# Faithfully Quadratic Rings, a Summary of Results

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The text below contains a listing of our principal results on the subject matter of the title, presented in full in the monograph [DM11] (submitted) [1]. It is intended as a bare summary of statements, for the benefit of the reader already possessing some familiarity with the subject, containing few or no comments. For motivation and informal explanations of a general character, the reader can consult the introduction of our work published in these seminar proceedings, [DM12]; this introduction still gives, on the whole, a meaningful guide to our results, although our treatment of the subject has evolved since its publication.

In all that follows, "ring" stands for commutative unitary ring where 2 is a unit (invertible), while "quadratic form" stands for *diagonal quadratic form with unit coefficients*.

To this end, we shall employ our **theory of special groups (SG)**, cf. [DM2]. [2]

We shall formulate axioms such that if $\langle A, T \rangle$ is a preordered ring, then its associated proto-special group ($\pi$-SG), $G_T(A)$, is a **special group**; further, these axioms satisfy the following conditions:

* The axioms are "elementary" and closely connected to fundamental concepts of the algebraic theory of quadratic forms. Moreover, their logical form makes them amenable to model-theoretic treatment and automatically entails a number of preservation results (see 1.2, 5.5 – 5.7).

* Ring-theoretic representation and isometry of forms of arbitrary dimension are faithfully coded by representation and isometry in $G_T(A)$;

* If $T = A^2$, the mod 2 algebraic $K$-theory of $A$ ([Gu]) is naturally isomorphic to that of $G(A)$.

In the case of preorders, one of the difficulties was to obtain an **intrinsic** characterization of $T$-isometry, i.e., a characterization depending only on $T$, the ring operations and the ring's general linear group. As is well-known, the usual treatment in fields uses signatures. A smooth theory ensues by using the concept of $T$-isometry introduced in Definition 3.9.

Once the basic theory is settled (in sections 1 – 5), considerable effort is devoted to establish that certain significant and well-known classes of rings are faithfully quadratic: rings with many units (section 7), various classes of f-rings (section 8), Archimedean -rings with bounded inversion (section 9). In section 10, we give some applications to the theory of quadratic forms over these classes of rings.

[1] A full copy of [DM11] can be obtained both at www.ime.usp.br/~miraglia/fq-rings.pdf or at www.maths.manchester.ac.uk/raag/index.php?preprint=0320.

[2] The applicability of the theory of SGs to rings – beyond the fundamental and motivating case of fields – was initially envisaged by M. Knebush.

## 1 Geometric and Horn-Geometric Theories

**Definition 1.1** *Let $L$ be a first-order language with equality and let $\overline{z} = (z_1, \ldots, z_n)$ be variables in $L$. A $L$-formula on the free variables $\overline{z}$ is*

*a)* **positive primitive (pp)** *if it is of the form $\exists\, \overline{v}\ \varphi(\overline{v}; \overline{z})$, where $\varphi$ is a conjunction of atomic formulas.*

*b)* **geometric** *if it is the negation of an atomic formula* **or** *of the form*

$$\forall \overline{v}(\exists \overline{y} \varphi_1(\overline{y}, \overline{v}; \overline{z}) \rightarrow \exists \overline{w} \varphi_2(\overline{w}, \overline{v}; \overline{z})),$$

*where $\varphi_1$, $\varphi_2$ are positive and quantifier free.*

*A theory is* **geometric** *if it has a set of geometric axioms.*

*c) A $L$-formula is* **Horn-geometric** *if it is the negation of an atomic formula* **or** *of the form $\forall \overline{v}(\varphi_1(\overline{z}) \rightarrow \varphi_2(\overline{z}))$, where $\varphi_1$ and $\varphi_2$ are pp-formulas in $L$.*

*A* **Horn-geometric theory in $L$** *is a theory having a set of Horn-geometric axioms.*

Clearly, the theory of unitary commutative rings ($1 \neq 0$) is Horn-geometric. Other important examples in our context shall appear below.

The following is classical:

**Theorem 1.2** *Let $T$ be a theory in a first-order language with equality.*

*a) If $T$ is Horn-geometric, then it preserved by arbitrary reduced products. In particular, it is preserved under non-empty products.*

*b) If $T$ is geometric, then it is preserved under arbitrary right-filtered inductive limits.* ∎

A topological space $X$ is **partitionable** if it is $T_1$ and every open covering $X$ has a disjoint open refinement. Discrete spaces and Boolean spaces are partitionable; any topological sum of partitionable spaces is partitionable.

We also have

**Theorem 1.3** *Let $T$ be a Horn-geometric $L$-theory. Let $\mathfrak{M}$ be a sheaf of $L$-structures over a partitionable space $X$. If every stalk of $\mathfrak{M}$ is a model of $T$, then $\mathfrak{M}(X) \models T$, where $\mathfrak{M}(X)$ is the $L$-structure of global sections of $\mathfrak{M}$.* ∎

## 2 Proto-Special Groups

A **proto-special group ($\pi$-SG)** is a triple

$$G = \langle G, \equiv_G, -1 \rangle,$$

such that

$*$ $G$ is a group of exponent two, (written multiplicatively; 1 is its identity), with a distinguished element $-1$. Set $-x = -1 \cdot x$;

$*$ A binary relation (isometry) $\equiv_G$ on $G \times G$, such that

[SG 0] : $\equiv_G$ is an equivalence relation on $G \times G$;

[SG 1] : $\langle a, b \rangle \equiv_G \langle b, a \rangle$; [SG 2] : $\langle a, -a \rangle \equiv_G \langle 1, -1 \rangle$;

[SG 3] : $\langle a, b \rangle \equiv_G \langle c, d \rangle \quad \Rightarrow \quad ab = cd$;

[SG 5] : $\langle a, b \rangle \equiv_G \langle c, d \rangle \quad \Rightarrow \quad \langle xa, xb \rangle \equiv_G \langle xc, xd \rangle$;

$G$ is **reduced ($\pi$-RSG)** if $1 \neq -1$ and

[red] : $\langle a, a \rangle \equiv_G \langle 1, 1 \rangle \quad \Rightarrow \quad a = 1$.

