

RT-MAE 2008-04

*A MIXED HAMMERSTEIN
INTEGRAL EQUATION*

J. C. Miranda

Palavras-Chave: Existence and uniqueness of solution, integral equation, Volterra integral equation, Fredholm integral equation, Hammerstein integral equation, positive solutions, fixed point theorem.

Classificação : 45 GLO

A mixed Hammerstein integral equation

José Carlos Simon de Miranda
Institute of Mathematics and Statistics
University of São Paulo *
São Paulo , São Paulo, Brazil
email: *simon@ime.usp.br*

Abstract

A sufficient condition for the existence and uniqueness of a continuous solution of the integral equation

$$f(x) = G \left(x, h_1(x) + \int_D K_1(x, y) H_1(y, f(y)) dy, h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy \right)$$

is established under regularity conditions on the functions G , h_1 , h_2 , H_1 , H_2 , and on the kernels K_1 and K_2 where D is a subset of \mathbb{R}^n and $(-\infty, x]$, $x \in \mathbb{R}^n$, is a simplified notation for the interval $\prod_{i=1}^n (-\infty, x_i] \subset \mathbb{R}^n$.

Keywords: Existence and uniqueness of solution, integral equation, Volterra integral equation, Fredholm integral equation, Hammerstein integral equation, positive solutions, fixed point theorem.

1 Introduction

The non linear integral equation $f(x) = \int_a^b K(x, y) H(y, f(y)) dy$ has been studied by R.Iglish [1], A.Hammerstein, [2], M.Golomb, [3] and C.L.Dolph [4]. Under restrictive conditions on the kernel and controlling the non linearity of the function H , they succeeded in finding sufficient conditions either to existence and uniqueness of a solution or solely to the existence of solutions to this integral equation. Non linearities of the type $H(y, f(y)) = 1/y$ lead to singular integral equations. The integral equation $f(x) \int_0^1 f(y) K(x, y) dy = 1$ that arises in the theory of communication systems was studied in [5] by P. Nowosad where the existence and uniqueness of continuous positive real solutions was established for positive semi-definite symmetric non-negative kernels, $K(x, y)$, $0 \leq x, y \leq 1$, such that $\int_0^1 K(x, y) dy \geq \delta > 0$. An extension of this result was obtained by S. Karlin and L. Nirenberg in [6]. In their work they prove the existence of continuous positive solutions of the equation $f(x) \int_0^1 f(y)^\alpha K(x, y) dy = 1$ where α is a fixed positive parameter and $K(x, y)$ is a non-negative continuous function on $[0, 1]^2$ such that $K(x, x) > 0$ for all $x \in [0, 1]$. They also showed the uniqueness of continuous positive solutions in case the parameter α belongs to $(0, 1]$. The Schauder fixed point theorem was used to derive the existence of solutions. Further extensions of P. Nowosad's result can be found in [7] where positive solutions are established for the integral equation $f(x) = g(x) + \int_0^1 K(x, y) \left(\frac{1}{(f(y))^\alpha} + h(f(y)) \right) dy$, $\alpha > 0$, $x \in [0, 1]$.

An existence theorem of integrable solutions to the integral equation $f(x) = g(x) + \lambda \int_D K(x, y) H(y, f(y)) dy$ where $D \subset \mathbb{R}^n$ is a compact set and g, K , and H are functions with values in finite dimensional Banach spaces is obtained by G.Emmanuele in [8]. Conditions for the existence of nonzero solutions of integral equations of the form $f(x) = \int_D K(x, y) H(y, f(y)) dy$, D compact subset of \mathbb{R}^n , where K is a real valued function that changes sign and may be discontinuous, and H satisfies Caratheodory conditions, are presented by G.Infante and J.R.L.Webb in [9]. The existence of integrable solutions to the non linear integral equation of Hammerstein - Volterra type $f(x, t) = \int_0^1 K(x, y) H(y, f(y, t)) dy + \int_0^t F(t, z) f(x, z) dz$ is obtained by M.A.Abdou, W.G.El-Sayed, and E.I.Deeks in [10]. Also of interest are monotone solutions to integral equations. In [11], J.Banas, J. Caballero, and K. Sadaragani established the existence of nondecreasing continuous solutions on a bounded and closed interval I to the nonlinear integral equation of Volterra type $f(x) = a(x) + (Tf)(x) \int_0^x v(x, y, f(y)) dy$, $y \in I$, under a set of conditions on the functions a, v , and on the continuous operator $T: C(I) \rightarrow C(I)$. A similar result is presented by W.G.El-Sayed and B.Rzepka in [12] for the quadratic integral equation of Urysohn type with the form $f(x) = a(x) + H(x, f(x)) \int_0^1 u(x, y, f(y)) dy$, $y \in I$. Due to plenty of practical applications, numerical methods

*This work was partially supported by FAPESP grant 03/10105-2.

for solving integral equations are of great interest. Recently, S.Yousefi and M.Razzaghi, [13] and K.Maleknejad and H.Derili, [14] applied wavelet methods to obtain numerical solutions to Volterra - Fredholm and Hammerstein type integral equations.

