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A SERIES REPRESENTATION OF A COHERENT SYSTEM

by

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A series representation of a coherent system

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Abstract Willing to work in reliability theory under dependence conditions, we intend to represent a coherent system as a series system through its components compensator transform .

Keywords: Martingale methods in reliability theory, compensator process, coherent systems, series systems.

1.Introduction

In reliability theory the lifetime T of a coherent system is related with its components lifetimes through a structure function Φ, (see Barlow and Proschan (1981)). If the components lifetimes are denoted by T_i, $i = 1, ..., n$ the lifetime of a series system is represented by $T = \min_{1 \leq i \leq n} T_i$. The parallel system lifetime is given by $T = \max_{1 \leq i \leq n} T_i$.

Even in despite of its simples structures, the series and parallel system are essential in reliability theory since that any coherent structure can be decomposed in a parallel-series structure using its minimal path set, a minimal set of components whose functioning insures the functioning of the system. If $P_j, 1 \leq j \leq p$ denotes the minimal path sets, the system lifetime T is

$$T = \Phi(\mathbf{T}) = \max_{1 \leq j \leq p} \min_{i \in P_j} T_i,$$

where $\mathbf{T} = (T_1, ..., T_n)$.

Furthermore, the performance of any coherent system is always better than the performance of a series system and worse than the performance of a parallel system; a parallel (active) redundancy at components level is always **better** than a parallel redundancy at system level; etc.

In the case of statistically dependent components the system reliability at time t, defined by the survival function $\overline{F}(t) = P(T > t)$ is not easy to calculate, involving intricate operations with multivariate distributions. Navarro et al. (2007) used the concept of hyperminimal distribution to calculate the system reliability of dependent components as a generalized mixture of series systems of such components. Our approach,to represent a coherent system as a series system, we apply martingale calculus in reliability theory.

In this paper,in Section 2, we give some preliminaries. In Section 3, we specialize in analyze a general coherent system and in particular, for a parallel system, the dual of a series system. We give some applications in Section 4.

2.Preliminaries.

In order to simplify the notation, we assume that relations such as $\subset, =, \leq, <, \neq$ between random variables and measurable sets, always hold with probability one.

Consider a coherent system of n components with lifetime T_i, $1 \leq i \leq n$, being positive random variables defined on a complete probability space $(\Omega, \Im, P)$ with $P(T_i = T_j) = 0$, $1 \leq i, j \leq n$, that is, the components can be dependent but simultaneous failures are null-sets. As in Barlow and Proschan (1981), a coherent system lifetime has a parallel-series decomposition

$$T = \Phi(\mathbf{T}) = \max_{1 \leq j \leq p} \min_{i \in P_j} T_i,$$

where $P_j, 1 \leq j \leq p$ are the minimal path sets.

The system is monitored through a family of sub σ-algebras of $\Im$, denoted $(\Im_t)_{t \geq 0}$, where

$$\Im_t = \sigma(1_{\{T_i > s\}}, 1 \leq i \leq n, \ s \leq t)$$

satisfies the Dellacherie conditions of right continuity and completeness. Let $\Im_0 = \{\Omega, \emptyset\}$.

The counting measure $(N_i(t))_{t \geq 0}$, $N_i(t) = 1_{\{T_i \leq t\}}$, is a submartingale and from the Doob-Meyer decomposition, there exists a unique right continuous, $\Im_t$-predictable, nondecreasing and integrable process $(A_i(t))_{t \geq 0}$, with $A_i(0) = 0$ and such that $(N_i(t) - A_i(t))_{t \geq 0}$ is a zero mean $\Im_t$-martingale. $A_i(t)$ is called the $\Im_t$-compensator of $N_i(t)$.

In what follows we assume that the lifetimes T_i are totally inaccessible $\Im_t$-stopping time which implies that the compensators $A_i(t)$ are continuous and differentiable.

If we denote the conditional survival functions of T_i as $\overline{F}_i(t) = P(T_i > t|\Im_t)$, it follows from Arjas and Yashin (1988) that the $\Im_t$-compensator processes of $N_i(t) = 1_{\{T_i \leq t\}}$ are given by $A_i(t) = -\ln P(T_i > t|\Im_t) = -\ln(\overline{F}_i(t \wedge T_i))$. As T_i, $1 \leq i \leq n$ are totally inaccessible $\Im_t$-stopping time the compensator processes are continuous.

