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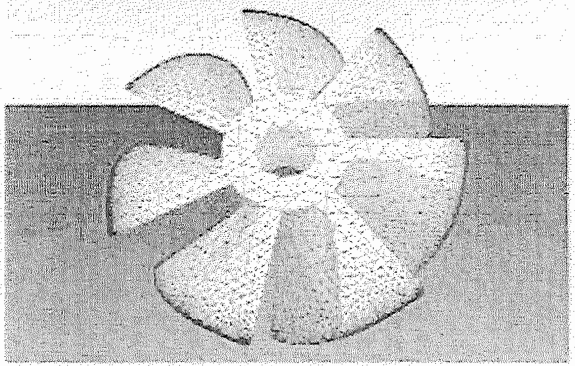

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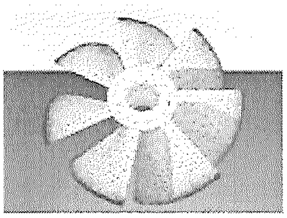

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Paper 124

An Alternative BEM Formulation for Pile/Layered Soil Interaction

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Abstract

In this article we propose to use the boundary element method (BEM) to analyze soil-foundation interactions. The soil structure is modeled as several dissimilar strata placed one on top of the other, composing a sandwich-like profile. Any of these layers may contain components of the foundations. Each region occupied by a soil layer or by a foundation component is handled as a 3D isotropic, elastic and homogeneous domain, and is analyzed by BEM.

As a consequence of the positioning of the various subregions in this model, the technique of successive rigidity can be applied directly, resulting in a considerable reduction in the volume of data being stored and manipulated throughout the analysis.

Results of two case-studies are reported, both to demonstrate that this technique is more efficient than the conventional subregion procedure.

Keywords: boundary element methods, heterogeneous soil, soil-foundation interaction, method of successive rigidity, soil-foundation-system, finite medium.

1 Introduction

In soil mechanics, the soil is generally taken to be a semi-infinite homogeneous, isotropic and linear-elastic continuum. However, the idea of the soil as a homogeneous layer supported at infinity on an immovable stratum is not a realistic representation of its structure; rather, an inhomogeneous medium with a rigid base at a finite depth would be nearer to what is found in nature. For this reason, several techniques have been proposed to model the finite soil, which are based, broadly speaking, on four different approaches.

In the first, known as Winkler's model, the continuum is replaced by a system of equivalent, discrete springs. When a pile is present, it is idealized as a beam element, with or without the possibility of axial deformation, supported by a series of discrete springs that represent the soil. The great advantage of this model is its simplicity and relative ease of implementation on the computer, while the most serious disadvantage is the difficulty of choosing the elastic moduli of the springs for a given combination of pile size and soil type. Normally, these constants are estimated empirically and the values chosen may lead to uncertain and inaccurate solutions. This line of research has been developed in the work of Cheung & Zienkiewicz [1], Randolph & Wroth [2], Witt [3], Lee [4] and Mylonakis & Gazetas [5].

The second approach commences with the application of equations, developed within elasticity theory, to a homogeneous continuum in each soil layer, after which equilibrium and compatibility conditions are applied at the contact surfaces between adjacent layers, to obtain expressions that represent the heterogeneous system. This technique has the disadvantage that the solutions to the equations can only be used in certain cases, as general solutions cannot be found; moreover, its handling of piles immersed in the soil is cumbersome.

Burmister [6, 7] is responsible for pioneering work in this area, in which integral transforms are used to obtain semi-analytical solutions for the displacements and stresses in a heterogeneous medium, without taking foundations into account. Poulos [8] uses Burmister's solutions to calculate influence factors for the cases of line, strip and sector loading. Chan *et al.* [9] and Davies & Banerjee [10] extend Burmister's procedure to the case of a vertical and/or horizontal force applied at a point inside a two-layer medium. Others working along the same lines are Gibson [11] and Ueshita & Meyerhof [12].

The third approach is known as the *finite layer method* (FLM), which can be used to reduce the 3D problem to one involving only two dimensions by combining the Fourier transform technique with the *finite element method* (FEM). The FLM, which can be simply and efficiently implemented on the computer, generates elastic solutions for the soil, whether homogeneous or not and isotropic or anisotropic. When piles are incorporated into the system, the method can still be used, by introducing equilibrium and compatibility conditions at the interfaces between the two media. A disadvantage of the FLM is that it only applies to elastic problems. For further information, refer to the seminal work of Small & Booker [13], Booker *et el.* [14], Lee & Small [15], Southcott & Small [16] and Ta & Small [17]. A variant of the FLM, known as the *infinite layer method*, has been proposed by Cheung *et al.* [18].

The fourth line of research makes use of two powerful numerical methods: FEM and the boundary element method (BEM). FEM is, generally speaking, the most versatile and powerful tool available to solve numerical problems in mechanics. However, when it is applied to infinite domains, such as the soil, the preparation of data is complicated and requirements for data storage and processor time are heavy, especially when the problem is formulated with 3D elements. This is because FEM needs a representation of the entire domain. Hence, few research workers have employed FEM to analyze the soil continuum, homogeneous or otherwise, but the work of Ottaviani [19] and Chow & Teh [20] should be mentioned. BEM has proved to be the most efficient and practical means of analyzing infinite-domain problems in elastostatics, owing to intrinsic features of the weighted functions that ensure the boundary conditions are satisfied at large distances. A number of researchers have employed Mindlin's solution [21] for the semi-infinite continuum and the simplified Steinbrenner model [8] for finite media. However, the use of Mindlin's fundamental solution in a BEM application, even when the depth of the immovable plane is included, is only valid for a single homogeneous medium. It is not possible to introduce other media with distinct physical properties because of the boundary conditions imposed *a priori* by the formulation of the solution. According to Poulos [22], under certain conditions of inhomogeneity of the soil and pile system, the displacements generated are inadequate.

Hence, BEM must be used to represent infinite-domain. The Kelvin solution (Love [23]) provides a basis for the most general formulae for an isotropic body of arbitrary shape. This solution is the one most often used to couple various regions with different properties, with the boundary surfaces satisfying equilibrium and compatibility conditions so as to maintain continuity throughout the heterogeneous medium. Thus, Banerjee [24], Banerjee & Davies [25] and Maier & Novati [26] use the fundamental Kelvin solution in BEM simulations of a finite heterogeneous medium. The latter authors [26] developed a new way of analyzing stratified soils by BEM, the *successive stiffness method* (SSM), but did not take the influence of piles into consideration.