In this short article we study the integral equation:

$$f(x) = G \left(x, h_1(x) + \int_D K_1(x, y)H_1(y, f(y))dy, h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y)H_2(y, f(y))dy \right)$$

where $f : D \subset \mathbb{R}^n \rightarrow \mathcal{A}$ is a function with values in a complete normed finite dimensional algebra \mathcal{A} , $K_i : D^2 \rightarrow \mathcal{A}$, $h_i : D \rightarrow \mathcal{A}$, $H_i : D \times \text{Im}(G) \rightarrow \mathcal{A}$, for $i \in \{1, 2\}$ and $G : D \times \mathcal{R}_1 \times \mathcal{R}_2 \rightarrow \mathcal{A}$, $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{A}$ satisfy regularity conditions. Throughout this work we denote the \mathbb{R}^n interval $\prod_{i=1}^n (a_i, b_i]$ by $(a, b]$ where $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$. Also, $a_i \wedge b_i$ will denote $\min\{a_i, b_i\}$ and $a_i \vee b_i$ equals $\max\{a_i, b_i\}$. For vectors this is done componentwise: $a \wedge b = (a_1 \wedge b_1, \dots, a_n \wedge b_n)$ and $a \vee b = (a_1 \vee b_1, \dots, a_n \vee b_n)$. It is not required neither that the algebra contains a unit element nor that, in case it has one, that its norm be one. These are common requirements to call \mathcal{A} a Banach algebra. Being $(\mathcal{A}, +, \cdot, \times, \|\cdot\|)$ an algebra over a field \mathcal{K} endowed with an absolute value $|\cdot|$, we suppress the notational use of \cdot and \times and use xy and αx instead of $x \times y$ and $\alpha \times x$ respectively. We assume that for all $x, y \in \mathcal{A}$, $\|xy\| \leq \|x\|\|y\|$. We use the notation $B[x, r]$ to mean the closed ball centered at x with radius r in a metric space.

In section 2 we present the main result: an existence and uniqueness of solution theorem based only on Banach's fixed point theorem. In section 3 some related results and corollaries are obtained, and, in section 4, we conclude this work with some examples and final remarks.

2 Main results

The following elementary lemma is used in the proof of Theorem 2.2.

Lemma 2.1 *Let $(\mathbb{R}^n, \Lambda, \ell)$ be \mathbb{R}^n endowed with Lebesgue measure and σ -algebra, $A \in \Lambda$, $f : A \rightarrow \mathbb{R}_+$ such that $\int_A f d\ell < \infty$ and $k \in \mathbb{N}$. Then, for all Lebesgue measurable sets $C \subset \bigcup_{i=1}^k C_i$ where, for all $i, 1 \leq i \leq k$, $C_i = H_i \times \{x_i + tn_i : 0 \leq t \leq \delta_i\}$, H_i a hyperplane, $x_i \in H_i$ and n_i one of its unitary normals, we have*

$$\int_C f d\ell \rightarrow 0 \quad \text{as} \quad \max\{\delta_i : 1 \leq i \leq k\} \rightarrow 0.$$

Proof : If A is bounded this is a direct consequence of the fact that, for $C \subset A$, we have $\lim_{\ell(C) \rightarrow 0} \int_C f d\ell = 0$ whenever $\int_A f d\ell < \infty$. In case A is not bounded, observe that for all $\epsilon > 0$ there exists $r > 0$ such that $\int_{\mathbb{R}^n \setminus [-r, r]^n} f d\ell < \epsilon/2$ and, taking into account that $\ell(C \cap [-r, r]^n) < k \max\{\delta_i : 1 \leq i \leq k\} (2r\sqrt{n})^{n-1}$, we guarantee that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\max\{\delta_i : 1 \leq i \leq k\} < \delta$ then $\int_{C \cap [-r, r]^n} f d\ell < \epsilon/2$, and, consequently, we conclude that

$$\forall \epsilon > 0 \exists \delta > 0 \quad \max\{\delta_i : 1 \leq i \leq k\} < \delta \implies \int_C f d\ell < \epsilon.$$

Our main result, an application of the fixed point theorem for contraction mappings, is the following:

Theorem 2.1 *Let D be a compact subset of \mathbb{R}^n , \mathcal{A} be a finite dimensional complete normed algebra, $K_i : D^2 \rightarrow \mathcal{A}$, $h_i : D \rightarrow \mathcal{A}$, $H_i : D \times \mathcal{A} \rightarrow \mathcal{A}$, for $i \in \{1, 2\}$, and $G : D \times \mathcal{A}^2 \rightarrow \mathcal{A}$ be functions such that:*

1. $\forall i \in \{1, 2\} \forall x \in D \quad \lim_{z \rightarrow x} \int_D \|K_i(z, y) - K_i(x, y)\| dy = 0.$
2. $\forall i \in \{1, 2\} \quad \|\int_D \|K_i(x, y)\| dy\|_\infty < \infty.$
3. $\forall i \in \{1, 2\} \quad \sup\{\|H_i(x, z)\| : x \in D, z \in \text{Im}(G)\} < \infty$
4. $\forall i \in \{1, 2\} \exists \iota_i \iota_i \geq 0 \forall z \in D \forall x, y \in \text{Im}(G) \quad \|H_i(z, x) - H_i(z, y)\| \leq \iota_i \|x - y\|.$
5. G is continuous in $D \times (\text{Im}(h_1) + B[0, \nu_1 \kappa_1]) \times (\text{Im}(h_2) + B[0, \nu_2 \kappa_2]).$
6. $\forall i \in \{1, 2\} \exists \mu_i \geq 0 \forall x \in D \forall y_i, z_i \in \text{Im}(h_i) + B[0, \nu_i \kappa_i], \quad \|G(x, y_1, z_1) - G(x, y_2, z_2)\| \leq \mu_1 \|y_1 - y_2\| + \mu_2 \|z_1 - z_2\|.$

7. $\forall i \in \{1, 2\}, h_i$ is continuous.

Denote, for $i \in \{1, 2\}$, $\|f_D\| \|K_i(x, y)\| dy\|_\infty$ by κ_i and $\sup\{\|H_i(x, z)\| : x \in D, z \in \text{Im}(G)\}$ by ν_i .

Then whenever $\mu_1 \kappa_1 \nu_1 + \mu_2 \kappa_2 \nu_2 < 1$ there exists one and only one continuous function $f : D \rightarrow \mathcal{A}$ such that

$$f(x) = G \left(x, h_1(x) + \int_D K_1(x, y) H_1(y, f(y)) dy, h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy \right).$$

Before the proof of this theorem is given, we observe that the functions $H_i : D \times \mathcal{A} \rightarrow \mathcal{A}$, for $i \in \{1, 2\}$, and $G : D \times \mathcal{A}^2 \rightarrow \mathcal{A}$ could be replaced by $H_i : D \times \text{Im}(G) \rightarrow \mathcal{A}$, for $i \in \{1, 2\}$, and $G : D \times (\text{Im}(h_1) + B[0, \nu_1 \kappa_1]) \times (\text{Im}(h_2) + B[0, \nu_2 \kappa_2]) \rightarrow \mathcal{A}$

Proof :

The possible continuous solutions to the integral equation belong to $S := C(D; \mathcal{A})$ which is a complete metric space with distance given by the supremum norm.