A $\pi$-SG $G$ is a **pre-special group (p-SG)** if, in addition, it satisfies,

[SG 4] : $\langle a, b \rangle \equiv_G \langle c, d \rangle \quad \Rightarrow \quad \langle a, -c \rangle \equiv_G \langle -b, d \rangle$.

$\varphi = \langle a_1, \ldots, a_n \rangle \in G^n$ is called a ***n*-form** over $G$.

Let $G$ be a $\pi$-SG. Binary isometry in $G$ can be extended to $n$-forms, $n \geq 1$, still written $\equiv_G$, as follows:

* $\langle a \rangle \equiv_G \langle b \rangle \quad \Leftrightarrow \quad a = b$;

* for $n = 2$, $\equiv_G$ is the primitive relation on $G$;

* for $n \geq 3$, $\langle a_1, \ldots, a_n \rangle \equiv_G \langle b_1, \ldots, b_n \rangle$ iff there are $x, y, z_3, \ldots, z_n \in G$ such that

(1) $\langle a_1, x \rangle \equiv_G \langle b_1, y \rangle$;

(2) $\langle a_2, \ldots, a_n \rangle \equiv_G \langle x, z_3, \ldots, z_n \rangle$;

(3) $\langle b_2, \ldots, b_n \rangle \equiv_G \langle y, z_3, \ldots, z_n \rangle$.

A p-SG is a **special group (SG)** if it verifies

[SG 6] : Isometry of forms of dimension 3 is transitive.

Proto-SGs, p-SGs, SGs and their reduced counterparts are **Horn-geometric theories** in the first-order language $L_{SG} = \langle \cdot, 1, -1, \equiv, = \rangle$.

If $G$, $H$ are $\pi$-SGs, a map $f : G \longrightarrow H$ is a **morphism** if $f$ is a group morphism, such that $f(-1) = -1$ and $\langle a, b \rangle \equiv_G \langle c, d \rangle \quad \Rightarrow \quad \langle fa, fb \rangle \equiv_H \langle fc, fd \rangle$.

Write **$\pi$-SG**, **SG** and **RSG** for the categories of proto-SGs, SGs and reduced SGs, respectively.

# 3 Preordered rings and Proto Special Groups

**3.1 Conventions and Notation.** Recall our standing hypothesis that all rings are unitary commutative rings, in which 2 is a unit.

a) If $R$ is a ring, $D \subseteq R$ and $x \in R$

(1) $R^\times$ = groups of units in $R$; (2) $D^\times = D \cap R^\times$;

(3) $D^2 = \{d^2 \in R : d \in D\}$;

(4) $\Sigma D^2 = \{\Sigma_{i=1}^n d_i^2 : \{d_1, \ldots, d_n\} \subseteq D, n \geq 1\}$.

(5) $\mathrm{GL}_n(R)$ = invertible $n \times n$ $R$-matrices.

b) A **preorder** of $R$ is subset $T \subseteq R$, closed under addition and multiplication and containing $R^2$. $T$ is **proper** if $-1 \notin T$.

The smallest preorder on $R$ is $\Sigma R^2$; if it is proper, then $R$ is said to be **semi-real**.

c) Unless explicitly stated otherwise, $A$ **is assumed semi-real** and **all its preorders are assumed proper**. ■

**Definition 3.2** *a) A* **preordered ring (p-ring)** *is a pair* $\langle A, T \rangle$ *such that* $A$ *is a ring and* $T$ *is a preorder of* $A$.

*b) The language of p-rings is* $L = \langle +, \cdot, 0, 1, -1, T \rangle$, *i.e., the first-order language of rings , with a unary predicate,* $T$, *satisfying the axioms of a preorder.*

*c) A* **morphism of p-rings**, $f : \langle A, T \rangle \longrightarrow \langle A', T' \rangle$, *is a ring morphism,* $f : A \longrightarrow A'$, *such that* $f(T) \subseteq T'$.

*d) Write* **p-Ring** *for the category of p-rings and their morphisms.*

The reader will have noticed that the theory of p-rings is Horn-geometrical.

**3.3** To a p-ring $\langle A, T \rangle$, we associate:

a) A group of exponent two

$$G_T(A) \;=\; A^\times / T^\times \;=\; \{a^T : a \in A^\times\},$$

writing 1 for $1^T$ and $-1$ for $(-1)^T$;

b) For $a, b \in A^\times$,

$$D^v_T(a, b) \;=\; \{x \in A^\times : \exists\, s, t \in T \text{ s.t. } x = sa + tb\},$$

is the set of units **value represented** by $\langle a, b \rangle$.

c) Define

$$\langle a^T, b^T \rangle \equiv_T \langle c^T, d^T \rangle \quad \Leftrightarrow \quad \begin{cases} a^T b^T = c^T d^T \\ \text{and} \\ D^v_T(a, b) = D^v_T(c, d). \end{cases}$$

d) If $h : \langle A_1, T_1 \rangle \longrightarrow \langle A_2, T_2 \rangle$ is a p-ring morphism, let $h^\pi : G_{T_1}(A_1) \longrightarrow G_{T_2}(A_2)$ be given by $h^\pi(a^{T_1}) = h(a)^{T_2}$. ∎

e) If $T = \Sigma A^2$, write $G_{red}(A)$ for the associated $\pi$-SG.

