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Decomposing highly connected graphs into paths of length five

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ABSTRACT

Barát and Thomassen (2006) posed the following decomposition conjecture: for each tree T, there exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a decomposition into copies of T. In a series of papers, Thomassen verified this conjecture for stars, some bistars, paths of length 3, and paths whose length is a power of 2. We verify this conjecture for paths of length 5.

1. Introduction

A *decomposition* $\mathcal{D}$ of a graph G is a set $\{H_1, \ldots, H_k\}$ of pairwise edge-disjoint subgraphs of G whose union is G. If each subgraph H_i, $1 \leq i \leq k$, is isomorphic to a given graph H, then we say that $\mathcal{D}$ is an *H-decomposition* of G.

A well-known result of Kotzig (see [7,20]) states that a connected graph G admits a decomposition into paths of length 2 if and only if G has an even number of edges. Dor and Tarsi [16] proved that the problem of deciding whether a graph has an H-decomposition is NP-complete whenever H is a connected graph with at least 3 edges. It is then natural to consider special classes of graphs H, and look for sufficient conditions for a graph G to admit an H-decomposition. One class of graphs that has been studied from this point of view is that of paths, in special when the input graph G is regular. A pioneering work on this topic dates back to 1957, and although some others have followed, a number of questions remain open [14,17,18,20]. For the special case in which H is a tree, Barát and Thomassen [3] proposed the following conjecture.

Conjecture 1.1. *For each tree T, there exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a T-decomposition.*

Barát and Thomassen [3] proved that Conjecture 1.1 in the special case T is the claw $K_{1,3}$ is equivalent to Tutte's weak 3-flow conjecture, posed by Jaeger [19]. They also observed that this conjecture is false if, instead of a tree, we consider a graph that contains a cycle.

Since 2008 many results on this conjecture have been found by Thomassen [28–32]. He has verified that this conjecture holds for paths of length 3, stars, a family of bistars, and paths whose length is a power of 2. In this paper we prove Conjecture 1.1 for paths of length 5. We will focus on the following version of Conjecture 1.1 for bipartite graphs.

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Conjecture 1.2. *For each tree T, there exists a natural number k'_T such that, if G is a k'_T-edge-connected bipartite graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a T-decomposition.*

Recently, Barát and Gerbner, and Thomassen independently proved that Conjectures 1.1 and 1.2 are equivalent. The next theorem states this result precisely.

Theorem 1.3 (*Barát–Gerbner [2]; Thomassen [31]*). *Let T be a tree with m edges, where $m > 3$. The following two statements are equivalent.*

(i) *There exists a natural number k'_T such that, if G is a k'_T-edge-connected bipartite graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a T-decomposition.*
(ii) *There exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a T-decomposition.*

Furthermore, $k_T \leq 4k'_T + 16m^{6m+1}$ and, if in addition T has diameter at most 3, then $k_T \leq 4k'_T + 16(m+1)m$.

In this paper we verify Conjecture 1.2 (and Conjecture 1.1) in the special case T is the path of length five. More specifically, we prove that $k'_{P_5} \leq 48$.

In our proof we use a generalization of the technique used by Thomassen [28] to obtain an initial decomposition into trails of length 5. Then, inspired by the edge-switching technique used in [13], we obtain a result that allows us to "disentangle" the undesired trails of this initial decomposition and construct a pure path decomposition. The main idea uses the following fact: since in a bipartite graph, a trail T of length 5 that is not a path contains a cycle C of length 4, there are two edges of C that can be switched with other elements of the decomposition in such a way that T becomes a path. In [13] only one switching is necessary to "improve" the initial decomposition, but in this paper we need a sequence of switchings to achieve this improvement.

An extended abstract [11] of this work was presented at the conference LAGOS 2015. Further improvements were obtained since then, and these are incorporated into this work. In special, a bound for k'_{P_5} was improved from 134 to 48. Moreover, we [8,10] have been able to generalize some of the ideas presented here to prove that Conjecture 1.1 holds for paths of any given length. We consider that the ideas and techniques presented in this paper are easier to be understood, and they can be seen as a first step towards obtaining more general results not only for paths of fixed length, but also for other types of results [9,12]. As the generalization is not so straightforward, we believe that those interested on the more general case will benefit reading this work first.

The paper is organized as follows. In Section 2 we give some definitions, establish the notation and state some auxiliary results needed in the proof of our main result, presented in Section 4. In Section 3 we prove that a highly edge-connected graph admits a "canonical" decomposition into paths and trails of length 5 satisfying certain properties. In Section 4 we show how to switch edges between the elements of the above decomposition and obtain a decomposition into paths of length 5. We finish with some concluding remarks in Section 5.

2. Notation and auxiliary results

The basic terminology and notation used in this paper follows [6,15]. A *graph* has no loops or multiple edges. A *multigraph* may have multiple edges but no loops. A *directed graph* (resp. *directed multigraph*) is a graph (resp. multigraph) together with an orientation of its edges. More precisely, a directed graph (resp. multigraph) is a pair $\vec{G} = (V, A)$ consisting of a vertex-set V and a set A of ordered pairs of distinct vertices, called *directed edges* (or, simply, *edges*). When a pair (V, A) that defines a (directed) graph G is not given explicitly, such a pair is assumed to be $(V(G), A(G))$. Given a directed graph $\vec{G}$, the set of edges obtained by removing the orientation of the directed edges in $A(\vec{G})$ is denoted by $\hat{A}(\vec{G})$ and is called the *underlying edge-set* of $A(\vec{G})$. We denote by G the *underlying graph* of $\vec{G}$, that is, the graph with vertex-set $V(\vec{G})$ and edge-set $\hat{A}(\vec{G})$. We say that $\vec{G}$ is k-edge-connected if G is k-edge-connected. We denote by $G = (A \cup B, E)$ a bipartite graph G on vertex classes A and B.

We denote by $Q = v_0v_1 \cdots v_k$ a sequence of vertices of a graph G such that $v_iv_{i+1} \in E(G)$, for $i = 0, \ldots, k-1$. If the edges v_iv_{i+1}, $i = 0, \ldots, k-1$, are all distinct, then we say that Q is a *trail*; and if all vertices in Q are distinct, then we say that Q is a *path*. The *length* of Q is k (the number of its edges). A path of length k is denoted by P_k, and is also called a *k-path*. Note that this notation is not standard. If $\vec{Q} = v_0v_1 \cdots v_k$ is a sequence of vertices of a directed graph $\vec{G}$, we say that $\vec{Q}$ is a *path* (resp. *trail*) if Q is a path (resp. trail) in G.

We say that a directed graph $\vec{H}$ is a *copy* of a graph G if H is isomorphic to G. We say that a set $\{H_1, \ldots, H_k\}$ of graphs is a *decomposition* of a graph G if $\bigcup_{i=1}^{k} E(H_i) = E(G)$ and $E(H_i) \cap E(H_j) = \emptyset$ for all $1 \leq i < j \leq k$. For a directed graph, the definition is analogous. Let $\mathcal{H}$ be a family of graphs. An *$\mathcal{H}$-decomposition* $\mathcal{D}$ of $\vec{G}$ is a decomposition of $\vec{G}$ such that each element of $\mathcal{D}$ is a copy of an element of $\mathcal{H}$. If $\mathcal{H} = \{H\}$ we say that $\mathcal{D}$ is an *H-decomposition*.

