RT-MAT 2008 - 09

DECOMPOSABILITY OF BRANCHED COVERINGS ON THE PROJECTIVE PLANE

Natalia A. Viana Bedoya and
Daciberg Lima Gonçalves

Novembro 2008

Esta é uma publicação preliminar ("preprint").

Decomposability of branched coverings on the projective plane

Natalia A. Viana Bedoya * and Daciberg Lima Gonçalves

Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010, CEP 05508-090, São Paulo, SP, Brasil.
naviana@ime.usp.br
Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010, CEP 05508-090, São Paulo, SP, Brasil.
dlgoncal@ime.usp.br

Abstract

In this work we will study the decomposability property of primitive branched coverings of the projective plane. The main result is to show that any branching data which is realizable it can be realized by an indecomposable if the degree d is even. Also we classify the branching coverings with one branching point from the view point of decomposibility. For more than one ramification point let d be odd. We have two cases. One is when the defect is $d-1$ and we give examples where in one case there is an indecomposable and in the other case there is no an indecomposable. When the defect is $> d-1$ few examples are computed and they give support to the conjecture that in this case there is always a positive answer to the question.

Key words: branched coverings, primitive groups, imprimitive groups, permutation groups, projective plane

2000 MSC: 57M12, 57M10, 54C10.

---

*The begin of this work appears im my PhD thesis written under supervision of the Professor Daciberg Lima Gonçalves and supported by FAPESP process 03/12309-4.

1 Introduction

Let (M, ϕ, S, B_ϕ, d) be a branched covering and $\mathscr{D}$ a branch data. The question of classify which branch data admits a relization which is indecomposable and which branched data admits a realization which is decomposable has been considered for many years. See for example [4], [3], [7] and [16]. Recently, in [3], this problema has been completely solved in the case where S is a closed surface different from the sphere S^2 and the projective plane $\mathbb{R}P^2$. The purpose of this work is study this question for the projective plane. Let us point out that the realization problem for the projective plane has been solved some years ago in [9], i.e. one knows exactly which branch data can be realized. The realization problem for the sphere is still an open question.

From now on let $S = \mathbb{R}P^2$, the projective plane and $\phi : M \to \mathbb{R}P^2$ a primitive branching covering, i.e. the induced homomorphism $\pi_1(M) \to \pi_1(\mathbb{R}P^2)$ is surjective. If the homomorphism is not surjective the maps admits an obvious decomposition.

Let (M, ϕ, S, B_ϕ, d) be a branched covering and $\mathscr{D}$ a branch data. Since ϕ is orientation-true then M is nonorientable. From [9] this data is realizable if and only if the total defect satisfies:

$$d - 1 \leq \nu(\mathscr{D}) = d - \chi(M) \equiv 0 \pmod 2. \quad (1)$$

2 Branch coverings with one branch point

This case is quite simple. From the Hurwitz conditions given in [9] follows that the total defect of these brahching data are $d-1$. We will show that the branching data is not realizable by an indecomposable, in contrast with the case where the base is not $\mathbb{R}P^2$. The total defect of these brahching datum are $d - 1$. So the branch covering is of the form $(\mathbb{R}P^2, \phi, \mathbb{R}P^2, \{x\}, d)$ and $\mathscr{D} = [d]$.

Proposition 2.1. *A branched coverings like $(\mathbb{R}P^2, \phi, \mathbb{R}P^2, \{x\}, d)$ is always decomposable if d is not prime.*

Proof. Look, if $\mathscr{D}$ is the branch data, by (HC) $\nu(\mathscr{D}) = d-1 \equiv 0 \pmod 2$, therefore d is odd and $\mathscr{D} = [d]$. Thus, every non-trivial factorization of $d = uw$ defines a factorization of $\mathscr{D}$ by non-trivial realizable datas $\mathscr{U} = [u]$ and $\mathscr{W} = [w]$ on $\mathbb{R}P^2$ and by Theorem 1 from [3] $\mathscr{D}$ is decomposable.

In another way, notice that the representation

$$\begin{aligned} \rho : \pi_1(\mathbb{R}P^2 - \{x\}) = \langle a, \mathbf{u_x} | a^2 u_x = 1 \rangle &\longrightarrow \Sigma_d \\ a &\longmapsto \alpha \\ \mathbf{u_x} &\longmapsto \gamma \end{aligned}$$

such that $\alpha^2 = \gamma^{-1}$ is a d-cycle, the group $G = Im\rho = \langle\gamma, \alpha\rangle = \langle\alpha\rangle$ is always imprimitive, because if α^2 is a d-ciclo, necessarly α is a d-cycle, then every isotropy subgroup is trivial and since d is no prime, it is contained in a proper subgroup of $Im\rho$ and our affirmation is a consequence of Corollary 1.5A in [8]. □

Now we will move to the cases where we have more than one branched point.