Other ways of formulating the problem have been adopted which were not described above, as there are very few published accounts of the procedures employed. Examples are the models of Chin & Chow [27], Fraser & Wardle [28], Sadecka [29] and Pan [30]. In one particular line, some authors have endeavored to integrate Winkler's system of springs with continuous models. This is accomplished by inserting continuity between the discrete springs by means of structural elements, or by imposing some simplifications, calibrated with real displacements and stresses, on the continuous models. The following foundation models share this approach: Hetenyi's [31]; Pasternak's, described in Kerr [32] and Wang *et al.* [33]; generalized [32] and Kerr's [34].

The fact remains that modeling the finite heterogeneous soil medium represents a highly complex problem in solid mechanics, by virtue of the absence of any control over the formation of the soil, in contrast with manufactured components such as steel or reinforced concrete. The soil stratum was produced by the weathering of rocks under diverse conditions, which created a highly anisotropic and heterogeneous medium, chaotic in appearance with discontinuities visible throughout its bulk.

In light of these considerations, we proposed to develop a computer-based method with which to analyze the mechanics of heterogeneous soils, including the influence of piles, in such a way that both these structural elements would be modeled in 3D by BEM. The technique used to model the layered structure was the SSM of Maier & Novati [26], but with the effects of piles added. It should be stressed that the new formulation is not restricted to simple pile foundations, but can also be applied to piles with enlarged tip, shallow foundations, etc.

2 The boundary element method applied to problems of elastostatics

In the absence of volume forces, the Navier-Cauchy equations are given by:

$$u_{i,jj}(s)+\frac{1}{1-2\cdot\upsilon}\cdot u_{j,ji}(s)=0,\quad i,j=1,2,3 \tag{1}$$

where $u_i(s)$ is the displacement in the orthogonal direction i from the point s inside the solid and satisfies certain boundary conditions, and v is Poisson's ratio. These domain equations can further be expressed as surface equations, which are represented by the Somigliana Identity:

$$u_i(p)+\int_{\Gamma}p_{ij}^{*}(p,S)\cdot u_j(S)\partial\Gamma(S)=\int_{\Gamma}u_{ij}^{*}(p,S)\cdot p_j(S)\partial\Gamma(S) \tag{2}$$

where p and S are, respectively, the source point where a unit force is applied and a boundary point at the surface, u_i and p_i are, respectively, the real displacement field and surface forces at the boundary point S in the ith direction, while u_{ij}^{*} and p_{ij}^{*} represent weighted field coefficients which indicate the response, in the direction j at S, to a force applied in the direction i at the point p. This identity is based on Betti's reciprocal theorem and weighted or fundamental solutions given by u_{ij}^{*} and p_{ij}^{*} represent particular solutions of the partial differential Equations (1) for a given boundary condition.

The strategy to obtain the boundary integral equations involves transforming p, which is inside the body, to P on the boundary. Thus, Expression (2) can be written as follows:

$$C_{ij}(P)\cdot u_j(P)+\int_{\Gamma}p_{ij}^{*}(P,S)\cdot u_j(S)\cdot\partial\Gamma(S)=\int_{\Gamma}u_{ij}^{*}(P,S)\cdot p_j(S)\cdot\partial\Gamma(S) \tag{3}$$

where the integral in (3) is defined in the sense of the Cauchy principal value [35] and C_{ij} are coefficients that depend on the geometry of the problem [36]. The fundamental solutions used herein are the known Kelvin solutions presented in [23] for the three-dimensional case.

Since the analytical solutions of Expression (3) are not given in closed form, they have to be estimated numerically. Hence, the Boundary Element Method (BEM) is based on the assembly of a system of algebraic equations resulting from boundary integral equations, Equation (3), written in terms of nodal parameters that are approximated to the boundary values using shape functions. The integral equations of (3) are then written, without considering the domain forces, as:

$$C_{ij}(P)\cdot u_j(P)+\sum_{k=1}^{NE}|J|\cdot\int_{\Gamma}p_{ij}^{*}(P,S)\cdot\Psi(S)\partial\xi(S)\cdot(U_i)^k=\sum_{k=1}^{NE}|J|\cdot\int_{\Gamma}u_{ij}^{*}(P,S)\cdot\Psi(S)\partial\xi(S)\cdot(P_i)^k \tag{4}$$

where NE, ψ, J are, respectively, the number of boundary elements, the shape function and the Jacobian transformation. In this paper, we will adopt linear shape functions of the form $\Psi_i(\xi_1,\xi_2,\xi_3)=\xi_i$, where ξ_i are homogeneous coordinates defined for the flat triangular element [37] into which the surface is discretized.

The integrals proposed in (4) cannot, however, be solved analytically for any generic surface; hence the use of numerical techniques such as those given in [38] and [39]. In the present paper, the integral equations are calculated numerically by using a three-dimensional triangular quadrature integral [37].

It is thus possible to assemble the shape matrices of equation (4), which takes on the following form:

$$[H]\cdot\{U\}=[G]\cdot\{P\} \tag{5}$$

where the Dirichlet or Neumann boundary conditions for each problem must be satisfied at each nodal point.

3 Analysis of layered soil/foundations by the successive stiffness method (SSM)

3.1 Single soil layer equations

Expression (5) is here extended to a homogeneous, isotropic and linear solid. Inhomogeneous problems can be solved by considering a combination of problems of adjacent homogeneous domains while applying the necessary equilibrium and compatibility conditions at the interfaces of the domains.

Consider a layered soil with η layers (Figure 1). For any given layer i, it is possible to write the correlation between the influence matrices of Eq. (5) as:

$$[[H_t^i]\quad[H_b^i]\quad[H_s^i]]\cdot\begin{bmatrix}U_t^i\\U_b^i\\U_s^i\end{bmatrix}=[[G_t^i]\quad[G_b^i]\quad[G_s^i]]\cdot\begin{bmatrix}P_t^i\\P_b^i\\P_s^i\end{bmatrix} \tag{6}$$

where the subscripts t, b and s represent quantities pertaining to the upper (top), lower (bottom) and side boundaries and U and P are, respectively, nodal displacements and tractions of the top, bottom or side boundaries.