In order to give some application in Section 4, we introduce the definitions of Arjas (1981a), discuss briefly the notion of stochastic order and introduce some related terminology.

Let θ_t be a shift in time and define for $1 \leq i \leq n$

$$\theta_t T_i = (T_i - t)^+ = \max\{T_i - t, 0\}.$$

We may think of $\theta_t T_i$ as the residual lifetime of T_i at time t. Let $\theta_t \mathbf{T} := (\theta_t T_1, .., \theta_t T_n)$.

A Borel set $U \subset \Re^n$ is called an upper set if for any $\mathbf{x}, \mathbf{y} \in \Re^n$, $\mathbf{x} \leq \mathbf{y}$, $\mathbf{x} \in U$ implies that $\mathbf{y} \in U$. In the univariate case, U is equal to either (u, ∞) or $[u, \infty)$, $u \in \Re$.

If $\mathbf{T}$ and $\mathbf{S}$ are two $\Re^n$-valued random vectors, defined respectively on $(\Omega, \Im, P)$ and $(\Omega^*, \Im^*, P^*)$, then we say that $\mathbf{T}$ is stochastically smaller than $\mathbf{S}$, denoted by $\mathbf{T} \leq^{st} \mathbf{S}$, if $\forall U$,

$$P\{\mathbf{T} \in U\} \leq P^*\{\mathbf{S} \in U\}.$$

Using a stochastic process approach to multivariate reliability systems, Arjas (1981a) introduced the following concepts of classes of non-parametric distributions based on conditional stochastic order.

Definition 2.2 We say that $\mathbf{T}$ is multivariate increasing failure rate relative to $(\Im_t)_{t \geq 0}$, denoted by MIFR$|\Im_t$, if for all $0 \leq s \leq t$ and all open upper sets $U \in \Re^n$

$$P\{\theta_t \mathbf{T} \in U|\Im_t\} \leq P\{\theta_s \mathbf{T} \in U|\Im_s\}.$$

In order to consider some equivalent conditions, let $U_{\mathbf{q}} := \prod_{i=1}^{n} (q_i, \infty)$ be an open upper set with corner point $\mathbf{q} := (q_1, ..., q_n)$, where $q_i, 1 \leq i \leq n$ are positive rational numbers and let U_Q be the class of finite unions of such upper sets. Arjas (1981a) proved the following

Lemma 2.3 The following statements are equivalent:

(i) T is MIFR$|\Im_t$.

(ii) For all rational $0 \leq s \leq t$ and all $U \in U_Q$

$$P\{\theta_t \mathbf{T} \in U | \Im_t\} \leq P\{\theta_s \mathbf{T} \in U | \Im_s\}.$$

(iii) For all real numbers $0 \leq s \leq t$ and all bounded and increasing Borel measurable functions $f : \Re^n \rightarrow \Re$ we have

$$E[f(\theta_t \mathbf{T}) | \Im_t] \leq E[f(\theta_s \mathbf{T}) | \Im_s].$$

Next, we introduce the association definition which was initially formulated by Esary, J.D., Proschan, F. (1970) and Esary, J.D., Proschan, F., Walkup, D. W. (1967) and it is useful to produce upper and lower bounds for system reliability.

Definition 2.6 The random variables $T_1, ..., T_n$ (or the corresponding random vector T) are associated, if for all upper sets U_1 and U_2 of $\Re^n$

$$P\{\mathbf{T} \in U_1 \cap U_2\} \geq P\{\mathbf{T} \in U_1\} P\{\mathbf{T} \in U_2\}.$$

If we consider the uniformly integrable martingales, $M_t^1 = E_P[1_{U_1}(\mathbf{T}) | \Im_t]$ and $M_t^2 = E_P[1_{U_2}(\mathbf{T}) | \Im_t]$, Bueno, V.C. 2008 proved that T is associated if, and only if,

$$E_P[< M^1, M^2 >_\infty] \geq 0.$$

where $< M^1, M^2 >_t$ is the covariation process of M_t^1 and M_t^2.

