the identity n-function, namely

$$f_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}$$

and

$$f_\sigma : \sigma \longrightarrow \sigma \quad \text{for all } \sigma \in \mathcal{V}$$

are identity functions, then the function $\phi: X \to X$ generated by f is also the identity function.

Furthermore, if we compose two n-functions f,g, their composition carries over the generated functions, namely, if f generates

ϕ and g generates ψ, then $f \circ g$ generates $\phi \circ \psi$. All these results are due to A.V. Jansen [7].

Proposition 2. Let X be a regular space and

$$f : (X,\mathcal{V}) \longrightarrow (Y,\mathcal{V}')$$

be continuous with $\mathcal{V}$ cofinal in Cov X and suppose that f generates a map ϕ: $X \to Y$. Then for every $x \in X$ and any $\sigma \in \mathcal{V}$ and any $A \in \sigma$ with $x \in \overline{A}$ we have

$$\phi(x) \in \overline{f_\sigma(A)}$$

Proof: See [7] p. 5, Lemma 3, after comparison with lemma 1, p. 4.

Proposition 2' - The same conclusion of Proposition 2 holds if f is fully regular.

Proof: See [7] p. 5, Lemma 3.

Definition I An n-dynamical system

$$f: (X \times P,\mathcal{W}) \to (X,\mathcal{V})$$

is pointwise cofinal if the same holds for f.

2. Let us now study the relations between n-dynamical systems and usual semi-groups of tranformations and also connections between n-semi-groups and usual semi-groups. By usual semi-group of transformations of a topological space X having a semi-group P as space of parameters we understand a function

$$\phi: X\times P \to X$$

such that:

(G_1) for the identity 1 of P and any $x \in X$ we have

$$\phi(x,1) = x$$

(G_2) for any $t,t' \in P$ and any $x \in X$ we have

$$\phi(\phi(x,t),t') = \phi(x,t\cdot t')$$

where the dot $\cdot$ is the multiplication in P.

If P is also a topological semi-group we say that the semi-group of transformations is continuous. All n-functions considered in this paragraph are supposed to be continuous, unless stated otherwise.

Theorem 1 Let X be a regular space and

$$f: (X\times P,\mathcal{W}) \to (X,\mathcal{V})$$

a pointwise cofinal n-dynamical system with $\mathcal{W}$ cofinal in Cov $(X\times P)$. Then f defines a usual semi-group of transformations of X

$$\psi: X\times[(P,\mathcal{V}_P)] \to X$$

which is continuous, having as space of parameters the semi-group of germs of $(P,\mathcal{V}_P)$ with the topology defined in §I, 3.

Proof: Let ϕ be the continuous function

$$\phi: X\times P \to X$$

as given by Proposition 1 and define by

$$\psi(x,p_P) = \phi(x,<p_P>)$$

for any $x \in X$ and any $p_P \in [(P,\mathcal{V}_P)]$, what makes sense because being cofinal in Cov(X×P) implies that $\mathcal{V}_P$ is cofinal in Cov P and so all germs in $(P,\mathcal{V}_P)$ are simple (§I, 2, Prop. 1).

(a) ψ is continuous. Indeed, take any $(x,p_P) \in X \times [(P,\mathcal{V}_P)]$ and $V(y)$ an arbitrary neighbourhood of $y = \psi(x,p_P)$. As ϕ is continuous, there is a neighbourhood $V(x,t)$, with $t = <p_P>$, such that

$$\phi[V(x,t)] \subset V(y).$$

Select $V(x)$ and $V(t)$ with

$$V(x) \times V(t) \subset V(x,t)$$

and define $V(p_P)$ as the set of all germs q_P such that $<q_P> \in V(t)$, which is a neighbourhood of p_P in the topology of $[(P,\mathcal{V}_P)]$. Then for any $q_P \in V(p_P)$ and any $z \in V(x)$ we have

$$\psi(z,q_P) = \phi(z,<q_P>) \in V(y)$$

and so Ψ is continuous.

(b) If e_P is the identity in $(P,\mathcal{V}_P)$, $\psi(x,e_P) = x$. Indeed, let

$$e_P = \{A^o_{\sigma_P}\}_{\sigma_P} \in \mathcal{V}_P$$

and look to

$$f^{e_P}: (X,\mathcal{V}) \to (X,\mathcal{V}) .$$

We have for any $\sigma \in \mathcal{V}$ and $A \in \sigma$,

$$f_\sigma^{e_P}(A) = f_{\sigma_{\mathcal{W}}}(A\times A^o_{\sigma_P}) = A$$

because f is an n-D.S. (Def V, D_3). Hence f^{e_P} is the identity n-function and generates the identity map. Let us show that f^{e_P} also generates $\phi(x,t)$, $t = \langle e_P\rangle$, regarded as a map, $X \to X$. Take $V(y)$ an arbitrary neighbourhood of $y = \phi(x,t)$ for an arbitrary $x \in X$. As f generates ϕ, there is $\sigma_{\mathcal{W}} \in \mathcal{W}$ and $(x,t) \in A_{\mathcal{W}} = A\times A_{\sigma_P} \in \sigma_{\mathcal{W}}$ auch that

$$\text{(i) } f_{\sigma_{\mathcal{W}}}(A\times A_P) \subset V(y)$$

$$\text{(ii) } y \in \overline{f_{\sigma_{\mathcal{W}}}(A\times A_P)}.$$

As $\mathcal{W}_P$ is cofinal in Cov P, there is $\tau_P > \sigma_P$ and $A^o_{\tau_P} \in \tau_P$ with $A_{\tau_P} \subset A_P$ and so by Proposition 2, with $\tau = \Lambda^{-1}(\tau_P)$,

$$f_\tau^{e_P}(A) = f_{\sigma_{\mathcal{W}}}(A\times A^o_{\tau_P}) \subseteq f_{\sigma_{\mathcal{W}}}(A\times A_P) \subset V(y)$$

and

$$y \in \overline{f_{\sigma_{\mathcal{W}}}(A\times A^o_{\tau_P})} = \overline{f_\tau^{e_P}(A)}$$

what proves that f^{e_P} generates $\phi(x,t)$. Now, by the unity of the generated function we have that $\phi(x,t)$ is the identity map in X and so for any $x \in X$,

$$\phi(x,t) = x$$

what implies that

$$\psi(x,e_P) = \phi(x,<e_P>) = \phi(x,t) = x.$$

(c) For any $x \in X$ and p_P, $q_P \in [(P,\mathcal{V}_P)]$ we have

(1) $\psi[\psi(x,p_P),q_P] = \psi[x,p_P \cdot q_P]$.

Indeed, by definition we have

$$(2)\begin{cases} \psi[\psi(x,p_P),q_P] = \phi[\phi(x,<p_P>),<q_P>] \\ \psi(x,p_P \cdot q_P) = \phi(x,<p_P \cdot q_P>) \end{cases}.$$

Now f^{p} generates $\phi(x,<p_P>)$ and f^{q_P} generates $\phi(x,<q_P>)$, as can be proved in the same way we did for e_P above, what implies that $f^{q_P} \circ f^{p_P}$ generates $\phi[\phi(x,<p_P>),<q_P>]$, by A.V. Jansen's results as pointed out before (.) But as f is a n-D.S.,

$$f^{q_P} \circ f^{p_P} = f^{p_P \cdot q_P}$$

and as $f^{p_P \cdot q_P}$ generates $\phi(x,<p_P \cdot q_P>)$ we have that

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(.) Recall that according to remark of §II, 1 f^{p_P} and f^{q_P} are fully regular.

$$\phi(x, \langle p_P \cdot q_P \rangle) = \phi[\phi(x, \langle p_P \rangle), \langle q_P \rangle],$$

which by (2) gives (1).

Therefore, in conclusion ψ satisfies all the requirements to be a usual group of transformations and the theorem is proved.

Theorem 2: Let X be a regular space and

$$f: (X \times X, \mathcal{W}) \to (X, \mathcal{V})$$

a n-group with $\mathcal{W}$ cofinal in Cov(X×X) and f pointwise cofinal. Then the map

$$\phi : X \times X \to X$$

generated by f induces in X a structure of topological group and the function

$$\phi: [(X, \mathcal{V})] \to X$$

given by

$$\phi(p) = \langle p \rangle$$

for all $p \in [(X, \mathcal{V})]$ is a homomorphism of topological groups.

Proof: As $\mathcal{W}$ is cofinal in Cov(X×X), $\mathcal{V}$ is also cofinal in Cov X and so all germs in $(X, \mathcal{V})$ are simple. We have to show that ϕ, regarded as an algebraic operation, has identity, inverse and is associative. To do this we take e the identity germ in $(X, \mathcal{V})$ and as seen in the proof of theorem 1 we have for any $x \in X$,

$$\phi(x,\langle e\rangle) = x = \phi(\langle e\rangle,x)$$

so that $\langle e\rangle$ is the identity of the operation ϕ. Consider now any germ p in $(X,\mathcal{V})$ with $\langle p\rangle = x$ and let $y = \langle p^{-1}\rangle$. If 1 is the identity of ϕ as just defined, i.e., $1 = \langle e\rangle$, we have, for any $\sigma \in \mathcal{V}$ and $A^{-1} \in \sigma$ with $A \in p$, $A^{-1} \in p^{-1}$,

$$f_{\sigma \times \sigma}(A\times A^{-1}) = A^{0} \in \sigma,\ A^{0} \in e\ ,$$

what is equivalent to say, because $\mathcal{V}$ is cofinal in Cov X, that any neighbourhood of 1 in X intersects all sets $f_{\sigma\times\sigma}(B\times C)$ for all $\sigma \in \mathcal{V}$ and B, C $\in \sigma$, $x \in B$, $y \in C$. By the way ϕ is defined in Proposition 1, we conclude that

$$\phi(x,y) = 1\ ,$$

what shows that every $x \in X$ has an inverse $y \in X$ by ϕ. Finally to show that ϕ is associative we define the n-function

$$F\colon (X\times X\times X,\tilde{\mathcal{W}}) \to (X,\mathcal{V})$$

by considering

(i) $\tilde{\mathcal{W}} = \{\sigma\times\sigma\times\sigma,\ \sigma \in \mathcal{V}\}$,

(ii) $F_{\tilde{\mathcal{W}}}(\sigma\times\sigma\times\sigma) = \sigma$, all $\sigma \in \mathcal{V}$,

(iii) $F_{\sigma\times\sigma\times\sigma}(A\times B\times C) = f_{\sigma\times\sigma}(A\times f_{\sigma\times\sigma}(B\times C)) =$

$= f_{\sigma\times\sigma}[f_{\sigma\times\sigma}(A\times B)\times C]$, for all $\sigma \in \mathcal{V}$ and $A,B,C \in \sigma$.