Now, for all $x \in D$, we have for every f

$$\begin{aligned} 0 &\leq \left\| \int_D K_1(x, y) H_1(y, f(y)) dy \right\| \leq \int_D \|K_1(x, y)\| \|H_1(y, f(y))\| dy \leq \\ &\int_D \|K_1(x, y)\| \sup\{\|H_1(s, t)\| : s \in D, t \in \text{Im}(G)\} dy = \nu_1 \int_D \|K_1(x, y)\| dy \leq \\ &\nu_1 \sup\left\{ \int_D \|K_1(x, y)\| dy : x \in D \right\} = \nu_1 \|K_1\|_\infty = \nu_1 \kappa_1. \end{aligned}$$

Analogously,

$$0 \leq \left\| \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy \right\| \leq \nu_2 \kappa_2.$$

Thus, for all $x \in D$,

$$h_1(x) + \int_D K_1(x, y) H_1(y, f(y)) dy \in (\text{Im}(h_1) + B[0, \nu_1 \kappa_1])$$

and

$$h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy \in (\text{Im}(h_2) + B[0, \nu_2 \kappa_2]).$$

Let, for all $f \in S$, $T(f) : D \rightarrow \mathcal{A}$ be given by

$$Tf(x) := (T(f))(x) = G \left(x, h_1(x) + \int_D K_1(x, y) H_1(y, f(y)) dy, h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy \right).$$

Note that $T(S) \subset S$ for if $f \in S$ we have

$$\begin{aligned} \|Tf(x) - Tf(z)\| &= \left\| G \left(x, h_1(x) + \int_D K_1(x, y) H_1(y, f(y)) dy, h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy \right) - \right. \\ &\quad \left. G \left(z, h_1(z) + \int_D K_1(z, y) H_1(y, f(y)) dy, h_2(z) + \int_{D \cap (-\infty, z]} K_2(z, y) H_2(y, f(y)) dy \right) \right\| \end{aligned}$$

Now, the continuity of G in $D \times (\text{Im}(h_1) + B[0, \nu_1 \kappa_1]) \times (\text{Im}(h_2) + B[0, \nu_2 \kappa_2])$ and that of h_1 and h_2 in D together with the inequalities and limits below

$$\begin{aligned} \left\| \left(\int_D K_1(x, y) H_1(y, f(y)) dy \right) - \left(\int_D K_1(z, y) H_1(y, f(y)) dy \right) \right\| &\leq \int_D \|H_1(y, f(y))\| \|K(x, y) - K(z, y)\| dy \leq \\ &\sup\{\|H_1(y, f(y))\| : y \in D\} \int_D \|K_1(x, y) - K_1(z, y)\| dy \leq \end{aligned}$$

$$\sup\{\|H_1(s, t)\| : s \in D, t \in Im(G)\} \int_D \|K_1(z, y) - K_1(x, y)\| dy =$$

$$\nu_1 \int_D \|K_1(z, y) - K_1(x, y)\| dy \rightarrow 0 \text{ as } z \rightarrow x$$

and

$$\left\| \left(\int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy \right) - \left(\int_{D \cap (-\infty, z]} K_2(z, y) H_2(y, f(y)) dy \right) \right\|$$

$$\leq \int_{D \cap (-\infty, x \wedge z]} \|H_2(y, f(y))\| \|K_2(z, y) - K_2(x, y)\| dy + \int_{D \cap (-\infty, x] \setminus D \cap (-\infty, x \wedge z]} \|H_2(y, f(y))\| \|K_2(x, y)\| dy +$$

$$\int_{D \cap (-\infty, z] \setminus D \cap (-\infty, x \wedge z]} \|H_2(y, f(y))\| \|K_2(z, y)\| dy \leq$$

$$\nu_2 \left(\int_D \|K_2(z, y) - K_2(x, y)\| dy + \int_{D \cap (-\infty, x] \setminus D \cap (-\infty, x \wedge z]} \|K_2(x, y)\| dy + \int_{D \cap (-\infty, z] \setminus D \cap (-\infty, x \wedge z]} \|K_2(z, y)\| dy \right) \rightarrow 0$$

as $z \rightarrow x$ guarantee that Tf is continuous.

Observe that the terms

$$\int_{D \cap (-\infty, x] \setminus D \cap (-\infty, x \wedge z]} \|K_2(x, y)\| dy \text{ and } \int_{D \cap (-\infty, z] \setminus D \cap (-\infty, x \wedge z]} \|K_2(z, y)\| dy$$

go to zero as $z \rightarrow x$ as a consequence of assumption 2 and the fact that the Lebesgue measure of the sets $D \cap (-\infty, x] \setminus D \cap (-\infty, x \wedge z]$ and $D \cap (-\infty, z] \setminus D \cap (-\infty, x \wedge z]$ are bounded above by $n\|x - z\| (\text{diam}(D))^{n-1}$, where $\text{diam}(D) < \infty$ stands for the diameter of the compact set D .

Now let us show that T is a contraction in S .