The results in the paper are related to Girsanov Theorem, Bremaud (1981).

Theorem 2.1 (Girsanov) Let $T_i, 1 \leq i \leq n$ be totally inaccessible $\Im_t$-stopping time representing the components lifetimes. Let $(\alpha_i(t))_{t \geq 0}, 1 \leq i \leq n$, be nonnegative, $\Im_t$-predictable processes such that for all $t \geq 0$ and all $i, 1 \leq i \leq n$,

$$A_i^*(t) = \int_0^t \alpha_i(s) dA_i(s) < \infty,$$

and denote $\alpha(t) = (\alpha_1(t), ..., \alpha_n(t))$ and $\alpha(\infty) = \alpha$, then

$$L_{\alpha(t)}(t) = \prod_{i=1}^{n} [\alpha_i(T_i)]^{N_i(t)} \exp[A_i(t) - A_i^*(t)]$$

is a nonnegative $\Im_t$-local martingale and a nonnegative $\Im_t$-super-martingale. Furthermore, if $E[L_\alpha(\infty)] = 1$, $A_i^*(t)$ is the unique $\Im_t$-compensator of $N_i(t)$ under the probability measure Q_α defined by the Radon Nikodyn derivative

$$\frac{dQ_\alpha}{dP} = L_\alpha(\infty).$$

3 A series representation of a coherent system.

We attained at the parallel series decomposition of the coherent system

$$T = \Phi(\mathbf{T}) = \max_{1 \le j \le p} \min_{i \in P_j} T_i$$

Follows that, for a realization $w \in \Omega$ and a fixed time t,

$$P(T > t | \Im_t) = P(\cup_{j=1}^{p} \cap_{i \in P_j} \{T_i > t\} | \Im_t) =$$

$$\sum_{k=1}^{p} (-1)^{k-1} \sum_{1 \le j_1 < ... < j_k \le p} P(\cap_{l=1}^{k} \cap_{i \in P_{j_l}} \{T_i > t | \Im_t\} =$$

$$\sum_{k=1}^{p} (-1)^{k-1} \sum_{1 \le j_1 < ... < j_k \le p} \exp[- \sum_{i \in \cup_{l=1}^{k} P_{j_l}} A_i(t)] =$$

$$\exp[- \sum_{j=1}^{n} A_j(t)] \{ \sum_{k=1}^{p} (-1)^{k-1} \sum_{1 \le j_1 < ... < j_k \le p} \exp[\sum_{i \in \cap_{l=1}^{k} P_{j_l}^c} A_i(t)] \}.$$

Therefore

$$- \ln P(T > t | \Im_t) = \sum_{j=1}^{n} A_j(t)] - ln(D(t))$$

where

$$D(t)=\sum_{k=1}^{p}(-1)^{k-1}\sum_{1\le j_1<\ldots<j_k\le p}\exp[\sum_{i\in\cap_{l=1}^{k}P_{j_l}^c}A_i(t)].$$

Follows that

$$dD(t)=\sum_{k=1}^{p}(-1)^{k-1}\sum_{1\le j_1<\ldots<j_k\le p}\exp[\sum_{i\in\cap_{l=1}^{k}P_{j_l}^c}A_i(t)]\sum_{i\in\cap_{l=1}^{k}P_{j_l}^c}dA_i(t)=$$

$$\sum_{m=1}^{n}\sum_{k=1}^{p}(-1)^{k-1}\sum_{1\le j_1<\ldots<j_k\le p,\ m\in P_{j_l}^c}\exp[-\sum_{i\in\cap_{l=1}^{k}P_{j_l}^c-\{m\}}A_i(t)]\exp[A_m(t)]dA_m(t)$$

For $1\le m\le n$, we are going to consider the compensator transforms $A_m^*(t)=\int_0^t\alpha_m(s)dA_m(s)$ where

$$\alpha_m(s)=\frac{\sum_{k=1}^{p}(-1)^{k-1}\sum_{1\le j_1<\ldots<j_k\le p\ \ m\in P_{j_l}^c}\exp[\sum_{i\in\cap_{l=1}^{k}P_{j_l}^c}A_i(t)]}{D(s)}.$$