Now if we take $x,y,z \in X$ arbitrary and $A,B,C \in \sigma$ with $x \in A$, $y \in B$, $z \in C$, by Proposition 2 we conclude that

$$\phi[x,\phi(y,z)] \in \overline{f_{\sigma\times\sigma}\ A\times f_{\sigma\times\sigma}(B\times C)]}$$

and

$$\phi[\phi(x,y),z] \in \overline{f_{\sigma\times\sigma}[f_{\sigma\times\sigma}\ (A\times B)\times C]},$$

what by the definition of f implies that it generates both $\phi[x,\phi(y,z)]$ and $\phi[\phi(x,y),z]$ and therefore by the unity of the generated function

$$\phi[x,\phi(y,z)] = \phi[\phi(x,y),z]$$

and hence ϕ is associative and define a group structure in X. We say that the group (X,ϕ) is generated by the n-group $\{(X,\mathcal{V}),f\}$.

Now by Proposition 3, §I, we can regard f as a n-D.S. and look to

$$\psi: X\times[(X,\mathcal{V})] \to X$$

as defined in Theorem 1. To prove that Φ is a homomorphism we know already as seen above that $\langle e\rangle = 1$, i.e.,

$$\Phi(e) = 1 .$$

Now considering any $p,q \in [(X,\mathcal{V})]$ we have

$$\psi[\psi(x,p),\ q] = \phi[\phi(x,\langle p\rangle),\langle q\rangle] =$$

$$= \phi(x,\langle p\rangle\cdot\langle q\rangle) = \phi(x,\Phi(p)\cdot\Phi(q))$$

and

$$\psi(x,p\cdot q) = \phi(x,\langle p\cdot q\rangle) = \phi(x,\Phi(p\cdot q)) .$$

As

$$\psi[\psi(x,p),q] = \psi(x,p\cdot q)$$

we have that

$$\phi(x,\Phi(p)\cdot\Phi(q)) = \phi(x,\Phi(p\cdot q)) .$$

and as ϕ is a group operation

$$\phi(p)\cdot\Phi(q) = \Phi(p\cdot q)$$

what shows that ϕ is a homomorphism.

Finally we have to show that the product in $[(X,\mathcal{V})]$ is continuous. Let $V(\langle p\cdot p\cdot\rangle)$ be an arbitrary neighbourhood of $\langle p\cdot q\rangle$ in X. As $\mathcal{V}$ is cofinal in Cov X, there is $\sigma \in \mathcal{V}$ such that any $D \in \sigma$ with $\langle p\cdot q\rangle \in D$ implies that $D \subset V(\langle p\cdot q\rangle)$. By definition of the product $p\cdot q$ if $A \in p$, $B \in q$, then

$$\langle p\cdot q\rangle \in f_{\sigma\times\sigma}(A\times B) \in \sigma$$

and so

$$f_{\sigma\times\sigma}(A\times B) \subset V(\langle p\cdot q\rangle).$$

Let

$$V(p) = \{p' : \langle p'\rangle \in A\}$$

$$V(q) = \{q' : \langle q'\rangle \in B\}$$

be two neighbourhoods of p and q in $[(X,\mathcal{V})]$. Again, as $\mathcal{V}$ is cofinal in Cov X, if $p' = \{A_\sigma^{p'}\}_{\sigma \in \mathcal{V}'}$, $q' = \{B_\sigma^{q'}\}_{\sigma \in \mathcal{V}'}$ there is $\sigma \in \mathcal{V}$ such that

$$A_\sigma^{p'} \subset A, \quad B_\sigma^{q'} \subset B,$$

what implies that

$$f_{\sigma\times\sigma}(A_\sigma^{p'} \times B_\sigma^{q'}) \subset f_{\sigma\times\sigma}(A\times B) \subset V(\langle p\cdot q\rangle).$$

Therefore

$$\langle p'\cdot q'\rangle \in V(\langle p\cdot q\rangle).$$

what proves that

$$V(p)\cdot V(q) \subset V(p\cdot q)$$

and so the product in $[(X,\mathcal{V})]$ is continuous and $[(X,\mathcal{V})]$ is a topological group. The continuity of Φ is immediately from the definition of the topology in $[(X,\mathcal{V})]$. Therefore the theorem is proved.

<u>Theorem 3</u>. Let X be a regular and

$$f : (X \times P,\mathcal{W}) \to (X,\mathcal{V})$$

be a pointwise cofinal n-dynamical system with $\mathcal{W}$ cofinal in Cov(X×P). Then the map ϕ

$$\phi : X \times P \longrightarrow X$$

generated by f is a semi-group of transformation of X in the usual sense.

<u>Proof</u> - By theorem 2, P has a semi-group structure generated by the structure of n-semi-group of $(P, \mathcal{V}_P)$ and by theorem 1 we have a usual semi-group of transformations

$$\psi : X \times [(P, \mathcal{V}_P)] \longrightarrow X.$$

Then if 1 is the identity of P we have, with $1 = \langle e_P \rangle$,

$$\phi\ (x,1) = \phi(x,e_P) = x$$

and for any $t, t' \in P$ and $p_P, p'_P \in [(P, \mathcal{V}_P)]$, with $t=\langle p_P \rangle$, $t'=\langle p'_P \rangle$, we have

$$\phi[\phi(x,t),\ t'] = \psi[\ \psi(x,p_P),\ p'_P\ ] =$$

$$= \psi\ (x,p_P \cdot p'_P).$$

As Φ defined in theorem 2 is a homomorphism we have

$$\phi(x,t \cdot t') = \phi(x,\langle p_P \rangle \cdot \langle p'_P\ \rangle) = \phi(x,\Phi(p_P) \circ \Phi(p'_P)) =$$

$$= \psi(x,\Phi(p_P \cdot p'_P)) = \phi(x,\langle p_P \cdot p'_P \rangle) =$$

$$= \psi(x,p_P \cdot p'_P) = \phi[\phi(x,t),t']$$

and this proves the theorem.

Remark - We have seen that in all three theorems proved before it is important for f to be pointwise cofinal and $\mathcal{W}$ cofinal and X regular. This justifies the definition.

Definition II - A n-dynamical system

$$f : (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

is regular if f is pointwise cofinal, $\mathcal{W}$ is cofinal in Cov(X×P) and X is a regular Hausdorff space.

Remark - In A.V. Jansen's thesis other theorems showing that n-functions generate usual functions are proved, which instead of pointwise cofinal use regular. So we believe that theorems similar to those proved in this paragraph might also be true under the hypothesis of regular instead of pointwise cofinal for the n-functions involved. However we do not treat these questions here.

§ III

Limit sets and related concepts

1. In the classical theory of dynamical systems after the basic definition one start introducing several concepts which approach the theory more and more of its important applications to physics and other branches of science. These are the concepts of point of equilibrium, limit sets, centre of attraction, etc., which are all very appealing to physical sciences. Our intention in this paragraph is to show how most of these concepts have a natural extension to n-dynamical system.

In this paragraph all n-semi-groups $(P, \mathcal{V}_P)$ are supposed to be with $P = R$, the set of real numbers with its usual topological and algebraic structures. Also the coverings σ_P of $\mathcal{V}_P$ will be made of open intervals.

Definition I - A germ $p \in [(X, \mathcal{V})]$ is called an equilibrium or a rest position of a n-D.S.

$$f: (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

if for any $\sigma_P \in \mathcal{V}_P$ and any $A_P \in \sigma_P$ we have with $\sigma_{\mathcal{W}} = \sigma \times \sigma_P$,

$$f_{\sigma_{\mathcal{W}}} (A \times A_P) = A \in \sigma \quad , \quad A \in p.$$

Definition II - A motion f^p, $p \in [(X, \mathcal{V})]$ of a n-D.S.

$$f: (X \times R, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

is called a periodic motion if:

(P_1) there is a number $K > 0$ such that for any $\sigma_P \in \mathcal{V}_P$ and for any $t \in R$ and $A_P, B_P \in \sigma_P$, with $t \in A_P$, $t \pm mK \in B_P$ $m \geq 0$, we have

$$f^P_{\sigma_P}(A_P) = f^P_{\sigma_P}(B_P);$$

(P_2) the infimum of all K satisfying (P_1) is a positive number T, called the <u>period of the motion f^P</u>. <u>The frequency of f^P</u> is the number $\nu = 1/_T$.

<u>Theorem 1</u> - Let $p \in [(X, \mathcal{V})]$ be an equilibrium position. Let q be any germ in $(X, \mathcal{V})$ and suppose that for some $A_P \in \sigma_P$ we have

$$f^q(A_P) = A \in p.$$

Then for any other $B_P \in \sigma_P$ we also have

$$f^q(A_P \cdot B_P) = A$$

<u>Proof</u> - We have, for any $A_P \in \sigma_P$:

$$f^P_{\sigma_P}(A_P) = f_{\sigma_W}(A \times A_P) = A.$$

Also for any $B_P \in \sigma_P$ and $B \in \sigma$, $B \in p$ we have

$$f^{q}_{\sigma_P}(A_P \cdot B_P) = f_{\sigma_{\mathcal{W}}}(B \times A_P \cdot B_P) = f_{\sigma_{\mathcal{W}}}[f_{\sigma_{\mathcal{W}}}(B \times A_P) \times B_P] =$$

$$= f_{\sigma_{\mathcal{W}}}[f^{q}_{\sigma_P}(A_P) \times B_P] =$$

$$= f_{\sigma_{\mathcal{W}}}(A \times B_P) = f^{p}_{\sigma_P}(B_P) = A.$$

The intuitive meaning of this theorem is that whenever we attain a point of equilibrium we stay there. If $(P, \mathcal{V}_P)$ is a n-group then we can also prove that a position of equilibrium cannot either be attained. Indeed suppose that for some $A_P \in \sigma_P$ we have

$$f^{q}_{\sigma_P}(A_P) = A \in p , \qquad A \in \sigma .$$

Then we have

$$f^{p}_{\sigma_P}(A_P^{-1}) = f_{\sigma_{\mathcal{W}}}(A \times A_P^{-1}) = A \in p .$$

By another side as for any $B \in q$, $B \in \sigma$,

$$f^{q}_{\sigma_P}(A_P) = f_{\sigma_{\mathcal{W}}}(B \times A_P)$$

we have

$$f^{p}_{\sigma_P}(A_P^{-1}) = f_{\sigma_{\mathcal{W}}}(A \times A_P^{-1}) = f_{\sigma_{\mathcal{W}}}[f^{q}_{\sigma_P}(A_P) \times A_P^{-1}] =$$

$$= f_{\sigma_{\mathcal{W}}}[f^{q}_{\sigma_{\mathcal{W}}}(B \times A_P^{-1} = f_{\sigma_{\mathcal{W}}} B \times (A_P \cdot A_P^{-1})] =$$

$$= f_{\sigma_{\mathcal{W}}}(B \times A^{\circ}_P) = B \in q,$$

which is a contradiction, because $p \cap q = \phi$.