We have

**Theorem 3.4** *a) If* $\langle A, T \rangle$ *is a p-ring, then* $G_T(A)$ *is a* $\pi$*-SG, which is reduced iff* $T$ *is proper.*

*b) If* $h$ *is a p-ring morphism,* $h^\pi$ *is morphism of* $\pi$*-SGs, yielding a covariant functor from* **p-Ring** *to* $\pi$**-SG**. ∎

*c) This functor preserves arbitrary non-empty products and all right-filtered inductive limits.*

**Remark 3.5** a) The preceding construction also holds with $A^2$ in place of $T$. In this case, write $G(A) = A^\times / A^{2\times}$ for the associated $\pi$-SG. ∎

**Fact 3.6** Let $A$ be a ring and let $T = A^2$ or a preorder of $A$. If $A$ satisfies **2-transversality** with respect to $T$, i.e., for all $a, b \in A^\times$

$$D^v_T(a, b) \;=\; \{c \in A^\times : \underline{\exists\, s, t \in T^\times} \text{ so that } c = sa + tb\},$$

then $G_T(A)$ is a pre-special group, i.e., ∎

$$\langle a, b \rangle \equiv_T \langle c, d \rangle \;\Rightarrow\; \langle a, -c \rangle \equiv_T \langle -b, d \rangle.$$

**Remark 3.7** In fact, the constructions we are presenting hold for **$T$-subgroups** of $\langle A, T \rangle$, i.e., subgroups $S \subseteq A^\times$, such that $T^\times \subseteq S$ and $-1 \in S$. If $T = A^2$, these are called **$q$-subgroups** of $A$. This generalization is important: we have shown that **any** RSG group is isomorphic to one of the type $G(S)$, where $S$ is a $q$-subgroup of the ring $\mathbb{C}(X)$, $X$ a Boolean space ([DM10]). To ease presentation, we only deal here with the case $S = A^\times$. ∎

**3.8 Diagonal $A^\times$-quadratic forms** Let $n \geq 1$ be an integer and $A$ be a ring.

a) To an $n$-form over $A^\times$, $\varphi = \langle a_1, \ldots, a_n \rangle \in A^{\times n}$ we associate

(1) A diagonal quadratic form over $A^n$, written $\varphi$: for $X = \langle X_1, \ldots, X_n \rangle$, $\varphi(X) = \sum_{i=1}^n a_i X_i^2$.

(2) A diagonal matrix in $\mathrm{GL}_n(R)$, $\mathcal{M}(\varphi)$, whose non-zero entries are $a_1, \ldots, a_n$;

(3) The **discriminant** of $\varphi$ is $d(\varphi) = \det \mathcal{M}(\varphi) = a_1 \cdots a_n \in A^\times$.

b) If $\varphi$, $\psi$ are $n$-forms over $A^\times$, $\varphi \approx \psi$ iff

$$\exists\, M \in GL_n(R) \text{ such that } M\mathcal{M}(\varphi)M^t = \mathcal{M}(\psi).$$

Clearly, $d(\varphi)\, det(M)^2 = d(\psi)$.

The relation $\approx$, **matrix isometry**, is an equivalence relation. It has the usual properties, e.g., preserves orthogonal sums and tensor products of forms. ∎

**Definition 3.9** *Let $\langle A, T \rangle$ be a p-ring and let $\varphi$, $\psi$ be forms of dimension $n \geq 1$ over $A^\times$. We say that $\varphi$ is* **T-isometric** *to $\psi$, written $\varphi \approx_T \psi$, if there is a sequence of n-dimensional forms over $A^\times$, $\varphi_0, \varphi_1, \ldots, \varphi_k$, such that (recall that $\approx$ is matrix isometry)*

*(i) $\varphi_0 = \varphi$ and $\varphi_k = \psi$;* **and**

*(ii) $\forall\, 1 \leq i \leq k$, either $\varphi_i \approx \varphi_{i-1}$,* **or** *$\varphi_i = \langle t_1 x_1, \ldots, t_n x_n \rangle$, with $t_1, \ldots, t_n \in T^\times$ and $\varphi_{i-1} = \langle x_1, \ldots, x_n \rangle$.*

**Remark 3.10** If $A$ is a field, $\approx_T$ is equivalent to the signature version of isometry for the reduced theory mod $T$ of quadratic forms with coefficients in $A^\times$. We shall see below that the same result holds for $T$-faithfully quadratic rings (Theorem 5.3). ∎

In the setting of $\pi$-SG associated to rings there are several notions of *representation* that must be distinguished. In the field case all these notions coincide.