Lemma 3.1 Under the above hypothesis and notation we have

$$\sum_{l=1}^{n}A_l^*=\sum_{l=1}^{n}A_l-\ln D(t).$$

Proof The compensator transformed $A_m^*(t)$ is equal to

$$\int_0^t\frac{\sum_{k=1}^{p}(-1)^{k-1}\sum_{1\le j_1<\ldots<j_k\le p\ \ m\in P_{j_l}^c}\exp[-\sum_{i\in\cap_{l=1}^{k}P_{j_l}^c}A_i(t)]}{D(s)}dA_m(s)=$$

$$\int_0^t\{1-\frac{\sum_{k=1}^{p}(-1)^{k-1}\sum_{1\le j_1<\ldots<j_k\le p\ \ m\in P_{j_l}^c}\exp[-\sum_{i\in\cap_{l=1}^{k}P_{j_l}^c-\{m\}}A_i(t)\exp[A_m(t)]]}{D(s)}dA_m(s)$$

$$A_m(t) - \int_0^t \frac{\sum_{k=1}^{p}(-1)^{k-1} \sum_{1\leq j_1<\ldots<j_k\leq p} \sum_{m\in P_{j_l}^c} \exp[-\sum_{i\in\cap_{l=1}^{k} P_{j_l}^c - \{m\}} A_i(t)\exp[A_m(t)]]}{D(s)} dA_m(s)$$

Therefore

$$\sum_{l=1}^{n} A_l^{\bullet} = \sum_{l=1}^{n} A_l - \int_0^t \frac{dD(s)}{D(s)} = \sum_{l=1}^{n} A_l - \ln D(t).$$

Our main result follows from an adaptation of Girsanov Theorem.

Theorem 3.2 The following process

$$L(t) = \prod_{m=1}^{n} (\alpha_m(T_m))^{1\{T_m\leq t\}} \exp[\sum_{j=1}^{n} A_j(t) - \sum_{j=1}^{n} A_j^{\bullet}(t)] =$$

$$\prod_{m=1}^{n} (\alpha_m(T_m))^{1\{T_m\leq t\}} D(t) =$$

$$\prod_{m=1}^{n} (\alpha_l(T_m))^{1\{T_m\leq t\}} \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\leq i_1<\ldots<i_{n-k}\leq n} \exp[\sum_{j=1}^{n-k} A_{i_j}(t)]$$

is a nonnegative $\Im_t$-local martingale with $E[L(\infty)] = 1$.
Proof The process

$$\alpha_m(s) = \frac{\sum_{k=1}^{p}(-1)^{k-1} \sum_{1\leq j_1<\ldots<j_k\leq p} \sum_{m\in P_{j_l}^c} \exp[\sum_{i\in\cap_{l=1}^{k} P_{j_l}^c} A_i(t)]}{D(s)}$$

is an $\Im_t$-predictable process because so are the components compensator processes. As $0 < \alpha_m(s) \leq 1$ we have

$$\int_0^t \alpha_m(s) dA_m(s) \leq A_m(t) < \infty, P - a.s.$$

and we can apply Girsanov Theorem.

Therefore, we are looking for a probability measure Q, such that, under Q, $C^*(t) = \sum_{i=1}^n A_i^*(t)$ becomes the $\Im_t$-compensator of $1_{T\leq t\}}$ with respect to this modified probability measure. By Girsanov Theorem the desired measure Q is given by the Radon Nikodyn derivative $\frac{dQ}{dP} = L(\infty)$.

Corollary 3.3 Under the above hypothesis , in which $\frac{dQ}{dP} = L(\infty)$, we have

$$Q(\min_{1\leq i\leq n} T_i > t) = P(T > t).$$

Proof

$$Q(\min_{1\leq i\leq n} T_i > t) = E_Q[1_{\{\min_{1\leq i\leq n} T_i > t\}}] = E_P[L_\infty 1_{\{\min_{1\leq i\leq n} T_i > t\}}] =$$

$$E_P[E_P[L_\infty 1_{\{\min_{1\leq i\leq n} T_i > t\}}|\Im_t] = E_P[1_{\{\min_{1\leq i\leq n} T_i > t\}} E_P[L_\infty|\Im_t]] =$$

$$E_P[1_{\{\min_{1\leq i\leq n} T_i > t\}} L(t)] = D(t) P(\min_{1\leq i\leq n} T_i > t) =$$

$$D(t)\exp[-\sum_{j=1}^n A_j(t)] = P(T > t)$$

Remark: The n component parallel system.