Expansion of Equation (6), gives:

$$\begin{bmatrix}[H_{tt}^i] & [H_{tb}^i] & [H_{ts}^i]\\ [H_{bt}^i] & [H_{bb}^i] & [H_{bs}^i]\\ [H_{st}^i] & [H_{sb}^i] & [H_{ss}^i]\end{bmatrix}\cdot\begin{Bmatrix}U_t^i\\U_b^i\\U_s^i\end{Bmatrix}=\begin{bmatrix}[G_{tt}^i] & [G_{tb}^i] & [G_{ts}^i]\\ [G_{bt}^i] & [G_{bb}^i] & [G_{bs}^i]\\ [G_{st}^i] & [G_{sb}^i] & [G_{ss}^i]\end{bmatrix}\cdot\begin{Bmatrix}P_t^i\\P_b^i\\P_s^i\end{Bmatrix} \tag{7}$$

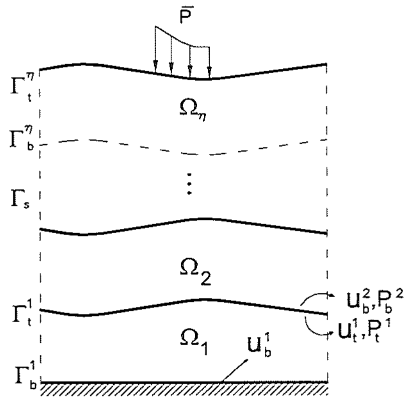

Figure 1: Layered soil subjected to forces on the surface, bottom, top and side

Equilibrium and compatibility conditions can then be imposed on displacements and stresses along the boundary between the *i*th and *(i+1)*th layers. For cases in which there are no relative movements between contact nodes, i.e., the case of ideal friction without forces being applied at the interface and with an undisturbed side boundary, the conditions can be expressed as:

$$\{u_t^i\} = \{u_b^{i+1}\} \tag{8.1}$$

$$\{p_t^i\} = -\{p_b^{i+1}\} \tag{8.2}$$

$$\{u_s\} = \{0\} \tag{8.3}$$

$$\{p_s\} = \{0\} \tag{8.4}$$

with i varying from 1 to $\eta\text{-}1$.

Assuming that the lateral surface is sufficiently remote, it is possible to use Equations (8.3) and (8.4) in (7) and obtain the influence of each layer, which is given by:

$$\begin{Bmatrix} P_t^i \\ P_b^i \end{Bmatrix} = \begin{bmatrix} [K_{tt}^i] & [K_{tb}^i] \\ [K_{bt}^i] & [K_{bb}^i] \end{bmatrix} \cdot \begin{Bmatrix} U_t^i \\ U_b^i \end{Bmatrix} \tag{9}$$

Hence, applying Equation (9) to a given layer i, and invoking Equations (8.1) and (8.2), this layer can easily be related to those above and below it.

In the lowest layer, where $i=1$, the displacement at the base is null, giving a fixed medium for which Expression (9) can be re-written as:

$$\{P_t^1\} = [K_{tt}^1] \cdot \{U_t^1\} \tag{10}$$

$$\{P_b^1\} = [K_{bt}^1] \cdot \{U_t^1\} \tag{11}$$

For $i=2$, we have, from expression (9):

$$\{P_t^2\} = [K_{tt}^2] \cdot \{U_t^2\} + [K_{tb}^2] \cdot \{U_b^2\} \tag{12}$$

$$\{P_b^2\} = [K_{bt}^2] \cdot \{U_t^2\} + [K_{bb}^2] \cdot \{U_b^2\} \tag{13}$$

Applying the conditions (8.1), (8.2) and (10) to Equation (13), gives:

$$\{U_b^2\} = -[(K_{tt}^1 + K_{bb}^2)^{-1} \cdot K_{bt}^2] \cdot \{U_t^2\} \tag{14}$$

and substitution of (14) in (12) leads to:

$$\{P_t^2\} = \left[[K_{tt}^2] - [K_{tb}^2] \cdot ([K_{tt}^1] + [K_{bb}^2])^{-1} \cdot [K_{bt}^2]\right] \cdot \{U_t^2\} \tag{15}$$

which may be written:

$$\{P_t^2\} = [\hat{K}^2] \cdot \{U_t^2\} \tag{16}$$

The above equation includes the influence of layers 1 and 2. Hence, applying equations (8.1), (8.2) and (9) to the layers i and $i+1$, we have:

$$\{P_t^i\} = \left[[K_{tt}^i] - [K_{tb}^i] \cdot ([\hat{K}^{i-1}] + [K_{bb}^i])^{-1} \cdot [K_{bt}^i]\right] \cdot \{U_t^i\} \tag{17}$$

Thus, for the topmost layer, where $i = \eta$:

$$\{P_t^\eta\} = [\hat{K}^\eta] \cdot \{U_t^\eta\} \tag{18}$$

where P_t^η and U_t^η are the nodal parameters at the soil surface.

At this point, the influence of the inhomogeneous soil is entirely expressed by Equation (18), which can be solved directly by applying the given loading conditions on the surface or by coupling the superstructure, using FEM or BEM.

3.2 Soil-foundation equations

When the influence of piles within a generic stratum has to be taken into account, the SSM can still be applied, but the unknown quantities located at points along the shaft are manipulated algebraically to be substituted by variables associated with the top and bottom of that stratum.