Definition III - Let f^P be a motion of a n-D.S.

$$f:(X\times P,\mathcal{W})\longrightarrow(X,\mathcal{V}).$$

The trajectory or orbit of f^P is the subset of X given by

$$T(f^P) = \bigcap_{\sigma_P \in \mathcal{V}_P} \overline{\bigcup_{A \in \sigma_P} f^P_{\sigma_P}(A)}$$

In previous work a similar motion was associated with the idea of particle as can be seen at [1] and it helps in giving more geometric insight into many questions connected with n-D.S., as we shall see in the following pages.

For our next theorem we need a result which is interesting by itself given in the proposition below.

Proposition 1 - Let

$$f:(X,\mathcal{V})\longrightarrow(X,\mathcal{V}')$$

be a cofinal in Cov X, n-function with Y paracompact and $\mathcal{V}$ made up of finite open coverings. Then T(f), as given by definition III, i.e.,

$$T(f) = \bigcap_{\sigma \in \mathcal{V}} \overline{\bigcup_{A \in \sigma} f_\sigma(A)},$$

is compact subset of Y.

Proof - If T(f) is empty nothing is to be proved. Otherwise let Γ_c

be an open covering of $T(f)$. As Y is paracompact it is normal and as $T(f)$ is closed, there is an open set V, such that:

$$\text{(i)} \quad V \supset T(f)$$

$$\text{(ii)} \quad \overline{V} \subset \bigcup_{E \in \Gamma_c} E$$

Define Γ as open covering of Y given by Γ_c together with the open set $Y-\overline{V}$. As f is cofinal in Cov Y, there is $\sigma \in \mathcal{V}$ with

$$f_{\mathcal{V}}(\sigma) > \Gamma$$

and as every $f_\sigma(A)$, $A \in \sigma$, intersects $\overline{V}$ they must all be contained in sets of Γ_c and so there are finitely many open sets of Γ_c, $E_1, E_2 \ldots E_n$ such that

$$\bigcup_{A \in \sigma} f\,(A) \subset \bigcup_{i=1}^{n} E_i \quad ,$$

because σ is finite. As Y is paracompact every open covering Γ has an open refinement Γ' such that for every set $E' \in \Gamma'$ there is $E \in \Gamma$ with $\overline{E'} \in E$. Therefore we can assume also that

$$\bigcup_{A \in \sigma} \overline{f_\sigma(A)} \subset \bigcup_{i=1}^{n} E_i \; .$$

this implies that

$$\bigcup_{i=1}^{n} E_i \supset \overline{\bigcup_{A \in \sigma} f_\sigma(A)} \supset T(f),$$

what proves that T(f) is compact.

Corollary 1 - Let

$$f: (X,\mathcal{V}) \longrightarrow (Y,\mathcal{V}')$$

be a n-function, cofinal in Cov Y, with $\mathcal{V}$ cofinal in Cov X, X compact and Y paracompact. Then T(f) is a compact subset of Y.

Proof - This corollary reduces to Proposition 1 if we show that T(f) can be defined by using only finite subcoverings of every covering $\sigma \in \mathcal{V}$. Indeed, to every $\sigma \in \mathcal{V}$ let us associate a covering $\tilde{\sigma}$ by selecting finitely many sets of σ, what is possible because X is compact. We have

$$\bigcup_{A\in\sigma} f_\sigma(A) \subset \bigcup_{A\in\tilde{\sigma}} f_\sigma(A).$$

Indeed, as $\mathcal{V}$ is cofinal in Cov X there is $\tau \in \mathcal{V}$ with $\tau > \tilde{\sigma}$ so that

$$T(f) \subset \overline{\bigcup_{B\in\tau} f_\tau(B)} \subset \overline{\bigcup_{A\in\tilde{\sigma}} f_\sigma(A)}$$

what implies that

$$T(f) = \bigcap_{\sigma\in\mathcal{V}} \overline{\bigcup_{A\in\tilde{\sigma}} f_\sigma(A)}$$

Given a pair $(X,\mathcal{V})$ we say that $\mathcal{V}$ is <u>cofinal relatively to compact subsets of X</u>, if given any compact subset K of X the family of all coverings of K given by

$$\sigma_K = \{A \in \sigma;\ A \cap K \neq \phi\}$$

for each $\sigma \in \mathcal{V}$ is cofinal in Cov K.

<u>Theorem 2</u> - Let f^p, $p \in [(X,\mathcal{V})]$ be a periodic motion of the n-D.S.

$$f: (X \times R, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

where f^p is cofinal in Cov X, X is paracompact and in the pair $(R, \mathcal{V}_p)$, $\mathcal{V}_p$ is cofinal relatively to compact subsets of R. Then $T(f^p)$ is a compact subset of X.

<u>Proof</u> - As f^p is periodic we can consider f^p as a n-function

$$f^p: (I, \mathcal{V}_I) \longrightarrow (X, \mathcal{V})$$

when I is a closed and bounded interval of R and $\mathcal{V}_I$ is given by all coverings

$$\sigma_I : \{A_p \in \sigma_p;\ A_p \cap I \neq \phi\}$$

But now, under the hypothesis of the theorem, f^p satisfies all conditions of Proposition 1 and its corollary and so $T(f^p)$ is compact, what proves the theorem.

<u>Remark</u> - A similar theorem can be proved by assuming that for any $\sigma_p \in \mathcal{V}_p$, σ_I is finite. We use now Proposition 1 only, to prove the theorem. For reference we state this result explicity:

Theorem 2' - Let f^p, $p \in [(X,\mathcal{V})]$ be a periodic motion of the n-D.S.

$$f: (X\times R, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

where f^p is cofinal in Cov X, X is paracompact and in the pair $(R, \mathcal{V}_P)$, for every $\sigma_P \in \mathcal{V}_P$ we have for each bounded and closed interval I of R that

$$\sigma_I = (A_P \in \sigma_P;\ A_P \cap I \neq \phi)$$

is finite. Then $T(f^p)$ is a compact subset of X.

2. In this section we investigate the main properties of invariant sets and limit sets of a n-D.S.

Definition IV - A set M of germs of $(X,\mathcal{V})$ is an invariant set of the n-D.S.

$$f: (X\times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

is for any germ $p \in [(X,\mathcal{V})]$, with $p \in M$, we have

$$f^p(p_P) \in M$$

for any $p_P \in [(P, \mathcal{V}_P)]$. The union of all $\langle p\rangle$, $p \in M$ is called the nucleus of M and is indicated by $\langle M\rangle$.

Definition V - A germ $p \in [(X,\mathcal{V})]$ is called a L^+-limit germ of a motion f^q, $q \in [(X,\mathcal{V})]$ of a n-D.S.

$$f: (X\times R, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

if there is a sequence $t_n \in R$, $n \geq 1$, $\lim_{n\to\infty} t_n = +\infty$ and a sequence of germs $\{p_P^n\}_{n\geq 1}$ in $(R, \mathcal{V}_P)$ with

(L_I) $p_P^n \cap [t_n, +\infty] \neq \phi$, $n = 1, 2, \ldots,$

(L_{II}) for any neighborhood $V_E(p)$ in the canonical topology of $[(X, \mathcal{V})]$

$$\sigma \in \mathcal{V} \text{ and } A \in \sigma,\ A \in p$$

there is an integer $n_o(\sigma) > 0$ such that

$$n > n_o(\sigma) \Longrightarrow f^q(p_P^n) \cap A_\sigma \neq \phi.$$

<u>Notation</u> - If $p_n = f^q(p_P^n)$ we write

$$p = (L^+) - \lim_{n\to\infty} p_n \ .$$

<u>Definition IV</u> - The (L^+) - limit set of a motion f^q is the set of all $p \in [(X, \mathcal{V})]$ which are (L^+) - limit germs of f^q. We use the nota tion $L^+(f^q)$.

If we start with a sequence $t_n \longrightarrow -\infty$ we can define in the same way (L-) - limit germ and $L^-(f^q)$.

From now on we prove all results only for (L^+)-limit sets and germs.

The <u>nucleus of $L^+(f^q)$</u>, denoted by $\langle L^+(f^q)\rangle$ is the subset of

X defined by

$$< L^+ (f^q) > = \bigcup_{p \in L^+ (f^q)} \overset{<p>}{} .$$

For our next theorem we need some special properties of the n-semi-group $(R, \mathcal{V}_P)$ which up to now was left arbitrary, i.e., for questions dealing with limit sets and related concepts the n-semi-group structure of the space of parameters has influence on the results.

We say that the n-semi-group structure of $(R, \mathcal{V}_P)$ is compatible with the ordering of R if the following happens: let $\{t_n\}_{n \geq 1}$, $\lim t_n = +\infty$ be a sequence of numbers in R and let $\{p_P^n\}_{n \geq 1}$ be a sequence of germs in $(R, \mathcal{V}_P)$ with

$$p_P^n \cap [t_n, +\infty) \neq \phi .$$

Let now p_P be any other germ in $(R, \mathcal{V}_P)$. Then there is a sequence $\{\tau_n\}_{n \geq 1}$, $\lim\limits_{n \to \infty} \tau_n = +\infty$, of numbers in R such that

$$p_P^n \circ p_P \cap [\tau_n , + \infty) \neq \phi .$$

This is a special case of ordered n-semi-group, whose study we do not consider now which intuitively corresponds to the usual case of addition for the reals, i.e., if we add any number to sequence of real numbers with limit $+\infty$ we get a similar type of sequence.

Theorem 3 - Let

$$f: (X \times R, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

be a n-D.S. with f a regular n-function, where $(R, \mathcal{V}_P)$ has a n-semi group structure compatible with the ordering of R. Then for any motion f^q, $q \in [(X, \mathcal{V})]$, the set $L^+(f^q)$ is an invariant set of f.

