Jordan canonical form: an elementary proof

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1 Introduction

In this work we present, via linear system of differential equations, a natural and elementary proof for the Jordan canonical form. The idea of this work appeared during the preparation of a class on resolution of linear system of ordinary differential equations for a course that we taught in the second semester of 1994 in the Instituto de Matemática e Estatística da Universidade de São Paulo. We were considering a linear system of ordinary differential equations $\dot{\mathrm{x}} = A\mathrm{x}$, where A is an $n \times n$ complex matrix, x a column vector of coordinates $x_1, x_2, \ldots, x_n$ and $\dot{\mathrm{x}} = \frac{d\mathrm{x}}{dt}$, t real. It is well known (see [5]) that there is a variable change that transforms $\dot{\mathrm{x}} = A\mathrm{x}$ in a system in a triangular inferior form. Now, if the system is in triangular inferior form, how do we put it in the *Jordan canonical form*? This was the natural question that came to our mind. Thus we started to play around with some examples. Miracle! Very soon we felt that we had in hands a natural and elementary proof for the *Jordan canonical form*. The proof is constructive and we think that the idea of the proof is new. In the references appear other interesting proofs for the Jordan canonical form.

We dedicate this work to the responsibles for it: our students.

2 Definition of Jordan partial chain

Let A be an $n \times n$ complex matrix, triangular inferior and with a unique eingenvalue λ. Let us consider the system $\dot{\mathbf{x}} = A\mathbf{x}$, $A = (a_{ij})$. Suppose that there exist $p_1 < p_2 < \cdots < p_s$, $s \geq 1$, such that

$$a_{ip_s} = 0 \text{ for } p_s < i \leq n$$

and for $k = 2, 3, \ldots, s$, since $s > 1$,

$$a_{p_k p_{k-1}} = 1 \ ,$$
$$a_{p_k j} = 0 \text{ for } 1 \leq j < p_k \text{ and } j \neq p_{k-1} \ ,$$

and

$$a_{ip_{k-1}} = 0 \text{ for } p_{k-1} < i \leq n \text{ and } i \neq p_k.$$

In these conditions, we say that the set of the s equations of the system $\dot{\mathbf{x}} = A\mathbf{x}$:

$$\begin{array}{llll} \dot{x}_{p_1} = & \sum_{j=1}^{q} a_{p_1 j} x_j & +\lambda x_{p_1} & \\ \dot{x}_{p_2} = & & x_{p_1} & +\lambda x_{p_2} \\ \vdots & & & \\ \dot{x}_{p_s} = & & & x_{p_{s-1}} \quad +\lambda x_{p_s} \end{array}$$

where $q = p_1 - 1$, is a *Jordan partial chain of size s starting at* λx_{p_1}. If we have also $a_{p_1 j} = 0$ for $j = 1, 2, \ldots, q$, we say that the Jordan partial chain is a *Jordan chain starting at* λx_{p_1}.

Example 2.1 Consider the system

$$\begin{array}{llllllll} \dot{x}_1 = & \lambda x_1 & & & & & & \\ \dot{x}_2 = & 2x_1 & +\lambda x_2 & & & & & \\ \dot{x}_3 = & x_1 & +x_2 & +\lambda x_3 & & & & \\ \dot{x}_4 = & & & & \lambda x_4 & & & \\ \dot{x}_5 = & & & x_3 & & +\lambda x_5 & & \\ \dot{x}_6 = & & & & x_4 & & +\lambda x_6 & \\ \dot{x}_7 = & & & & & x_5 & & +\lambda x_7 \end{array}$$

We have a Jordan partial chain of size 3 starting at λx_3 and a Jordan chain of size 2 starting at λx_4.

3 Proof of the Jordan canonical form for a particular case

In this section we will prove the theorem on the Jordan canonical form for an $n \times n$ complex matrix A that admits a uniqur eingenvalue λ. For this we need three lemmas, whose proofs are immediate.