In what follows, we present some concepts and auxiliary results that will be used in the forthcoming sections. We assume here that the set of natural numbers does not contain zero.

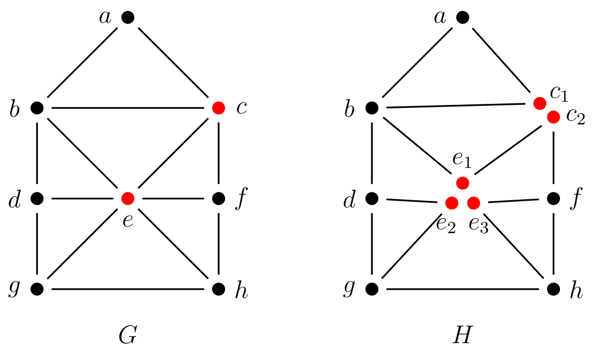

Fig. 1. A graph G and a graph H that is an $\{S_c, S_e\}$-detachment of G, where $S_c = \{2, 2\}$ and $S_e = \{2, 2, 2\}$.

2.1. Vertex splittings

Let $G = (V, E)$ be a graph and x a vertex of G. A set $S_x = \{d_1, \ldots, d_{s_x}\}$ of s_x natural numbers is called a *subdegree sequence for* x if $d_1 + \cdots + d_{s_x} = d_G(x)$. We say that a graph G' is obtained by an (x, S_x)*-splitting* of G if G' is composed of $G - x$ together with s_x new vertices $x_1, \ldots, x_{s_x}$ and $d_G(x)$ new edges satisfying the following conditions:

- $d_{G'}(x_i) = d_i$, for $1 \le i \le s_x$;
- $\bigcup_{i=1}^{s_x} N_{G'}(x_i) = N_G(x)$.

Let G be a graph and consider a set $V' = \{v_1, \ldots, v_r\}$ of r vertices of G. Let $S_{v_1}, \ldots, S_{v_r}$ be subdegree sequences for $v_1, \ldots, v_r$, respectively. Let $H_1, \ldots, H_r$ be graphs obtained as follows: H_1 is obtained by a (v_1, S_{v_1})-splitting of G, the graph H_2 is obtained by a (v_2, S_{v_2})-splitting of H_1, and so on, up to H_r, which is obtained by a (v_r, S_{v_r})-splitting of H_{r-1}. In this case, we say that each graph H_i is a $\{S_{v_1}, \ldots, S_{v_i}\}$*-detachment* of G. Roughly, a detachment of a graph G is a graph obtained by successive applications of splitting operations on vertices of G (see Fig. 1).

The next result provides sufficient conditions for the existence of $2k$-edge-connected detachments of $2k$-edge-connected graphs.

Lemma 2.1 (*Nash-Williams [25]*). *Let k be a natural number, and G be a $2k$-edge-connected graph with $V(G) = \{v_1, \ldots, v_n\}$. For every v in $V(G)$, let $S_v = \{d_1^v, \ldots, d_{s_v}^v\}$ be a subdegree sequence for v such that $d_i^v \ge 2k$ for $i = 1, \ldots, s_v$. Then, there exists a $2k$-edge-connected $\{S_{v_1}, \ldots, S_{v_n}\}$-detachment of G.*

2.2. Edge liftings

Let $G = (V, E)$ be a graph that contains vertices u, v, w such that $uv, vw \in E$. The multigraph $G' = \big(V, (E \setminus \{uv, vw\}) \cup \{uw\}\big)$ is called a uw*-lifting* (or, simply, a *lifting*) at v. If for all distinct pairs $x, y \in V \setminus \{v\}$, the maximum number of edge-disjoint paths between x and y in G' is the same as in G, then the lifting at v is called *admissible*. If v is a vertex of degree 2, then the lifting at v is always admissible. This lifting together with the deletion of v is called a *suppression* of v.

The next result, known as Mader's Lifting Theorem, presents conditions for a multigraph to have an admissible lifting.

Theorem 2.2 (*Mader [22]*). *Let G be a finite multigraph and let v be a vertex of G that is not a cut-vertex. If $d_G(v) \ge 4$ and v has at least 2 neighbors, then there exists an admissible lifting at v.*

The next lemma will be useful to apply Mader's Lifting Theorem. For two vertices x, y in a graph G, we denote by $p_G(x, y)$ the maximum number of edge-disjoint paths between x and y in G.

Lemma 2.3. *Let k be a natural number. If G is a multigraph and v is a vertex in G such that $d(v) < 2k$ and $p_G(x, y) \ge k$ for any two distinct neighbors x, y of v, then v is not a cut-vertex of G.*

Proof. Let k, G and v be as in the hypothesis of the lemma. Suppose, by contradiction, that v is a cut-vertex. Let G_x and G_y be two components of $G - v$. Let $x \in V(G_x)$ and $y \in V(G_y)$ be two neighbors of v. By hypothesis, G has at least k edge-disjoint paths joining x to y. Since v is a cut-vertex, each of these paths must contain v. Thus, $d(v) \ge 2k$, a contradiction. □

2.3. Some consequences of high connectivity

If G is a graph that contains $2k$ pairwise edge-disjoint spanning trees, then, clearly, G is $2k$-edge-connected.
The converse is not true, but as the following result shows, every $2k$-edge-connected graph contains k such trees.

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Theorem 2.4 (*Nash-Williams [24]; Tutte [33]*)**.** *Let k be a natural number. If G is a $2k$-edge-connected graph, then G contains k pairwise edge-disjoint spanning trees.*

We state now a result (Theorem 2.5) that we shall use in the proof of Lemma 2.6. The latter allows us to treat highly edge-connected bipartite graphs as regular bipartite graphs; it is a slight generalization of Proposition 2 in [31]. Given an orientation O of a graph G, we denote by $d^+_O(v)$ the outdegree of v in O.

Theorem 2.5 (*Lovász–Thomassen–Wu–Zhang [21]*)**.** *Let $k \geq 3$ be an odd natural number and G a $(3k-3)$-edge-connected graph. Let $p : V(G) \to \{0, \ldots, k-1\}$ be such that $\sum_{v \in V(G)} p(v) \equiv |E(G)| \pmod{k}$. Then there is an orientation O of G such that $d^+_O(v) \equiv p(v) \pmod{k}$, for every vertex v of G.*

Lemma 2.6. *Let $k \geq 3$ and r be natural numbers, k odd. If $G = (A_1 \cup A_2, E)$ is a $(6k-6+4r)$-edge-connected bipartite graph and $|E|$ is divisible by k, then G admits a decomposition into two spanning r-edge-connected graphs G_1 and G_2 such that, the degree in G_i of each vertex of A_i is divisible by k, for $i = 1, 2$.*

Proof. Let k, r and $G = (A_1 \cup A_2, E)$ be as stated in the lemma. By Theorem 2.4, G contains $3k-3+2r$ pairwise edge-disjoint spanning trees. Let H_1 be the union of r of these trees, let H_2 be the union of other r of these trees, and let $H_3 = G - E(H_1) - E(H_2)$. Thus, H_1 and H_2 are r-edge-connected, and H_3 is $(3k-3)$-edge-connected.