Proof - We must show that for any germ $p \in L^+(f^q)$ if we look to the motion f^p, then for every germ $p_P \in [(R, \mathcal{V}_P)]$ we have that the germ

$$r = f^p(p_P) \in L^+(f^q).$$

By definition of $L^+(f^q)$ there is a sequence $\{t_n\}_{n \geq 1}$, with $\lim_{n \to \infty} t_n = +\infty$ and a sequence of germs $\{p_P^n\}$ of $(R, \mathcal{V}_P)$ such that for any $\sigma \in \mathcal{V}$ and $A_\sigma \in p$ there is $n_o(\sigma)$ such that

$$n > n_o(\sigma) \Longrightarrow f^q(p_P^n) \cap A_\sigma \neq \phi,$$

when, for all $n \geq 1$,

$$p_P^n \cap [t_n, +\infty) \neq \phi$$

Consider now $\sigma_P \in \mathcal{V}_P$ and $A_\sigma \in \sigma = \Lambda^{-1}(\sigma_P)$, where the map $\Lambda : \sigma \longrightarrow \sigma_P$ is part of the definition of f, and look to $A_{\sigma_P} \in p_P^n$, $n > n_o(\sigma)$. Then

$$f^q_{\sigma_P}(A^n_{\sigma_P}) \cap A_\sigma \neq \phi \qquad (A_\sigma \in p).$$

As f is regular the same holds for f^{p_P}, hence

(1) $$f_\sigma^{p_P}[f^q_{\sigma_P}(A^n_{\sigma_P})] \cap f_\sigma^{p_P}(A_\sigma) \neq \phi .$$

By definition of f^{p_P}, for all $A_{\sigma_P} \in p_P$ we have

$$f_{\sigma}^{p_P}(A_{\sigma}) = f_{\sigma\times\sigma_P}(A_{\sigma} \times A_{\sigma_P}) = f^{p}_{\sigma_P}(A_{\sigma_P}).$$

Now for all $A_{\sigma}^{q} \in q$, $A_{\sigma}^{q} \in \sigma = \Lambda^{-1}(\sigma_P)$, we have,

$$f^{q}_{\sigma_P}(A^{n}_{\sigma_P}) = f_{\sigma\times\sigma_P}(A^{q}_{\sigma} \times A^{n}_{\sigma_P}).$$

Hence,

$$f_{\sigma}^{p_P}[f^{q}_{\sigma_P}(A^{n}_{\sigma_P})] = f_{\sigma}^{p_P}[f_{\sigma\times\sigma_P}(A^{q}_{\sigma} \times A^{n}_{\sigma_P})] =$$

$$= f_{\sigma\times\sigma_P}[f_{\sigma\times\sigma_P}(A^{q}_{\sigma} \times A^{n}_{\sigma_P}) \times A_{\sigma_P}] =$$

$$= f_{\sigma\times\sigma_P}[A^{q}_{\sigma} \times (A^{n}_{\sigma_P} \times A_{\sigma_P})] =$$

$$= f^{q}_{\sigma_P}(A^{n}_{\sigma_P} \times A_{\sigma_P}) = B^{n}_{\sigma}.$$

This show that there is a sequence of germs in $(R, \mathcal{V}_P)$,

$$q^{n}_{P} = \{A_{\sigma_P} \cdot A_{\sigma_P}\}_{\sigma_P} \in \mathcal{V}_P,$$

such that, for $n > n_0(\sigma)$, by (1) above,

(2) $$f^q(q_P^n) \cap f^P_{\sigma_P}(A_{\sigma_P}) \neq \phi \,.$$

Now as $A^n_{\sigma_P} \cap [t_n, +\infty) \neq \phi$ we have also

$$A^n_{\sigma_P} \cdot A_{\sigma_P} \cap [\tau_n, +\infty) = \phi \,.$$

for some convenient τ_n, by the definition of product in $(R, \mathcal{V}_P)$.

As $\lim_{n\to\infty} \tau_n = +\infty$ and

$$q_P^n \cap [\tau_n, +\infty) \neq \phi \,.$$

we have that, due do (2),

$$\lim_{n\to\infty} f^q(q_P^n) = f^P(p_P) = r$$

and consequently $r \in L^+(f^q)$.

Therefore $L^+(f^P)$ is an invariant set of f. A similar theo_ rem is true for $L^-(f^P)$ assuming that the n-semi-group structure of $(R, \mathcal{V}_P)$ is compatible with the ordering of R relatively to sequences $\{t_n\}_{n\geq 1}$, with $\lim_{n\to\infty} t_n = -\infty$.

<u>Theorem 4</u> - If f^P, $p \in [(X, \mathcal{V})]$ is a periodic motion then for any $p_P \in [(R, \mathcal{V}_P)]$,

$$f^P(p_P) \in L^+(f^P) = L^-(f^P).$$

ourselves in dealing only with some basic concepts. In the future a detailed study of this important theory of stability of n-D.S. will be published elsewhere.

Let us introduce the notion of positive and negative trajectory of a motion f^p, denoted by $T^+(f^p)$ and $T^-(f^p)$ respectively and whose definitions are

$$T^+(f^p) = \bigcap_{\sigma_p \in \mathcal{V}_p} \overline{\bigcup_{A_p \in \sigma_p} f^p_{\sigma_p}(A_p)}$$

$$A_p \cap [0, +\infty) \neq \phi$$

$$T^-(f^p) = \bigcap_{\sigma_p \in \mathcal{V}_p} \overline{\bigcup_{A_p \in \sigma_p} f^p_{\sigma_p}(A_p)}$$

$$A_p \cap [0, -\infty) \neq \phi$$

Clearly

$$T(f^p) = T^+(f^p) \cup T^-(f^p) .$$

Definition V - A motion f^p is positively Lagrange-stable if $T^+(f^p)$ is compact and negatively Lagrange-stable if $T^-(f^p)$ is compact. It is Lagrange-stable if both $T^+(f^p)$ and $T^-(f^p)$ are compact.

Theorem 5 - Let f^p be a positively Lagrange stable motion in $(X, \mathcal{V})$ with $\mathcal{V}$ cofinal in Cov X. Then

$$T^+(f^p) \supset \langle L^+(f^p) \rangle .$$

Suppose that there is $q \in L^+(f^p)$ with $\langle q \rangle \not\subset T^+(f^p)$. As $\mathcal{V}$ is cofinal in Cov X all germs in $(X,\mathcal{V})$ are simple and so we can write

$$\langle q \rangle \not\in T^+(f^p) .$$

As $T^+(f^p)$ is compact, it is closed in X (X is (Hausdorff!) and so there is $B_\sigma \in \sigma \in \mathcal{V}$, $B_\sigma \in q$, for a convenient $\sigma \in \mathcal{V}$, such that

$$B_\sigma \cap T^+(f^p) = \phi .$$

But now, for any $\sigma_p \in \mathcal{V}_p$ and $A_p \in \sigma_p$ we have

$$f^p_{\sigma_p}(A_p) \cap B_\sigma = \phi .$$

and so q cannot be (L^+)-limit germs of f^p, what is a contradiction and proves the theorem. A similar theorem can be proved for negatively Lagrange stable motions and hence for Lagrange stable motions.

<u>Theorem 6</u> - If f^p is a periodic motion in $(X,\mathcal{V})$ with $\mathcal{V}$ cofinal in Cov X and satisfying all hypothesis of theorem 2, we have

$$T(f^p) = \langle L^+(f^p) \rangle .$$

<u>Proof</u> - By theorem 5 above we know that

$$T(f^p) \supset \langle L^+(f^p) \rangle .$$

because by theorem 2, $T(f^P)$ is compact and so every periodic motion is also Lagrange stable. To prove the opposite inclusion, i.e.,

$$T(f^P) \subset \langle L^+(f^P)\rangle$$

let us suppose that there is some $x \in T(f^P)$ and $x \notin \langle L^+(f^P)\rangle$. In this case for any germ q in $(X,\mathcal{V})$ with $x \in \langle q\rangle$, there is $\sigma \in \mathcal{V}$, $A_\sigma \in \sigma$, $A_\sigma \in q$ and $t(\sigma) \in R$ such that for any germ p_P in $(R,\mathcal{V}_P)$ with $p_P \cap [t(\sigma), +\infty) \neq \phi$ we have, for some $\tau_P \in \mathcal{V}_P$,

$$(1) \qquad f^P_{\tau_P}(A_{\tau_P}) \cap A_\sigma = \phi .$$

But as f^P is periodic (1) is actually true for any $A_{\tau_P} \in \tau_P$ so that

$$x \notin \overline{\bigcup_{A_{\tau_P} \in \tau_P} f_{\tau_P}(A_{\tau_P})}$$

what implies that $x \notin T(f^P)$, a contradiction.

<u>Remark</u> - Analysing the proof of the theorem 6 above we see that to prove the inclusion

$$(2) \qquad T(f^P) \subset \langle L^+(f^P)\rangle$$

we do not use all the hypothesis of the theorem. Indeed, (2) is true for any periodic motion, as seen by the proof above. By another side if f^P is cofinal in Cov X the opposite inclusion is also easy to establish, without further hypothesis and therefore we can state the theorem:

Theorem 7 - If f^P is a periodic motion, with f^P cofinal in Cov X,

$$f^P: (R, \mathcal{V}_P) \longrightarrow (X, \mathcal{V}),$$

then

$$T(f^P) = \langle L^+(f)\rangle = \langle L^-(f)\rangle .$$

Theorem 8 - If f^P is positively Lagrange stable, $L^+(f^P)$ is not empty. Similary if f^P is negatively Lagrange stable $L^-(f)$ is not empty.

Proof - Let $\{t_n\}_{n\geq 1}$ be a sequence of germs in $(R, \mathcal{V}_P)$ with

$$p_P^n \cap [t_n, +\infty) \neq \phi .$$

Look to the sequence of germs

$$p^n = f^P(p_P^n) , \quad n \geq 1 ,$$

in $(X, \mathcal{V})$ and select a point $x_n \in \langle p^n\rangle$ for each $n \geq 1$. As all $\langle p^n\rangle$ are contained in $T(f^P)$ and this set is compact, there is a point $x \in T(f^P)$ which is the limit of a subsequence of $\{x_n\}_{n\geq 1}$, which for simplicity we assume that $x = \lim\limits_{n\to\infty} x_n$. Let q be any germ in $(X, \mathcal{V})$ whose nucleus contains x and look to any $\sigma \in \mathcal{V}$, $A_\sigma^q \in q$, $A_\sigma^q \in \sigma$. As x is the limit of $\{x_n\}$, there is an integer $n(\sigma)$ such that for

$n > n(\sigma)$ we have $x_n \in A^q_\sigma$ and consequently

$$p^n \cap A^q \neq \phi \quad .$$

This shows that

$$q = (L^+) - \lim_{n\to\infty} f^p(p^n_p)$$

and so $L^+(f^p) \neq \phi$.

4. To finish this paragraph we want to say a few words about centre of attraction and related subjects.

Definition VI - A closed subset $M \subset X$ is centre of attraction of the motion f^p, $p \in [(X,\mathcal{V})]$ of the n-D.S.

$$f:(X \times R,\mathcal{W}) \longrightarrow (X,\mathcal{V})$$

if for any open set $E \supset M$ there is an integer $n(E) > 0$ such that for any germ $p_p \in [(R,\mathcal{V}_p)]$, with $p_p \cap [n(E), +\infty) \neq \phi$ we have

$$f^p(p_p) \cap E \neq \phi \quad .$$

M is a minimal centre of attraction if no proper closed subset of M is centre of attraction.

For our next theorem we need the concept of locally cofinal family $\mathcal{V}$ of coverings of a space X. This is defined as follows: $\mathcal{V}$ is a family of open coverings of X satisfying the property that for any $\sigma \in \mathcal{V}$ and any $A \in \sigma$, there is $\tau \in \mathcal{V}$ and $B \in \tau$ with $\tau > \sigma$ and

$B \subseteq A$. Every cofinal family $\mathcal{V}$ in Cov X is also locally cofinal but not otherwise as for instance the canonical family $\mathcal{V}_R$ in R.