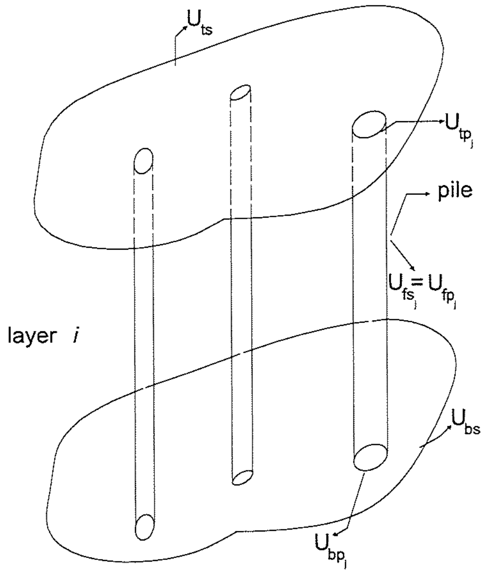

Figure 2: Generic stratum with three piles and their top (t), bottom (b) and shaft (f) variables

Thus, considering a general layer i (Figure 2) with m piles, equation (5) for a general pile j, in expanded form, becomes:

$$\begin{bmatrix} [H_{fp_jfp_j}] & [H_{fp_jtp_j}] & [H_{fp_jbp_j}] \\ [H_{tp_jfp_j}] & [H_{tp_jtp_j}] & [H_{tp_jbp_j}] \\ [H_{bp_jfp_j}] & [H_{bp_jtp_j}] & [H_{bp_jbp_j}] \end{bmatrix} \cdot \begin{Bmatrix} U_{fp_j} \\ U_{tp_j} \\ U_{bp_j} \end{Bmatrix} = \begin{bmatrix} [G_{fp_jfp_j}] & [G_{fp_jtp_j}] & [G_{fp_jbp_j}] \\ [G_{tp_jfp_j}] & [G_{tp_jtp_j}] & [G_{tp_jbp_j}] \\ [G_{bp_jfp_j}] & [G_{bp_jtp_j}] & [G_{bp_jbp_j}] \end{bmatrix} \cdot \begin{Bmatrix} P_{fp_j} \\ P_{tp_j} \\ P_{bp_j} \end{Bmatrix} \quad (19)$$

while the matrix equation for the layer of soil, including the influence of pile j, is:

$$\begin{bmatrix} [H_{tsts}] & [H_{tsbs}] & \cdot\cdot & [H_{tsfs_j}] & \cdot \\ [H_{bsts}] & [H_{bsbs}] & \cdot\cdot & [H_{bsfs_j}] & \cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot \\ [H_{fs_jts}] & [H_{fs_jbs}] & \cdot\cdot & [H_{fs_jfs_j}] & \cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot \end{bmatrix} \cdot \begin{Bmatrix} U_{ts} \\ U_{bs} \\ \cdot\cdot \\ U_{fs_j} \\ \cdot\cdot \end{Bmatrix} = \begin{bmatrix} [G_{tsts}] & [G_{tsbs}] & \cdot\cdot & [G_{tsfs_j}] & \cdot \\ [G_{bsts}] & [G_{bsbs}] & \cdot\cdot & [G_{bsfs_j}] & \cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot \\ [G_{fs_jts}] & [G_{fs_jbs}] & \cdot\cdot & [G_{fs_jfs_j}] & \cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot \end{bmatrix} \cdot \begin{Bmatrix} P_{ts} \\ P_{bs} \\ \cdot\cdot \\ P_{fs_j} \\ \cdot\cdot \end{Bmatrix} \quad (20)$$

Isolating the forces on the surface of the shaft and applying the conditions of displacement compatibility and force equilibrium at the interface between the soil and the shaft of pile j, we arrive at equation (21) for the coupling between the soil and piles in layer i:

$$
\begin{bmatrix}
H_{11} & H_{12} & \cdots & \cdots & H_{1(2j+1)} & H_{1(2j+2)} & \cdots & \cdots & H_{1(2m+3)} & \cdots & H_{1(3m+2)} \\
H_{21} & H_{22} & \cdots & \cdots & H_{2(2j+1)} & H_{2(2j+2)} & \cdots & \cdots & H_{2(2m+3)} & \cdots & H_{2(3m+2)} \\
 & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & [0] & & [0] & H_{(2j+1)(2j+1)} & H_{(2j+1)(2j+2)} & & [0] & [0] & H_{(2j+1)(2m+2+j)} & [0] \\
 & [0] & & [0] & H_{(2j+2)(2j+1)} & H_{(2j+2)(2j+2)} & & [0] & [0] & H_{(2j+2)(2m+2+j)} & [0] \\
 & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
H_{(2m+3)1} & H_{(2m+1)2} & H_{(2m+3)3} & H_{(2m+3)4} & \cdots & H_{(2m+3)(m+2)} & & \cdots & & & H_{(2m+3)(3m+2)} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
H_{(3m+2)1} & H_{(3m+2)2} & H_{(3m+2)3} & H_{(3m+3)4} & \cdots & H_{(3m+3)(m+2)} & & \cdots & & & H_{(3m+3)(3m+2)}
\end{bmatrix}
\bullet
\begin{Bmatrix} U_{ts} \\ U_{bs} \\ \vdots \\ U_{tp_j} \\ U_{bp_j} \\ \vdots \\ U_{fs_1} \\ \vdots \\ U_{fs_m} \end{Bmatrix}
=
\begin{bmatrix}
G_{11} & G_{12} & \cdots & \cdots & G_{1(2j+1)} & G_{1(2j+2)} & \cdots & \cdots \\
H_{21} & H_{22} & \cdots & \cdots & H_{2(2j+1)} & H_{2(2j+2)} & \cdots & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
 & [0] & & [0] & G_{(2j+1)(2j+1)} & G_{(2j+1)(2j+2)} & & [0] \\
 & [0] & & [0] & G_{(2j+2)(2j+1)} & G_{(2j+2)(2j+2)} & & [0] \\
 & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
G_{(2m+3)1} & G_{(2m+1)2} & G_{(2m+3)3} & G_{(2m+3)4} & \cdots & G_{(2m+3)(m+2)} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
G_{(3m+2)1} & G_{(3m+2)2} & G_{(3m+2)3} & G_{(3m+3)4} & \cdots & G_{(3m+3)(m+2)} & &
\end{bmatrix}
\bullet
\begin{Bmatrix} P_{ts} \\ P_{bs} \\ \vdots \\ P_{tp_j} \\ P_{bp_j} \\ \vdots \end{Bmatrix}
\tag{21}
$$

