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Graphs with asymmetric Ramsey properties

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Abstract

Given positive integers k and ℓ we write $G \to (K_k, K_\ell)$ if every 2-colouring of the edges of G yields a red copy of K_k or a blue copy of K_ℓ. We prove that for every integer $k \geq 3$, there exists a graph G such that $G \nrightarrow (K_k, K_k)$ while $G \to (K_{k+1}, K_{k-1})$. This result can be viewed as a variation of a classical theorem of Nešetřil and Rödl [11], who proved that for every $k \geq 2$ there is a graph G such that $K_k \nsubseteq G$ and $G \to K_{k-1}$. Our construction combines probabilistic methods and hypergraph container techniques to produce graphs exhibiting this Ramsey behavior.

Keywords: Random graphs; Asymmetric Ramsey; Ramsey numbers

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1. Introduction

Given positive integers k and ℓ, we say a graph G is *Ramsey* for the pair (K_k, K_ℓ) if every red/blue colouring of the edges of G yields either a red copy of K_k or a blue copy of K_ℓ and we write $G \to (K_k, K_\ell)$ to indicate this property. Furthermore, we simply write $G \to K_k$ when $k = \ell$.

A central problem in Ramsey theory is determining the *Ramsey number* $R(k)$, which denotes the smallest n such that $K_n \to K_k$. It is notoriously difficult to estimate $R(k)$. In fact, classical bounds due to Erdős [7] and Erdős–Szekeres [8] give $2^{k/2} \leq R(k) \leq 2^{2k}$, and despite several improvements the exponents remained nearly unchanged for decades, until few years ago, when there were some breakthroughs (see, e.g., [5, 6, 12]) and very recently Campos, Griffiths, Morris and Sahasrabudhe [4] obtained a fantastic result that provides an exponential improvement over the upper bound of Erdős and Szekeres by proving that there is an $\varepsilon > 0$ such that $R(n) \leqslant (4-\varepsilon)^n$ for sufficiently large n. More generally, one can think of the Ramsey number $R(k, \ell)$ which is the minimum n such that $K_n \to (K_k, K_\ell)$. Also recently, Mattheus and Verstraete showed that $R(4, t) = \Omega(t^3/(\log^4 t))$ [10], also a major breakthrough.

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While most efforts focus on estimating Ramsey numbers, a parallel and rich direction of research explores the structure of graphs that are Ramsey for given pairs of graphs. In this context, we study Ramsey phenomena of the form $G \to (K_{k+1}, K_{k-1})$.

In a classical result, Nešetřil and Rödl [11] proved that for every $k \geq 2$ there are graphs with no copies of K_k which are Ramsey for K_{k-1}.

Theorem 1 (Nešetřil & Rödl, 1976)**.** *For every $k \geq 2$ there is a graph G such that $K_k \nsubseteq G$ and $G \to K_{k-1}$.*

Our result, which we state below, can be seen as a variation of Theorem 1. We prove that for any $K \geq 3$ there exists a graph G that is not Ramsey for K_k but it is Ramsey for the pair (K_{k+1}, K_{k-1}), i.e., we replace the condition $K_k \nsubseteq G$ in Theorem 1 with the weaker condition $G \nrightarrow K_k$, which allows G to contain copies of K_k, but still there is a colouring of $E(G)$ avoiding monochromatic copies of K_k. Also, we strengthen the conclusion $G \to K_{k-1}$ by showing that $G \to (K_{k+1}, K_{k-1})$, by which we mean in any {red,blue}-colouring of $E(G)$ one can find a red copy of K_{k+1} or a blue copy of K_{k-1}.

Theorem 2. *For every integer $k \geq 3$, there exists a graph G such that $G \nrightarrow K_k$ but $G \to (K_{k+1}, K_{k-1})$.*

We remark that Theorem 2 also relates to the theory of Ramsey equivalence. Szabó, Zumstein, and Zürcher [14] defined the following relation between graphs: two graphs H_1 and H_2 are said to be *Ramsey-equivalent* if for every graph G, we have $G \to H_1$ if and only if $G \to H_2$ (see [2, 9]) for more results on Ramsey equivalence). Two pairs of graphs (F_1, H_1) and (F_2, H_2) are said to be *Ramsey-equivalent* if for every graph G we have $G \to (F_1, H_1)$ if and only if $G \to (F_2, H_2)$. In other words, the two pairs share exactly the same family of Ramsey graphs. When $F_1 = H_1$ and $F_2 = H_2$, this reduces to the classical notion of Ramsey equivalence between graphs H_1 and H_2. In this direction, our result implies that pairs (K_k, K_k) and (K_{k+1}, K_{k-1}) are not Ramsey-equivalent.