Theorem 9 - Let f^P be a continuous motion in $(X, \mathcal{V})$ cofinal in Cov X, with $\mathcal{V}_P$ locally cofinal and X, a normal space. The necessary and sufficient condition for a closed subset $M \subset X$ to be a centre of attraction of f^P is that for any open set $E \supset M$, there is an integer $n(E) > 0$ and $\sigma_P \in \mathcal{V}_P$ such that for all $\tau_P \in \mathcal{V}_P$, $\tau_P > \sigma_P$ and all $A_{\tau_P} \in \tau_P$, with $A_{\tau_P} \cap [n(E), +\infty) \neq \phi$, we have

$$f^P_{\tau_P}(A_{\tau_P}) \subset E .$$

Proof - First we show that the condition is necessary. Indeed, if M is center of attraction of f^P there is an integer $n(E) > 0$ such that for all $\sigma_P \in \mathcal{V}_P$ and all $A_{\sigma_P} \in \sigma_P$, $A_{\sigma_P} \cap [n(E), +\infty) = \phi$, implies that,

$$f^P_{\sigma_P}(A_{\sigma_P}) \cap E \neq \phi .$$

For any open set E, as in the hypothesis of the theorem, let us define an open covering Γ_E of X as follows: by the normality of X, there are two open sets W_1 and W_2 such that

(i) $W_1 \cap W_2 = \phi$

(ii) $M \subset W_1 \subset E$

(iii) $X - E \subset W_2$.

Define Γ_E as made up of the open sets:

$$E - M, \; W_1 \text{ and } W_2.$$

Suppose we have selected σ_p above such that

$$f^p_{\mathcal{V}_p}(\sigma_p) > \Gamma_E$$

what is possible because f^p is cofinal in Cov X. Then any set A_{σ_p} as above will imply that if

$$(1) \qquad f^p_{\sigma_p}(A_{\sigma_p}) \not\subset E$$

$$(2) \qquad f^p_{\sigma_p}(A_{\sigma_p}) \subset W_2 .$$

Suppose that the condition is not necessary. Then there is an open set $E \supset M$ with the following property: whatever is the integer $n > 0$ and whatever is the covering $\sigma_p \in \mathcal{V}_p$, there is always a covering $\tau_p \in \mathcal{V}_p$, $\tau_p > \sigma_p$, and an open set $A_{\tau_p} \in \sigma_p$ with

$$A_{\tau_p} \cap [n, +\infty) \neq \phi \text{ and } f^p_{\tau_p}(A_{\tau_p}) \not\subset E.$$

Let $\nu_p \in \mathcal{V}_p$ with $\nu_p > \tau_p$ and

$$f^p_{\mathcal{V}_p}(\nu_p) > \Gamma_E .$$

We can also assume that for some open set $B_{\nu_p} \in \nu_p$,

$$B_{\nu_p} \cap [n, +\infty) \neq \phi \quad \text{and} \quad f^p_{\nu_p}(B_{\nu_p}) \not\subset E.$$

As seen above we must have

$$f^p_{\nu_p}(B_{\nu_p}) \subset W_2 .$$

Now for any $\gamma_p \in \mathcal{V}_p$, $\gamma_p > \nu_p$, by the continuity of f^p we have a $C_{\gamma_p} \in \gamma_p$ with

$$C_{\gamma_p} \subset B_{\nu_p} \Longrightarrow f^p_{\gamma_p}(C_{\gamma_p}) \subset W_2 .$$

Look to the set W_1 above. We have $W_1 \supset M$, but for any $n>0$ we can always build $\gamma_p \in \mathcal{V}_p$ as above having sets like C_{γ_p} for which

$$f^p_{\gamma_p}(C_{\gamma_p}) \subset W_2 .$$

and so

$$f^p_{\gamma_p}(C_{\gamma_p}) \cap W_1 = \phi .$$

But this contradicts that M is a centre of attraction, what proves that the condition is necessary.

To prove the sufficiency we proceed as follows: Let $E \supset M$ and $n(E)$ as in the hypothesis of the theorem. Take any $\mu_p \in \mathcal{V}_p$ and any $D_{\mu_p} \in \mathcal{V}_p$, $D_{\mu_p} \cap [n(E), +\infty) \neq \phi$. We are assuming that there is a $\sigma_p \in \mathcal{V}_p$ such that for all $\tau_p \in \mathcal{V}_p$, $\tau_p > \sigma_p$ and all $A_{\tau_p} \in \tau_p$ with $A_{\tau_p} \cap [n(E), +\infty)$ we have

$$f^p_{\tau_p}(A_{\tau_p}) \subset E.$$

As $\mathcal{V}_p$ is locally cofinal for any $D_{\mu_p} \in \mu_p$ as above there is $\tau_p > \sigma_p$ and $A_{\tau_p} \in \tau_p$ with $A_{\tau_p} \subset D_{\mu_p}$. Then by the continuity of f^p we have

$$f^p_{\tau_p}(A_{\tau_p}) \subset f^p_{\mu_p}(D_{\mu_p})$$

and so

$$f^p_{\mu_p}(D_{\mu_p}) \cap E \neq \phi$$

what proves that the condition is sufficient.

§ IV

n-D.S. and Differential Equations

1. In this section we discuss the connections between n-D.S. and differential equations under the point of view of non-deterministic mathematics. Essentially, we want to show that every n-D.S. system under certain conditions can be considered as a family of solutions of a certain type of differential equations, what, in a certain sense, bring us back to the origins of the theory of dynamical systems as inaugurated by the work of S. Lie and H. Poincaré.

For this purpose we need to introduce the concept of partial derivatives of a n-deterministic function and this is done here by the first time, as this concept has not yet been introduced before. Several properties of partial derivatives are studied and also related ideas and this is the purpose of the next paragraph. A knowledge of the theory of derivatives of n-functions is necessary to understand this section, what can be seen at [1].

There is only one point where our nomenclature differs from that in [1], which is connected with the concept of real n- function. The concept of derivative of n-function requires that we introduce a particular pair $[R, \mathcal{V}_R]$, where R is the set of real numbers and $\mathcal{V}_R$ a family of collections σ_R of open intervals or points in R, i.e., $\sigma_R \in \mathcal{V}_R$ then σ_R is made up of open intervals or points, in R, which do not necessarily cover R. Such a pair have been called in previous work by special pair and we can define, similarly as we did for n-function, what we shall call real n-function, previously called special g-function, as follows:

$$f: (X,\mathcal{V}) \longrightarrow [R,\mathcal{V}_R]$$

is given by:

I). A function

$$f_{\mathcal{V}}: \mathcal{V} \longrightarrow \mathcal{V}_R$$

II) for any $\sigma \in \mathcal{V}$, a function

$$f_\sigma: \sigma \longrightarrow \sigma_R = f_{\mathcal{V}}(\sigma).$$

It is continuous if in addition we have:

III) a) $\tau > \sigma \Longrightarrow f(\tau) > f(\sigma)$, $\sigma, \tau \in \mathcal{V}$

b) If $B \subset A$, $B \in \tau$, $A \in \sigma$, $\tau > \sigma$, then

i) $f_\tau(B) \subset f_\sigma(A)$ if $f_\tau(B)$ and $f_\sigma(A)$ are both open intervals or points. In the case of points, clearly, $\subset$ becomes $=$.

ii) $f_\tau(B) \subset \overline{f_\sigma(A)}$ if $f_\tau(B)$ is a point and $f_\sigma(A)$ is an open interval.

iii) The case, $f_\tau(B)$ interval and $f_\sigma(A)$ point is excluded.

The reason for such a rather sophisticated definition is purely technical and cannot be discussed here in detail.

2. Consider the pairs $(X, \mathcal{V})$, $(Y, \mathcal{V}')$, $(Z, \mathcal{V}_Z)$ with Gauss structures $\mathcal{F}_X, \mathcal{F}_Y, \mathcal{F}_Z$ respectively, and Gauss transformations

$$G_X : \mathcal{F}_X \longrightarrow \mathcal{F}_Z$$

$$G_Y : \mathcal{F}_Y \longrightarrow \mathcal{F}_Z$$

Let

$$f : (X \times Y, \mathcal{W}) \longrightarrow (Z, \mathcal{V}_Z)$$

be a continuous n-function, with $\mathcal{W} \subset \mathcal{V} \times \mathcal{V}'$.

Definition I - The partial derivative of f relative to $(X, \mathcal{V})$ is a real n-function

$$\frac{\partial f}{\partial X} : (X \times Y, \mathcal{W}) \to [R, \mathcal{V}_R]$$

defined as follows: consider any $\sigma_W = \sigma \times \sigma' \in \mathcal{W}$ and take any $A_W \in \sigma_W$. Then $A_W = A \times A'$, with $A \in \sigma \in \mathcal{V}$ and $A' \in \sigma' \in \mathcal{V}'$. Consider any $\tau_W > \sigma_W$, $\tau_W = \tau \times \tau' \in \mathcal{W}$ and any $B_W \in \tau_W$, $B_W \subset A_W$. So

$$B_W = B \times B', \; B \in \tau, \; B' \in \tau', \; B \subset A, \; B' \subset A'.$$

Define

$$\overline{\left(\frac{\partial f}{\partial X}\right)}_{\tau_W}(B_W) = \overline{\lim_{\alpha_X \in \mathcal{F}_X}} \frac{n(f_{\tau_W}(B_W), \alpha_Z)}{n(B, \alpha_X)}$$

where $\alpha_Z = G_X(\alpha_X)$. Analoguous definition, by changing $\overline{\lim}$ to $\underline{\lim}$

will be given for

$$\underline{\left(\frac{\partial f}{\partial X}\right)}_{\tau_W} (B_W) = \lim_{\alpha_X \in \tau_X} \frac{n(f_{\tau_W}(B_W), \alpha_Z)}{n(B, \alpha_X)} .$$

Now define

$$\left(\frac{\partial f}{\partial X}\right)_{\tau_W} (A_W) = A_R$$

where A_R is the interior of the smallest interval in R containing all numbers

$$\overline{\left(\frac{\partial f}{\partial X}\right)}_{\tau_W} (B_W) \quad \text{and} \quad \underline{\left(\frac{\partial f}{\partial X}\right)}_{\tau_W} (B_W)$$

for all $\tau_W \in \mathcal{W}$, $\tau_W > \sigma_W$ and all $B_W \in \tau_W$, $B_W \subset A_W$. If all these numbers are equal we take A_R to be their common value. So A_R is either an open interval or a point in R.