i) In a parallel system the path sets P_j are unit sets $P_j = \{j\}$ and the configurations $\cap_{l=1}^k P_{j_l}^c$ are sets of $n-k$ components over all configurations $1 \leq j_1 < ... < j_k \leq p$ and, therefore, we have $T = \max_{1\leq i\leq n}\{T_i\}$, and

$$P(T > t|\Im_t) = \exp[-\sum_{j=1}^n A_j(t)] \sum_{k=1}^n (-1)^{k-1} \sum_{1\leq i_1<...<i_{n-k}\leq n} \exp[\sum_{j=1}^{n-k} A_{i_j}(t)]$$

and, in the set $\{t < T\}$ the $\Im_t$- compensator of T is $A_\Phi(t) = -\ln[P(T > t|\Im_t)] =$

$$\sum_{j=1}^n A_j(t) - \ln\{\sum_{k=1}^n (-1)^{k-1} \sum_{1\leq i_1<...<i_{n-k}\leq n} \exp[\sum_{j=1}^{n-k} A_{i_j}(t)]\}.$$

Furthermore

$$\alpha_l(s) = \frac{\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_{n-k}\le n; i_j\ne l} \exp[\sum_{j=1}^{n-k} A_{i_j}(s)]}{D(s)}$$

where

$$D(t) = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_{n-k}\le n} \exp[\sum_{j=1}^{n-k} A_{i_j}(t)].$$

ii) From Theorem 3.2.3 we have

$$L(\infty) = \prod_{l=1}^{n}(\alpha_l(T_l)) \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_{n-k}\le n} \exp[\sum_{j=1}^{n-k} A_{i_j}(T)].$$

iii) If the components are dependent but identically distributed, we have

$$D(t) = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_{n-k}\le n} \exp[\sum_{j=1}^{n-k} A_{i_j}(t)] =$$

$$\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_{n-k}\le n} \exp[(n-k)A(t)] =$$

$$\sum_{k=1}^{n}(-1)^{k-1} \binom{n}{n-k} \exp[(n-k)A(t)] =$$

$$\sum_{j=0}^{n-1}(-1)^{n-j} \binom{n}{j} \exp[jA(t)] = \exp[nA(t)] - (\exp[A(t)]-1)^n.$$

and

$$\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_{n-k}\le n; i_j\ne l} \exp[\sum_{j=1}^{n-k} A_{i_j}(s)] =$$

$$\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j}\exp[jA(t)] = (\exp[A(t)]-1)^{n-1}.$$

Therefore

$$\alpha_l(t) = \frac{(\exp[A(t)]-1)^{n-1}}{\exp[nA(t)]-(\exp[A(t)]-1)^n}.$$

iv) If the components are dependent but identically distributed with $n = 2$, the component compensator transform is

$$A^{*}(t) = \int_0^t \frac{1-e^{A(s)}}{1-2e^{A(s)}}dA(s)$$

and the compensator of $1_{\{T_1 \vee T_2 \leq t\}}$ is

$$\int_0^t \frac{2-2e^{-A(s)}}{2-e^{-A(s)}}dA(s).$$

under the measure Q such that

$$\frac{dQ}{dP} = 2 - 2\exp[-A(T)].$$

which is used in Bueno and Carmo (2007) to define active redundancy operation when the component and the spare are dependent but identically distributed.

4. Applications.

At this point we deserve to ask about the reliability properties of the vector lifetime $\mathbf{T} = (T_1, T_2, ..., T_n)$, under measure P which are preserved under the measure Q.

4.1. The association property.

In many reliability situations, we encounter structures of coherent systems where components share a load, so that a failure of one component results in increased load on each of the remaining components. In such cases, the random variables of interest are not independent but rather associated.

Therefore, it is very interesting to verify whether the association properties of the lifetimes $T_1, ..., T_n$ under P are preserved under Q.