Lemma 3.1 *Consider the system* $\dot{\mathrm{x}} = A\mathrm{x}$, *where* $A = (a_{ij})$ *is triangular inferior, with a unique eingenvalue* λ *and given by*

$$
\begin{array}{lcllll}
\dot{x}_1 & = & \lambda x_1 & & & \\
\dot{x}_2 & = & a_{21}x_1 & +\ \lambda x_2 & & \\
\dot{x}_3 & = & a_{31}x_1 & +a_{32}x_2\ \ +\lambda x_3 & & \\
\vdots & & & & & \\
\dot{x}_{q_1} & = & \sum_{j=1}^{r} a_{q_1 j}x_j & +\lambda x_{q_1} & & \\
\dot{x}_{q_2} & = & \sum_{j=1}^{r} a_{q_2 j}x_j & & +\lambda x_{q_2} & \\
\vdots & & & & & \\
\dot{x}_{q_{k-1}} & = & \sum_{j=1}^{r} a_{q_{k-1} j}x_j & & +\lambda x_{q_{k-1}} & \\
\dot{x}_{q_k} & = & \sum_{j=1}^{r} a_{q_k j}x_j & +x_{q_1} & & +\lambda x_{q_k} \\
\vdots & & & & &
\end{array}
$$

where $r = q_1 - 1$, λ*is a complex number,* $q_m = q_{m-1} + 1$ *for* $m = 2, 3, \ldots, k$ *and* $a_{iq_1} = 0$ *for* $q_k < i \leq n$. *Then such a system can be transformed in a system where the* q_1*th and* q_k*th equations are, respectively,*

$$\dot{y}_{q_1} = \sum_{j=1}^{r} b_{q_1 j}x_j \ +\lambda y_{q_1}$$

and

$$\dot{x}_{q_k} = \qquad y_{q_1} \quad +\lambda x_{q_k}$$

and the other equations remain unchanged. The elements $b_{q_1 j}$, $j = 1, 2, \ldots, r$ *and* y_{q_1} *are given by*

$$b_{q_1 j} = a_{q_1 j} + \sum_{i=j+1}^{r} a_{ij}a_{q_k i}\ , \quad j = 1, 2, \ldots, r-1 \ , \quad b_{q_1 r} = a_{q_1 r}$$

and

$$y_{q_1} = \sum_{j=1}^{r} a_{q_k j}x_j + x_{q_1}\ .$$

Proof. It is enough to multiply the pth equation, $p = 1,2,\ldots,r$ by $a_{q_k p}$ to add with the q_1th and to make

$$y_{q_1} = \sum_{j=1}^{r} a_{q_k j} x_j + x_{q_1} \qquad \square$$

The main idea of this work is contained in the next lemma.

Lemma 3.2 *Let $A = (a_{ij})$ be an $n \times n$ complex matrix, in triangular inferior form that admits a unique eingenvalue λ. Suppose that the p_1th equation of the system $\dot{\mathbf{x}} = A\mathbf{x}$ has the form*

$$\dot{x}_{p_1} = \sum_{j=1}^{k} a_{p_1 j} x_j + \lambda x_{p_1}$$

and the q_1th has the form

$$\dot{x}_{q_1} = \sum_{j=1}^{k} a_{q_1 j} x_j + \lambda x_{q_1}$$

with $k < \min\{p_1, q_1\}$ and $a_{q_1 k} = 1$. Suppose also there is a Jordan partial chain of size r starting at λx_{p_1} and other of size s starting at λx_{q_1}, with $r \leq s$. Then the given system can be transformed in a system where the p_1th equation is

$$\dot{u}_{p_1} = \sum_{j=1}^{k-1} b_{p_1 j} x_j + \lambda u_{p_1}$$

where $b_{p_1 j} = a_{p_1 j} - a_{p_1 k} a_{q_1 j}$, $j = 1,2,\ldots,k-1$, the p_ith equation, $i = 2,3,\ldots,r$, is

$$\dot{u}_{p_i} = u_{p_{i-1}} + \lambda u_{p_i}$$

where $x_{p_i} - a_{p_1 k} x_{q_i} = u_{p_i}$, $i = 1,2,\ldots,r$, and the other equations remain unchanged.

Proof. Multiplying the q_1th equation by $-a_{p_1 k}$ adding with the p_1th and making $x_{p_1} - a_{p_1 k} x_{q_1} = u_{p_1}$ we have

$$\dot{u}_{p_1} = \sum_{j=1}^{k-1} b_{p_1 j} x_j + \lambda u_{p_1}$$

where $b_{p_1 j} = a_{p_1 j} - a_{p_1 k} a_{q_1 j}$, $j = 1,2,\ldots,k-1$.