Take $p\colon V(H_3) \to \{0, \ldots, k-1\}$ such that $p(v) \equiv (k-1)d_{H_1}(v) \pmod{k}$ if v is a vertex of A_1, and $p(v) \equiv (k-1)d_{H_2}(v) \pmod{k}$ if v is a vertex of A_2. Thus, the following holds, where the congruences are taken modulo k.

$$\begin{aligned}\sum_{v \in V(G)} p(v) &= \sum_{v \in A_1} p(v) + \sum_{v \in A_2} p(v)\\ &\equiv (k-1)(|E(H_1)| + |E(H_2)|)\\ &\equiv (k-1)(|E| - |E(H_3)|)\\ &\equiv k\,(|E| - |E(H_3)|) - |E| + |E(H_3)|\\ &\equiv |E(H_3)|.\end{aligned}$$

Since H_3 is a $(3k-3)$-edge-connected spanning subgraph of G, by Theorem 2.5 there is an orientation O of H_3 such that $d^+_O(v) \equiv p(v) \pmod{k}$ for every $v \in V(H_3) = V(G)$. For $i = 1, 2$, let G_i be the graph H_i together with the edges of H_3 that leave A_i in the orientation O (note that, $E = E(G_1) \cup E(G_2)$). Thus, $d_{G_i}(v) = d_{H_i}(v) + d^+_O(v) \equiv k\,d_{H_i}(v) \equiv 0 \pmod{k}$ for every vertex v in A_i, and moreover, G_i is r-edge-connected (because it contains H_i). □

We note that in Lemma 2.6 we have k odd and the $(6k-6+4r)$-edge-connectivity of G is a consequence of the $(3k-3)$-edge-connectivity in the statement of Theorem 2.5. When k is even, we can also prove an analogous result, changing the edge-connectivity of G to $6k-4+4r$. For that, we only have to use a slightly weaker form of Theorem 2.5 for k even, according to which, as stated in [21], one may change the bound $(3k-3)$ to $(3k-2)$.

Given a graph G and a natural number r, an *r-factor* in G is an r-regular spanning subgraph of G. The following two results on r-factors in regular multigraphs will be used later.

Theorem 2.7 (*Von Baebler [34] (See also [1, Theorem 2.37])*)**.** *Let $r \geq 2$ be a natural number, and G be an $(r-1)$-edge-connected r-regular multigraph of even order. Then G has a 1-factor.*

Theorem 2.8 (*Petersen [26]*)**.** *If G is a $2k$-regular multigraph, then G admits a decomposition into 2-factors.*

3. Fractional factorizations and canonical decompositions

In this section we prove that every 4-edge-connected bipartite graph $G = (A \cup B, E)$ such that the degree of each vertex in A is divisible by 5 admits a special decomposition, which we call "fractional factorization" (see Section 3.1). Moreover, if G is 6-edge-connected, then such a factorization guarantees that we can construct a decomposition of G into trails of length 5 with some special properties (see Section 3.2).

3.1. Fractional factorizations

To simplify notation, if F is a set of edges of a graph G, we write $d_F(v)$ to denote the degree of v in $G[F]$, the subgraph of G induced by F. If F is a set of edges of a directed graph $\vec{G}$, we write $d^+_F(v)$ (resp. $d^-_F(v)$) to denote the outdegree (resp. indegree) of v in $\vec{G}[F]$.

Definition 3.1. Let $\vec{G}$ be a bipartite directed graph with vertex classes A and B, and such that the degree of each vertex in A is divisible by 5. We say that $\vec{G}$ admits a *fractional factorization* (M, F, H) for A if $A(\vec{G})$ can be decomposed into three edge-sets M, F and H such that the following holds.

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(i) Every edge in M is directed from B to A;
(ii) For every $v \in A$, we have $d_F^-(v) = d_F^+(v) = d_H^-(v) = d_H^+(v) = d_M^-(v) = d(v)/5$;
(iii) For every $v \in B$, we have $d_F^-(v) = d_F^+(v)$ and $d_H^-(v) = d_H^+(v)$.

Lemma 3.2. *Let $G = (A \cup B, E)$ be a 4-edge-connected bipartite graph such that the degree of each vertex in A is divisible by 5. Then, G is the underlying graph of a directed graph $\vec{G}$ that admits a fractional factorization (M, F, H) for A.*

Proof. Let $G = (A \cup B, E)$ be as stated in the hypothesis of the lemma. First, we want to apply Lemma 2.1 to G and obtain a 4-edge-connected graph G' with maximum degree 7. To do this, for every vertex $v \in B$, we take integers $s_v \geq 1$ and $0 \leq r_v < 4$ such that $d(v) = 4s_v + r_v$. We put $d_1^v = 4 + r_v$ and $d_2^v = \cdots = d_{s_v}^v = 4$. Furthermore, for every vertex $v \in A$, we put $s_v = d(v)/(5)$ and $d_i^v = 5$ for $1 \leq i \leq s_v$. By Lemma 2.1, applied with parameters $k = 2$ and the integers s_v, d_i^v $(1 \leq i \leq s_v)$ for every $v \in V(G)$, there exists a 4-edge-connected bipartite graph G' obtained from G by splitting each vertex v of A into s_v vertices of degree 5, and each vertex v of B into a vertex of degree $4 + r_v < 8$ and $s_v - 1$ vertices of degree 4. Let A' and B' be the set of vertices of G' obtained from the vertices of A and B, respectively. For ease of notation, if $v \in (A' \cup B') \setminus (A \cup B)$ we also denote by v the original vertex in $(A \cup B)$ at which we applied splitting.

The next step is to obtain a 5-regular multigraph G^* from G' by using lifting operations. For this, we will add some edges to A' and remove the even-degree vertices of B' by successive applications of Mader's Lifting Theorem as follows.

Let $G'_0, G'_1, \ldots, G'_\lambda$ be a maximal sequence of graphs such that $G'_0 = G'$ and (for $i \geq 0$) G'_{i+1} is the graph obtained from G'_i by the application of an admissible lifting at an arbitrary vertex v of degree in $\{4, 6, 7\}$.

Recall that given any two vertices of G', say x and y, we denote by $p_{G'}(x, y)$ the maximum number of pairwise edge-disjoint paths joining x and y in G'. We claim that $p_{G'_i}(x, y) \geq 4$ for any x, y in A' and every $i \geq 0$. Clearly, $p_{G'_0}(x, y) \geq 4$ holds for any x, y in A', since G' is 4-edge-connected. Fix $i \geq 0$ and suppose $p_{G'_i}(x, y) \geq 4$ holds for any x, y in A'. Let x, y be two vertices in A'. Since G'_{i+1} is a graph obtained from G'_i by the application of an admissible lifting at a vertex v in B', we have $p_{G'_{i+1}}(x, y) \geq p_{G'_i}(x, y) \geq 4$.