Theorem 4.1.2 If $\mathbf{T}$ is associated under P, then also under Q.

Proof We consider the upper sets in $\Re^n$, U_1 and U_2, and the uniformly integrable $\Im_t$-martingales, $M_t^1 := E_P[1_{U_1}(\mathbf{T})|\Im_t]$ and $M_t^2 := E_P[1_{U_2}(\mathbf{T})|\Im_t]$.

It follows that there exists an unique $\Im_t$-predictable process, such that the covariance process $< M^1, M^2 >_t$ is increasing, right continuous, with $< M^1, M^2 >_0 = 0$ and such that $M_t := M_t^1 M_t^2 - < M^1, M^2 >_t$ is an $\Im_t$-martingale.

Bueno 2008) proved that $\mathbf{T}$ is associated if, and only if,

$$E_P[< M^1, M^2 >_\infty] \geq 0.$$

Now,

$$E_Q[< M^1, M^2 >_\infty] = E_P[L_\infty < M^1, M^2 >_\infty] \geq 0$$

for $L_\infty \geq 0$ P-a.s..

Follows that the probability measure P is stochastically smaller than the probability measure Q:

$$Q\{\mathbf{T} \in U\} = E_Q[1_{\{\mathbf{T}\in U\}}] = E_P[L_\infty 1_{\{\mathbf{T}\in U\}}] \geq$$

$$E_P[1_{\{\mathbf{T}\in U\}}] = P\{\mathbf{T} \in U\}.$$

Therefore the lower bounds for the system reliability, as in Barlow and Proschan (1981), can be sharper:

If the components $\mathbf{T} = (T_1, ..., T_n)$ is associated, the system reliability is always bigger than the reliability of a series system,that is:

$$P(T > t) \geq P(\min_{i=1}^{n} T_i > t) \geq \prod_{i=1}^{n} P(T_i > t).$$

Actually

$$P(T > t) = Q(\min_{i=1}^{n} T_i > t) \geq \prod_{i=1}^{n} Q(T_i > t) \geq \prod_{i=1}^{n} P(T_i > t).$$

4.2. The increasing failure rate property.

Classes of non-parametric distributions , such as increasing (decreasing) failure rate (IFR (DFR)) distributions, new better(worse) than used (NBU (NWU)) distributions and others, have been extensively investigated in reliability theory. They can be used to achieve the benefit of a maintenance operation or to derive bounds on system reliability.

Several extensions of these concepts appeared in the literature, e.g. Harris (1970), Barlow and Proschan (1981), Marshall (1975) and others. However, they all have in common that they don't order the lifetime vectors in the sense of stochastic order as the univariate concept does. Arjas (1981a), considered to observe the components, continuously in time, based on a family of sub σ-algebras $(\Im_t)_{t\geq 0}$. Arjas (1981a), introduced the notion of increasing failure rate distribution (new better than used distribution) relative to the σ-algebras $(\Im_t)_{t\geq 0}$, IFR$|\Im_t$ (NBU$|\Im_t$), generalizing the conventional definition of IFR (NBU) and extending these classes into a multivariate form, denoted by MIFR$|\Im_t$ (MNBU$|\Im_t$). See the Arjas definition in Section 2.

To generalize the classical results using the series structure, we have to prove that the MIFR$|\Im_t$ class under the measure P is preserved under the measure Q.

Theorem 4.2.3 If $\mathbf{T}$ is multivariate IFR relative to $(\Im_t)_{t\geq 0}$, under P, then also under Q.

Proof First, we show that T_i is IFR$|\Im_t$. under Q. Since, by hypothesis

$$\int_A P\{(T_i - t)^+ > s|\Im_t\}dP = E_P[1_A 1_{\{T_i > t+s\}}] \downarrow t$$

for all $A \in \Im_t$. As L_t is $\Im_t$-measurable and increasing, it can be approximated by steps functions in the form $\sum a_n 1_{\{L_t > b_n\}}$ and then

$$Q(A \cap \{T_i > t + s\}) = E_Q[1_A 1_{\{T_i > t+s\}}] = E_P[L_t(1_A 1_{\{T_i > t+s\}})] \downarrow t.$$

Therefore T_i is IFR$|\Im_t$. under Q. In order to prove that the vector $\mathbf{T}$ is multivariate increasing failure rate relative to $(\Im_t)_{t\geq 0}$, we use the relation

$$\{(T_1 - t)^+ > s, ..., ((T_n - t)^+ > s\} = \bigcap_{1\leq i\leq n} \{((T_i - t)^+ > s\}$$

and the case ii) of Lemma 2.3.