In (21), shaft variables have been eliminated, leaving only unknowns related to the top and bottom of both piles and soil. A relation for the general layer i, equivalent to equation (9), can thus be written:

$$
\begin{bmatrix}
K^i_{tsts} & K^i_{tsbs} & K^i_{tstp_1} & K^i_{tsbp_1} & \cdots\cdots & K^i_{tstp_m} & K^i_{tsbp_m} \\
K^i_{bsts} & K^i_{bsbs} & K^i_{bstp_1} & K^i_{bsbp_1} & \cdots\cdots & K^i_{bstp_m} & K^i_{bsbp_m} \\
K^i_{tp_1ts} & K^i_{tp_1bs} & K^i_{tp_1tp_1} & K^i_{tp_1bp_1} & \cdots\cdots & K^i_{tp_1tp_m} & K^i_{tp_1bp_m} \\
K^i_{bp_1ts} & K^i_{bp_1bs} & K^i_{bp_1tp_1} & K^i_{bp_1bp_1} & \cdots\cdots & K^i_{bp_1tp_m} & K^i_{bp_1bp_m} \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot\ \cdot\cdot & \cdot\cdot & \cdot\cdot \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot\ \cdot\cdot & \cdot\cdot & \cdot\cdot \\
K^i_{tp_mts} & K^i_{tp_mbs} & K^i_{tp_mtp_1} & K^i_{tp_mbp_1} & \cdots\cdots & K^i_{tp_mtp_m} & K^i_{tp_mbp_m} \\
K^i_{bp_mts} & K^i_{bp_mbs} & K^i_{bp_mtp_1} & K^i_{bp_mbp_1} & \cdots\cdots & K^i_{bp_mtp_m} & K^i_{bp_mbp_m}
\end{bmatrix}
\cdot
\begin{Bmatrix} U^i_{ts} \\ U^i_{bs} \\ U^i_{tp_1} \\ U^i_{bp_1} \\ \cdot \\ \cdot \\ U^i_{tp_m} \\ U^i_{bp_m} \end{Bmatrix}
=
\begin{Bmatrix} P^i_{ts} \\ P^i_{bs} \\ P^i_{tp_1} \\ P^i_{bp_1} \\ \cdot \\ \cdot \\ P^i_{tp_m} \\ P^i_{bp_m} \end{Bmatrix}
\tag{22}
$$

Hence, the *stiffness matrix* for layer i has been obtained, incorporating m piles. Adjacent layer boundary conditions of compatible displacements and force equilibrium can now be applied in a procedure analogous to that presented in section 3.1.

3.3.1 A priori calculation of the number of computer operations involved

SSM may appear more costly in terms of processing time than the conventional sub-region technique [37], because the former requires the inversion and multiplication of many matrices associated with each stratum. Moreover, when the stratum contains parts of the foundations, additional processing is needed to condense their respective displacement values into variables located at the top and bottom of the stratum.

To resolve this question, the cost of each stage involved in both these techniques can be computed, recalling that the number of floating-point operations used in the following procedures is:

$n^3 \rightarrow$ product of two square matrices of the identical dimensions $n \times n$;
$n \cdot m \cdot p \rightarrow$ product of two rectangular matrices of dimensions A(n,m) and B(m,p);
$\frac{4}{3} \cdot n^3 \rightarrow$ inversion of a matrix of dimension n;
$\frac{1}{3} \cdot n^3 \rightarrow$ solution of a system with very large n by Gaussian elimination;
$n \cdot B^2 - \frac{2}{3} \cdot B^3 \rightarrow$ solution of a system containing a matrix of half bandwidth B.
In what follows, the total dimension of the set of variables on a surface, $n = 3 \cdot N$, where N is the number of nodes on that surface, and η is the number of layers of soil.

Successive stiffness method (SSM)
i) $[G^i]^{-1}$: $O(i) = \frac{4}{3} \cdot (2n)^3 \cdot \eta = 288 \cdot N^3 \cdot \eta$

ii) $[G^i]^{-1} \cdot [H^i]$: $O(ii) = n^3 \cdot \eta = 216 \cdot N^3 \cdot \eta$

iii) Condense variables located on shaft: $O(iii) = 2 \cdot f(\alpha) \cdot [6 \cdot N + 3 \cdot N_\alpha]^3 \cdot \eta$

iv) Compute operations for soil-foundation equilibrium condition. Here, the numbers of operations used to find the product of rectangular matrices and to invert a matrix $[G_{fp_j fp_j}]$ are calculated for each element of the foundation, assuming an identical number *NE* of these elements within each stratum:
$O(iv) = 27 \cdot (N_\alpha)^2 \cdot NE \cdot \eta \cdot [10 \cdot N + 29{,}33 \cdot N_\alpha - {}^{3 \cdot N_\alpha}/_{NE}]$

v) $[K^i_{tsts} + K^{i+1}_{bsbs}]^{-1}$: $O(v) = \frac{4}{3} \cdot n^3 \cdot (\eta - 1) = 36 \cdot N^3 \cdot (\eta - 1)$

vi) $[K^2_{tb}] \cdot ([K^1_{tt}] + [K^2_{bb}])^{-1} \cdot [K^2_{bt}]$: $O(vi) = 2 \cdot n^3 \cdot (\eta - 1) = 2 \cdot (3 \cdot N)^3 \cdot (\eta - 1)$

vii) Solve the system: $O(vii) = \frac{1}{3} \cdot n^3 = 9 \cdot N^3$

Total number of operations:

Without pile elements $(\alpha = 0)$: $O(SSM) = N^3 \cdot \{594 \cdot \eta - 81\}$

With pile elements $(\alpha = \frac{1}{3})$: $O(SSM) = N^3 \cdot \{797 \cdot \eta + 59 \cdot \eta \cdot NE - 81\}$

Conventional sub-region technique (SR):
$B = 9 \cdot N + (3 \cdot N_\alpha) \cdot NE$

$n = 3 \cdot N \cdot (2 \cdot \eta) + 3 \cdot N_\alpha \cdot (2 \cdot \eta) \cdot NE$

Total number of operations:

Without pile elements ($\alpha = 0$) : $O(SR) = N^3 \cdot (486 \cdot \eta - 486)$

With pile elements ($\alpha = \frac{1}{3}$) : $O(SR) = 2 \cdot N^3 \cdot (9 + NE)^2 \cdot [(3 + NE) \cdot \eta - \frac{1}{3} \cdot (9 + NE)]$

Where $N_\alpha = \alpha \cdot N$, N_α being the number of nodes in each element of the foundation and α is given the value 1/3.