Now considering all $A_W \in \sigma_W$ we define a collection σ_R in R made up of open intervals or points. Hence, we define

$$\left(\frac{\partial f}{\partial X}\right)_{\mathcal{W}} : \mathcal{W} \longrightarrow \mathcal{V}_R ,$$

where $\mathcal{V}_R$ is the collection of all σ_R as σ_W runs in $\mathcal{W}$, by

$$\left(\frac{\partial f}{\partial X}\right)_{\mathcal{W}} (\sigma_W) = \sigma_R .$$

Therefore the real n-function, $\frac{\partial f}{\partial X}$ is completely defined. We have analoguous definition for $\frac{\partial f}{\partial Y}$. Observe that if f is continuous the same is true for both $\frac{\partial f}{\partial X}$, $\frac{\partial f}{\partial Y}$.

Analyzing definition I above we see that the process described therein can be iterated, so we can talk about

$$\frac{\partial^2 f}{\partial X^2} = \frac{\partial}{\partial X}\left(\frac{\partial f}{\partial X}\right), \quad \frac{\partial^n f}{\partial X^p \partial Y^p} = \frac{\partial^p}{\partial X^p}\left(\frac{\partial^p}{\partial X^q}\right), \quad p+n = n \text{ etc.}$$

The first questions arising is naturally, what are the relations among

$$\frac{\partial f}{\partial X}, \quad \frac{\partial f}{\partial Y} \quad \text{and} \quad Df .$$

To give some information in that direction we introduce a Gauss structure $\mathcal{F}$ in $X \times Y$ and call

$$G: \mathcal{F} \longrightarrow \mathcal{F}_Z$$

the Gauss transformation used to define Df. To have connections among the derivatives above, it is natural to expect that there must be some relations among $\mathcal{F}_X$, $\mathcal{F}_Y$, $\mathcal{F}_Z$ and $\mathcal{F}$, as well as, among G, G_X and G_Y.

Hence, we assume to begin with, that

$$\mathcal{F} = \mathcal{F}_X \times \mathcal{F}_Y ,$$

namely, any $\sigma \in \mathcal{F}$ is given by $\alpha_X \times \alpha_Y$, $\alpha_X \in \mathcal{F}_X$ and $\alpha_Y \in \mathcal{F}_Y$, this meaning that for any $F \in \alpha$ we have

$$F = F_X \times F_Y, \; F_X \in \alpha_X, \; F_Y \in \alpha_Y.$$

Secondly, we introduce two new Gauss transformations

$$G'_X : \mathcal{F} \longrightarrow \mathcal{F}_X$$

$$G'_Y : \mathcal{F} \longrightarrow \mathcal{F}_Y \; ,$$

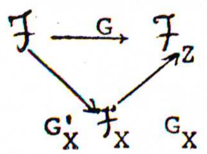
defined as follows: take $\alpha \in \mathcal{F}$, $\alpha = \alpha_X \times \alpha_Y$ and put

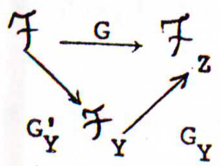
$$G'_X(\alpha) = \alpha_X \; , \; G'_Y(\alpha) = \alpha_Y$$

and as easily seen G'_X and G'_Y so defined are indeed Gauss transformations, i.e.

$$\alpha < \beta, \; \alpha,\beta \in \mathcal{F} \Longrightarrow \begin{cases} G'_X(\alpha) < G'_X(\beta) \\ G'_Y(\alpha) < G'_Y(\beta) \end{cases}$$

Finally, we assume that the diagrams





commute. Under these circunstances we say that the Gauss structure of

$(X, \mathcal{F}_X)$, $(Y, \mathcal{F}_Y)$, $(Z, \mathcal{F}_Z)$ are compatible and then we investigate relations among the derivatives

$$\frac{\partial f}{\partial X}, \quad \frac{\partial f}{\partial Y} \quad \text{and} \quad Df.$$

Let us look back to the definition of

$$\overline{\left(\frac{\partial f}{\partial X}\right)}_{\tau_W} (B_W)$$

considered before and let us write for simplicity

$$f_{\tau_W}(B_W) = B_Z \in \sigma_Z = f_{\mathcal{W}}(\sigma_W) \in \mathcal{V}_Z .$$

We have, with $\alpha_X = G'_X(\alpha)$, $\alpha_Z = G(\alpha)$, $\alpha_Y = G'_Y(\alpha)$, for any $\alpha \in \mathcal{F}$,

$$(1) \qquad \frac{n(B_Z, \alpha_Z)}{n(B_W, \alpha)} = \frac{n(B_Z, \alpha_Z)}{n(B, \alpha_X) \cdot n(B', \alpha_Y)} .$$

and we notice also that

$$\alpha_Z = G_X \circ G'_X(\alpha) = G_X(\alpha_X)$$

$$\alpha_Z = G_Y \circ G'_Y(\alpha) = G_Y(\alpha_Y) .$$

Therefore, if for instance

$$\left(\overline{\frac{\partial f}{\partial X}}\right)_{\tau_W} (B_W) = \overline{\lim_{\alpha_X \in \mathcal{F}_X}} \frac{n(B_Z, \alpha_Z)}{n(B, \alpha_X)} < +\infty$$

we have that

$$\overline{Df}_{\tau_W} (B_W) = \overline{\lim_{\alpha \in \mathcal{F}}} \frac{n(B_Z, \alpha_Z)}{n(B_W, \alpha)} = 0 ,$$

because $n(B', \alpha_X)$ increases indefinitely. Hence we conclude :

<u>Theorem 1</u> . If the Gauss structures of $(X, \mathcal{F}_X)$, $(Y, \mathcal{F}_Y)$, $(Z, \mathcal{F}_Z)$ are compatible and if one of the partial derivatives is finite, namely, all sets

$$\left(\frac{\partial f}{\partial X}\right)_{\sigma_W} (A_W) \quad \text{or} \quad \left(\frac{\partial f}{\partial Y}\right)_{\sigma_W} (A_W)$$

are bounded subsets of R for all $\sigma_W \in \mathcal{W}$ and all $A_W \in \sigma_W$, then

$$Df = 0 .$$

There is a natural interpretation of this result as follows: if we consider the particular case where $X = Y = Z = R$ and all Gauss structure involved are the canonical structures as considered in [1] we know from results referred to in [1] due to V. Buonomano, that Df generates the jacobian determinant $J(\phi)$ of the function ϕ generated by f. Then clearly any differentiable map of R^2 into R must have $J(\phi) = 0$ for otherwise by the implicity function theorem there would exist a local diffeomorphism between a region of R^2 and an interval in R, what is absurd. Therefore, theorem 1 is, in a certain sense, a generalization of this result.

To finish this paragraph we give a theorem relating, for every germ p' in $(Y, \mathcal{V}')$,

$$Df^{p'} \quad \text{with} \quad \left(\frac{\partial f}{\partial X}\right)^{p'} ,$$

when f is Λ-regulated.

Theorem 2 . Let

$$f: (X \times Y, \mathcal{W}) \longrightarrow (Z, \mathcal{V}_Z)$$

be a Λ-regulated, continuous n-function and $\mathcal{F} = \mathcal{F}_X \times \mathcal{F}_Y$ with $\mathcal{F}_X$, $\mathcal{F}_Y$, $\mathcal{F}_Z$ compatible. Then, for each germ p' in $(Y, \mathcal{V}')$ we have

$$Df^{p'} = \left(\frac{\partial f}{\partial X}\right)^{p'}$$

Proof . Let $\sigma \in \mathcal{V}$ and $A \in \sigma$ be arbitrarily given. To compute $Df^{p'}_{\sigma}(A)$ we start by looking to al $\tau > \sigma$ and $B \subset A$, $B \in \tau$ and consider

$$\frac{n(B_Z, \alpha_Z)}{n(B, \alpha_X)}$$

where $\alpha_X \in \mathcal{F}_X$, $\alpha_Z = G_X(\alpha_X)$, $B_Z = f^{p'}_{\sigma}(B) = f_{\tau_W}(B \times B')$, $B' \in p'$, $\tau_W = \tau \times \tau'$, $\tau' = \Lambda(\tau)$.

By another side, to compute

$$\left(\frac{\partial f}{\partial X}\right)^{p'} \quad (A)$$

we have to look at

$$\frac{n(f_{\tau_W}(B_W),\ \alpha_Z)}{n(B,\ \alpha_X)} = \frac{n(f_W(B\times B'),\alpha_Z)}{n(B,\ \alpha_X)} = \frac{n(B_Z,\alpha_Z)}{n(B,\alpha_X)} .$$

Therefore, when we consider $\overline{\lim}$ and $\underline{\lim}$ in either case we get the same set in R and this completes the proof.

In the same way we also have for every $p \in [(X, \mathcal{V})]$,

$$D_f p = \left(\frac{\partial f}{\partial Y}\right)^{p}$$

We can state this result in a short form a follows: let p' stand for the operation which takes f into $f^{p'}$ and the same for D and $\partial/\partial X$. Then our theorem can be stated simply as

$$(D \circ p')(f) = \left(p' \circ \frac{\partial}{\partial X}\right)(f) .$$

which is easy to memorize and indicates that the operators D and $\partial/\partial X$ commute with the operator p'. I found this result particularly elegant.

3. Now we are prepared to study the relations between n-D.S. and differential equation in the sense of non-deterministic mathematics.

In a first moment, we want to investigate conditions under which a given n-D.S. might be considered as a family of solutions of a differential equation of the type

$$(1) \qquad Df = \Phi \circ f$$

where Φ is a n-field. More precisely a n-field is a real n-function

$$\Phi : (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

and f is a n-function

$$f : (X, \mathcal{V}_P) \longrightarrow (X, \mathcal{V})$$

Of course, we can also ask if a n-D.S. is solution of other types of equations different from (1) and all this will be discussed below.

<u>Definition II</u> - A n-D.S.

$$f : (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

derives from a n-field if for any $p \in [(X, \mathcal{V})]$, $p = \{A^p_\sigma\}$ $\sigma \in \mathcal{V}$, we have

$$\left(\frac{\partial f}{\partial X}\right)_{\sigma_W} [\, f^p_{\sigma_P}(A_P) \times A_P \,] = \left(\frac{\partial f}{\partial p}\right)_{\sigma_W} (A^p_\sigma \times A_P)$$

for any $\sigma_W \in \mathcal{W}$ and any $A_P \in \sigma_P$, where $\sigma_W = \sigma \times \sigma_P$.