The proof of Theorem 2 combines probabilistic methods with the hypergraph container framework [1, 13] and it is inspired in ideas from [3]. In Section 2, we construct the hypergraph $\mathcal{H}$ that will serve as base for obtaining the desired graph G such that $G \nrightarrow K_k$ but $G \to (K_{k+1}, K_{k-1})$. Then we give a complete overview of the proof of Theorem 2 in Section 3 and we finish with some concluding remarks in Section 4.

2. Preparing a hypergraph for the construction of the desired graph

To prove Theorem 2, we construct a graph G such that $G \nrightarrow K_k$ while $G \to (K_{k+1}, K_{k-1})$. First, recall that a hypergraph $\mathcal{H}$ is linear if every pair of hyperedges shares at most one vertex. The graph G is obtained from a suitable linear subhypergraph $\mathcal{H}$ of a random $(k+1)$-uniform hypergraph, which is obtained by deleting a small number of hyperedges to eliminate certain undesirable substructures. Then, we obtain G by turning all hyperedges of $\mathcal{H}$ into edges. The deletion step used to obtain $\mathcal{H}$ is crucial to ensure $G \nrightarrow K_k$.

Before defining these substructures, we introduce the notion of *clique decomposition* of a complete graph K_k. Given a copy of K_k with vertex set S, a set $C = \{V_1, \ldots, V_\ell\}$ of subsets of S is called a clique decomposition of K_k if $|V_i| \geq 2$ for each $1 \leq i \leq \ell$ and every edge of this K_k belongs to exactly one of the cliques in C.

The following definition describes the kind of sets induces subhypergraphs we aim to eliminate from $\mathcal{H}$.

Definition 1 (Shattered sets of vertices)**.** *Let $k \geq 3$ and let $\mathcal{H}$ be a linear $(k+1)$-uniform hypergraph. A k-element set $S \subseteq V(\mathcal{H})$ is called* shattered by $\mathcal{H}$ *(or simply* shattered*) if $C := \{E \cap S : E \in E(\mathcal{H})\}$ is a clique decomposition of a copy of K_k on S.*

We say a linear $(k+1)$-uniform hypergraph $\mathcal{H}$ if *shattered-free* if $\mathcal{H}$ contains no shattered sets S. We will obtain a subhypergraph from $\mathcal{H}_{k+1}(n, \tilde{p})$, which is the n-vertex $(k+1)$-uniform hypergraph obtained from including each possible hyperedge independently with probability $\tilde{p}$. We will use $\tilde{p} = pn^{-k+1}$ and $p = Cn^{-1/m_2(K_{k-1})}$ for some sufficiently large constant C, where $m_2(K_{k-1})$ stands for the *maximum 2-density* of K_{k-1}. Here the maximum 2-density of a graph H with at least 3 vertices is defined as

$$m_2(H) = \max\left\{\frac{e(J) - 1}{v(J) - 2} \ : \ J \subset H, \ v(J) \geq 3\right\}.$$

where $v(J)$ and $e(j)$ are the number of vertices and edges in the graph J, respectively. Note that we have $m_2(K_{k-1}) = k/2$.

Writing $f(n) \sim g(n)$ to indicate that $f(n) = \Theta g(n)$, by standard Chernoff bounds, one can check that with high probability we have

$$e(\mathcal{H}_{k+1}(n, \tilde{p})) \sim \tilde{p}n^{k+1} \sim pn^2.$$

We will construct a linear shattered-free subhypergraph $\mathcal{H} \subseteq \mathcal{H}_{k+1}(n, \tilde{p})$, by deleting $o(pn^2)$ hyperedges from the original random hypergraph. This sparsification is key to ensuring that $G \nrightarrow K_k$.

On the other hand, since only a small fraction of hyperedges are removed, the remaining structure retains enough density to guarantee that $G \rightarrow (K_{k+1}, K_{k-1})$.

The following lemma formalizes this deletion step, showing that $\mathcal{H}$ can be made both linear and shattered-free with high probability.

Lemma 1. *Let $k \geq 3$ be an integer and $\tilde{p} \sim pn^{-k+1}$ with $p = Cn^{-1/m_2(K_{k-1})}$ for some sufficiently large C. Then with high probability the random hypergraph $\mathcal{H}_{k+1}(n, \tilde{p})$ can be made linear and shattered-free by deleting $o(pn^2)$ hyperedges.*

Proof sketch of Lemma 1. We aim to construct a linear shattered-free $(k + 1)$-uniform hypergraph $\mathcal{H}$ on n vertices, sampled as subhypergraph from the random model $\mathcal{H}_{k+1}(n, \tilde{p})$. Our goal is to ensure that such a structure arises with high probability after removing only $o(pn^2)$ hyperedges, where $p = n^{-1/m_2(K_{k-1})}$ and $\tilde{p} \sim pn^{-k+1}$. The proof proceeds in two parts.