We claim that if v is a vertex in B', then $d_{G'_\lambda}(v) \in \{2, 5\}$. Suppose by contradiction that there is a vertex v in B' such that $d_{G'_\lambda}(v) \notin \{2, 5\}$. Note that $d_{G'_i}(u) \geq d_{G'_{i+1}}(u) \geq 2$ for every vertex u of G and every $0 \leq i \leq \lambda$. Since $d_{G'}(u) \leq 7$ for every vertex u in V', we have $2 \leq d_{G'_i}(u) \leq 7$ for every $0 \leq i \leq \lambda$. Therefore, $d_{G'_i}(v) \in \{4, 6, 7\}$. Since $d_{G'_\lambda}(v) \leq 7$ and for any two neighbors x, y of v we have $p_{G'_\lambda}(x, y) \geq 4$, Lemma 2.3 implies that v is not a cut-vertex of G'_λ. Then, by Mader's Lifting Theorem (Theorem 2.2) in G'_λ, there is an admissible lifting at v. Therefore, $G'_0, G'_1, \ldots, G'_\lambda$ is not maximal, a contradiction.

In G'_λ we may have some vertices in B' that have degree 2. For every such vertex v, if u and w are the neighbors of v, we apply a uw-lifting at v, and remove the vertex v, i.e., we perform a suppression of v. Let G^* be the graph obtained by this process. Note that the number of pairwise edge-disjoint paths joining two distinct vertices of A' remains the same, and thus, $p_{G^*}(x, y) = p_{G'_\lambda}(x, y) \geq 4$ for every x, y in A'. Furthermore, the set of vertices of G^* that belong to B' is an independent set; we denote it by B^* (eventually, $B^* = \emptyset$). Note that, if B^* is nonempty, every vertex of B^* has degree 5.

Claim 3.3. *G^* is 4-edge-connected.*

Proof. Let $Y \subseteq V(G^*)$. Suppose there is at least one vertex x of A' in Y and at least one vertex y of A' in $V(G^*) - Y$. Since there are at least 4 edge-disjoint paths joining x to y, there are at least 4 edges, each one with vertices in both Y and $V(G^*) - Y$. Now, suppose that $A' \subset Y$ (otherwise $A' \subset V(G^*) - Y$ and we take $V(G^*) - Y$ instead of Y), and then $V(G^*) - Y \subseteq B^*$. Since B^* is an independent set, all edges with a vertex in $V(G^*) - Y$ must have the other vertex in A'. Since every vertex in B^* has degree 5, there are at least 5 edges, each one with vertices in both Y and $V(G^*) - Y$. □

We conclude that G^* is a 4-edge-connected 5-regular multigraph with vertex-set $A' \cup B^*$, where B^* is an independent set.

Now we work on the multigraph G^*. Since G^* is 5-regular, G^* has even order. Thus, by Theorem 2.7, G^* contains a perfect matching M^*. The multigraph $J^* = G^* - M^*$ is a 4-regular multigraph. By Theorem 2.8, J^* admits a decomposition into 2-factors with edge-sets, say F^* and H^*. Thus, M^*, F^*, and H^* define a partition of $E(G^*)$.

Now let us go back to the bipartite graph G. Let xy be an edge of G^*. If xy joins a vertex of A' to a vertex of B^*, then xy corresponds to an edge of G. If xy joins two vertices of A', then there is a vertex v_{xy} of B' and two edges of G' incident to it, xv_{xy} and $v_{xy}y$, such that xy was obtained by an xy-lifting at v_{xy} (either by an application of Mader's Lifting Theorem or by the suppression of vertices of degree 2). Thus, each edge of G^* represents an edge of G or a 2-path in G such that the internal vertices of these 2-paths are always in B. For every edge $xy \in E(G^*)$, define $f(xy) = \{xy\}$ if xy joins a vertex of A' to a vertex of B^*, and $f(xy) = \{xv_{xy}, v_{xy}y\}$ if xy joins two vertices of A'. Note that, for every edge xy of G^*, we have $f(xy) \subset E(G)$. For a set S of edges of G^*, put $f(S) = \cup_{e \in S} f(e)$. The partition of $E(G^*)$ into M^*, F^* and H^* induces a partition of $E(G)$ into $M = f(M^*)$, $F = f(F^*)$ and $H = f(H^*)$.

Now we construct an Eulerian orientation of $G[F]$ and $G[H]$ induced by any Eulerian orientation of $G^*[F^*]$ and $G^*[H^*]$. Let xy be an edge of $G^* - M^*$ oriented from x to y. If xy joins a vertex of A' to a vertex of B', let xy be oriented from x to y in $G - M$. Otherwise, recall that $f(xy) = \{xv_{xy}, v_{xy}y\}$, and let xv_{xy} be oriented from x to v_{xy} in $G - M$, and $v_{xy}y$ be oriented from v_{xy} to y in $G - M$. The obtained orientation of $G - M$ is Eulerian. Finally, orient all edges of M from B to A. Let $\vec{G}$ be the directed graph obtained by such an orientation of G.

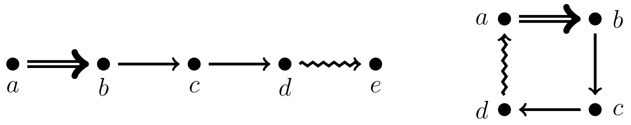

Fig. 2. An $\mathcal{F}$-basic path and an $\mathcal{F}$-basic cycle.

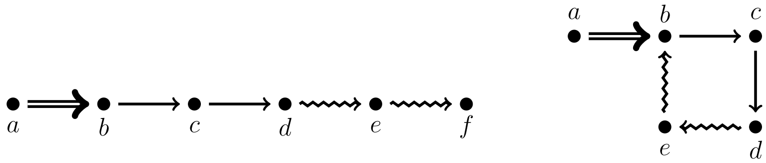

Fig. 3. An $\mathcal{F}$-canonical path and an $\mathcal{F}$-canonical trail.

Let us prove that (M, F, H) is a fractional factorization of $\vec{G}$ for A. Let v be a vertex of A in G of degree $5d'(v)$. The vertex v is represented by $d'(v)$ vertices in G^*. Since M^* is a perfect matching in G^*, there are $d'(v)$ edges of M entering v. Since $G^*[F^*]$ (resp. $G^*[H^*]$) is a 2-factor in G^*, there are $d'(v)$ edges of F (resp. H) entering v and $d'(v)$ edges of F (resp. H) leaving v. Since $G^*[F^*]$ (resp. $G^*[H^*]$) is a 2-factor in G^*, we have $d_F^+(v) = d_F^-(v) = d_H^+(v) = d_H^-(v)$, concluding the proof. $\square$

3.2. Canonical decompositions

In this subsection we show that if a 6-edge-connected bipartite directed graph admits a fractional factorization, then it admits a very special trail decomposition. We make precise what are the properties of such a special trail decomposition.

Let $\vec{G}$ be a directed graph such that $A(\vec{G})$ is the union of pairwise disjoint sets of directed edges M, F and H. The following definitions refer to the triple $\mathcal{F} = (M, F, H)$. Let $T = abcde$ be a trail of length 4 in $\vec{G}$, where $ab \in M$, $bc, cd \in F$ and $de \in H$. We say that T is an $\mathcal{F}$*-basic path* if T is a path; and T is an $\mathcal{F}$*-basic cycle* if T is a cycle (see Fig. 2). Furthermore, let $T = abcdef$ be a trail in $\vec{G}$ such that $abcde$ is an $\mathcal{F}$-basic path. We say that T is an $\mathcal{F}$*-canonical path* if T is a path; and an $\mathcal{F}$*-canonical trail*, otherwise (see Fig. 3). We say that a decomposition $\mathcal{D}$ of $\vec{G}$ is an $\mathcal{F}$*-basic decomposition* if each element of $\mathcal{D}$ is an $\mathcal{F}$-basic path or an $\mathcal{F}$-basic cycle. Analogously, $\mathcal{D}$ is an $\mathcal{F}$*-canonical decomposition* if each element of $\mathcal{D}$ is an $\mathcal{F}$-canonical path or an $\mathcal{F}$-canonical trail.