If $r = 1$ the lemma is proved. Otherwise,

$$\dot{x}_{p_2} = u_{p_1} + a_{p_1 k} x_{q_1} + \lambda x_{p_2}\ .$$

Remembering that $s \geq r$, multiplying the q_2th equation by $-a_{p_1 k}$, adding with the p_2th and making $x_{p_2} - a_{p_1 k} x_{q_2} = u_{p_2}$ we get

$$\dot{u}_{p_2} = u_{p_1} + \lambda u_{p_2} .$$

If $r = 2$ the lemma is proved. Otherwise it is enough to repeat the process. □

Example 3.1 Consider the system

$$\begin{aligned}
\dot{x}_1 &= \lambda x_1 \\
\dot{x}_2 &= x_1 + \lambda x_2 \\
\dot{x}_3 &= 2x_1 + x_2 + \lambda x_3 \\
\dot{x}_4 &= x_1 + 2x_3 + \lambda x_4 \\
\dot{x}_5 &= 2x_1 + x_2 + x_3 + \lambda x_5 \\
\dot{x}_6 &= x_5 + \lambda x_6 \\
\dot{x}_7 &= x_4 + \lambda x_7 .
\end{aligned}$$

The Jordan partial chain starting at λx_4 has size 2 and the one starting at λx_5 has also size 2. So, applying lemma 3.2 we get

$$\begin{aligned}
\dot{x}_1 &= \lambda x_1 \\
\dot{x}_2 &= x_1 + \lambda x_2 \\
\dot{x}_3 &= 2x_1 + x_2 + \lambda x_3 \\
\dot{u}_4 &= -3x_1 - 2x_2 + \lambda u_4 \\
\dot{x}_5 &= 2x_1 + x_2 + x_3 + \lambda x_5 \\
\dot{x}_6 &= x_5 + \lambda x_6 \\
\dot{u}_7 &= u_4 + \lambda u_7 .
\end{aligned}$$

where $x_4 - 2x_5 = u_4$ and $x_7 - 2x_6 = u_7$. Now applying lemma 3.1 we obtain

$$\begin{aligned}
\dot{x}_1 &= \lambda x_1 \\
\dot{x}_2 &= x_1 + \lambda x_2 \\
\dot{u}_3 &= 3x_1 + x_2 + \lambda u_3 \\
\dot{u}_4 &= -3x_1 - 2x_2 + \lambda u_4 \\
\dot{x}_5 &= u_3 + \lambda x_5 \\
\dot{x}_6 &= x_5 + \lambda x_6 \\
\dot{u}_7 &= u_4 + \lambda u_7 .
\end{aligned}$$

where $2x_1 + x_2 + x_3 = u_3$.

The next lemma is suggested by the following example.

Example 3.2 Consider the system

$$\begin{aligned}
\dot{x}_1 &= \lambda x_1 \\
\dot{x}_2 &= \quad +\lambda x_2 \\
\dot{x}_3 &= \quad +\lambda x_3 \\
\dot{x}_4 &= \quad x_2 \quad +\lambda x_4 \\
\dot{x}_5 &= \quad x_4 \quad +\lambda x_5 \\
\dot{x}_6 &= \quad x_3 \quad +\lambda x_6 \,.
\end{aligned}$$

We have a Jordan chain of size 1 starting at λx_1, a Jordan chain of size 3 starting at λx_2 and a Jordan chain of size 2 starting at λx_3. The system can be rewritten in the following form

$$\begin{aligned}
\dot{x}_2 &= \lambda x_2 \\
\dot{x}_4 &= x_2 \quad +\lambda x_4 \\
\dot{x}_5 &= \quad x_4 \quad +\lambda x_5 \\
\dot{x}_3 &= \quad +\lambda x_3 \\
\dot{x}_6 &= \quad x_3 \quad +\lambda x_6 \\
\dot{x}_1 &= \quad \lambda x_1
\end{aligned}$$

which is the *Jordan canonical form* of the given system.

In the following, we indicate by $J_{\lambda,k}$ a square matrix of order n_k given by

$$J_{\lambda,k} = \lambda I + E$$

where I is the identity matrix of order n_k and $E = (e_{ij})$ is a square matrix of order n_k given by $e_{ij} = 1$ if $j = i - 1$ and $e_{ij} = 0$ if $j \neq i - 1$. So

$$J_{\lambda,k} = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 \\ \vdots & & & \cdot & & \\ 0 & 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$$

The matrix $J_{\lambda,k}$ is called a *Jordan block of order* n_k.