**Definition 3.11** *Let $T = A^2$ or a proper preorder on a ring $A$. Let $\varphi = \langle b_1, \ldots, b_n \rangle$ and $\varphi^T = \langle b_1^T, \ldots, b_n^T \rangle$ be a $A^\times$-form and its correspondent form in $G_T(A)$.*

*a) $D_T(\varphi) = \{a \in A^\times : \exists\, a_2, \ldots, a_n \in A^\times$ such that $\varphi^T \equiv_T \langle a^T, a_2^T, \ldots, a_n^T \rangle\}$*

*are the elements* **isometry-represented** *by $\varphi^T$ in $G_T(A)$.*

*b) $D_T^v(\varphi) = \{a \in A^\times : \exists\, x_1, \ldots, x_n \in T$ such that $a = \sum_{i=1}^n x_i b_i\}$,*

*is the set of elements* **value-represented mod $T$ by** *$\varphi$.*

*c) $D_T^t(\varphi) = \{a \in A^\times : \exists\, z_1, \ldots, z_n \in T^\times$ such that $a = \sum_{i=1}^n z_i b_i\}$*

*is the set of elements* **transversally represented mod $T$ by** *$\varphi$.*

*Clearly, $D_T^t(\varphi) \subseteq D_T^v(\varphi)$.*

*d) Define $\mathfrak{D}_T(\varphi)$ as follows:*

* If $n = 2$, $\mathfrak{D}_T(\varphi) = D^v_T(b_1, b_2)$;
* If $n \geq 3$, $\mathfrak{D}_T(\varphi) = \bigcap_{k=1}^{n} \bigcup \{D^v_T(b_k, u) : u \in D^v_T(b_1, \ldots, \check{b}_k, \ldots, b_n)\}$,

*where* $\left\langle b_1, \ldots, \check{b}_k, \ldots, b_n \right\rangle$ *denotes the* $(n-1)$*-dimensional form obtained by omitting the* $k^{th}$*-entry.*

*e) If* $T = \Sigma A^2$ *is a proper preorder on* $A$ *(i.e.,* $A$ *is semi-real), the corresponding representation sets will be written* $D_\Sigma$, $D^v_\Sigma$, $D^t_\Sigma$ *and* $\mathfrak{D}_\Sigma$.

*f) If* $T = A^2$*, the corresponding representation sets will be written* $D$, $D^v$, $D^t$ *and* $\mathfrak{D}$.

## 4 The Axioms

Let $T$ be a preorder of a ring $A$ or $T = A^2$. Consider the following conditions:

[T-FQ 1] : (2-tranversality) For all $a, b \in S$, $D^v_T(a, b) = D^t_T(a, b)$.

[T-FQ 2] : For all $n \geq 2$ and all $n$-forms $\varphi$ over $S$, $D^v_T(\varphi) = \mathfrak{D}_T(\varphi)$.

[T-FQ 3] : (1-Witt-cancellation) For all integers $n \geq 1$, all $a \in A^\times$ and all $n$-forms $\varphi, \psi$ over $A^\times$, $\langle a \rangle \oplus \varphi \approx_T \langle a \rangle \oplus \psi \Rightarrow \varphi \approx_T \psi$.

We then have

**Theorem 4.1** *Let* $A$ *be a ring and let* $T$ *be* $A^2$ *or a preorder of* $A$*. If* $A \models$ [T-FQ 1], [T-FQ 2] *and* [T-FQ 3], *then*

*a) For all* $A^\times$*-forms,* $\varphi$, $D_T(\varphi) = D^v_T(\varphi)$*, i.e.,* $a \in A^\times$ *is value represented iff it is isometry represented in* $G_T(A)$.

*b) For all* $A^\times$*-forms* $\varphi, \psi$ *of the same dimension,* $\varphi \approx_T \psi \Leftrightarrow \varphi^T \equiv_T \psi^T$. *(in* $G_T(A)$*)*

*c)* $G_T(A) = \langle G_T(A), \equiv_T, -1 \rangle$ *is a SG, faithfully coding* $T$*-isometry and value representation of diagonal quadratic forms over* $A^\times$. ∎

Theorem 4.1 has a partial converse:

**Theorem 4.2** *Let* $A$ *be a ring and let* $T$ *be a preorder of* $A$*, or* $T = A^2$.

*If* $A \models$ [T-FQ 1], *the following are equivalent:*

(1) $G_T(A)$ *is a SG such that for all* $A^\times$*-forms of the same dimension,* $\varphi, \psi$,

$(*)$ $\varphi \approx_T \psi \Leftrightarrow \varphi^T \equiv_T \psi^T$; $(**)$ $D^v_T(\varphi) = D_T(\varphi^T)$. ∎

(2) $A \models$ [T-FQ 2] *and* $A \models$ [T-FQ 3].

With respect to $K$-theory we obtain the following very general result:

**Theorem 4.3** *If* $A$ *is a ring verifying* [FQ 1], *there is a natural graded ring isomorphism between Milnor's mod 2* $K$*-theory of* $A$ *and that of the pre-special group* $G(A)$ [3]. ∎

The preceding results justify the following

[3] Milnor's $K$-theory of rings is developed in [Gu].

**Definition 4.4** *Let $A$ be a ring and $T$ be $A^2$ or a preorder of $A$.*

*a) $A$ is* **$T$-faithfully quadratic** *if it satisfies axioms* [T-FQ 1], [T-FQ 2] *and* [T-FQ 3].

*b) If $T = A^2$ write* [FQ $i$] *for* [T-FQ $i$] *($i$ = 1, 2, 3), and call $A$* **faithfully quadratic**.

*c) If $T = \Sigma A^2$, write* [$\Sigma$-FQ $i$] *for* [T-FQ $i$] *($i$ = 1, 2, 3), and call $A$* **$\Sigma$-faithfully quadratic**.