To prove the next lemma, we use some ideas inspired by the techniques in [28].

Lemma 3.4. *Let $\vec{G}$ be a 6-edge-connected bipartite directed graph. If $\vec{G}$ admits a fractional factorization $\mathcal{F}$ for A, then $\vec{G}$ admits an $\mathcal{F}$-canonical decomposition.*

Proof. Let $\vec{G}$ be a bipartite directed graph with vertex classes A and B that admits a fractional factorization $\mathcal{F} = (M, F, H)$ for A. Let $H^+(A)$ be the set of edges of H leaving vertices of A, and let $H^-(A)$ be the set of edges of H entering vertices of A. Note that $\mathcal{F}' = (M, F, H^+(A))$ decomposes the edge-set of $G' = G[M \cup F \cup H^+(A)]$.

We start by proving that G' admits an $\mathcal{F}'$-basic path decomposition. For that, we first show that G' admits an $\mathcal{F}'$-basic decomposition and after we prove that there is an $\mathcal{F}'$-basic decomposition without cycles.

By item (iii) of Definition 3.1, for every $v \in B$, we have $d_F^-(v) = d_F^+(v)$. Then, the subgraph of G' induced by the edges of F admits a P_2-decomposition such that the endpoints of the elements of the decomposition are in A. Let $\mathcal{D}_2$ be a P_2-decomposition of $G'[F]$. By item (ii) of Definition 3.1, for every $v \in A$, we have $d_M^-(v) = d_F^+(v)$ and $d_F^-(v) = d_H^+(v)$. Therefore, one can extend $\mathcal{D}_2$ to an $\mathcal{F}'$-basic decomposition of G' by adding two edges to each element of $\mathcal{D}_2$. Precisely, for each path xyz that is an element of $\mathcal{D}_2$, it is possible to extend it to either an $\mathcal{F}'$-basic path or an $\mathcal{F}'$-basic cycle by adding one edge of M to x and one edge of H^+ to z.

For each $\mathcal{F}'$-basic decomposition $\mathcal{D}$ of G', let $\rho(\mathcal{D})$ be the number of $\mathcal{F}'$-basic cycles in $\mathcal{D}$. Let $\mathcal{D}$ be an $\mathcal{F}'$-basic decomposition of G' that minimizes $\rho(\mathcal{D})$ over all $\mathcal{F}'$-basic decompositions of G'. If $\rho(\mathcal{D}) = 0$ then $\mathcal{D}$ is an $\mathcal{F}'$-basic path decomposition of G'. Thus, suppose $\rho(\mathcal{D}) > 0$.

By definition, every element T of an $\mathcal{F}'$-basic decomposition contains exactly one directed path P of length two on the edges of F (see Fig. 2), which we call the *center* of T. Moreover, suppose that P starts at a vertex x and ends at a vertex y. We say that x and y are the *starting* and *ending* vertices of T, and we denote them $\mathrm{start}(T)$ and $\mathrm{end}(T)$, respectively. Note that $x, y \in A$.

Since G is 6-edge-connected and every vertex in A has degree divisible by 5, every vertex in A has degree at least 10. Then, since for every $v \in A$ we have $d_F^-(v) = d_F^+(v) = d_H^-(v) = d_H^+(v) = d_M^-(v)$, we conclude that every $v \in A$ contains at least two incoming edges of F and two outgoing edges of F. Therefore, given an element T_2 of $\mathcal{D}$, there exists an element T_1 of $\mathcal{D}$ such that $\mathrm{start}(T_1) = \mathrm{start}(T_2)$ and there exists an element T_3 of $\mathcal{D}$, such that $\mathrm{end}(T_3) = \mathrm{end}(T_2)$ (note that possibly $T_3 = T_1$). Then, there is a maximal sequence $S = T_0, T_1, T_2, \ldots$ of elements of $\mathcal{D}$ such that T_0 is an $\mathcal{F}'$-basic cycle and, for every $k \geq 0$, we have $\mathrm{end}(T_{2k}) = \mathrm{end}(T_{2k+1})$ and $\mathrm{start}(T_{2k+1}) = \mathrm{start}(T_{2k+2})$ (see Fig. 4 for an example).

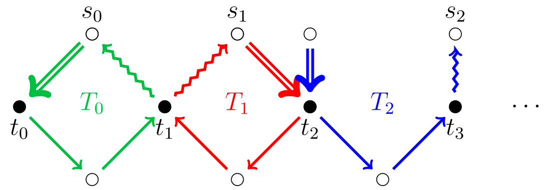

Fig. 4. Example of a sequence $T_0, T_1, T_2, \ldots$ such that T_0 is an $\mathcal{F}'$-basic cycle and, for every $k \geq 0$, we have $\mathrm{end}(T_{2k}) = \mathrm{end}(T_{2k+1})$ and $\mathrm{start}(T_{2k+1}) = \mathrm{start}(T_{2k+2})$.

Consider the sequence $R = t_0, t_1, t_2, \ldots$ of vertices of A that belong to elements of S, i.e., for every $k \geq 0$, we have $t_{2k} = \mathrm{start}(T_{2k})$ and $t_{2k+1} = \mathrm{end}(T_{2k+1})$. Since G is finite, $t_j = t_i$ for some $0 \leq i < j$. Therefore, there exists a "cycle" of elements of $\mathcal{D}$ in the sequence S. Let i be the minimum integer such that $t_i = t_j$ for some $j > i$. Note that if $i \neq 0$, then $T_{i-1} \neq T_{j-1}$. For each element T_k of S, let s_k be the vertex of T_k such that either $s_k t_{k+1} \in E(T_k) - F$ or $t_{k+1} s_k \in E(T_k) - F$, i.e., s_k is the vertex of T_k that is neighbor of t_{k+1} and is not incident to the edges in $E(T_k) \cap F$. We claim that $s_k \neq s_0$ for some $k > 0$. If $i = 0$, then $t_j = t_0$. Since T_0 is an $\mathcal{F}'$-basic cycle, we have $s_0 t_0 \in E(T_0) - F$, from where we conclude that $s_0 t_j \notin E(T_{j-1})$, implying that $s_{j-1} \neq s_0$. Thus, suppose $i > 0$. Note that, since $T_{i-1} \neq T_{j-1}$ and $t_i = t_j$, we have $s_i \neq s_j$. Thus, at least one vertex in $\{s_i, s_j\}$ is different from s_0.