1. Linearity. We first consider the number X of pairs of hyperedges in $\mathcal{H}_{k+1}(n, \tilde{p})$ that share at least two vertices. Standard calculations show that the expected value of X is $o(pn^2)$ from our choice of parameters. Hence, by Markov's inequality, we can delete at most $o(pn^2)$ hyperedges to make the hypergraph *linear*, i.e., every pair of hyperedges shares at most one vertex.

2. Destroying shattered sets. We determine an upper bound on the expected number of sets S that are shattered by $\mathcal{H}_{k+1}(n, \tilde{p})$. Each such configuration is determined by a set $S \subseteq V(\mathcal{H}_{k+1}(n, \tilde{p}))$ of size k along with a clique decomposition $C = \{V_1, \ldots, V_\ell\}$ of a copy of K_k with vertex set S. Each V_i in the decomposition is required to extend to a hyperedge $\mathcal{E}_i$ such that $\mathcal{E}_i \cap S = V_i$.

To control the total number of such structures, we consider all possible clique decompositions and group them according to the sizes of the sets V_i. A key step is to observe that any decomposition involving larger cliques can be transformed into one using only 2-vertex cliques, with negligible impact on the count. This reduction allows us to focus on an "extremal" configuration where all cliques have size 2, for which we compute a tight upper bound of $o(pn^2)$.

Therefore, by linearity of expectation and a union bound over all clique decompositions, we deduce that with high probability the number of sets shattered by $\mathcal{H}_{k+1}(n, \tilde{p})$ is also $o(pn^2)$. Hence, removing one hyperedge from each such occurrence yields a linear shattered-free hypergraph. Since the total number of edges in $\mathcal{H}_{k+1}(n, \tilde{p})$ is $\Theta(pn^2)$, the resulting hypergraph $\mathcal{H}$ still contains $\Theta(pn^2)$ edges.

This completes the construction of a shattered-free linear hypergraph $\mathcal{H}$, as required. □

3. Monochromatic copies of complete graphs

In this section, we prove our main result, Theorem 2, which asserts that for every integer $k \geq 3$, there exists a graph G such that $G \nrightarrow K_k$ but $G \rightarrow (K_{k+1}, K_{k-1})$.

Given a linear $(k + 1)$-uniform hypergraph $\mathcal{H}$, we define the graph $G[\mathcal{H}]$ on the same vertex set as $\mathcal{H}$ by placing a complete graph K_{k+1} on the vertex set of each hyperedge of $\mathcal{H}$. That is, for each hyperedge $W \in \mathcal{H}$, the graph $G[\mathcal{H}]$ contains all $\binom{k+1}{2}$ edges between the vertices in W. Theorem 2 follows directly from the result below.

Theorem 3. *Let $k \geq 3$ be an integer and $\tilde{p} \sim pn^{-k+1}$ with $p = Cn^{-1/m_2(K_{k-1})}$ for a sufficiently large C. Then with high probability, there exists a subhypergraph $\mathcal{H} \subset \mathcal{H}_{k+1}(n, \tilde{p})$ such that*

$$G[\mathcal{H}] \nrightarrow K_k \quad \text{and} \quad G[\mathcal{H}] \rightarrow (K_{k+1}, K_{k-1}).$$

The hypergraph $\mathcal{H}$ we obtain in Theorem 3 is precisely the shattered-free subhypergraph constructed in Lemma 1. A hypergraph $\mathcal{H}$ as the one obtained in this lemma inherits some "random" properties from the hypergraph from which it was obtained. These are the properties that will ensure we can apply the Container Lemma to ensure that $G[\mathcal{H}] \rightarrow (K_{k+1}, K_{k-1})$.

Also important, note that by destroying all hypergraphs shattered by some set S to obtain $\mathcal{H}$ we guarantee that there every copy of K_k in $G[\mathcal{H}]$ is contained in a hyperedge of $\mathcal{H}$ and this fact ensures we have $G[\mathcal{H}] \nrightarrow K_k$.

Theorem 3 follows from the next two lemmas. Lemma 2 ensures that $G[\mathcal{H}] \nrightarrow K_k$, and Lemma 3 guarantees that $G[\mathcal{H}] \rightarrow (K_{k+1}, K_{k-1})$.

Lemma 2. *Let $k \geq 3$ be an integer. If a shattered-free $(k+1)$-uniform hypergraph $\mathcal{H}$, then*

$$G[\mathcal{H}] \nrightarrow K_k.$$

Lemma 2 follows directly from the fact that, since $\mathcal{H}$ is shattered-free, any copy of K_k is contained in a hyperedge of $\mathcal{H}$. Then, it is enough to, for each hyperedge of $\mathcal{H}$, pick two pairs of vertices in it and give colour red in $G[\mathcal{H}]$, and colour the other pairs in this hyperedge with colour blue. This clearly avoids monochromatic copies of K_k. The following result is proved in the next subsection.