# 5 $T$-isometry and Signatures

**Definition 5.1** *Let $\langle A, T \rangle$ be a p-ring.*

*a) A* **T-signature** *on $A$ is a group morphism, $\tau : A^{\times} \longrightarrow \mathbb{Z}_2 = \{\pm 1\}$, such that $\tau(-1) = -1$ and for all $a \in A^{\times}$, $a \in ker\, \tau \Rightarrow D^{v}_{T}(1, a) \subseteq ker\, \tau$.*

*Write $Z_{A,T}$ for the set of $T$-signatures on $A$.*

*b) If $\varphi = \langle a_1, \ldots, a_n \rangle$ is a form over $A^{\times}$, and $\tau \in Z_{A,T}$, $sgn_{\tau}(\varphi) = \sum_{i=1}^{n} \tau(a_i)$ is the* **signature** *of $\varphi$ at $\tau$.*

If $\langle A, T \rangle$ is a p-ring, an ordering $\alpha \in \mathrm{Spec}_R(A, T)$ (the real spectrum of $\langle A, T \rangle$, cf. 6.1.(d)) gives rise to a signature

$$\tau_{\alpha} : A^{\times} \longrightarrow \mathbb{Z}_2, \text{ given by } \tau_{\alpha}(a) = 1 \quad \text{iff} \quad a \in \alpha.$$

For p-rings satisfying [T-FQ 2], we have:

**Proposition 5.2** *If $\langle A, T \rangle \models$ [T-FQ 2], then for all $\tau \in Z_{A,T}$ there is $\alpha \in \mathrm{Spec}_R(A, T)$ such that $\tau = \tau_{\alpha}$.* ■

We now state

**Theorem 5.3 (Pfister's local-global principle)** *If $\langle A, T \rangle$ is a $T$-faithfully quadratic p-ring and $\varphi$, $\psi$ are forms of the same dimension over $A^{\times}$, the following are equivalent:*

(1) *$\varphi \approx_T \psi$;*

(2) *For all $\tau \in Z_{A,T}$, $sgn_{\tau}(\varphi) = sgn_{\tau}(\psi)$.*

(3) *For all $\alpha \in \mathrm{Spec}_R(A, T)$, $sgn_{\tau_\alpha}(\varphi) = sgn_{\tau_\alpha}(\psi)$.* ■

As will be seen forthwith, $T$-quadratic faithfulness is preserved by a number of important operations; we here register its preservation by localization at an idempotent:

**Theorem 5.4** *Let $A$ be a ring and let $T$ be $A^2$ or a proper preorder of $A$. If $A$ is $T$-faithfully quadratic and $e$ is an idempotent in $A$ (i.e., $e^2 = e$), then the ring $Ae$ is $Te$-faithfully quadratic.* ■

Concerning the elementary character of the axioms:

**Theorem 5.5** *a) The theory of faithfully quadratic rings is Horn-geometric in the language of unitary rings, $\{+, \cdot, 0, 1, -1\}$.*

*b) The theory of $T$-faithfully quadratic rings is Horn-geometric in the language of unitary rings, with an additional unary predicate symbol, $T$ (interpreted as a preorder).* ■

We now obtain the following preservation results:

**Corollary 5.6** *a) Let* $\mathscr{A} = \langle\langle A_\lambda, T_\lambda\rangle; \{f_{\lambda\mu} : \lambda \leq \mu \text{ in } \Lambda\}\rangle$ *be an inductive system of* $T_\lambda$*-faithfully quadratic p-rings over the right-directed poset* $\langle \Lambda, \leq \rangle$*. Let* $\langle A, T\rangle = \varinjlim \mathscr{A}$ *be the inductive limit of* $\mathscr{A}$*. Then,*

* $A$ *is* $T$*-faithfully quadratic;*
* $G_T(A) = \varinjlim_{\lambda\in\Lambda} G_{T_\lambda}(A_\lambda)$.

*A similar statement holds for faithfully quadratic rings.*

*b) Let* $\{\langle A_i, T_i\rangle : i \in I\}$ *be a non-empty family of p-rings and let* $D$ *be a filter on* $I$*. Let*

$$\langle A_D, T_D\rangle = \langle \textstyle\prod_D A_i, \prod_D T_i\rangle$$

*be the reduced product of the* $\langle A_i, T_i\rangle$ *modulo* $D$*. If for all* $i \in I$*,* $A_i$ *is* $T_i$*-faithfully quadratic, then* $\langle A_D, T_D\rangle$ *is* $T_D$*-faithfully quadratic. Moreover,* $G_{T_D}(A_D) = \prod_D G_{T_i}(A_i)$. ∎

*An analogous statement holds for non-empty families of faithfully quadratic rings.*

**Corollary 5.7** *Let* $X$ *be a partitionable space and let* $B(X)$ *be the BA of clopens in* $X$*. If* $\mathfrak{A}$ *is a sheaf of rings over* $X$*, the following are equivalent:*

(1) $\mathfrak{A}(X)$ *is faithfully quadratic;*

(2) *For all* $U \in B(X)$*,* $\mathfrak{A}(U)$ *is faithfully quadratic;*

(3) *For all* $x \in X$*,* $\mathfrak{A}_x$ *(the stalk of* $\mathfrak{A}$ *at* $x$*) is faithfully quadratic.* ∎

*An analogous statement holds for a sheaf of p-rings,* $\langle \mathfrak{A}, \mathfrak{T}\rangle$*, over* $X$*.*

**Remark 5.8** A sheaf of p-rings over $X$ is a pair, $\langle \mathfrak{A}, \mathfrak{T}\rangle$, where $\mathfrak{A}$ is a sheaf of rings over $X$ and $\mathfrak{T}$ is a **sub-sheaf** of $\mathfrak{A}$, such that $\mathfrak{T}(U)$ is a preorder of $\mathfrak{A}(U)$, for all open $U$ in $X$. ∎

# 6 Rings with Bounded Inversion

**Definition 6.1** *Let* $A$ *be a ring, let* $J$ *be an ideal in* $A$ *and let* $T$ *be a preorder of* $A$*.*

*a)* $J$ *is* **$T$-convex** *if for all* $s, t \in T$*,* $s + t \in J \Rightarrow s, t \in J$*.*

*b)* $J$ *is* **$T$-radical** *if for all* $a \in A$ *and* $t \in T$*,* $a^2 + t \in J \Rightarrow a \in J$*.*

*A* $\Sigma A^2$*-radical ideal is called* **real**.