Let k^* be the minimum integer such that $s_{k^*} \neq s_0$. We want to disentangle the elements of $\mathcal{D}$ to obtain an $\mathcal{F}'$-basic decomposition with fewer copies of $\mathcal{F}'$-basic cycles than $\mathcal{D}$. For that, consider the following notation for the elements of $\mathcal{D}$. For $0 \leq \ell \leq k^*$, let $T_\ell = a_0^\ell\, a^{\ell_1}\, a^{\ell_2}\, a^{\ell_3}\, a^{\ell_4}$ such that $a_0^\ell a^{\ell_1} \in M$, $a^{\ell_1} a^{\ell_2}$, $a^{\ell_2} a_3^\ell \in F$ and $a_3^\ell a_4^\ell \in H$. Thus, note that $a_1^\ell = t_\ell$ and $a_3^\ell = t_{\ell+1}$ if ℓ is even, and that $a_1^\ell = t_{\ell+1}$ and $a_3^\ell = t_\ell$ if ℓ is odd. Let

$$T'_0 = a_0^0\, a_1^0\, a_2^0\, a_3^0\, \boldsymbol{a_4^1};$$

$$T'_\ell = \begin{cases} \boldsymbol{a_0^{\ell+1}} a_1^\ell\, a_2^\ell\, a_3^\ell\, \boldsymbol{a_4^{\ell-1}}, & \text{if } \ell \text{ is odd,} \\ \boldsymbol{a_0^{\ell-1}} a_1^\ell\, a_2^\ell\, a_3^\ell\, \boldsymbol{a_4^{\ell+1}}, & \text{if } \ell \text{ is even,} \end{cases} \quad \text{for } 0 < \ell < k^*;$$

$$T'_{k^*} = \begin{cases} a_0^{k^*} a_1^{k^*} a_2^{k^*} a_3^{k^*} \boldsymbol{a_4^{k^*-1}}, & \text{if } k^* \text{ is odd,} \\ \boldsymbol{a_0^{k^*-1}} a_1^{k^*} a_2^{k^*} a_3^{k^*} a_4^{k^*}, & \text{if } k^* \text{ is even.} \end{cases}$$

Then, $\mathcal{D}' = \mathcal{D} - T_0 - T_1 \cdots - T_{k^*} + T'_0 + T'_1 \cdots + T'_{k^*}$ is an $\mathcal{F}'$-basic decomposition (see Fig. 5 for an example). Furthermore, $\rho(\mathcal{D}') < \rho(\mathcal{D})$, contradicting the minimality of $\rho(\mathcal{D})$. Therefore, G' admits an $\mathcal{F}'$-basic path decomposition $\mathcal{D}$.

To finish the proof we extend the $\mathcal{F}'$-basic path decomposition $\mathcal{D}$ of G' to an $\mathcal{F}$-canonical decomposition of G by using the edges of $H^-(A)$. Note that each $\mathcal{F}$-basic path in $\mathcal{D}$ is a directed path ending with an edge of $F_2^+(A)$ and at a vertex of B. But since, by item (iii) of Definition 3.1, $d_H^-(v) = d_H^+(v)$ for every $v \in B$, we can easily extend $\mathcal{D}$ to an $\mathcal{F}$-canonical decomposition of G by adding one edge of $H^-(A)$ to each one of its $\mathcal{F}'$-basic paths, concluding the proof. □

Combining Lemmas 3.2 and 3.4 we obtain the following corollary.

Corollary 3.5. *Let $G = (A \cup B, E)$ be a 6-edge-connected bipartite graph such that the vertices in A have degree divisible by 5. Then, G is the underlying graph of a directed graph $\vec{G}$ that admits a fractional factorization $\mathcal{F}$ and an $\mathcal{F}$-canonical decomposition.*

4. Proof of the main theorem

In this section we manage to "disentangle" the trails of a canonical decomposition to obtain a decomposition into paths of length 5. Denote by T_5 the only bipartite trail of length 5 that is not a path. We recall that a $\{P_5, T_5\}$-decomposition $\mathcal{D}$ of a directed graph $\vec{G}$ is a decomposition of $\vec{G}$ such that every element of $\mathcal{D}$ is either a copy of P_5 or a copy of T_5.

Let $\vec{G}$ be a directed graph and ab an edge of $\vec{G}$. Let $\mathcal{D}$ be a decomposition of $\vec{G}$, and let T be the element of $\mathcal{D}$ that contains ab. We say that ab is *inward* in $\mathcal{D}$ if $d_T(a) = 1$. Suppose that $\vec{G}$ admits a fractional factorization $\mathcal{F} = (M, F, H)$. Let $\mathcal{D}$ be a $\{P_5, T_5\}$-decomposition of $\vec{G}$. We say that $\mathcal{D}$ is *M-complete* if every edge of M is inward in $\mathcal{D}$. Note that if T is an $\mathcal{F}$-canonical path or an $\mathcal{F}$-canonical trail, then the edge of M in T is inward in $\mathcal{D}$. Therefore, if $\mathcal{D}$ is an $\mathcal{F}$-canonical decomposition, then $\mathcal{D}$ is M-complete. The next theorem is our main result.

Theorem 4.1. *There exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(G)|$ is divisible by 5, then G admits a P_5-decomposition.*

Our main theorem follows directly from Theorem 1.3 and the next result.

Theorem 4.2. *If G is a 48-edge-connected bipartite graph and $|E(G)|$ is divisible by 5, then G admits a P_5-decomposition.*

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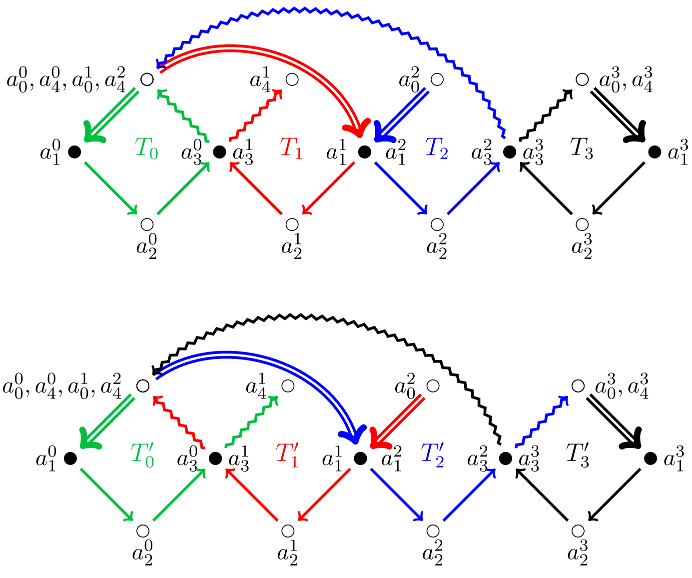

Fig. 5. Example of a sequence T_0, T_1, T_2, T_3 and the corresponding paths T_0', T_1', T_2', T_3'.

Proof. Let $G = (A \cup B, E)$ be a 48-edge-connected bipartite graph such that $|E|$ is divisible by 5. By Lemma 2.6 (taking $r = 6$ and $k = 5$), G can be decomposed into graphs G_1 and G_2 such that G_1 is 6-edge-connected and $d_{G_1}(v)$ is divisible by 5 for every $v \in A$, and G_2 is 6-edge-connected and $d_{G_2}(v)$ is divisible by 5 for every $v \in B$. Thus, by Corollary 3.5, G_i is the underlying graph of a directed graph $\vec{G}_i$ that admits a fractional factorization $\mathcal{F}_i = (M_i, F_i, H_i)$ and an $\mathcal{F}_i$-canonical decomposition $\mathcal{D}_i$, for $i = 1, 2$.