Remark Arjas (1981a) discuss the preservation of the MIFR$|\Im_t$-property under formation of monotone systems. In fact the MIFR$|\Im_t$-property of component lifetimes is carried over to system lifetime if the family $(\Im_t)_{t\geq 0}$ is retained. In our context, from Theorem 4.2.3, the MIFR$|\Im_t$-property under P of $\mathbf{T}$ imply that $\mathbf{T}$ has the MIFR$|\Im_t$-property under Q and this is carried over the series system lifetime T.

Corollary 4.2.4 If $\mathbf{T}$ is MIFR$|\Im_t$, then the series representation T, under Q is IFR$|\Im_t$.

Proof We use condition (iii) of Lemma 2.3. Note that

$$\theta_t\Phi(\mathbf{T}) = \theta_t(\min_{1\leq i\leq n} T_i) = [\min_{1\leq i\leq n} T_i - t]^+ = [\min_{1\leq i\leq n}(T_i - t)]^+ = \Phi(\theta_t\mathbf{T}).$$

Since the function $\mathbf{T} \to \Phi(\mathbf{T})$ is increasing for all bounded and increasing Borel measurable function $f : \Re^m \to \Re$, we have

$$f(\theta_t\Phi(\mathbf{T})) = f(\Phi(\theta_t\mathbf{T}) = (fo\Phi)(\theta_t\mathbf{T}).$$

Moreover, since the composite function $fo\Phi : \Re^m \to \Re$ is increasing, we may conclude that by the assumed MIFR$|\Im_t$ of $\mathbf{T}$ and Theorem 4.2.3

$$E_Q[(fo\Phi)(\theta_t\mathbf{T})|\Im_t)] \leq E_Q[(fo\Phi)(\theta_s\mathbf{T})|\Im_s)],$$

for all $s \leq t$, so that T is IFR$|\Im_t$.

Remark Corollary 4.2.4 contradicts with the following well known property: Monotone systems with independent IFR component lifetimes need not to be IFR. Under Q we can use the IFR$|\Im_t$ property of the series system.

Remark From Arjas (1981b), we known that if $\mathbf{T}$ is MIFR$|\Im_t$ under P, the sample paths of the $\Im_t$-compensator process $(A_i(t))_{t\geq 0}$ are $P - a.s.$ convex for $t \in (0, T_i]$.

From this result and Theorem 4.2.2 we conclude the following result:

Corollary 4.2.5 Let $\mathbf{T}$ be the random vector representing the component lifetimes of a coherent system. If $\mathbf{T}$ is MIFR$|\Im_t$ under P, then the sample paths of the $\Im_t$-compensator process of $N_i(t) = 1_{\{t_i \leq t\}}$ are $Q - a.s.$ convex for $t \in (0, T_i]$.

It is remarkable (Norros (1986)) that total hazards experienced by different components are actually independent . This hold no matter how dependent the actual lifetimes are and what the history, as long as simultaneous failure are ruled out. In our framework we have that, even if T_i, $1 \leq i \leq n$ are dependent, $A_i(T_i)$,$(A_i^*(T_i))$, $1 \leq i \leq n$ are, under P, (Q), independent and identically distributed unit exponential random variables .