Lemma 3.3 *Let $A = (a_{ij})$ be an $n \times n$ complex matrix, in triangular inferior form, with a unique eingenvalue λ and let r be the number of linearly independent eingenvectors associated with λ, that is, $r = \dim\{v \in \mathbb{C}^n \mid Av = \lambda\}$. Suppose, if $r < n$, there are*

$j_1, j_2, \ldots, j_{n-r}$, *with* $j_\alpha \neq j_\beta$ *for* $\alpha \neq \beta$, $j_k \in \{1, 2, \ldots, n\}$ *and* $j_k < r+k$, $k = 1, 2, \ldots, n-r$, *such that*

$$a_{(r+k)j_k} = 1 \;, \quad k = 1, 2, \ldots, n-r,$$

and

$$a_{ij} = 0 \text{ for } j < i \text{ and } (i,j) \notin \{(r+k, j_k) | k = 1, 2, \ldots, n-r\} \,.$$

In these conditions, there exists an $n \times n$ *nonsingular matrix* P *such that* $P^{-1}AP = J_\lambda$ *where* $J_\lambda = \mathrm{diag}(J_{\lambda,1}, J_{\lambda,2}, \ldots, J_{\lambda,r})$, $\sum_{k=1}^{r} n_k = n$ *and* $n_1 \geq n_2 \geq \cdots \geq n_r$, *where* n_k, $k = 1, 2, \ldots, r$, *is the order of the matrix* $J_{\lambda,k}$.

Proof. If $r = n$ the lemma is immediate. So, let us suppose $1 \leq r < n$. It is immediate that we have a Jordan chain starting at λx_i, $i = 1, 2, \ldots, r$, and if $r < m \leq n$, there is $i \in \{1, 2, \ldots, r\}$ such that the mth equation belongs to the Jordan chain starting at λx_i. Now, let $i_k \in \{1, 2, \ldots, r\}$ and let n_k be the size of the Jordan chain starting at λx_{i_k}. It is clear that we can choose i_k, $k = 1, 2, \ldots, r$, such that $n_1 \geq n_2 \geq \cdots \geq n_r$. Now, interchanging conveniently the positions of the equations (and changing the names of the variables) our system can be rewritten in the following form:

$$\begin{aligned} \dot{\mathbf{y}}_1 &= J_{\lambda,1}\mathbf{y}_1 \\ \dot{\mathbf{y}}_2 &= \qquad J_{\lambda,2}\mathbf{y}_2 \\ &\vdots \\ \dot{\mathbf{y}}_r &= \qquad\qquad J_{\lambda,r}\mathbf{y}_r \end{aligned}$$

where $\mathbf{y}_k$, $k = 1, 2, \ldots, r$, is a column vector of coordinates $y_{k,1}, y_{k,2}, \ldots, y_{k,n_k}$. To close, let $\mathbf{y}$ be the column vector of coordinates $y_{1,1}, y_{1,2}, \ldots, y_{1,n_1}, y_{2,1}, \ldots, y_{2,n_2}, \ldots, y_{r,n_r}$. It is clear that the matrix P such that $\mathbf{x} = P\mathbf{y}$ is nonsingular. From $\dot{\mathbf{x}} = A\mathbf{x}$, it follows that $P\dot{\mathbf{y}} = AP\mathbf{y}$ and therefore $\dot{\mathbf{y}} = P^{-1}AP\mathbf{y}$. So $J_\lambda = P^{-1}AP$. □

In the following we enunciate and prove the theorem on the Jordan canonical form for a matrix A with a unique eingenvalue λ.

Theorem 3.1 (Jordan canonical form) *Let A be an $n \times n$ complex matrix with a unique eingenvalue λ. Then there exists an $n \times n$ nonsingular matrix M such that*

$$M^{-1}AM = J_\lambda$$

where $J_\lambda = \mathrm{diag}(J_{\lambda,1}, J_{\lambda,2}, \ldots, J_{\lambda,r})$, $r = \dim\{\mathrm{v} \in \mathbb{C}^n | A\mathrm{v} = \lambda\mathrm{v}\}$, $\sum_{k=1}^{r} n_k = n$ *and* $n_1 \geq n_2 \geq \cdots \geq n_r$.