The numbers of operations involved in the analysis of a layered medium, with and without stiffening elements, by each of the techniques, are plotted in Figure 3. Note that in the absence of any foundation elements, the conventional zone method requires fewer operations than SSM. Nevertheless, if 2 such elements are included in the model, the method described here (SSM) becomes slightly more efficient than the conventional one, and as the number of elements rises this advantage becomes much more marked.

Finally, as well as the reduction in processing time achieved with SSM, another big advantage is the smaller storage space required for the final system matrix. This occurs because in the zone technique it is necessary to store in the final matrix explicit information about all the zones that exist in the problem. By contrast, the SSM procedure arrives at a final matrix that combines the influence of all the layers and thus represents the entire body of soil, which may or may not be stiffened by piles, and yet contains only quantities located at the free surface. Furthermore, this condensed influence matrix can easily be coupled to matrices representing the superstructure supported by the foundations.

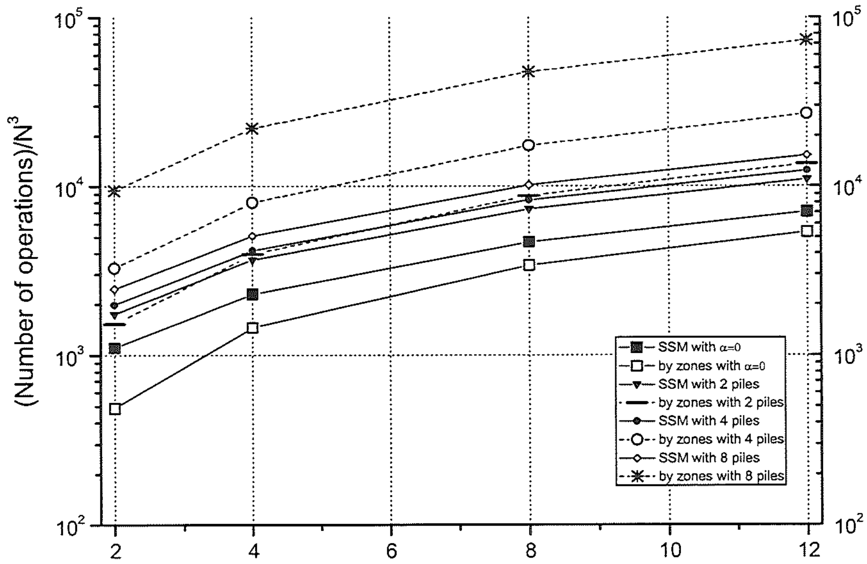

Figure 3: Number of computer operations, divided by N^3, (log scale) *vs* number of layers, using two techniques: SSM and zones method

4 Numerical examples

To implement the formulation introduced in the previous sections, computer code was written in Fortran90, using the IMSL libraries and routines from BLAS 1, BLAS 2 and BLAS 3. Some examples follow, to validate the technique.

4.1 Finite layer with linear variation of modulus

This example presents an analysis of the soil considered as a heterogeneous, linear isotropic medium. For the specific case of a uniformly distributed load applied to a circular region on the soil surface, the stiffness modulus is considered to increase linearly with depth (see Fig. 4). The results given in Table 1 demonstrate the robustness of the formulation, even when the ratio h/a is varied, as long as additional layers are included as this ratio increases, in order to represent more closely the linear variation of the soil's rigidity. The relative errors in the central surface displacement are obtained by comparing the values found by this formulation with those calculated by the semi-analytical expression of Burmister [6,7].

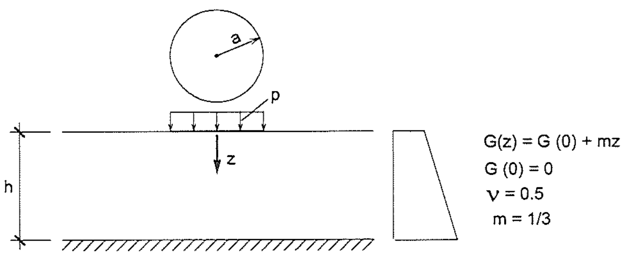

Figure 4: Finite stratum under uniform circular loading

Thickness factor h/a	Layers	Error (%)
2	2	13.36
2	3	2.20
2	5	1.54
8	3	52.96
8	7	19.44
8	20	0.71

Table 1: Percentage error of the central surface displacement

4.2 Single pile in a finite medium

This example is taken from Ottaviani [19], who used FEM with 3D elements to analyze the displacements and tensions resulting from contact between a pile and the surrounding soil. That author analyzed a prismatic region whose side boundary, presumed immovable, was 13m from the center of the pile, and employed 2700 brick-shaped elements, with 8 nodes each, in a mesh containing 3300 nodes in all.

The results from [19] will be compared with those obtained by the present method, evaluating in particular the effect of varying the depth between the rigid floor of the soil and the end of the pile. Also, the influence of the pile/soil stiffness ratio ($\lambda = E_{pile}/E_{soil}$) on the consistency of the methods will be considered.

The layout is shown in Fig.5. Figure 6 plots the values of maximum shear stress in the soil adjacent to the pile, given by $\tau_{max} = 0.5 \cdot (\sigma_1 - \sigma_3)$, at points down the shaft. The number of nodes used in the present work 571, with 820 plane elements, i.e. almost 6 times fewer nodes than those needed in [19]. Stresses have been converted into dimensionless units, by means of the expressions: $\sigma^0 = \frac{\sigma \cdot A}{P}$ and $\tau^0 = \frac{\tau \cdot A}{P}$, where A is the cross-sectional area of the pile.

In Figure 7, the displacements calculated by the two formulations are compared, for various combinations of thickness and stiffness ratio. The difference between the two sets of results stayed below 5% at points along the shaft, indicating the consistency of the present method. Figure 8 compares the estimates of displacements at points along the shaft for 3 cases: in case 1 and 2, the present formulation was used, with 346 and 571 nodes respectively, while in case 3 the commercial package ANSYS 5.4 was used, with 16,000 nodes.