By definition, $\mathcal{D}_1$ is an M_1-complete decomposition of G_1 and $\mathcal{D}_2$ is an M_2-complete decomposition of G_2. Let $M = M_1 \cup M_2$ and $\mathcal{F} = (M, F_1 \cup F_2, H_1 \cup H_2)$. Then, $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is an M-complete $\mathcal{F}$-canonical decomposition of $\vec{G}$, where $\vec{G} = \vec{G}_1 \cup \vec{G}_2$. Note that, for every vertex v of $\vec{G}$, there is at least one edge of M pointing to v. Moreover, since an $\mathcal{F}$-canonical path is a copy of P_5, and an $\mathcal{F}$-canonical trail is a copy of T_5, we have that any $\mathcal{F}$-canonical decomposition of $\vec{G}$ is also a $\{P_5, T_5\}$-decomposition of $\vec{G}$. Therefore, $\mathcal{D}$ is an M-complete $\{P_5, T_5\}$-decomposition of $\vec{G}$.

Let $\mathcal{D}$ be an M-complete $\{P_5, T_5\}$-decomposition of $\vec{G}$ with minimum number of copies of T_5. If there is no copy of T_5 in $\mathcal{D}$, then $\mathcal{D}$ is a P_5-decomposition of $\vec{G}$ and the proof is complete. Therefore, we may suppose that there is at least one copy of T_5 in $\mathcal{D}$. In what follows, we aim for a contradiction.

Let $T = v_0v_1v_2v_3v_4v_5$ with $v_5 = v_1$ be a copy of T_5 in $\mathcal{D}$. Recall that there exists an edge uv_2 of M pointing to v_2. Let B_1 be the element of $\mathcal{D}$ that contains uv_2. Since $\mathcal{D}$ is M-complete, $d_{B_1}(u) = 1$. Therefore, we may suppose that $B_1 = b_0b_1b_2b_3b_4b_5$, where $b_1 = v_2$, and, possibly, $b_1 = b_5$.

We divide the proof in two cases, depending on whether v_1 belongs or not to $V(B_1)$.

Case 1: $v_1 \notin V(B_1)$.

Let $T' = v_0v_1v_4v_3v_2b_0$, $B_1' = v_1b_1b_2b_3b_4b_5$, and $\mathcal{D}' = \mathcal{D} - T - B_1 + T' + B_1'$. We claim that T' is a path, B_1' is of the same type of element as B_1 (i.e., the underlying graphs of B_1' and B_1 are isomorphic), and the edges of M in $A(T') \cup A(B_1')$ are inward in $\mathcal{D}'$. Thus $\mathcal{D}'$ is an M-complete decomposition with fewer copies of T_5 than $\mathcal{D}$, a contradiction.

First, let us prove that T' is a path. Note that $b_0 \neq v_0$ and $b_0 \neq v_4$, otherwise $b_0b_1v_1$ would induce a triangle in G, a contradiction. We also know that $b_0 \neq v_1$ and $b_0 \neq v_3$, since G has no parallel edges. Furthermore, $b_0 \neq v_2$, since G has no loops. Since $v_1 \notin V(B_1)$, if B_1 is a path, then B_1' is a path; and B_1' is a copy of T_5, otherwise.

It is left to prove that every directed edge in M is inward in $\mathcal{D}'$. We just need to prove this for the directed edges in $M \cap \big(A(T') \cup A(B_1')\big)$. Note that the only edges in $M \cap \big(A(T') \cup A(B_1')\big)$ are b_0v_2 and, possibly, v_0v_1 and b_5b_4. Since $d_{T'}(b_0) = 1$ and $d_{T'}(v_0) = 1$, the edges b_0b_1 and v_0v_1 are inward in $\mathcal{D}'$. If b_5b_4 is an edge of M, then B_1 is a path ending at b_5. Therefore, B_1' is a path ending at b_5, and b_5b_4 is inward in $\mathcal{D}'$.

Case 2: $v_1 \in V(B_1)$.

Consider a sequence $\mathcal{B} = B_1B_2 \dots B_{k-1}$ of elements of $\mathcal{D}$, where $b_1^1 = v_2$, $B_i = b_0^ib_1^ib_2^ib_3^ib_4^ib_5^i$ for $i \leq k-1$. We say that $\mathcal{B}$ is a *coupled sequence centered at* v_1 if the following properties hold (see Fig. 6).

(i) $b_0^ib_1^i \in M$, for $1 \leq i \leq k-1$;
(ii) $b_1^i = b_3^{i-1}$, for $2 \leq i \leq k-1$;
(iii) $b_4^i = v_1$, for $1 \leq i \leq k-1$.

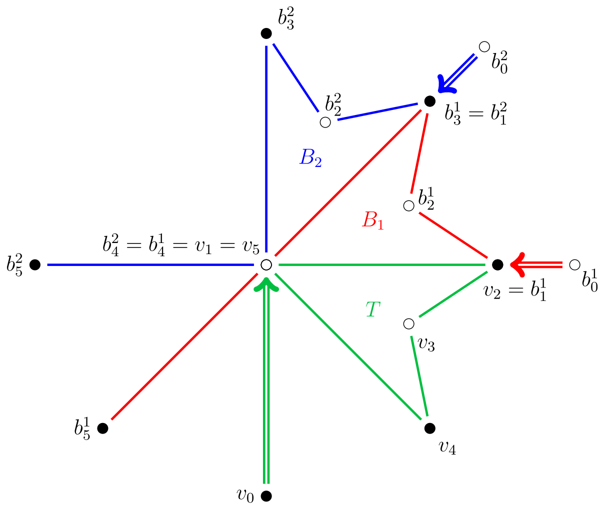

Fig. 6. Example of a trail $T = v_0v_1v_2v_3v_4v_5$ with $v_5 = v_1$, and a coupled sequence B_1, B_2 centered at v_1.

Note that, by hypothesis, v_1 is a vertex of B_1. Since G is a bipartite graph, $v_1 = b_4^1$. Therefore, B_1 is a coupled sequence centered at v_1 with only one element (that is, $k = 2$). Thus, we may suppose that there is a maximal coupled sequence $\mathcal{B}$ centered at v_1.