Proof. Without lost of generality we can suppose that A is in triangular inferior form. If $r = n$ the theorem is immediate. So we can suppose $1 \leq r < n$. From lemma 3.3, it is sufficient to prove there exists an $n \times n$ nonsingular matrix Q such that $Q^{-1}AQ = B$ where $B = (b_{ij})$ is triangular inferior and given by: there are $j_1, j_2, \ldots, j_{n-r}$, with $j_\alpha \neq j_\beta$ for $\alpha \neq \beta$, $j_k \in \{1, 2, \ldots, n\}$ and $j_k < r + k$, $k = 1, 2, \ldots, n - r$, such that

$$b_{(r+k)j_k} = 1 \ , \quad k = 1, 2, \ldots, n - r$$

and

$$b_{ij} = 0 \text{ for } j < i \text{ and } (i, j) \notin \{(r + k, j_k) | k = 1, 2, \ldots, n - r\} \ .$$

Let

$$p = \max\{j \mid \text{there is } i, i > j, \text{ with } a_{ij} \neq 0\} \ .$$

(such a p exists because $r < n$). Without lost of generality we can suppose $a_{np} = 1$. The Jordan partial chain starting at λx_{p+k}, $k = 1, 2, \ldots, n - p$, has size 1. From lemmas 3.1 and 3.2, the system $\dot{\mathbf{x}} = A\mathbf{x}$ can be transformed in a system $\dot{\mathbf{y}} = S\mathbf{y}$, $S = (s_{ij})$, where

$$s_{np} = 1 \ ,$$
$$s_{nj} = 0 \text{ for } j \neq p \text{ and } j < n$$

and

$$s_{ij} = 0 \text{ for } p \leq j < i < n.$$

It is clear that the $n \times n$ matrix M_1 such that $\mathbf{x} = M_1\mathbf{y}$ is nonsingular. By induction, suppose now there is an $n \times n$ nonsingular matrix M_2 such that $M_2^{-1}AM_2 = D$, $D = (d_{ij})$, where D is given by: there are $j_1^*, j_2^*, \ldots, j_{n-q}^*$, with $q \geq r$, $j_\alpha^* \neq j_\beta^*$ for $\alpha \neq \beta$, $j_1^* = \min\{j_1^*, j_2^*, \ldots, j_{n-q}^*\}$, $j_k^* \in \{1, 2, \ldots, n\}$ and $j_k^* < q + k$ for $k = 1, 2, \ldots, n - q$ such that

$$d_{(q+k)j_k^*} = 1 \text{ for } k = 1, 2, \ldots, n-q,$$
$$d_{ij} = 0 \text{ for } i \geq q+1,\ j < i \text{ and } (i,j) \notin \{(q+k, j_k^*) | k = 1, 2, \ldots, n-q\}$$

and

$$d_{ij} = 0 \text{ for } j_1^* \leq j < i < q+1.$$

Observe that for $q = n-1$ such a matrix M_2 there exists: $M_2 = M_1$. Consider the system $\dot{\mathbf{u}} = D\mathbf{u}$. If $q = r$ the theorem is proved. So, let us suppose $q > r$. Let

$$h = \max\{j < j^* \mid \text{there is } i, j < i \leq q, \text{ with } d_{ij} \neq 0\}.$$

(Such an h exists because $q > r$.) There is $q_1 \in \{h + i | i = 1, 2, \ldots, q-h\}$ such that $d_{q_1 h} \neq 0$ and the size of the Jordan partial chain starting at λu_{q_1} is $\geq$ to the size of the Jordan partial chain starting at λu_{h+i}, $i = 1, 2, \ldots, q-h$, with $d_{(h+i)h} \neq 0$. Without lost of generality we can suppose $d_{q_1 h} = 1$. Now, interchanging the positions of the qth and q_1th equations and applying lemmas 3.1 and 3.2 our system $\dot{\mathbf{u}} = D\mathbf{u}$, where $\mathbf{x} = M_2\mathbf{u}$, is transformed in $\dot{\mathbf{z}} = H\mathbf{z}$, $H = (h_{ij})$, where H is given by: there are $\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_{n-m}$, with $m = q - 1 \geq r$, $\bar{j}_\alpha \neq \bar{j}_\beta$ for $\alpha \neq \beta$, $\bar{j}_1 = h$, $\bar{j}_1 = \min\{\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_{n-m}\}$, $\bar{j}_k \in \{1, 2, \ldots, n\}$ and $\bar{j}_k < m + k$ for $k = 1, 2, \ldots, n-m$ such that

$$h_{(m+k)\bar{j}_k} = 1 \text{ for } k = 1, 2, \ldots, n-m,$$

$$h_{ij} = 0 \text{ for } i \geq m+1 = q,\ j < i \text{ and } (i,j) \notin \{(m+k, \bar{j}_k) | k = 1, 2, \ldots, n-m\}$$

and

$$h_{ij} = 0 \text{ for } h \leq j < i < q.$$

It is clear that $\mathbf{u} = M_3\mathbf{z}$ where the $n \times n$ matrix M_3 is nonsingular. If $m > r$, it is enough to repeat the process. □

4 Jordan canonical form: the general case

In order to prove the general case we need three lemmas, whose proofs are immediate and will be omitted.