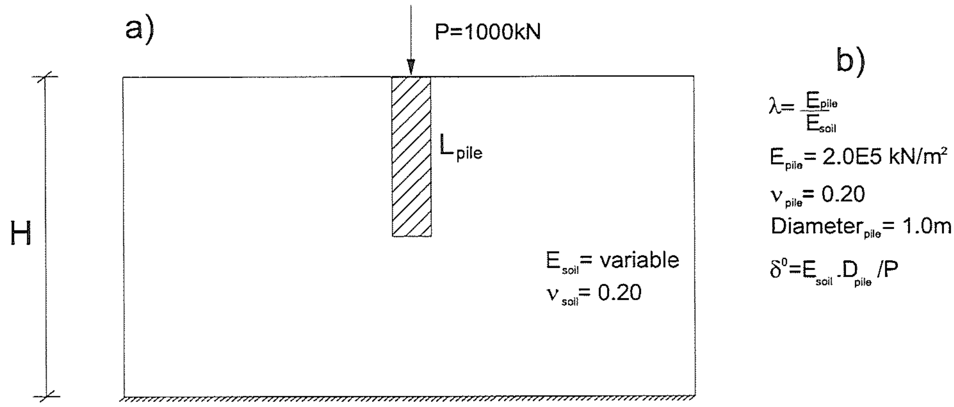

Figure 5: Material and geometric configuration for the soil-pile problem

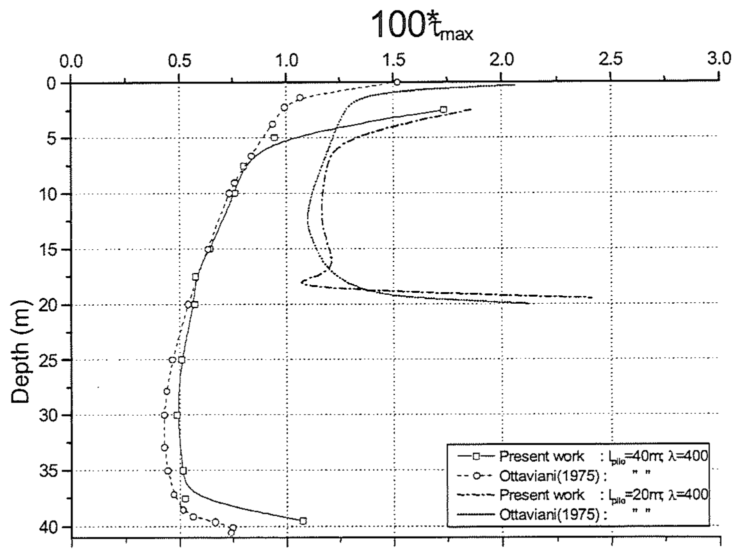

Figure 6: Maximum shear stress in soil adjacent to pile plotted against depth

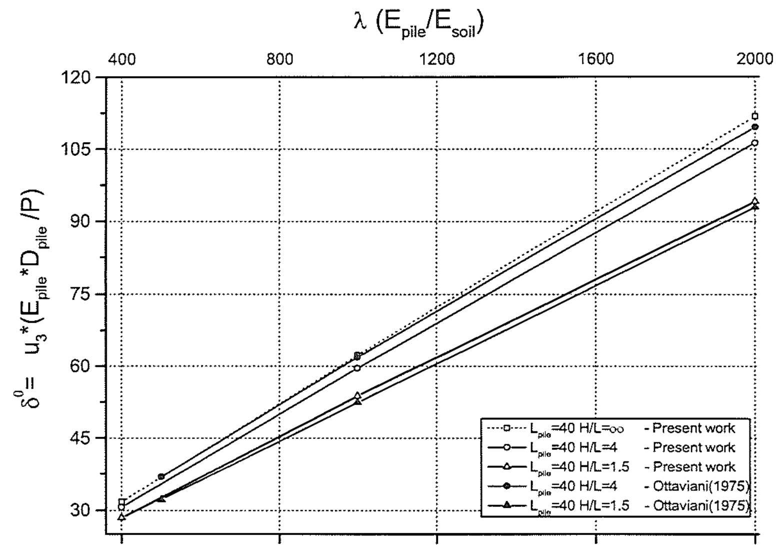

Figure 7: Settlement of pile plotted against $\lambda=E_{pile}/E_{soil}$

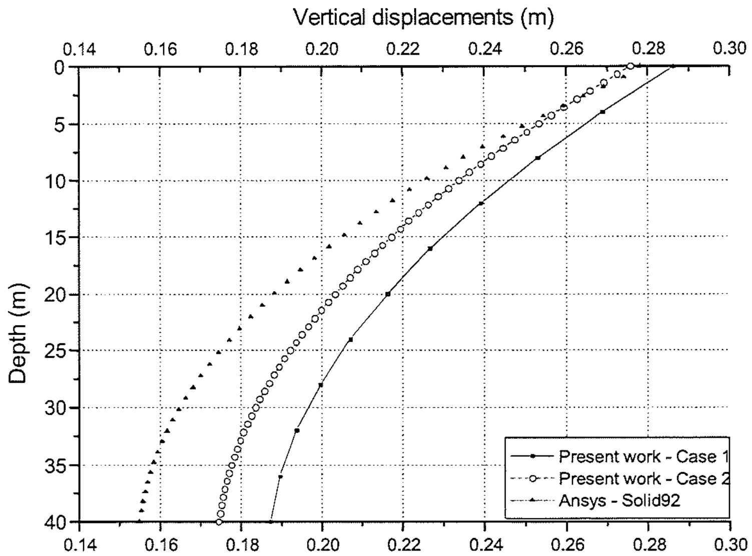

Figure 8: Depth against settlement of the single pile for 3 different configurations

5 Conclusion

An analysis was made of inhomogeneous soil-foundation interactions by a BEM-BEM combination. The method of successive stiffness proposed in [9] was extended to 3D problems, including the influence of piles that could be immersed in one (homogeneous) layer or pass across several layers of different properties.

In the method of successive stiffness, the influence of the several subdomains was incorporated into condensed variables, offering two computational advantages over the standard boundary element method by zones (subregions), namely: (1) fewer computational operations, as shown in the graph depicted in Figure 3, and (2) lower storage memory requirements for equations of the final soil system. It should also be pointed out that the influence of each stratum can be computed independently, so that this technique can be used in distributed-memory computers with the advantage of achieving high efficiency and loading balance naturally.