Lemma 3. *Let $k \geq 3$ be an integer and let $\tilde{p} = pn^{-k+1}$ and $p = Cn^{-1/m_2(K_{k-1})}$ for sufficiently large C. Then, for every subhypergraph $\mathcal{H} \subset \mathcal{H}_{k+1}(n, \tilde{p})$ with $(1 - o(1))pn^2$ hyperedges, we have, with high probability,*

$$G[\mathcal{H}] \rightarrow (K_{k+1}, K_{k-1}).$$

3.1. Forcing a red copy of K_{k+1} or a blue copy of K_{k-1}

Before proving the result, let us state a tailored version of the hypergraph container lemma, due to Balogh-Morris-Samotij [1] and Saxton-Thomason [13]. We state a form suited to our application.

Lemma 4 (Container Lemma)**.** *For every graph F and $\varepsilon > 0$, there exist constants n_0 and $D > 0$ such that for every $n \geq n_0$, there are $t = t(n)$ pairwise distinct subsets $S_1, \ldots, S_t \subseteq E(K_n)$ and $C_1, \ldots, C_t \subseteq E(K_n)$ satisfying:*

1. $|S_i| \leq Dn^{2-1/m_2(F)}$ *for all i;*
2. *C_i contains at most $\varepsilon n^{v(F)}$ copies of F;*
3. *for every F-free graph G, there exists i such that $S_i \subseteq E(G) \subseteq C_i$.*

We refer to the sets C_i as *containers* and S_i as their corresponding *sources*. We are now ready to prove Lemma 3.

Proof sketch of Lemma 3. Let $\mathcal{H} \subset \mathcal{H}_{k+1}(n, \tilde{p})$ with $(1-o(1))pn^2$ hyperedges be the obtained hypergraph from Lemma 1 and let $G = G[\mathcal{H}]$ be the associated graph constructed by placing a clique K_{k+1} on each hyperedge. Consider an arbitrary 2-colouring of $E(G)$ into red and blue edges.

Suppose, for contradiction, that this colouring avoids both a red copy of K_{k+1} and a blue copy of K_{k-1}. Let G_r and G_b denote the subgraphs of G induced by the red and the blue edges, respectively.

Since every K_{k+1} in G must be coloured non-monochromatically, each such clique contains at least one blue edge. Intuitively, this creates many blue edges spread across G, potentially forming large monochromatic blue substructures.

To formalize this, we apply the Container Lemma to the class of K_{k-1}-free graphs. This gives a family of pairs of sources and containers (S_i, C_i) such that any K_{k-1}-free graph lies between $S_i \subseteq E(G_b) \subseteq C_i$ for some i, where each C_i contains few copies of K_{k-1}. We fix such a container C_i for G_b.

Now, observe that $G \cap \overline{C_i}$ (the edges not in C_i) are all red. Since G_r is K_{k+1}-free by assumption, so is $G \cap \overline{C_i}$. However, by averaging, one can check that $\overline{C_i}$ must contain many copies of K_{k+1} (since C_i has few copies of K_{k-1}).

Finally, we bound the probability that all these K_{k+1}-candidates are destroyed in G, using linearity of expectation and standard bounds on random graphs. By carefully estimating this probability and using the container parameters, we conclude that the total failure probability is $o(1)$, i.e., with high probability, $G[\mathcal{H}]$ contains a red K_{k+1} or a blue K_{k-1}, completing the proof. □

4. Concluding Remarks

In this work, we constructed, for every integer $k \geq 3$, a graph G such that $G \nrightarrow K_k$ but $G \rightarrow (K_{k+1}, K_{k-1})$, which can be viewed as a variation of the classical theorem of Nešetřil and Rödl, which showed the existence of graphs G with no copies of K_k but such that $G \rightarrow K_{k-1}$ In contrast, our random construction starts with a non-Ramsey graph for K_k and we show that one can enforce copies of K_{k+1} or K_{k-1} in the "right" colour.

Our approach combines probabilistic techniques with the hypergraph container method to obtain "pseudorandom" host graphs that exhibits some particular Ramsey behavior. The key idea was to encode the construction of G through a random $(k+1)$-uniform hypergraph, carefully pruned to eliminate certain configurations that could otherwise lead to monochromatic copies of K_k. The resulting graph $G[\mathcal{H}]$ then simultaneously avoids K_k in some colouring while forcing either a red copy of K_{k+1} or a blue copy of K_{k-1}.

There are several directions for future work. It would be interesting to find deterministic constructions of such graphs, or to impose additional structural constraints such as "bounded" degree or forbidding some subgraphs. More broadly, a natural question is to determine for which values of k the inequality $R(k-1, k+1) < R(k)$ is strict, and whether methods similar to ours can help characterize more generally when asymmetric pairs are not Ramsey-equivalent to the corresponding diagonal pair.

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