Lemma 4.1 *Let A, B and M be $n \times n$ complex matrices, with M nonsingular, such that $M^{-1}AM = B$. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{v}_1$ and $\mathbf{v}_2$ be column vectors of $\mathbb{C}^n$ such that $M\mathbf{e}_1 = \mathbf{v}_1$ and $M\mathbf{e}_2 = \mathbf{v}_2$. Suppose there is a complex number α such that $(B - \lambda I)\mathbf{e}_1 = \alpha \mathbf{e}_2$. Then, we have also $(A - \lambda I)\mathbf{v}_1 = \alpha \mathbf{v}_2$.*

Lemma 4.2 *Let J_λ be an $n \times n$ matrix given by $J_\lambda = \mathrm{diag}(J_{\lambda,1}, J_{\lambda,2}, \ldots, J_{\lambda,r})$. Let $\mathbf{e}_i$, $i = 1, 2, \ldots, n$ be the ith canonical column vector of $\mathbb{C}^n$, that is, $\mathbf{e}_i$ is the column vector of coordinates e_{ij}, $j = 1, 2, \ldots, n$, with $e_{ij} = 0$ for $i \neq j$ and $e_{ij} = 1$ for $i = j$. Then*

$$(J_\lambda - \lambda I)\mathbf{e}_i = \mathbf{e}_{i+1} \text{ for } i \notin \left\{ \sum_{q=1}^{k} n_q | k = 1, 2, \ldots, r \right\}$$

and

$$(J_\lambda - \lambda I)\mathbf{e}_i = 0 \text{ for } i \in \left\{ \sum_{q=1}^{k} n_q | k = 1, 2, \ldots, r \right\}$$

where n_q is the order of the matrix $J_{\lambda,q}$.

Lemma 4.3 *Let A be an $n \times n$ complex matrix with eingenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$. Let m_k, $k = 1, 2, \ldots, s$, denote the algebraic multiplicity of λ_k and for $k = 1, 2, \ldots, s$ let r_k be the dimension of $\{\mathbf{v} \in \mathbb{C}^n | A\mathbf{v} = \lambda_k \mathbf{v}\}$. Then, for each k, $k = 1, 2, \ldots, s$, there is an $n \times n$ nonsingular matrix M_k such that $M_k^{-1} A M_k = H_k$ with*

$$H_k = \begin{pmatrix} H_{nm} & O_{pq} \\ & J_{\lambda_k} \end{pmatrix}$$

where $m = n - m_k$, $p = n - m_k$, $q = m_k$, H_{nm} is a $n \times m$ complex matrix, O_{pq} a $p \times q$ matrix where all the elements are equal to zero and $J_{\lambda_k} = \mathrm{diag}(J_{\lambda_{k,1}}, J_{\lambda_{k,2}}, \ldots, J_{\lambda_{k,r_k}})$.

Now, let us consider the matrix H_k of the above lemma 4.3. Let $\mathbf{f}_{k,i}$ denote the canonical column vector $\mathbf{e}_{n-m_k+i}$, $k = 1, 2, \ldots, s$ and $i = 1, 2, \ldots, m_k$. From lemmas 4.2 and 4.3, it follows inmediately that

$$(H_k - \lambda_k I)\mathbf{f}_{k,i} = \mathbf{f}_{k,i+1} \text{ for } i \notin \left\{ \sum_{q=1}^{h} n_{k,q} | h = 1, 2, \ldots, r_k \right\}$$

and

$$(H_k - \lambda_k I)\mathbf{f}_{k,i} = 0 \text{ for } i \in \left\{\sum_{q=1}^{h} n_{k,q} | h = 1,2,\ldots,r_k\right\}$$