We have, for all $f, g \in S$

$$\|Tf - Tg\|_\infty = \|G \left(x, h_1(x) + \int_D K_1(x, y) H_1(y, f(y)) dy, h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy \right) -$$

$$G \left(x, h_1(x) + \int_D K_1(x, y) H_1(y, g(y)) dy, h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, g(y)) dy \right)\|_\infty \leq$$

$$\mu_1 \|h_1(x) - h_1(x) + \int_D K_1(x, y) H_1(y, f(y)) dy - \int_D K_1(x, y) H_1(y, g(y)) dy\|_\infty +$$

$$\mu_2 \|h_2(x) - h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy - \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, g(y)) dy\|_\infty \leq$$

$$\mu_1 \sup\{\|H_1(y, f(y)) - H_1(y, g(y))\| : y \in D\} \| \int_D \|K_1(x, y)\| dy \|_\infty +$$

$$\mu_2 \sup\{\|H_2(y, f(y)) - H_2(y, g(y))\| : y \in D \cap (-\infty, x]\} \| \int_{D \cap (-\infty, x]} \|K_2(x, y)\| dy \|_\infty \leq$$

$$\mu_1 \sup\{\iota_1 \|f(y) - g(y)\| : y \in D\} \| \int_D \|K_1(x, y)\| dy \|_\infty +$$

$$\mu_2 \sup\{\iota_2 \|f(y) - g(y)\| : y \in D \cap (-\infty, x]\} \| \int_{D \cap (-\infty, x]} \|K_2(x, y)\| dy \|_\infty \leq$$

$$\mu_1 \iota_1 \|f - g\|_\infty \| \int_D \|K_1(x, y)\| dy \|_\infty +$$

$$\mu_2 \iota_2 \|f - g\|_\infty \| \int_D \|K_2(x, y)\| dy \|_\infty$$

Thus

$$\|Tf - Tg\|_\infty \leq (\mu_1 \iota_1 \kappa_1 + \mu_2 \iota_2 \kappa_2) \|f - g\|_\infty,$$

a contraction whenever $\mu_1 \iota_1 \kappa_1 + \mu_2 \iota_2 \kappa_2 < 1$, and the theorem follows. \square

Theorem 2.2 Let D be a measurable subset of \mathbb{R}^n , \mathcal{A} be a finite dimensional complete normed algebra, $K_i : D^2 \rightarrow \mathcal{A}$, $h_i : D \rightarrow \mathcal{A}$, $H_i : D \times \mathcal{A} \rightarrow \mathcal{A}$, for $i \in \{1, 2\}$, and $G : D \times \mathcal{A}^2 \rightarrow \mathcal{A}$ be functions satisfying conditions 1 to 4, 6 and 7 in theorem 2.1. If G is continuous and bounded in $D \times (\text{Im}(h_1) + B[0, \nu_1 \kappa_1]) \times (\text{Im}(h_2) + B[0, \nu_2 \kappa_2])$, then whenever $\mu_1 \kappa_1 \nu_1 + \mu_2 \kappa_2 \nu_2 < 1$ there exists one and only one continuous function $f : D \rightarrow \mathcal{A}$ such that

$$f(x) = G \left(x, h_1(x) + \int_D K_1(x, y) H_1(y, f(y)) dy, h_2(x) + \int_{D \cap (-\infty, x]} K_2(x, y) H_2(y, f(y)) dy \right).$$

The same remark that follows theorem 2.1 is still applicable.

Proof : Follow the steps in theorem 2.1 proof. Observe that $S := B(D; \overline{\text{Im}(G)})$, the set of bounded functions from D to the closure of the image of G , is a Banach space and that $T(S) \subset S$. Use Lemma 2.1 to deal with the possibly unbounded sets $D \cap (-\infty, x] \setminus D \cap (-\infty, x \wedge z]$ and $D \cap (-\infty, z] \setminus D \cap (-\infty, x \wedge z]$.

3 Related results and corollaries

Theorem 3.1 Let $K_i : ([0, 1]^n)^2 \rightarrow \mathbb{R}_+$, $h_i : [0, 1]^n \rightarrow \mathbb{R}_+$, for $i \in \{1, 2\}$, be non-negative functions, β and δ be strictly positive real numbers. and $H_i : [0, 1]^n \times [0, \frac{1}{\beta}] \rightarrow \mathbb{R}_+$ for $i \in \{1, 2\}$ and $G : [0, 1]^n \times (\text{Im}(h_1) + [0, \nu_1 \kappa_1]) \times (\text{Im}(h_2) + [0, \nu_2 \kappa_2]) \rightarrow \mathbb{R}_+$ be functions such that:

1. $\forall i \in \{1, 2\} \forall x \in [0, 1]^n \lim_{z \rightarrow x} \int_{[0, 1]^n} |K_i(z, y) - K_i(x, y)| dy = 0$.
2. $\forall i \in \{1, 2\} \|\int_{[0, 1]^n} K_i(x, y) dy\|_\infty < \infty$.
3. $\forall i \in \{1, 2\} \sup\{H_i(x, z) : x \in [0, 1]^n, z \in [0, \frac{1}{\beta}]\} < \infty$.
4. $\forall i \in \{1, 2\} \exists \iota_i \geq 0 \forall z \in [0, 1]^n \forall x, y \in [\frac{1}{\beta}, \frac{1}{\delta}] |H_i(z, x) - H_i(z, y)| \leq \iota_i |x - y|$.
5. $0 < \delta = \inf\{G(x, y, z) : (x, y, z) \in [0, 1]^n \times (\text{Im}(h_1) + [0, \nu_1 \kappa_1]) \times (\text{Im}(h_2) + [0, \nu_2 \kappa_2])\}$ and $\beta = \sup\{G(x, y, z) : (x, y, z) \in [0, 1]^n \times (\text{Im}(h_1) + [0, \nu_1 \kappa_1]) \times (\text{Im}(h_2) + [0, \nu_2 \kappa_2])\} < \infty$.
6. $\forall i \in \{1, 2\} \exists \mu_i \geq 0 \forall x \in [0, 1]^n \forall y_i, z_i \in \text{Im}(h_i) + [0, \nu_i \kappa_i], \left| \frac{G(x, y_1, z_1) - G(x, y_2, z_2)}{G(x, y_1, z_1) G(x, y_2, z_2)} \right| \leq \mu_1 |y_1 - y_2| + \mu_2 |z_1 - z_2|$.
7. $\forall i \in \{1, 2\} h_i$ is continuous

Denote, for $i \in \{1, 2\}$, $\|\int_{[0, 1]^n} K_i(x, y) dy\|_\infty$ by κ_i and $\sup\{H_i(x, z) : x \in [0, 1]^n, z \in [0, \frac{1}{\beta}]\}$ by ν_i

Then whenever $\mu_1 \kappa_1 \nu_1 + \mu_2 \kappa_2 \nu_2 < 1$ there exists one and only one continuous function $f : [0, 1]^n \rightarrow \mathbb{R}$, such that

$$f(x) G \left(x, h_1(x) + \int_{[0, 1]^n} K_1(x, y) H_1(y, f(y)) dy, h_2(x) + \int_{[0, x]} K_2(x, y) H_2(y, f(y)) dy \right) = 1.$$

Clearly this function is strictly positive.