*c)* $T$ *has* **bounded inversion** *if* $1 + T \subseteq A^\times$*. We say that* $\langle A, T\rangle$ *is a* **BIR**. *A is said to have* **weak bounded inversion (WBIR)** *if* $1 + \Sigma A^2 \subseteq A^\times$*.*

*d)* $\mathrm{Spec}_R(A, T)$ *is the* **real spectrum** *of* $\langle A, T\rangle$*, i.e., the space of all (ring-theoretic) orderings of* $A$ *containing* $T$*.*

**Proposition 6.2** *If* $\langle A, T\rangle$ *is a p-ring and* $Y_T = \mathrm{Spec}_R(A, T)$*, the following are equivalent:*

(1) *Every maximal ideal of* $A$ *is* $T$*-convex;*

(2) $\langle A, T\rangle$ *is a BIR;*

(3) $\bigcap_{\alpha\in Y_T} \alpha \setminus (-\alpha) = T^\times$. ∎

Preorders satisfying a generalization of (3) above – called *unit-reflecting* – will play an important role in what follows.

**Theorem 6.3** (Tranversality for BIRs) *If $\langle A, T \rangle$ is a BIR, then for all $a_1, \ldots, a_n \in A^\times$,*

$$D^v_T(a_1, \ldots, a_n) \quad = \quad D^t_T(a_1, \ldots, a_n).$$ ■

* This generalizes a result of Mahé for $T = \Sigma A^2$ ([Ma2]).

* The proof requires the theory of real semigroups in [DP1], [DP2].

# 7 Rings with Many Units

**Definition 7.1** *Let $R$ be a ring.*

*a) A polynomial $f \in R[X_1, \ldots, X_n]$* **has local unit values** *if for every maximal ideal $\mathfrak{m}$ of $R$, there are $u_1, \ldots, u_n$ in $R$ such that $f(u_1, \ldots, u_n) \notin \mathfrak{m}$.*

*b) $R$ is a* **ring with many units** [4] *if for all $n \geq 1$, and all $f \in R[X_1, \ldots, X_n]$, if $f$ has local unit values, there is $\bar{r} \in R^n$ such that $f(\bar{r}) \in R^\times$.*

**7.2 Examples of Rings with Many Units.**

* Fields;

* semi-local rings;

* Commutative von Neumann regular rings;

* Arbitrary products of rings with many units;

* The ring of global sections of a sheaf of rings over a partitionable topological space, whose stalks are rings with many units. In particular, the following are rings with many units:

  (1) The ring of global sections of a sheaf of rings over a Boolean space, whose stalks are rings with many units;

  (2) $\mathbb{C}(X, \mathbb{R})$, where $X$ is a Boolean space. ■

**Theorem 7.3** *If $A$ is a ring with many units such that every residue field of $A$ has at least 7 elements, then $A$ is* **completely faithfully quadratic**, *i.e., it is faithfully quadratic ($T = A^2$) and $T$-faithfully quadratic for any preorder $T$ of $A$.* ■

The proof of Theorem 7.3 uses results in [Wa]. We also register

**Theorem 7.4** *Rings with many units are Horn-geometric axiomatizable in the first-order language of rings.* ■

There are important categories of rings that, in general, **do not have many units**:

* $\mathbb{C}(X) \quad = \quad \mathbb{C}(X, \mathbb{R})$, where $X$ is a (non-partitionable) completely regular space;

* The real holomorphy ring of a formally real field (the intersection of all of its real valuation rings).

We now turn to the analysis of quadratic faithfulness of general classes of rings suggested by these examples.

[4] Also called *local-global* rings; cf. [Mc].

# 8 Reduced f-rings

A partially ordered ring (po-ring), $\langle A, \leq \rangle$, is **lattice-ordered (lo)** if for all $a, b \in A$,

$$a \vee b = \sup\{a, b\} \quad \text{and} \quad a \wedge b = \inf\{a, b\}$$

exist in $A$ (join and meet with respect to $\leq$).

A lo-ring is an **f-ring** if it is isomorphic to a subdirect product of linearly ordered rings.

Recall that a ring $A$ is **reduced** if 0 is its only nilpotent element.

For a lo-ring $A$, the following are equivalent:

(1) $A$ is a reduced f-ring;

(2) $A$ is a subdirect product of linearly ordered domains.

**Definition 8.1** *An f-ring A comes equipped with a partial order, written $T_\sharp^A$, with respect to which it is lattice ordered. If A is clear from context, write $T_\sharp$ in place of $T_\sharp^A$.*

We then obtain (after quite a bit of work)

**Theorem 8.2** *If A is reduced f-ring containing $\mathbb{Q}$, then A is $T_\sharp$-faithfully quadratic. Moreover, its associated special group is the Boolean algebra of idempotents in A.* ∎

**Lemma 8.3** *Let $\langle A, T \rangle$ be a p-ring, let $Y_T$ be its real spectrum and let $Y_T^*$ be the compact Hausdorff space of closed points in $Y_T$. The following are equivalent:*

(1) $A^\times \cap \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha) = T^\times$;

(2) *For some non-empty $K \subseteq Y_T$,*

$$A^\times \cap \bigcap\nolimits_{\alpha \in K} \alpha \setminus (-\alpha) = T^\times.$$