3 Even degree

Let $(M, \phi, \mathbb{R}P^2, B_\phi, d)$ be primitive with branch data $\mathscr{D} = \{D_1, \ldots, D_t\}$, and notice that if d is even then by (1) the genus of M is even, therefore $\chi(M) \leq 0$. Moreover, by (1) we have $\chi(M) - |\phi^{-1}(B_\phi)| = d(1-t)$ and since $\chi(M) \leq 0$, necessarlly $t > 1$.

Proposition 3.1. *Let $d > 2$ be an even number and $\mathscr{D} = \{D_1, \ldots, D_t\}$ a finite collection of partitions of d such that $d \leq \nu(\mathscr{D}) \equiv 0 \pmod 2$ and containing a partition different of $[2, \ldots, 2]$. Then $\mathscr{D}$ can be realized by an indecomposable primitive branched covering.*

Proof. Without loss of generality, let us suppose $D_t \neq [2, \ldots, 2]$. Since $d \leq \nu(\mathscr{D}) \equiv 0 \pmod 2$, there is $q \geq 0$ such that $\nu(\mathscr{D}) = d + 2q$, then $\nu(D_1) + \nu(D_2) = d + 2q - \sum_{i=3}^{t} \nu(D_i)$. If $s = \sum_{i=3}^{t} \nu(D_i) - 2q$ is bigger than zero, applying Lemma 4.2 in [9] there are permutations $\gamma_1 \in D_1$, $\gamma_2 \in D_2$ such that $\langle\gamma_1, \gamma_2\rangle$ acts in $\{1, \ldots, d\}$ with s orbits and

$$\nu(\gamma_1\gamma_2) = d - s = d - \sum_{i=3}^{t} \nu(D_i) + 2q. \tag{2}$$

If $s \leq 0$, applying Lemma 4.3 in [9] for $k = -(1 - \sum_{i=3}^{t} \nu(D_i))$ there exist $\gamma_1 \in D_1$, $\gamma_2 \in D_2$ such that $\langle\gamma_1, \gamma_2\rangle$ acts trasitively on $\{1, \ldots, d\}$ and

$$\nu(\gamma_1\gamma_2) = (d-1) - k = d - \sum_{i=3}^{t} \nu(D_i). \tag{3}$$

Let D_{12} be the partition determined by $\gamma_1\gamma_2$ and by (2) implies $\nu(D_{12}) + \nu(D_3) = d + 2q - \sum_{i=4}^{t} \nu(D_i)$ and we can repeat the analisis before. On the other hand, the situation in (3) implies that $\nu(D_{12}) + \nu(D_3) = d - \sum_{i=4}^{t} \nu(D_i)$ and since $\sum_{i=4}^{t} \nu(D_i) > 0$, by Lemma 4.2 in [9], there are $\gamma_{12} \in D_{12}$, $\gamma_3 \in D_3$ such that $\langle\gamma_{12}, \gamma_3\rangle$ acts with $\sum_{i=4}^{t} \nu(D_i)$ orbits and $\nu(\gamma_{12}\gamma_3) = d - \sum_{i=4}^{t} \nu(D_i)$. It is clear that repeating the analisis before we will obtain the following situation:

$$\nu(D_{12\ldots t-1}) + \nu(D_t) = \begin{cases} d + 2q & \text{, applying Lemma 4.2[9]} \\ d & \text{, applying Lemma 4.3[9]} \end{cases} \tag{4}$$

where $D_{12\ldots j}$ denotes the partition dtermined by the cyclic structure of $\gamma_{1\ldots j-1}\cdot\gamma_j$ with $\gamma_{1\ldots j-1} \in D_{1\ldots j-1}$ and $\gamma_j \in D_j$, for $j = 2, \ldots, t-1$, where the γ_i's are obtained by succesive applications of Lemmas 4.2, 4.3 in [9]. Whatever the case, we are under the hypothesis of Lemma 4.5 in [9] $D_t \neq [2, \ldots, 2]$, the are permutations $\gamma_{12\ldots t-1} \in D_{12\ldots t-1}$, $\gamma_t \in D_t$ such that the group $\langle \gamma_{12\ldots t-1}, \gamma_t \rangle$ acts transitively on $\{1, \ldots, d\}$ and the product $\gamma_{12\ldots t-1}\gamma_t$ is a $d-1$ cycle, wich guarantees the existence of a permutation $\alpha \in \Sigma_d$ such that $\gamma_{12\ldots t-1}\gamma_t = \alpha^2$ and the permutation group $\langle \gamma_{12\ldots t-1}, \gamma_t \rangle$ is primitive.