where $n_{k,q}$ is the order of the matrix $J_{\lambda_{k,q}}$. Now, let us make

$$M_k \mathbf{f}_{k,i} = \mathbf{v}_{k,i} ,$$

$k = 1,2,\ldots,s$ and $i = 1,2,\ldots,m_k$. From lemma 4.1 it follows that

$$A\mathbf{v}_{k,i} = \lambda_k \mathbf{v}_{k,i} + \mathbf{v}_{k,i+1} \text{ for } i \notin \left\{\sum_{q=1}^{h} n_{k,q} | h = 1,2,\ldots,r_k\right\} \tag{1}$$

and

$$A\mathbf{v}_{k,i} = \lambda_k \mathbf{v}_{k,i} \text{ for } i \in \left\{\sum_{q=1}^{h} n_{k,q} | h = 1,2,\ldots,r_k\right\} . \tag{2}$$

By induction, one prove easily that the n column vectors $\mathbf{v}_{k,i}$, $k = 1,2,\ldots,s$ and $i = 1,2,\ldots,m_k$, are linearly independent.

Now let us consider the system $\dot{\mathbf{x}} = A\mathbf{x}$ and let us make the variable change given by

$$\mathbf{x} = \sum_{k=1}^{s}\sum_{i=1}^{m_k} y_{k,i}\mathbf{v}_{k,i} .$$

We have

$$\dot{\mathbf{x}} = \sum_{k=1}^{s}\sum_{i=1}^{m_k} \dot{y}_{k,i}\mathbf{v}_{k,i} \quad \text{and} \quad A\mathbf{x} = \sum_{k=1}^{s}\sum_{i=1}^{m_k} y_{k,i}A\mathbf{v}_{k,i} .$$

From the relations (1) and (2) above, it follows immediately that

$$\dot{\mathbf{y}}_k = J_{\lambda_k}\mathbf{y}_k , \quad k = 1,2,\ldots,s,$$

where $\mathbf{y}_k$ is the column vector of coordinates $y_{k,1}$, $y_{k,2}$, ..., y_{k,m_k}. So $\dot{\mathbf{y}} = J\mathbf{y}$ where $J = \mathrm{diag}(J_{\lambda_1}, J_{\lambda_2}, \ldots, J_{\lambda_s})$ and $\mathbf{y}$ is the column vector of coordinates $y_{1,1}$, $y_{1,2}$, ..., y_{1,m_1}, ..., $y_{s,1}$, $y_{s,2}$, ..., y_{1,m_s}. Observe that the variable change

$$\mathbf{x} = \sum_{k=1}^{s}\sum_{i=1}^{m_k} y_{k,i}\mathbf{v}_{k,i}$$

can be rewritten in the following form

$$\mathbf{x} = M\mathbf{y}$$

where M is the $n \times n$ nonsingular matrix whose column vectors are $\mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \ldots, \mathbf{v}_{1,m_1}, \mathbf{v}_{2,1}, \mathbf{v}_{2,2}, \ldots, \mathbf{v}_{2,m_2}, \ldots, \mathbf{v}_{s,1}, \mathbf{v}_{s,2}, \ldots, \mathbf{v}_{s,m_s}$. So $J = M^{-1}AM$. Thus we have proved the following theorem.

Theorem 4.1 (Jordan canonical form: general case) *Let A be an $n \times n$ complex matrix with eingenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$. Let m_k and r_k, $k = 1, 2, \ldots, s$, be where m_k is the algebraic multiplicity of λ_k and $r_k = \dim\{\mathbf{v} \in \mathbb{C}^n | A\mathbf{v} = \lambda_k \mathbf{v}\}$. Then there is an $n \times n$ nonsingular matrix M such that $M^{-1}AM = J$ where $J =$* $\mathrm{diag}(J_{\lambda_1}, J_{\lambda_2}, \ldots, J_{\lambda_s})$ *and, for $k = 1, 2, \ldots, s$, J_{λ_k} is a square matrix of order m_k given by $J_{\lambda_k} = \mathrm{diag}(J_{\lambda_k,1}, J_{\lambda_k,2}, \ldots, J_{\lambda_k,r_k})$. We have also, for $k = 1, 2, \ldots, s$, $\sum_{q=1}^{r_k} n_{k,q} = m_k$ and $n_{k,1} \geq n_{k,2} \geq \cdots \geq n_{k,r_k}$ where $n_{k,q}$ is the order of the Jordan block $J_{\lambda_k,q}$.*

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