(3) *For some non-empty $D \subseteq Y_T^*$,*

$$A^\times \cap \bigcap\nolimits_{\beta \in D} \beta \setminus (-\beta) = T^\times.$$

∎

**Definition 8.4** *A preorder T of a ring A is* **unit-reflecting (u.r.)** *if it satisfies the equivalent conditions in Lemma 8.3.*

**Remark 8.5** a) If $\langle A, T \rangle$ is a BIR, then $T$ is unit-reflecting.

b) Examples of unit-reflecting preorders that are not of bounded inversion will appear below, see Theorem 8.10.(a).

c) Any preorder of a ring with many units is unit-reflecting (a result due to Leslie Walters, Corollary 1.9, p. 33 in [Wa]). ∎

We then have

**Theorem 8.6** *Let A be a reduced f-ring containing $\mathbb{Q}$. If T is a unit-reflecting preorder such that $T_\sharp \subseteq T$, then A is T-faithfully quadratic.* ∎

**Definition 8.7** *A ring A is* **weakly real closed (WRCR)** *if it satisfies the following properties:*

[WRCR 1] : *$A$ is reduced;*

[WRCR 2] : *$A^2$ is the positive cone of a partial order $\leq$ on $A$, with which it is a f-ring;*

[WRCR 3] : *For all $a, b \in A$, $0 \leq a \leq b \Rightarrow b$ divides $a^2$.*

**Remarks 8.8** a) The missing axiom for **real closed rings** (in the sense of Prestel-Schwartz, [PS]) is :

For all primes $\mathfrak{p} \subseteq A$, the field of fractions of $A/\mathfrak{p}$ is real closed and $A/\mathfrak{p}$ is integrally closed in it.

All real closed rings are, of course, WRCR, as are all rings of real-valued continuous functions on a completely regular topological space. M. Tressl has shown: there are real closed rings that are not even *elementary equivalent* to any one of the type $\mathbb{C}(X)$, $X$ a completely regular space.

b) If $A$ is a WRCR, then it is

* **completely real**, i.e. all prime ideals in $A$ are real. In particular it is a BIR;
* A Pythagorean f-ring containing $\mathbb{Q}$ (cf. 9.1.(a)). ■

Our results on f-rings containing $\mathbb{Q}$ yield

**Theorem 8.9** *If A is weakly real closed ring, and T is any unit-reflecting preorder of A, then A is T-faithfully quadratic. In particular, A is faithfully quadratic.* ■

We also obtain:

**Theorem 8.10** *Let $X$ be a completely regular topological space and let $K \neq \emptyset$ be a closed set in $X$. Set*

$$P_K = \{f \in \mathbb{C}(X) : f \restriction K \geq 0\}.$$

*Then,*

*a) $P_K$ is a proper unit-reflecting preorder of $\mathbb{C}(X)$, which is of bounded inversion iff $K = X$.*

*b) $\mathbb{C}(X)$ is $P_K$-faithfully quadratic. In particular, $\mathbb{C}(X)$ is faithfully quadratic.* ■

# 9 Archimedean p-Rings with Bounded Inversion

We recall

**Definition 9.1** *Let A be a ring and let T be a preorder of A.*

*a) A is* **Pythagorean** *if $\Sigma A^2 = A^2$.*

*b) T is* **Archimedean** *if for all $a \in A$ there is $n \in \mathbb{N}$ such that $n - a \in T$.*

* Real holomorphy rings are Archimedean WBIRs;
* $\mathbb{C}(X)$ is a Pythagorean WBIR; it is Archimedean (with its natural partial order) iff $X$ is pseudo-compact.

We have

**Theorem 9.2** *If $\langle A, T \rangle$ is an Archimedean p-ring with bounded inversion and $P$ is a preorder containing $T$, then $A$ is $P$-faithfully quadratic. In particular, $A$ is $T$-faithfully quadratic.* ∎

**Remark 9.3** The proof of this result uses (among other things) the Becker-Schwartz version of the Kadison-Dubois Theorem, cf. [BS] (that may also be attributed to Stone and Krivine). ∎

**Corollary 9.4** *Let $K$ be a formally real field and let $H(K)$ be its real holomorphy ring. Then, $H(K)$ is $\Sigma$-faithfully quadratic. Moreover, $G_{red}(H(K))$ is isomorphic to the Boolean algebra of clopens of the compact Hausdorff space of real places of $H(K)$.* ∎

# 10 Some Applications to Quadratic Form Theory over Rings

**10.1 The Witt Ring and the Graded Witt Ring.** Let $\langle A, T \rangle$ is a p-ring and let $\varphi$, $\psi$ be forms over $A^\times$. We say that $\varphi$, $\psi$ are **Witt-equivalent** mod $T$ if there are integers $n$, $m \geq 0$ so that $\varphi \oplus m\langle 1, -1 \rangle \approx_T \psi \oplus n\langle 1, -1 \rangle$.

If $\langle A, T \rangle$ is $T$-faithfully quadratic, let

$$W_T(A) \;=\; \{\overline{\varphi} : \varphi \text{ is a form over } A\},$$

be the set of equivalence classes of forms over $A^\times$, under Witt-equivalence. With operations induced by $\oplus$ and $\otimes$, $W_T(A)$ is a commutative ring with identity $\langle 1 \rangle$, whose zero is the class of hyperbolic forms, **the Witt ring of $A$ mod $T$**.

* $W_T(A)$ is naturally isomorphic to $W(G_T(A))$, the Witt ring of the RSG $G_T(A)$, (by the map induced on Witt rings by $\varphi \longmapsto \varphi^T$).