Now, under Q, we consider the bijective correspondence (Norros (1986)) between the system lifetime distribution T and its $\Im_t$-compensator at its final point $A_\Phi(T) = \sum_{i=1}^n A_i^*(T)$ and that $A_\Phi(T)$ is an unit exponential random variable. More precisely, given the value of $A_\Phi(T)$, T can be generated. If E is an exponential unit random value, we have $T = \inf\{u : A_\Phi(u) \geq E\}$, in such a way that $\{T > u\}$ is equivalent to $\{A_\Phi(u) < E\}$. Therefore

$$\overline{F}(u|\Im_u) = P(T > u|\Im_u) = P(E > A_\Phi(u)) = \exp\{-A_\Phi(u)\}.$$

From such argument we can get the following lower bound for system reliability:

Theorem 4.2.6 If T_i is $IFR|\Im_t^i$ for all $1 \leq i \leq n$ and $P(T_i \neq T_j) = 1$ for all $1 \leq i, j \leq n$, then

$$\overline{F}(u) \geq \begin{cases} \exp -u \sum_{i=1}^n \frac{1}{E(T_i)}, & \text{if } u < m; \\ 0, & \text{otherwise.} \end{cases}$$

where $\Im_t^i = \sigma\{1_{\{T_j > s\}}, 1 \leq j \leq n, \quad j \neq i, \quad 0 \leq s \leq t\}$, $\overline{F}$ is the system reliability and $m = \min_{1 \leq i \leq n} \{E((T_i)\}$.

Proof $A_i^*((T_i)$, $1 \leq i \leq n$ are random variables independent and identically distributed with unit exponential distribution. Follows from Corollary 4.2.5 that the compensator $A_i^*(u)$ are Q-a.s. convex on $(0, T_i]$. We can use Jensen's Inequality to show that

$$1 = E\{A_i^*(T_i)\} \geq A_i^*\{E((T_i)\}$$

Now as the system $\Im_t$ compensator is $A_\Phi(u) = \sum_{i=1}^n A_i^*(u)$,

$$\frac{A_\Phi(u)}{u} = \frac{\sum_{i=1}^n A_i^*(u)}{u} \leq \sum_{i=1}^n \frac{A_i^*(E(T_i)}{E(T_i)} \leq \sum_{i=1}^n \frac{1}{E(T_i)}.$$

As

$$\overline{F}(u|\Im_u) = P(T > u|\Im_u) = \exp\{-A_\Phi(u)\},$$

the result follows taking the expectation values.

References

[1] Arjas, E.(1981a). The failure and hazard processes in multivariate reliability systems. Mathematics of Operations Research 6, 551 - 562.

[2] Arjas, E.(1981b). A stochastic process approach to multivariate reliability system: Notions based on conditional stochastic order, Mathematics of Operations Research 6, 263 - 276.

[2] Arjas, E. and Yashin, A. (1988). A note on random intensities and conditional survival functions. Journal of Applied Probability, 25, 630 - 635.

[3] Barlow, R., Proschan, F.(1981). **Statistical Theory of Reliability and Life Testing: Probability Models** . Mc Ardle Press, Inc. Silver Spring, MD.

[4] Bremaud, P. (1981). **Point Processes and Queues: Martingales Dynamics**. Springer Verlag, NY.

[5] Bueno, V.C. 2008. Reliability properties under a stopping time shift. European Journal of Operational Research 186:202-210.

[6] Bueno, V.C. and Carmo, I.M.(2007). Active redundancy allocation for a k-out-of-n:F system of dependent components. European Journal of Operational Research, 176, 1041 - 1051.

[7] Esary, J.D., Proschan, F. (1970). A reliability bound for system of maintained, interdependent components. Journal of American Statistical Association, 65, 713 - 717.

[8] Esary, J.D., Proschan, F., Walkup, D. W. (1967). Association of random variables with applications. The Annals of Mathematical Statistics, 38, 1466 - 1474.

[9] Harris, R.(1970). A multivariate definition for increasing hazard rate distribution functions. The Annals of Mathematical Statistics 37, 713 - 717.

[10] Navarro, J., Ruiz, J.M., Sandoval, C.J. (2007). Properties of coherent systems with dependent components. Communications in Statistics - Theory and Methods, 36, 175 - 191.

[11] Norros, I. (1986). Notes on Natvig measure of importance of systems components. Scand. J. Statist. 12, 43 - 54.

[12]Marshall, A. W.(1975). Multivariate distributions with monotone hazard rate. In *Reliability and Fault Tree Analysis, 259 - 284.* Edited by Barlow R.E., Fussel, J.B. and Singpurwalla, N.D., Copyright by Society for Industrial and Applied Mathematics. Philadelphia, Pa.

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