Theorem 3 - Let the n-D.S.

$$f: (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

derives from a n-field and suppose also that $\partial f/\partial X$ is independent of $(P, \mathcal{V}_P)$, namely,

$$\left(\frac{\partial f}{\partial X}\right)_{\sigma_W} (A \times A_P) = \left(\frac{\partial f}{\partial X}\right)_{\sigma_W} (A \times B_P)$$

for any $\sigma_W = \sigma \times \sigma_P \in \mathcal{W}$ and any $A \in \sigma$, A_P, $B_P \in \sigma_P$. Then there is a n-field

$$\Phi: (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

such that any motion f^P of the given n-D.S. is solution of the equation

(1) $$Dg = \Phi \circ g,$$

namely

$$Df^P = \Phi \circ f^P.$$

Proof - As $\partial f/\partial X$ is independent of $(P, \mathcal{V}_P)$ we can define for each $\sigma \in \mathcal{V}$ and any $A \in \sigma$

$$\Phi_\sigma(A) = \left(\frac{\partial f}{\partial X}\right)_{\sigma_W} (A \times A_P)$$

where $\sigma_W = \sigma \times \sigma_P \in \mathcal{W}$ and $A_P \in \sigma_P$ is arbitrary . This clearly defines a n-function

$$\Phi : (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R] \ .$$

Let now f^p be any motion of f, with $p = \{A^p_\sigma\}_{\sigma \in \mathcal{V}}$. We have, due to theorem 2,

$$(Df^p)_{\sigma_P} (A_P) = \left(\frac{\partial f}{\partial P}\right)^p_{\sigma_P} (A_P) = \left(\frac{\partial f}{\partial p}\right)_{\sigma_W} (A^P_\sigma \times A_P)$$

for any $\sigma_W = \sigma \times \sigma_P \in \mathcal{W}$ and any $A_P \in \sigma_P$.

By another side

$$\Phi_\sigma [f^p_{\sigma_P} (A_P) = \left(\frac{\partial f}{\partial X}\right)_\sigma [f^p_{\sigma_P} (A_P) \times A_P] \ .$$

But as f derives from a n-field we have

$$Df^p_{\sigma_P} (A_P) = \Phi_\sigma \ [f^p_{\sigma_P} (A_P)]$$

what shows that f^p is solution of (1).

<u>Definition III</u> - Let $p \in [(X, \mathcal{V})]$ where $p = \{A^p_\sigma\}_{\sigma \in \mathcal{V}}$. A section in $(X \times P, \mathcal{W})$ through p is a n-function

$$T^p : (P, \mathcal{V}_P) \longrightarrow (X \times P, \mathcal{W})$$

defined as follows:

(i) $$T^{p}_{\mathcal{V}_p} : \mathcal{V}_p \longrightarrow \mathcal{W}$$

is given by

$$T^{p}_{\mathcal{V}_p}(\sigma_p) = \Lambda^{-1}(\sigma_p) \times \sigma_p = \sigma \times \sigma_p = \sigma_w \in \mathcal{W}.$$

(ii) $$T^{p}_{\sigma_p} : \sigma_p \longrightarrow \sigma_w$$

is given by

$$T^{p}_{\sigma_p}(A_p) = A^{p}_{\sigma} \times A_p$$

<u>Theorem 4</u> - Given a n-D.S.

$$f: (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

there is a n-field

$$\Phi: (X \times P, \mathcal{W}) \longrightarrow [R, \mathcal{V}_p)$$

such that any motion f^p of f, with $p = \{A^p\}_{\sigma \in \mathcal{V}}$, satisfies the equation

$$Df^p = \Phi \circ T^p,$$

where T^p is a section in $(X \times P, \mathcal{W})$ through p.

<u>Proof</u> - Define

$$\Phi = \frac{\partial f}{\partial P} .$$

Using theorem 2, for any $\sigma_w = \sigma \times \sigma_P$ and $A_P \in \sigma_P$ we have

$$D_{\sigma_P} f^P (A_P) = \left(\frac{\partial f}{\partial P}\right)_{\sigma_w} (A_\sigma^P \times A_P).$$

But

$$\Phi_{\sigma_w} \circ T^P_{\sigma_P} (A_P) = \Phi_{\sigma_w} (A_\sigma^P \times A_P) = \left(\frac{\partial f}{\partial P}\right)_{\sigma_w} (A_\sigma^P \times A_P),$$

what proves the theorem.

Both theorems 3 and 4 above are rather trivial in the sense that the proof is easy and also it is not used the fact that f is a n-D.S. namely, condition D_1, D_2, D_3 of definition V, §I. But that is also true for the classical case, namely, to prove that a dynamical system, in the plane for instance, satisfies a system of equation of the type

$$(1) \qquad \begin{cases} \dfrac{dx}{dt} = X(x,y) \\[2ex] \dfrac{dy}{dt} = Y(x,y) \end{cases}$$

all we need, in the line of Lie's approach, is the fact that our dynamical system is defined by analytic functions and then equation (1) are simple consequence of the Taylor development of these functions.

4. Let us consider now the inverse problem, i.e., to find condition for the solution of a differential equation in non deterministic sense to define a n-D.S. For the classical case all we need is the theorem

of existence and unicity of solutions and if this is true the equations themselves are not important anymore because we can restrict ourselves to its family of solutions. In the case of non-deterministic mathematics we usually do not have unicity of solution and to these days nobody has yet attempted to find sufficient conditions for an equation, say of the type

$$(2) \qquad Df = \Phi \circ f$$

to have "unique solutions". Even the concept of "unicity" is not clearly estabilished and that is one of our first considerations now.

Let us precise the concept of unicity of solution of an equation

$$Dg = \Phi^{-} \circ f .$$

Definition IV - A family of n-functions $\{f^P\}$, indexed by a set of germs $I \subset [(X,\mathcal{V})]$

$$f^P : (P, \mathcal{V}_P) \longrightarrow (X, \mathcal{V})$$

where $(P, \mathcal{V}_P)$ is a continuous n-semi-group, is a continuous family of solutions of the differential equation

$$Dg = \Phi \circ f$$

satisfying unicity conditions if:

(U_I) there is a Λ-regulated continuous, germ preserving n-function

$$f: (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

such that f generates the family $\{f^p\}$, namely, for any $\sigma_W \in \mathcal{W}$, $\sigma_W = \sigma \times \sigma_P$,

(i) $f_{\mathcal{W}}(\sigma_W) = \sigma$

(ii) $f_{\sigma_W}(A_\sigma \times A_{\sigma_P}) = f^p_{\sigma_P}(A_{\sigma_P})$, $\forall A_\sigma \in p \in I$, $A_\sigma \in \sigma, A_{\sigma_P} \in \sigma_P$.

(U_{II}) $\forall$ $p \in I$, if $e_P = \{\mathring{A}_{\sigma_P}\}_{\sigma_P \in \mathcal{V}_P}$ is the identity in $(P, \mathcal{V}_P)$ then

$$f^{p^-}(e_P) = p.$$

<u>Theorem 5</u> - Let $\{f^p\}$, $p \in I$ be a family of n-functions satisfying conditions (U_I) and (U_{II}) of Definition IV and suppose that for any $p \in I$ and any $\sigma_P \in \mathcal{V}_P$ and $A_{\sigma_P} \in \sigma_P$ there is $p_P \in [(P, \mathcal{V}_P)]$ such that $A_{\sigma_P} \in p_P$ and

$$f^p(p_P) \in I.$$

Then the n-function f, generating $\{f^p\}$, $p \in I$, as in definition IV is a n-D.S.

<u>Proof</u> - We have to show that $\{f^p\}$ is a group of n-transformations of $(X, \mathcal{V})$. Consider any $\sigma_W = \sigma \times \sigma_P \in \mathcal{W}$ and any $A \in \sigma$, $A_{\sigma_P} \in \sigma_P$,

$B_{\sigma_P} \in \sigma_P$. We have

$$f_{\sigma_W}(f_{\sigma_W}(A \times A_{\sigma_P}) \times B_{\sigma_P}) = f^q_{\sigma_P}(B_{\sigma_P}) \qquad (1)$$

where q is a germ in $(X, \mathcal{V})$ given by

$$q = \{f_{\sigma_W}(A \times A_{\sigma_P})\}_{\sigma_P \in \mathcal{V}_P} ,$$

assuming that A belongs to some germ $p \in I$ and A_{σ_P} belongs to some germ in $(P, \mathcal{V}_P)$, such that , $f^p(p_P) \in I$. Clearly we have arbitrainess in the selection of q which depends an the arbitrary choice of p and p_P. However the first number of (1) is independent of these choices.

Now we have that

$$f_{\sigma_W}(A \times (A_{\sigma_P} \cdot B_{\sigma_P})) = f^p_{\sigma_P}(A_{\sigma_P} \cdot B_{\sigma_P}) \qquad (2)$$

and also

$$\begin{cases} f^q_{\sigma_P}(A^\circ_{\sigma_P}) = f_{\sigma_W}(A_P \times A_{\sigma_P}) = f^p_{\sigma_P}(A_{\sigma_P}) \\ f^p_{\sigma_P}(A_{\sigma_P} \cdot A^\circ_{\sigma_P}) = f^p_{\sigma_P}(A_{\sigma_P}) . \end{cases}$$

Let $r \in I$ be given by

$$r = f^p(p_P)$$

and define

$$f^r : (P, \mathcal{V}_P) \longrightarrow (X, \mathcal{V})$$

by $f^r_{\mathcal{V}_P}(\sigma_P) = \sigma$ and

$$\forall \ \sigma_P \in \mathcal{V}_P \ , \quad f^r_{\sigma_P}(B_{\sigma_P}) = f^p_{\sigma_P}(A_{\sigma_P} \cdot B_{\sigma_P}) \quad \forall \ B_{\sigma_P} \in \sigma_P \in \mathcal{V}_P \ .$$

This will give for $A^\circ_{\sigma_P} \in e_P$, the unity germ of $(P, \mathcal{V}_P)$,

$$f^r_{\sigma_P}(A^\circ_{\sigma_P}) = f^p_{\sigma_P}(A_{\sigma_P}) = f^q_{\sigma_P}(A^\circ_{\sigma_P})$$

and so $r = q$. Therefore the second member of (1) is equal to the second member of (2) for any $B_{\sigma_P} \in \sigma_P$ and as $A \in r$ was arbitrarily selected they are always equal for all sets in σ_w.

This shows that f is a n-D.S.

<u>Definition V</u> - A family of n-functions $\{f^p\}$ indexed by $I \subset [(X, \mathcal{V})]$,

$$f^r : (P, \mathcal{V}_P) \longrightarrow (X, \mathcal{V})$$

is called a normal family if:

(N_I) there is a bijection

$$\Lambda : \mathcal{V} \longrightarrow \mathcal{V}_P$$

such that, $\forall \ \sigma, \tau \in \mathcal{V}$,

$$\sigma < \tau \iff \Lambda(\sigma) < \Lambda(\tau) ,$$

(N_{II}) for any $\sigma \in \mathcal{V}$ and any $A_\sigma \in \sigma$ such that $A_\sigma \in p$ and $A_\sigma \in q$, $p,q \in I$, we have

$$f^p_{\sigma_P}(A_{\sigma_P}) = f^q_{\sigma_P}(A_{\sigma_P})$$

for any $A_{\sigma_P} \in \sigma_P = \Lambda(\sigma)$,

(N_{III}) for any $\sigma, \tau \in \mathcal{V}, \sigma < \tau$, and any $B_\tau \in \tau$, $A_\sigma \in \sigma$, $B_\tau \subset A_\sigma$, there is $p \in I$ such that A_σ , $B_\tau \in p$.