On the other hand, for $i = 1, \ldots, t-1$, there is $\lambda_i \in \Sigma_d$ such that $\gamma_{12\ldots t-1} = \gamma_1^{\lambda_1}\gamma_2^{\lambda_2}\ldots\gamma_{t-1}^{\lambda_{t-1}}$ (recall that $\gamma_i^{\lambda_i} = \lambda_i\gamma_i\lambda_i^{-1}$). Thus, we define the representation

$$\begin{aligned} \rho : \langle a, \mathrm{u}_1, ..., \mathrm{u}_t | a^2\Pi_{i=1}^t \mathrm{u}_i = 1]\rangle &\longrightarrow \Sigma_d \\ a &\longmapsto \alpha^{-1} \\ \mathrm{u}_i &\longmapsto \gamma_i^{\lambda_i} \\ \mathrm{u}_t &\longmapsto \gamma_t \end{aligned}$$

Then, there exists a primitive branched covering $(M, \phi, \mathbb{R}P^2, B_\phi, d)$ with M nonorientable realizing $\mathscr{D}$ as branch data and since $\langle \gamma_{12\ldots t-1}, \gamma_t \rangle < G = \mathrm{Im}\rho$, then G is a primitive permutation group and by Theorem 2.1 in [2], $(M, \phi, \mathbb{R}P^2, B_\phi, d)$ is indecomposable. $\square$

Proposition 3.2. *Let $d > 4$ be even and $\mathscr{D} = \{D_1, \ldots, D_t\}$ a finite collection of partitions of d such that $d \leq \nu(\mathscr{D}) \equiv 0 \pmod 2$ and $D_i = [2, \ldots, 2]$ for $i = 1, \ldots, t$. If $t > 2$ there is an indecomposable primitive branched covering $(M, \phi, \mathbb{R}P^2, B_\phi, d)$ realizing $\mathscr{D}$ as branch data.*

Proof. Suppose $t > 2$ and since $\nu(D_1) + \nu(D_2) = d$, by Lemma 4.5 in [9]

there are permutations $\gamma_1 \in D_1$, $\gamma_2 \in D_2$ such that $\langle \gamma_1, \gamma_2 \rangle$ is transitive and $\gamma_1\gamma_2 \in D_{12} = [d/2, d/2]$. Thus $\nu(D_{12}) + \nu(D_3) = d - 2 + d/2$.

If $d/2$ is even, we apply again Lemma 4.5 in [9] and we get permutations $\gamma_{12} \in D_{12}$, $\gamma_3 \in D_3$ such that $\langle \gamma_{12}, \gamma_3 \rangle$ is transitive and $\gamma_{12}\gamma_3 \in D_{123} = [d-1, 1]$ is a $(d-1)$-cycle (because $d/2 \neq 2$). Then, there exist $\alpha \in \Sigma_d$ such that $\gamma_{12}\gamma_3 = \alpha^2$ and $\langle \gamma_{12}, \gamma_3 \rangle$ is primitive. Since γ_{12} and $\gamma_1\gamma_2$ are conjugates, there is $\lambda \in \Sigma_d$ such that $\gamma_{12} = \lambda\gamma_1\gamma_2\lambda^{-1}$. If t is odd, we define the following representation:

$$\begin{aligned} \rho : \langle a, \{\mathrm{u}_j\}_{j=1}^t | a^2 \prod_{j=1}^t \mathrm{u}_j = 1\rangle &\longrightarrow \Sigma_d \\ a &\longmapsto \alpha^{-1} \\ \mathrm{u}_1 &\longmapsto \lambda\gamma_1\lambda^{-1} \\ \{\mathrm{u}_i\}_{i=2}^{t-1} &\longmapsto \lambda\gamma_2\lambda^{-1} \\ \mathrm{u}_t &\longmapsto \gamma_3 \end{aligned}$$