Proof :

Similar to that of theorem 2.1 . For all $x \in [0, 1]^n$ we have for every f

$$0 \leq \int_{[0, 1]^n} K_1(x, y) H_1(y, f(y)) dy \leq \int_{[0, 1]^n} \sup\{H_1(y, f(y)) : y \in [0, 1]^n\} K_1(x, y) dy \leq$$

$$\nu_1 \int_{[0, 1]^n} K_1(x, y) dy \leq \nu_1 \kappa_1.$$

Analogously,

$$0 \leq \int_{[0, x]} K_2(x, y) H_2(y, f(y)) dy \leq \nu_2 \kappa_2.$$

so that for all $x \in [0, 1]^n$ the solution of the integral equation satisfies

$$\frac{1}{\beta} \leq f(x) = \frac{1}{G\left(x, h_1(x) + \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy, h_2(x) + \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy\right)} \leq \frac{1}{\delta}.$$

Thus the possible continuous solutions to the integral equation belong to $S := C([0,1]^n; [\frac{1}{\beta}, \frac{1}{\delta}])$ which is a complete metric space with distance given by the supremum norm.

Let, for all $f \in C([0,1]^n; \mathbb{R}_+)$, $T(f) : [0,1] \rightarrow \mathbb{R}$ be given by

$$Tf(x) := (T(f))(x) = \frac{1}{G\left(x, h_1(x) + \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy, h_2(x) + \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy\right)}.$$

Note that $T(S) \subset S$ for $f \in S$ we have

$$0 \leq \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy \leq \nu_1 \kappa_1$$

and

$$0 \leq \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy \leq \nu_2 \kappa_2$$

from which

$$\frac{1}{\delta} \geq \left\| \frac{1}{G\left(x, h_1(x) + \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy, h_2(x) + \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy\right)} \right\|_{\infty} \geq \frac{1}{\beta}$$

and we get $\frac{1}{\beta} \leq \|Tf\|_{\infty} \leq \frac{1}{\delta}$.

Moreover, conditions 5 and 6 imply the continuity of $\frac{1}{G}$ on $[0,1]^n \times (Im(h_1) + [0, \nu_1 \kappa_1]) \times (Im(h_2) + [0, \nu_2 \kappa_2])$ and a similar argument to that in Theorem 2.1 proof shows that $|Tf(x) - Tf(z)| \rightarrow 0$ as $z \rightarrow x$; i.e. leads to the conclusion that Tf is continuous.

Observe that $\left| \frac{G(x,y_1,z_1) - G(x,y_2,z_2)}{G(x,y_1,z_1)G(x,y_2,z_2)} \right| = \left| \frac{1}{G(x,y_1,z_1)} - \frac{1}{G(x,y_2,z_2)} \right|$ so that condition 6 in this corollary is the same as condition 6 in Theorem 2.1 applied to $\frac{1}{G}$. Now follow the steps in Theorem 2.1 proof to show that T is a contraction in S . ■

We observe that the conditions on the kernels in theorems 2.1 and 3.1 hypothesis are implied by Kernel continuity. As a matter of fact, their continuity on the compact set D^2 or on $[0,1]^{2n}$ implies uniform continuity on D^2 or on $[0,1]^{2n}$ which, by its turn, implies condition 1; continuity on D^2 or on $[0,1]^{2n}$ also implies boundedness and integrability so that condition 2 is guaranteed.

Partial differentiability of H_i with respect to the second variable on $D \times Im(G)$ or on $[0,1]^n \times [\frac{1}{\beta}, \frac{1}{\delta}]$ and boundedness of this derivative on $D \times Im(G)$ or on $[0,1]^n \times [\frac{1}{\beta}, \frac{1}{\delta}]$ implies condition 4 as we can choose, by the mean value inequality, $u_i = \sup\{\|\partial_2 H_i(x,z)\| : (x,z) \in D \times Im(G)\}$ or $u_i = \sup\{\|\partial_2 H_i(x,z)\| : (x,z) \in [0,1]^n \times [\frac{1}{\beta}, \frac{1}{\delta}]\}$. Also continuous differentiability of H_i on $[0,1]^n \times [\frac{1}{\beta}, \frac{1}{\delta}]$ clearly implies 4.

Similarly, bounded partial differentiability with respect to the second and third variables of G on $D \times (Im(h_1) + B[0, \nu_1 \kappa_1]) \times (Im(h_2) + B[0, \nu_2 \kappa_2])$ or of $\frac{1}{G}$ on $[0,1]^n \times (Im(h_1) + [0, \nu_1 \kappa_1]) \times (Im(h_2) + [0, \nu_2 \kappa_2])$ or continuous differentiability of $\frac{1}{G}$ on $[0,1]^n \times (Im(h_1) + [0, \nu_1 \kappa_1]) \times (Im(h_2) + [0, \nu_2 \kappa_2])$ implies condition 6.

Clearly, the pure Hammerstein or Volterra - Hammerstein integral equations are particular cases of the complete mixed integral equation.