Claim 4.3. *B_i is a path of length 5, for $1 \le i \le k-1$.*

Proof. If for some $i \in \{1, \dots, k-1\}$, the element B_i is a copy of T_5, then $d_{B_i}(b_0^i) = 1$ and $b_5^i = b_1^i$, because (by item (i)) $b_0^ib_1^i$ is an edge of M and, since $\mathcal{D}$ is M-complete, $b_0^ib_1^i$ must be inward in $\mathcal{D}$. Since $v_1 \in V(B_i)$, we know that either $v_1 = b_2^i$ or $v_1 = b_4^i$, because G is bipartite. Note that the edge v_2v_1 is an edge of T. If $i = 1$, then $b_1^1v_1$ and v_2v_1 are parallel edges. If $i > 1$, then (by item (ii)) $b_3^{i-1}v_1 = b_1^iv_1$ must be an edge of B_{i-1} and of B_i, and $\mathcal{D}$ covers this edge twice. Therefore, for every $1 \le i \le k-1$, the element B_i is a copy of P_5. □

Claim 4.4. *$B_i \ne B_j$, for $1 \le i < j \le k-1$.*

Proof. Suppose, by contradiction, that $\mathcal{B}$ has repeated elements. Let B_i be the first element of $\mathcal{B}$ such that $B_i = B_j$ for some j with $i < j$. Since $b_0^ib_1^i \in M$ and $b_0^jb_1^j \in M$ (item (i)), and the elements of B belong to an M-complete decomposition, either $b_0^j = b_0^i$ or $b_0^j = b_5^i$. If $b_0^j = b_5^i$, then we know that $b_4^j = b_1^i = v_1$ (by item (iii)), from where we conclude that B_i contains the triangle $b_4^j\,b_3^j\,b_2^j\,b_4^j$, a contradiction. Therefore, assume that $b_0^j = b_0^i$. Note that $b_3^{j-1} = b_1^j = b_1^i$ (by item (ii)). Also, $i > 1$, otherwise $b_3^{j-1} = v_2$ and $b_3^{j-1}b_4^{j-1} = v_2v_1 \in E(B_{j-1})$, but $v_1v_2 \in E(T)$ and T and B_{j-1} are different, by the choice of i. Therefore, by item (iii), $b_4^{j-1} = b_4^{i-1} = v_1$, implying that $b_3^{i-1}b_4^{i-1} = b_3^{j-1}b_4^{j-1}$ and, then, $B_{i-1} = B_{j-1}$, a contradiction to the minimality of i. Therefore, $B_i \ne B_j$ for every $1 \le i < j \le k-1$. □

Recall that there is at least one edge e in M pointing to b_3^{k-1}. Let B_k be the element of $\mathcal{D}$ that contains e. We may suppose that $B_k = b_0^k\,b_1^k\,b_2^k\,b_3^k\,b_4^k\,b_5^k$, where $e = b_0^kb_1^k$. Note that $\mathcal{B}' = B_1B_2\cdots B_{k-1}B_k$ satisfies items (i) and (ii). Also, item (iii) holds for $1 \le i \le k-1$. Since $\mathcal{B}$ is maximal, $\mathcal{B}'$ is not a coupled sequence. Thus, item (iii) does not hold for $i = k$. Therefore, $b_4^k \ne v_1$.

Now consider the following elements:

- $T' = T - v_2v_1 + b_0^1b_1^1$.
- $B_1' = B_1 - b_0^1b_1^1 + v_2v_1 - b_3^1v_1 + b_0^2b_1^2$.
- $B_i' = B_i - b_0^ib_1^i + b_3^{i-1}v_1 - b_3^iv_1 + b_0^{i+1}b_1^{i+1}$, for $2 \le i \le k-1$.
- $B_k' = B_k - b_0^kb_1^k + b_3^{k-1}v_1$.

We claim that $T', B_1', \dots, B_{k-1}'$ are paths and B_k' is of the same type of element as B_k. The following arguments are very similar to the ones above, we present them for completeness.

To check that T' is a path, we prove that $b_0^1 \notin V(T) - v_0$. Note that $b_0^1 \ne v_0$ and $b_0^1 \ne v_4$, otherwise $b_0^1b_1^1v_1$ would induce a triangle in G. Also $b_0^1 \ne v_1$ and $b_0^1 \ne v_3$, because G has no parallel edges, and since G has no loops, $b_0^1 \ne v_2$. Therefore, T' is a path.

Let us check that B_i' is a path for $1 \le i \le k-1$. Since $V(B_i') = V(B_i) - b_0^i + b_0^{i+1}$, we just have to prove that $b_0^{i+1} \notin \{b_1^i, b_2^i, b_3^i, b_4^i, b_5^i\}$. If $b_0^{i+1} = b_1^i$, then $b_1^i\,b_2^i\,b_3^i\,b_0^{i+1}$ is a triangle in G. If $b_0^{i+1} = b_2^i$, then $b_3^ib_2^i$ and $b_1^{i+1}b_0^{i+1}$ are parallel edges.

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Since $b_3^i = b_1^{i+1}$ and $b_1^{i+1} \neq b_0^{i+1}$, we have $b_0^{i+1} \neq b_3^i$. If $b_0^{i+1} = b_4^i$, then $b_3^i b_4^i$ and $b_1^{i+1} b_0^i$ are parallel edges. If $b_0^{i+1} = b_5^i$, then $b_0^{i+1} b_3^i\, b_4^i\, b_5^i$ is a triangle in G. Therefore, $B_2', \ldots, B_{k-1}'$ are paths.

Now, let us prove that $v_1 \notin \{b_1^k, b_2^k, b_3^k, b_4^k, b_5^k\}$. Since $b_1^k = b_3^{k-1}$ and G is bipartite, we conclude that $v_1 \notin \{b_1^k, b_3^k, b_5^k\}$. Furthermore, since $b_1^k = b_3^{k-1}$ and $b_3^{k-1} v_1 \in E(G)$, we conclude that $b_2^k \neq v_1$. By the maximality of the sequence $\mathcal{B}$, we conclude that $b_4^k \neq v_1$. Thus, B_k' is a trail. If $b_5^k \neq b_1^k$, then B_k and B_k' are both paths of length five. If $b_5^k = b_1^k$, then B_k and B_k' are both copies of T_5. Therefore, B_k' is of the same type of element as B_k.

Let $\mathcal{D}' = \mathcal{D} - T - B_1 - \cdots - B_k + T' + B_1' + \cdots + B_k'$. Since the edges of M are $b_0^i b_1^i$ and, possibly $b_5^i b_4^i$, every edge of M is inward in $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an M-complete decomposition with fewer copies of T_5 than $\mathcal{D}$, a contradiction. □

5. Concluding remarks

The technique we have shown here (in Section 4) to disentangle elements of the canonical decomposition seems to be useful to deal with more general structures. Indeed, we have used it in [10] to show that Conjecture 1.1 holds for paths of any fixed length, and also in [12] to show a result that deals with P_ℓ-decompositions of regular graphs of prescribed girth. These results were obtained by combining ideas from this paper and a special result, which we named "Disentangling Lemma", that generalizes the ideas used in Section 4. In this paper, we use Lemma 3.2 to obtain a fractional factorization of a 4-edge-connected graph G, which is a decomposition consisting of three special elements. In [8], we extended Lemma 3.2 to obtain, for general length ℓ and an $(\ell-1)$-edge-connected graph, an analog of a fractional factorization with $\lceil \ell/2 \rceil$ elements. Refs. [8,9] contain complete proofs of the results mentioned in [10,12].

Recently, Merker [23] developed a stronger factorization technique with which he can prove that highly edge-connected graphs admit decompositions into graphs that can be obtained from (the given tree) T by identification of vertices. Using this technique, he was able to prove that Conjecture 1.1 holds for trees with diameter at most 4 [23] and for some trees with diameter at most 5, including P_5 [27]. Also, during the review period of this paper, Bensmail, Harutyunyan, Le, and Thomassé [5], using probabilistic arguments, gave an alternative proof of Conjecture 1.1 for paths of any length. Later, together with Merker, they extended their technique for every tree [4].

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