(N_{IV}) for any $p \in I$ and any $\sigma_P \in \mathcal{V}_P$ and $A_{\sigma_P} \in \sigma_P$, there is $p_P \in [(P, \mathcal{V}_P)]$ such that $A_{\sigma_P} \in p_P$ and

$$f^p(p_P) \in I .$$

<u>Theorem 6</u> - If $\{f^p\}$ is a normal family of continuous n-functions, indexed by

$$f^p : (P, \mathcal{V}_P) \longrightarrow (X, \mathcal{V})$$

then, there is a continuous n-function

$$f : (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

generating the family $\{f^p\}$.

Proof - We define $\mathcal{W}$ as the family of coverings of $X \times P$ given by

$$\mathcal{W} = \{\sigma \times \sigma_P : \sigma \in \mathcal{V},\ \sigma_P \in \mathcal{V}_P,\ \sigma_P = \Lambda(\sigma)\}$$

We define

$$f_{\mathcal{W}} : \mathcal{W} \longrightarrow \mathcal{V}$$

by

$$\forall\ \sigma_W = \sigma \times \sigma_P \in \mathcal{W}, \qquad f_{\mathcal{W}}(\sigma_W) = \sigma .$$

Now for any $\sigma_W \in \mathcal{W}$ and any $A_W = A_\sigma \times A_{\sigma_P} \in \sigma_W$ we define

$$f_{\sigma_W}(A_\sigma \times A_{\sigma_P}) = f^p_{\sigma_P}(A_{\sigma_P})$$

where p is any germ in I with $A_\sigma \in p$; the selection of p does not influenciate the result by condition N_{II} of Definition V.

Finally to stablish the continuity of f, let $\tau_W = \tau \times \tau_P > \sigma_W$ and let $B_\tau \subset A_\sigma$, $B_{\tau_P} \subset A_{\sigma_P}$. Let q be any germ in I containing A_σ and B_τ as stated in N_{III} and

$$f_{\sigma_W}(B_\tau \times B_{\tau_P}) = f^q_{\tau_P}(B_{\tau_P})$$

But again by condition N_I,

$$f^q_{\sigma_P}(A_{\sigma_P}) = f^p_{\sigma_P}(A_{\sigma_P})$$

and by the continuity of f^q

$$f_{\tau_w}(B_\tau \times B_{\tau_p}) = f^q_{\tau_p}(B_{\tau_p}) \subset f^q_{\sigma_p}(A_{\sigma_p}) = f_{\sigma_w}(A_\sigma \times A_{\sigma_p})$$

and therefore f is continuous, what proves the theorem.

Approaching theorem 5 and 6 we have:

Theorem 7 - Let $\{f^p\}$ be a normal family of continuous n-functions

$$f^p : (P, \mathcal{V}_p) \longrightarrow (X, \mathcal{V})$$

indexed by I , where $(P, \mathcal{V}_p)$ is a continuous a n-group.

Suppose that if e_p is the unity in $(P, \mathcal{V}_p)$ we have for each p;

$$f^p(e_p) = p .$$

Then there is a n-D.S.

$$f : (X \times P, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

whose family of motions is $\{f^p\}$.

Remark - (a) In both theorems 5 and 6 above the differential equation really does not have any influence in the results because all we need is the unicity of solutions. That is also the same for the classical case: once we know the unicity of solutions, the differential equation does not play any role in the results. However, in the classical case we have sufficient conditions for this to happen and in the n-deterministic case it has not been studied, so far, the question of finding sufficient conditions on Φ to garantee the unicity conditions for the equation

$$Dg = \Phi \circ g .$$

(b) In definition IV we can use the concepts of regular or fully regular for the n-functions involved and so we can speak of regular or fully regular family of solutions of the differential equation

$$Dg = \Phi \circ g$$

satisfying unicity conditions.

All theorems 5, 6, 7 remain true if instead of continuity we use regular or fully regular all over. This remark is very important in the applications.

(c) The continuity f the n-group $(P, \mathcal{V}_P)$ is not used in the proofs of theorems 5,6,7 and it is only included in the business because it is part of the definition of n-D.S.

5. Let us give an example of a n-D.S. in the plane which is related to the Brownian motion.

Consider first $\phi(x)$ a real function defined for all $x \in R$, continuous but nowhere differentiable. If $P(x,y)$ is a point in the plane, define

$$f: R^2 \times R \longrightarrow R^2$$

by

$$\forall P \in R^2, t \in R \quad , \quad f(P,t) = (x+t \; , \; \phi(x+t) + y - \phi(x)) \; .$$

As easily seen, f is a one-parameter group of continuous, nowhere differentiable transformations in R^2. Clearly, f cannot be identified with any family of solutions of any differential equation because for any fixed $P(x,y)$ the function $f(P,t)$ is nowhere differentiable, as a function of t.

We want to show how we can build a n-dynamical system using the function ϕ considered above. This will illustrate the use of the main theorems proved before.

Consider in R the family $\mathcal{V}$ of coverings defined as follows: for each $n \geq 1$ call σ_P^n the covering of R given by open intervals of length $1/2^n$.

The family of coverings $\mathcal{V}$ in R^2 is defined as follows: take any $\sigma_P^n \in \mathcal{V}_P$ and call ϕ_t a curve ϕ as considered before cutting the 0_y axis at the point t. For any $A_{\sigma_P}^n \in \sigma_P^n$ and any t define

$$A_\sigma^n = \{(x,y) : x \in A_{\sigma_P}^n , \phi_t(x) < y < \phi_{t+1/2^n}(x)\}$$

Changing t and $A_{\sigma_P}^n$ in all possible ways we get a covering σ_P^n of R^2 by open sets A_σ^n. Finally changing σ_P^n in $\mathcal{V}_P$ we get a family $\mathcal{V}$ of open coverings of R^2. Notice that there is a one-to-one correspondence between the $\mathcal{V}$ and $\mathcal{V}_P$ given by

$$\Lambda(\sigma^n) = \sigma_P^n$$

For simplicity we drop the "n" and write only σ, σ_P, etc. Call $[(R^2,\mathcal{V})]_C$ the set of germs of $(R^2,\mathcal{V})$ compatible with refinements and let us define a normal family $\{f^p\}$, $p \in [(R^2, \mathcal{V})]_C$ of continuous n-functions

$$f^p : (R, \mathcal{V}_p) \longrightarrow (R^2, \mathcal{V}).$$

Let $p \in [(R, \mathcal{V})]_C$ be given and define

$$f^p_{\mathcal{V}_p} : \mathcal{V}_p \longrightarrow \mathcal{V}.$$

by

$$f^p_{\mathcal{V}_p} = \Lambda^{-1} .$$

Now give any $\sigma_p \in \mathcal{V}_p$ and any $A_{\sigma_p} \in \sigma_p^n$. Consider $A^p_\sigma \in p$ and $A^p_\sigma \in \sigma = \Lambda^{-1}(\sigma_p)$. By definition of σ, the bottom of A^p_σ is bounded by one curve ϕ which we shall denote by $\underline{\phi}^{p,\sigma}$ and the same for the top with $\bar{\phi}^{p,\sigma}$. Define

$$f^p_{\sigma_p}(A_{\sigma_p}) = \{(x,y) : x \in A_{\sigma_p} , \underline{\phi}^{p,\sigma}(x) < y < \bar{\phi}^{p,\sigma}\}$$

Therefore

$$f^p_{\sigma_p} : \sigma_p \longrightarrow \sigma = \Lambda^{-1}(\sigma_p)$$

is defined and also f^p. Now it is immediate consequence from the definition that $\{f^p\}$ is a normal family of continuous n-functions indexed by $[(R, \mathcal{V}_p)]_C$, according to definition V, §IV.

Assume now that $(R, \mathcal{V}_p)$ is a n-semi-group with with identity

e_p. We shall define from $\{f^p\}$ another family $\{g^p\}$ satisfying the condition

$$g^p(e_p) = p .$$

Define

$$g^p_{\mathcal{V}_p} : (R, \mathcal{V}_p) \longrightarrow (R^2, \mathcal{V}) .$$

by

$$g^p_{\mathcal{V}_p}(\sigma_p) = \sigma = \Lambda^{-1}(\sigma_p) .$$

Now for any $A_{\sigma_p} \in \sigma_p$ define

$$g^p_{\sigma_p}(A_{\sigma_p}) = f^p_{\sigma_p}(B_{\sigma_p} \cdot A_{\sigma_p}) .$$

where $B_{\sigma_p} \in p_p$ and $p_p \in [(R, \mathcal{V}_p)]$ is such that

$$f^p(p_p) = p .$$

The existence of such p_p is a consequence of the definition of f^p.

Clearly now

$$g^p(e_p) = p .$$

Now the family $\{f^p\}$ satisfies all conditions of theorem 6, §IV and so there is a continuous n-function

$$f: (R^2 \times R, \mathcal{W}) \longrightarrow (R^2, \mathcal{V})$$

generating $\{f^p\}$. Finally the family $\{g^p\}$ satisfies all conditions of theorem 5, §IV and so it is generated by a n-D.S.g which is defined by f above. Clearly we do not worry yet if $\{g^p\}$ is or it is not a family of solutions of some differential equation. What matters now is that it satisfies conditions U_I and U_{II} of definition IV, §IV.

2.

As well known, by Wiener's approach, we can look to the Brownian motion as a collection of particles whose trajectories are curves like φ considered above. So we can start investigations of the Brownian motion by starting with a family $\{g^p\}$ as above and the corresponding n-D.S.

$$g: (R^2 \times R, \mathcal{W}) \longrightarrow (R^2, \mathcal{V}).$$

The first thing to do is to define Gauss structures in R and and R^2. We consider in them the canonical Gauss structures and the same for $R^2 \times R = R^3$. Then we can speak of velocity, acceleration, etc. of a particle indentified with some motion g^p of the n-D.S.g. In few words, in non-deterministic mathematics we can study the Brownian motions in the same spirit as we study motions of particles in classical mechanics.

When conditions of theorems discussed in §II are satisfied,

the n-D.S.

$$g : (R^2 \times R, \mathcal{W}) \longrightarrow (R^2, \mathcal{V})$$

generates a usual dynamical system which is not differentiable and Dg^p might also generate a usual continuous function

$$\psi^p : R \longrightarrow R$$

which can be taken as the field of velocities associated to the motion g^p for a germ $p \in [(R, \mathcal{V})]$.

In the future we discuss this question in more detail.

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