In this way one can write some corollaries, the weaker of them is:

Corollary 3.1 *Let $K : [0,1]^2 \rightarrow \mathbb{R}$ be a non-negative continuous function. Denote $\| \int_0^1 K(x,y)dy \|_{\infty}$ by κ and assume $\kappa < \infty$. Let also β and δ be strictly positive real numbers, and $H : [0, \frac{1}{\beta}] \rightarrow \mathbb{R}_+$ and $G : [0, \nu\kappa] \rightarrow \mathbb{R}_+$ be functions such that:*

1. $\nu = \sup\{H(z) : z \in [0, \frac{1}{\beta}]\} < \infty$.

2. H is continuously differentiable on $[\frac{1}{\beta}, \frac{1}{\delta}]$.

Denote $\iota = \sup\{|H'(z)| : z \in [\frac{1}{\beta}, \frac{1}{\delta}]\}$.

3. $0 < \delta = \inf\{G(z) : z \in [0, \nu\kappa]\}$ and $\beta = \sup\{G(z) : z \in [0, \nu\kappa]\} < \infty$.

4. G is continuously differentiable on $[0, \nu\kappa]$.

Denote $\mu = \sup\{|\left(\frac{1}{G(z)}\right)'| : z \in [0, \nu\kappa]\}$.

Then whenever $\mu\iota < 1$ there exists one and only one continuous function $f(x)$, $x \in [0, 1]$, such that

$$f(x) G\left(\int_0^1 K(x, y)H(f(y))dy\right) = 1.$$

Clearly this function is strictly positive.

Proof : Theorem 3.1 and remarks above. ■

4 Examples and Final Remarks

The following examples will show typical uses of the theorems and corollaries developed so far.

Concerning the integral equation

$$f(x) \exp\left(\int_0^1 f(y)^\gamma K(x, y)dy\right) = 1$$

we can obtain the following

Example 4.1 Let γ be a real positive number and $K : [0, 1]^2 \rightarrow \mathbb{R}$ be a non-negative continuous function such that

$$\| \int_0^1 K(x, y)dy \|_\infty = \kappa$$

where $\kappa < \frac{1}{\gamma}$ in case $\gamma \geq 1$ and $\kappa\gamma e^{(1-\gamma)\kappa} < 1$ in case $0 < \gamma < 1$.

Then there exists one and only one continuous solution $f(x)$, $x \in [0, 1]$, to the integral equation

$$f(x) \exp\left(\int_0^1 f(y)^\gamma K(x, y)dy\right) = 1.$$

This solution is strictly positive.

Proof :

Clearly, $\delta = \inf\{\exp(z) : z \in \mathbb{R}_+\} = \inf\{\exp(z) : z \in [0, a]\} = 1$, whatever $a > 0$ is, so that $\nu = \sup\{z^\gamma : z \in [0, \frac{1}{\delta}]\} = 1$ and $\beta = \sup\{\exp(z) : z \in [0, \nu\kappa]\} = e^\kappa$. Thus $\iota = \sup\{\gamma z^{\gamma-1} : z \in [\frac{1}{e^\kappa}, 1]\}$ which is equal to γ in case $\gamma \geq 1$ and to $\gamma e^{(1-\gamma)\kappa}$ in case $0 < \gamma < 1$. The exponential function is continuously differentiable and $\mu = \sup\{e^{-x} : x \in [0, \kappa]\} = 1$. Now apply Corollary 3.1. ■

The second example concerns integral equations for matrix valued functions.

Example 4.2 Consider the complete mixed Hammerstein integral equation on $M_{2 \times 2}(\mathbb{R})$ -valued functions of $[0, 1]^2$

$$f(x) = G\left(x, h_1(x) + \int_{[0,1]^2} K_1(x, y)H_1(y, f(y))dy, h_2(x) + \int_{[0,1]} K_2(x, y)H_2(y, f(y))dy\right)$$

where $\lambda_1, \lambda_2, \theta_1, \theta_2$ are real numbers,

$$K_1(x, y) = \exp(\lambda_1 \begin{pmatrix} x_1 & y_1 \\ y_2 & x_2 \end{pmatrix}), H_1(y, f(y)) = \frac{\theta_1 f(y)^2}{1 + \|f(y)\|^2} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, h_1(x) = \begin{pmatrix} 2 + x_1 & \frac{\sqrt{3}}{\sqrt{2}}x_1^2 \\ \frac{\sqrt{3}}{\sqrt{2}}x_2^2 & 2 - x_2 \end{pmatrix},$$

$$K_2(x, y) = \exp(\lambda_2 \begin{pmatrix} x_1^2 & y_1^2 \\ x_2^2 & y_2^2 \end{pmatrix}), \quad H_2(y, f(y)) = \frac{\theta_2 f'(y)^2}{1 + \|f'(y)\|^2} \begin{pmatrix} 0 & y_1 \\ y_2 & 0 \end{pmatrix}, \quad h_2(x) = \begin{pmatrix} x_1^2 & x_1 \\ x_1 & x_2^2 \end{pmatrix},$$

and

$G(x, y, z) = \|x\|^\alpha yz$, for positive α , and the norms of vectors and matrices are the euclidean ones. Then a sufficient condition for existence and uniqueness of a solution to this integral equation is

$$\sqrt{2} \left(R_2 e^{2|\lambda_1|} |\theta_1| + R_1 e^{2|\lambda_2|} |\theta_2| \right) \left(1 + \frac{R^2}{1 + R^2} \right) \left(\frac{R + (R \wedge (\sqrt{R^2 + 1} - R))}{1 + (R \wedge (\sqrt{R^2 + 1} - R))} \right) < 1$$

where $R_1 = (4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|})$, $R_2 = (2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|})$ and $R = R_1 R_2$.

Thus, the set of parameters $(\lambda_1, \lambda_2, \theta_1, \theta_2)$ for which the solution is unique contains an unbounded open neighbourhood of the origin.