The results obtained show good agreement with those found in the literature, and the robustness of the responses remained unaltered even when very thin soil layers were considered. The implemented model provides an efficient numerical tool for solving real problems such as the soil-foundation-structure-system.

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References

[1] Cheung YK, Zienkiewicz OC. Plates and tanks on elastic foundations – an application of finite element method. Int. J. Solids Struct. 1965, 1: 451-461.

[2] Randolph MF, Wroth CP. An analysis of the vertical deformation of pile groups. Géotechnique 1979, 29: 423-439.

[3] Witt M. Solutions of plates on a heterogeneous elastic foundation. Computers & Structures 1984, 18: 41-45.

[4] Lee CY. Pile group settlement analysis by hybrid layer approach. Journal of Geotechnical Engineering 1993, 119: 984-997.

[5] Mylonakis G, Gazetas G. Settlement and additional internal forces of grouped piles in layered soil. Géotechnique 1988; 48 (1): 55-72.

[6] Burmister DM. The general theory of stresses and displacements in layered systems I. Journal of applied physics 1945; 16: 89-96.

[7] Burmister DM. The general theory of stresses and displacements in layered systems III. Journal of applied physics 1945; 16:, 296-302.

[8] Poulos HG. Stresses and displacements in an elastic layer underlain by rough rigid base. Géotechnique 1967; 17: 378-410.

[9] Chan KS, Karasudhi P, Lee SL. Force at a point in the interior of a layered elastic half-space. Int. J. Solids Structs 1974; 10: 1179-99.

[10] Davies TG, Banerjee PK. The displacement field due to a point load at the interface of a two layer elastic half-space. Géotechnique 1978; 28 (1): 43-56.

[11] Gibson RE. Some results concerning displacements and stresses in a nonhomogeneous elastic half-space. Géotechnique 1967; 17: 58-64.

[12] Ueshita K, Meyerhof GG. Surface displacement of an elastic layer under uniformly distributed loads. Highway Reasearch Board Record 1968, 228: 1-10.

[13] Small JC, Booker JR. Finite layer analysis of layered elastic materials using a flexibility approach. Part 1 – Strip loadings. Int. J. for Numerical Methods in Engineering 1984, 20: 1025-1037.

[14] Booker JR, Carter JP, Small JC. Some recent applications of numerical methods to geotechnical analysis. Computers & Structures 1989, 31 (1): 81-92.

[15] Lee CY, Small JC. Finite Layer Analysis of Laterally Loaded Piles in Cross-Anisotropic Soils. International Journal for Numerical Methods in Geomechanics 1991, 15: 785-808.

[16] Southcott PH, Small JC. Finite layer analysis of vertically loaded piles and pile groups. Computer and Geotechnics 1996, 18 (1): 47-63.

[17] Ta LD, Small JC. Analysis and performance of piled raft foundations on layered soils-case studies. Soil and Foundations 1998, 38 (4): 145-150.

[18] Cheung YK, Tham LG, Guo DJ. Analysis of pile group by infinite layer method. Géotechnique 1988, 38 (3): 415-431.

[19] Ottaviani M. Three-dimensional finite element analysis of vertically loaded pile groups. Géotechnique 1975, 25 (2): 159-174.

[20] Chow YK, Teh CI. Pile-cap-pile-group interaction in nonhomogeneous soil. Journal of Geotechnical Engineering 1991, 117 (11): 1655-1668.

[21] Mindlin RD. Force at a point in the interior of a semi-infinite solid. J. Physics 1936, 7: 195-202.
[22] Poulos HG. Settlement of single piles in nonhomogeneous soil. Journal of Geotechnical Engineering Division, ASCE 1979, 105 (5): 627-641.
[23] Love AEH. A Treatise on the mathematical theory of elasticity. New York: Dover Publ., 1944.
[24] Banerjee PK. Integral equation methods for analysis of piece-wise nonhomogeneous three-dimensional elastic solids of arbitrary shape. Int. J. Mechanical Science 1976; 18: 293-303.
[25] Banerjee PK, Davies TG. Analysis of pile groups embedded in Gibson soil. In: Int. Conf. Soil Mechs Fdn Engng., 9th, Tokyo. Proc. 1977, 1: 381-386.
[26] Maier G, Novati G. Boundary element elastic analysis by a sucessive stiffness method. Int. J. for Numerical and Anal. Methods in Geomechanics 1987; 11: 435-47.
[27] Chin JT, Chow YK. Numerical analysis of axially loaded vertical piles and pile groups. Computers and Geotechnics 1990, 9: 273-290.
[28] Fraser RA, Wardle LJ. Numerical analysis of rectangular rafts on layered foundations. Géotechnique 1976; 26: 613-30.
[29] Sadecka L. A finite/infinite element analysis of thick plate on a layered foundation. Comp. and Struct. 2000; 76: 603-10.
[30] Pan E. Static Green's functions in multilayered half spaces. Applied Math. Modelling 1997, 21: 509-521.
[31] Hetenyi MA. A general solution for the bending of beams on an elastic foundations of arbitrary continuity. Journal Applied Physics 1950, 21: 55-58.
[32] Kerr AD. Elastic and viscoelastic foundation models. Journal Applied Mechanic Transactions 1964, ASME, 31: 491-498.
[33] Wang CM, Iang YX, Wang Q. Axisymmetric buckling of reddy circular plates on Pasternak foundation. Journal Engineering Mechanic 2001, ASCE, 127: 254-259.
[34] Kerr AD. A study of a new foundation model. Acta Mechanic 1965, 1: 135-147.
[35] París F, Cañas J. Boundary Element Method: Fundamental and Applications. Oxford University Press, New York, 1997.
[36] Hartmann F. Computing the C- matrix in non-smooth boundary points. In: Brebbia CA, editor. New developments in boundary elements methods. Southampton, UK: CMP Publ., 1980.
[37] Brebbia CA, Dominguez J. Boundary elements: An introductory Course. Computational Mech. Publ. Southampton, Boston, 1989.
[38] Aliabadi MH, Hall WS, Phemister TG. Taylor expansions for singular kernels in the boundary element method. Int. J. Numer. Meth. Engng 1985; 21: 2221-36.
[39] Telles JCF. A self-adaptative co-ordinate transformation for efficient numerical evaluation of general boundary element integrals. Int. J. Numer. Meth. Engng 1987; 24; 959-73.