* $I_T(A) = I(G_T(A))$ is the **fundamental ideal** of $W_T(A)$, consisting of the classes of even dimensional forms.

* For $n \geq 1$, $I^n_T(A) = I^n(G_T(A))$, the $n^{th}$-power of $I_T(A)$, consists of all linear combinations of Pfister forms of degree $n$ over $A$.

* The **graded Witt ring of $A$ mod $T$**, is the sequence

$$W_{T_g}(A) \;=\; \langle \mathbb{F}_2, \ldots, \overline{I^n_T}(A), \ldots \rangle,$$

where for $n \geq 1$, $\overline{I^n_T}(A) \;=\; I^n_T(A)/\, I^{n+1}_T(A)$. If $T = A^2$, we omit $T$ from the notation. ∎

With notation as above, we have

**Theorem 10.2** *Let $A$ be a Pythagorean ring.*

*a) If $A$ is an f-ring containing $\mathbb{Q}$, then for all $n \geq 1$, $k_n A \;\simeq\; \overline{I^n}(A) \;\simeq\; B(A)$, where $B(A)$ is the BA of idempotents in $A$. In particular, $A$ satisfies Milnor's mod 2 Witt ring conjecture.*

*b) If $A$ is an Archimedean BIR, then for all $n \geq 1$, $k_n A \;\simeq\; \overline{I^n}(A) \;\simeq\; B(Y^*)$, where $B(Y^*)$ is the BA of clopens of the subspace of closed points in $\mathrm{Spec}_R(A)$. In particular, $A$ verifies Milnor's mod 2 Witt ring conjecture.* ∎

* Examples of 10.2.(a): Weakly real closed rings, and $\mathbb{C}(Z)$, $Z$ a topological space.

* Examples of 10.2.(b): The real holomorphy ring of a formally real Pythagorean field, and $\mathbb{C}(X)$, $X$ a *compact* Hausdorff space.

**Theorem 10.3** (The Arason-Pfister Hauptsatz) *Let $\langle A, T \rangle$ be a $T$-faithfully quadratic p-ring. If $\varphi$ is a form over $A^\times$ such that* $\dim \varphi < 2^n$ *and* $\varphi \in I^n(A)$, *then $\varphi$ is $T$-hyperbolic ($\varphi \approx_T H$, $H$ a hyperbolic form). In particular,* $\bigcap_{n \geq 1} I^n(A) = \{0\}$. ■

The last set of results we shall mention is stated for f-rings containing $\mathbb{Q}$. However, the ones marked with an exponent $^\bullet$ also hold for Archimedean BIRs.

**Theorem 10.4** *Let $A$ be a f-ring containing $\mathbb{Q}$ and let $T_\sharp$ be its natural partial order. Let $T$ be an u.r.-preorder of $A$, such that $T_\sharp \subseteq T$.*

*a)*$^\bullet$ (Marshall's signature conjecture) *Let $\varphi$ be a form over $A^\times$ and let $n \geq 1$ be an integer. If for all $\alpha \in$* $\mathrm{Spec}_R(A, T)$, *$sgn_{T_\alpha}(\varphi) \equiv 0$ mod $2^n$, then $\varphi \in I^n_T(A)$.*

*b) For $n$-forms $\varphi = \langle a_1, \ldots, a_n \rangle$ and $\psi = \langle b_1, \ldots, b_n \rangle$ over $A^\times$, the following are equivalent:*

(1) $\varphi \approx_{T_\sharp} \psi$;

(2) (Local-global Sylvester's inertia law) *There is an orthogonal decomposition of $A$ into idempotents, $\{e_1, \ldots, e_m\}$, so that for each $1 \leq j \leq m$, we have (with $\underline{n} = \{1, \ldots, n\}$) :*

$$(i)\ \underline{n} = \{k \in \underline{n} : a_k e_j >_{T_\sharp} 0\} \ \cup\ \{k \in \underline{n} : a_k e_j <_{T_\sharp} 0\}$$
$$= \{k \in \underline{n} : b_k e_j >_{T_\sharp} 0\} \ \cup\ \{k \in \underline{n} : b_k e_j <_{T_\sharp} 0\},$$

*i.e., each entry of $\varphi$ and $\psi$ is either strictly positive or strictly negative in $Ae_j$, for all $j \in \underline{n}$.*

$$(ii)\ \mathrm{card}(\{k \in \underline{n} : a_k e_j <_{T_\sharp} 0\}) = \mathrm{card}(\{k \in \underline{n} : b_k e_j <_{T_\sharp} 0\}).$$

*c)*$^\bullet$ *Let $R$ be a f-ring containing $\mathbb{Q}$, let ${T_\sharp}^R$ be its natural partial order and let $P$ be a u.r.-preorder of $R$ containing ${T_\sharp}^R$. If $\langle A, T \rangle \xrightarrow{f} \langle R, P \rangle$ is a p-ring morphism, the following are equivalent, where $f \star \langle a_1, \ldots, a_n \rangle = \langle f(a_1), \ldots, f(a_n) \rangle$ :*

(1) *$f$ is complete, that is, for all $n$-forms $\varphi$, $\psi$ over $A^\times$,* $\varphi \approx_T \psi \Leftrightarrow f \star \varphi \approx_P f \star \psi$;

(2) *$f$ reflects isotropy (if $\varphi$ is a form over $A^\times$ and $f \star \varphi$ is $P$-isotropic, then $\varphi$ is $T$-isotropic);*

(3) $f^{-1}[P^\times] = T^\times$. ■

**Note:** As a guide to the interested reader, we have included the full reference list of the original [DM11].

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