Proof :

The algebra $M_{2 \times 2}(\mathbb{R})$ with usual operations and euclidean norm is complete and satisfies $\|xy\| \leq \|x\| \|y\|$. We have the following inequalities:

$$1. \forall x \in [0, 1]^2 \quad \|h_1(x)\| \leq 4, \quad \forall x \in [0, 1]^2 \quad \|h_2(x)\| \leq 2,$$

$$2. \|K_1(x, y)\| \leq \exp(|\lambda_1| \left\| \begin{pmatrix} x_1 & y_1 \\ y_1 & x_2 \end{pmatrix} \right\|), \text{ and}$$

$$\| \int_D \|K_1(x, y)\| dy \|_\infty \leq \sup_{x \in [0, 1]^2} \int_{[0, 1]^2} e^{|\lambda_1| \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}} dy \leq e^{2|\lambda_1|}$$

$$3. \|K_2(x, y)\| \leq \exp(|\lambda_2| \left\| \begin{pmatrix} x_1^2 & y_1^2 \\ y_1^2 & x_2^2 \end{pmatrix} \right\|), \text{ and}$$

$$\| \int_D \|K_2(x, y)\| dy \|_\infty \leq \sup_{x \in [0, 1]^2} \int_{[0, 1]^2} e^{|\lambda_2| \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}} dy \leq e^{2|\lambda_2|}$$

$$4. \|H_1(y, f(y))\| \leq |\theta_1| \frac{\|f(y)\|^2}{1 + \|f(y)\|^2} \sqrt{y_1^2 + y_2^2}, \quad \|H_2(y, f(y))\| \leq |\theta_2| \frac{\|f'(y)\|^2}{1 + \|f'(y)\|^2} \sqrt{y_1^2 + y_2^2} \text{ and}$$

$$\sup\{\|H_1(y, f(y))\| : y \in D, z \in \text{Im}(G)\} < \sup\{\|H_1(y, f(y))\| : y \in D, z \in \mathcal{A}\} \leq \sqrt{2}|\theta_1|$$

$$\sup\{\|H_2(y, f(y))\| : y \in D, z \in \text{Im}(G)\} \leq \sup\{\|H_2(y, f(y))\| : y \in D, z \in \mathcal{A}\} \leq \sqrt{2}|\theta_2|$$

Thus, $\kappa_1 \leq e^{2|\lambda_1|}$, $\kappa_2 \leq e^{2|\lambda_2|}$, $\nu_1 \leq \sqrt{2}|\theta_1|$, $\nu_2 < \sqrt{2}|\theta_2|$, and $\text{Im}(h_1) \subset B[0, 4]$ as well as $\text{Im}(h_2) \subset B[0, 2]$.

Now, $\|G(x, y_1, z_1) - G(x, y_2, z_2)\| = \|x\|^\alpha \|y_1 z_1 - y_2 z_2\| = \|x\|^\alpha \|y_1 z_1 - y_2 z_1 + y_2 z_1 - y_2 z_2\| \leq \|x\|^\alpha (\|y_1 - y_2\| \|z_1\| + \|y_2\| \|z_1 - z_2\|)$.

Thus, $\forall x \in D, \forall y_1, y_2 \in \text{Im}(h_1) + B[0, \nu_1 \kappa_1] \subset B[0, 4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|}] \forall z_1, z_2 \in \text{Im}(h_2) + B[0, \nu_2 \kappa_2] \subset B[0, 2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|}]$ we have

$$\|G(x, y_1, z_1) - G(x, y_2, z_2)\| \leq (2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|}) \|y_2 - y_1\| + (4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|}) \|z_2 - z_1\|$$

and $\mu_1 \leq 2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|}$ and $\mu_1 \leq 4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|}$,

$$G = \|x\|^\alpha yz \longrightarrow \text{Im}(G) \subset B[0, \sup\{\|x\|^\alpha yz : x \in D, y \in B[0, 4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|}], z \in B[0, 2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|}]\}] =$$

$$B[0, (4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|})(2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|})].$$

$$\|H_1(x, y) - H_1(x, z)\| = \left\| \frac{\theta_1 y^2}{1 + \|y\|^2} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} - \frac{\theta_1 z^2}{1 + \|z\|^2} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \right\| \leq \sqrt{2}|\theta_1| \frac{\|y^2(1 + \|z\|^2) - z^2(1 + \|y\|^2)\|}{(1 + \|z\|^2)(1 + \|y\|^2)} \leq$$

$$\sqrt{2}|\theta_1| \left(\left\| \frac{y^2 - z^2}{1 + \|y\|^2} \right\| + \left\| \frac{(\|z\|^2 - \|y\|^2)z^2}{(1 + \|z\|^2)(1 + \|y\|^2)} \right\| \right) \leq \sqrt{2}|\theta_1| \left(\frac{(\|y\| + \|z\|)(\|y - z\|)}{1 + \|y\|^2} + \frac{\|z\|^2(\|y\| + \|z\|)(\|y - z\|)}{(1 + \|z\|^2)(1 + \|y\|^2)} \right) =$$

$$\sqrt{2}|\theta_1| \left(1 + \frac{\|z\|^2}{1 + \|z\|^2} \right) \left(\frac{\|y\| + \|z\|}{1 + \|y\|^2} \right) \|y - z\|$$

Now, the maximization of $g(u, v) = \left(1 + \frac{u^2}{1+v^2} \right) \left(\frac{u+v}{1+v^2} \right)$ subjected to the constraint $(u, v) \in [0, R]^2$, for arbitrary R , furnishes $u = R$ and $v = R \wedge (\sqrt{R^2 + 1} - R)$ so that, letting $R \leftrightarrow (4 + \sqrt{2}|\theta_1|e^{2\lambda_1})(2 + \sqrt{2}|\theta_2|e^{2\lambda_2})$ we have

$$\epsilon_1 \leq \sqrt{2}|\theta_1| \left(1 + \frac{R^2}{1 + R^2} \right) \left(\frac{R + (R \wedge (\sqrt{R^2 + 1} - R))}{1 + (R \wedge (\sqrt{R^2 + 1} - R))^2} \right)$$

Analogously, since $z \in B[0, r] \longmapsto z^t \in B[0, r]$, we have

$$\|H_2(x, y) - H_2(x, z)\| \leq \sqrt{2}|\theta_2| \left(1 + \frac{\|(z^t)^2\|}{1 + \|z^t\|^2} \right) \left(\frac{\|y^t\| + \|z^t\|}{1 + \|y^t\|^2} \right) \|y^t - z^t\| \leq$$

$$\sqrt{2}|\theta_2| \left(1 + \frac{R^2}{1 + R^2} \right) \left(\frac{R + (R \wedge (\sqrt{R^2 + 1} - R))}{1 + (R \wedge (\sqrt{R^2 + 1} - R))^2} \right) \|y - z\|$$

and

$$\epsilon_2 \leq \sqrt{2}|\theta_2| \left(1 + \frac{R^2}{1 + R^2} \right) \left(\frac{R + (R \wedge (\sqrt{R^2 + 1} - R))}{1 + (R \wedge (\sqrt{R^2 + 1} - R))^2} \right).$$

In this way, by Theorem 2.1, existence and uniqueness of solution of the integral equation is implied by $\mu_1\kappa_1\epsilon_1 + \mu_2\kappa_2\epsilon_2 < 1$ and, consequently, by $s(\theta_1, \theta_2, \lambda_1, \lambda_2) :=$

$$\sqrt{2} \left((2 + \sqrt{2}|\theta_2|e^{2\lambda_2})e^{2\lambda_1}|\theta_1| + (4 + \sqrt{2}|\theta_1|e^{2\lambda_1})e^{2\lambda_2}|\theta_2| \right) \left(1 + \frac{R^2}{1 + R^2} \right) \left(\frac{R + (R \wedge (\sqrt{R^2 + 1} - R))}{1 + (R \wedge (\sqrt{R^2 + 1} - R))^2} \right) < 1$$

Now, observe that s is a continuous function and $s^{-1}((0, 1)) \supset \{(0, 0)\} \times \mathbb{R}^2$.

Finally, we remark that one can consider the situation where either K_i or H_i takes values in the field instead of in the algebra and obtain variants of the theorems presented thus far.

Acknowledgements

The author thanks our Lord and Saviour Jesus Christ.

References

- [1] R. Iglisch, Reelle Lösungsfelder der Elliptischen Differentialgleichung $\Delta u = F(u)$ und nichtlinearer Integralgleichungen, *Math. Ann.* ,**101** 1929, 98,119.
- [2] A. Hammerstein, Nichtlineare Integralgleichungen nebst Anwendungen, *Acta Mathematica* ,**54** 1930, 117,176.
- [3] M. Golomb, Zur theorie der nichtlinearen Integralgleichungen, Integralgleichungssysteme und allgemeinen Funktionalgleichungen, *Math Zeitschrift* ,**39**1935, 45,75.
- [4] C. L. Dolph, Non-Linear Integral Equations of the Hammerstein Type, *Proc. N.A.S* ,**31** 1945, 60,65.
- [5] P. Nowosad, On the Integral Equation $\kappa f = 1/f$ Arising in a Problem in Communication, *Journal of Mathematical Analysis and Applications*,**14** 1966, 484,492.
- [6] S. Karlin, L. Nirenberg, On a Theorem of P. Nowosad, *Journal of Mathematical Analysis and Applications*,**17** 1967, 61,67.
- [7] R. P. Agarwal, D. O'Regan, A Note on a Singular Integral Equation Arising in a Problem in Communications, *Zeitschrift für angewandte Mathematik und Mechanik* . **81** 2001, 499,504.
- [8] C. Emmanuele, An Existence Theorem for Hammerstein Integral Equations, *Portugaliae Mathematica* , vol. **51** Fasc.**4** 1994, 607,611.

- [9] G. Infante, J.R.L Webb, Nonzero Solutions of Hammerstein Integral equations with Discontinuous Kernels, *Journal of Mathematical Analysis and Applications* , **272** 2002, 30,42.
- [10] M. A. Abdou, W. G. El-Sayed, E. I. Deeb, A solution of a nonlinear integral equation, *Applied Mathematics and Computation* , **160** 2005, 1,14.
- [11] J. Banas , J. Caballero, J. Rocha, K. Sadaragani, Monotonic Solutions of a Class of Quadratic Integral Equations of Volterra Type, *Computers and Mathematics with Applications* , **49** 2005, 943,952.
- [12] W. G. El-Sayed, B. Rzepka , Nondecreasing Solutions of a Quadratic Integral Equation of Urysohn Type . *Computers and Mathematics with Applications* , **51** 2006, 1065,1074.
- [13] S. Yousefi, M. Razzaghi, Legendre wavelets method for nonlinear Volterra - Fredholm Integral Equations, *Mathematics and Computers in Simulation* , **70** 2005, 1,8.
- [14] K. Maleknejad, H. Derili, The Collocation Method for Hammerstein Equations by Daubechies Wavelets, *Applied Mathematics and Computation* , **172** 2006, 846,864.

ÚLTIMOS RELATÓRIOS TÉCNICOS PUBLICADOS

2008-01 - CABRAL, C.R.B.; BOLFARINE, H. Bayesian Inference for the Skew Student-t-Normal Model. 2008. 13p. (RT-MAE-2008-01)

2008-02 - CABRAL, C.R.B.; BOLFARINE, H. Bayesian Density Estimation Using Skew Student-t-Normal Mixtures. 2008. 18p. (RT-MAE-2008-02)

2008-03 - FARIAS, R.B.A.; ARENAS, G.M.; PATRIOTA, A.G. Redução de modelos na presença de parâmetros de perturbação. 2008. 19p. (RT-MAE-2008-03)

The complete list of "Relatórios do Departamento de Estatística", IME-USP, will be sent upon request.

*Departamento de Estatística
IME-USP
Caixa Postal 66.281
05311-970 - São Paulo, Brasil*