On the other hand, if t is even, then $\nu(D_{123}) + \nu(D_4) = d - 2 + d/2$ and again, applying Lemma 4.5 [9] we get permutations $\gamma_{123} \in D_{123}$ and $\gamma_4 \in D_4$ such that $\langle \gamma_{123}, \gamma_4 \rangle$ is transitive and $\gamma_{123}\gamma_4$ is a $(d-1)$-cycle. Then, there is $\alpha \in \Sigma_d$ such that $\gamma_{123}\gamma_4 = \alpha^2$ and $\langle \gamma_{123}, \gamma_4 \rangle$ is primitive. Notice that there exist $\lambda_1, \lambda_2, \lambda_3 \in \Sigma_d$ such that $\gamma_{123} = \gamma_1^{\lambda_1}\gamma_2^{\lambda_2}\gamma_3^{\lambda_3}$ and in this case we define the following representation:

$$
\begin{array}{rcl}
\rho : \langle a, \{\mathrm{u_j}\}_{j=1}^{t} | a^2 \prod_{j=1}^{t} \mathrm{u_j} = 1 \rangle & \longrightarrow & \Sigma_d \\
a & \longmapsto & \alpha^{-1} \\
\mathrm{u_1} & \longmapsto & \gamma_1^{\lambda_1} \\
\{\mathrm{u_i}\}_{i=2}^{t-2} & \longmapsto & \gamma_2^{\lambda_2} \\
\mathrm{u_{t-1}} & \longmapsto & \gamma_3^{\lambda_3} \\
\mathrm{u_t} & \longmapsto & \gamma_4
\end{array}
$$

Whatever the case $G = Im(\rho)$ is a primitive permutation group and by Theorem YO, the primitive branched covering associated to G is indecomposable.

If $d/2$ is odd, the hypothesis implies t even bigger than or equal to 4, thus $\nu(D_{12}) + \nu(D_3) = (d-1) + (d/2 - 1)$, and by Lemma 4.3 [9] there are $\gamma_{12} \in D_{12}$, $\gamma_3 \in D_3$ such that $\langle \gamma_{12}, \gamma_3 \rangle$ is transitive and $\gamma_{12}\gamma_3$ is a d-cycle. Let $D_{123} = [d]$, then $\nu(D_{123}) + \nu(D_4) = d + (d/2 - 1)$. we can implement Lemma 4.5 in [9] and we get permutations $\gamma_{123} \in D_{123}$, $\gamma_4 \in D_4$ such that $\langle \gamma_{123}, \gamma_4 \rangle$ is transitive and $\gamma_{123}\gamma_4$ is a $(d-1)$-cycle. Then, there is $\alpha \in \Sigma_d$ such that $\gamma_{123}\gamma_4 = \alpha^2$ and $\langle \gamma_{123}, \gamma_4 \rangle$ is primitive. For this case we define a representation like the last in the case before. □

Lemma 3.3. *Let $d \neq 2$ be even and $\alpha, \beta \in \Sigma_d$ are permutations with cyclic structure given by $[2, \ldots, 2]$ such that the group $G = \langle \alpha, \beta \rangle$ is transitive. Then G is imprimitive and unique up to conjugation.*

Proof. We will show that, up to conjugation, $\alpha = (1\ 2)(3\ 4)\ldots(d-1\ d)$ and $\beta = (2\ 3)(4\ 5)\ldots(d-2\ d-1)(d\ 1)$. Thus, the set $B = \{1, 3, \ldots, d-1\}$ will be a nontrivial block of $G = \langle \alpha, \beta \rangle$ (just applying α and β on it) wich makes it an imprimitive permutation group.

Consider $\epsilon, \delta \in \Sigma_d$ with cyclic structure given by $[2, \ldots, 2]$ such that $H = \langle \epsilon, \delta \rangle$ is transitive. Rearrange the cycles of ϵ and δ in the following way: if $(1\ i) \in \epsilon$ and $(i\ j) \in \delta$, we looking at ϵ the cycle containing j, $(j\ k)$, and we write it after $(1\ i)$. Then, we loking at δ the cycle containing k, $(k\ l)$, and we write it after $(i\ j)$. Then, we looking at ϵ the cycle containing l, $(l\ m)$, and we write it after $(j\ k)$ and we follow in the same way. We can move the cycles because they are disjoint and moreover, the process finish because d is finite. Thus $\epsilon = (1\ i)(j\ k)(l\ m)\ldots(y\ w)$, $\delta = (i\ j)(k\ l)\ldots(w\ 1)$, where it is clear that H and G are conjugates. □

Proposition 3.4. *Every primitive branched covering of even degree $d > 4$ on the projective plane, with two ramification points realizing the collection $\{[2,\dots,2],[2,\dots,2]\}$ as branch data is decomposable.*

Proof. If $(M,\phi,\mathbb{R}P^2,\{x,y\},d)$ is a primitive branched covering with branch data $\{[2,\dots,2],[2,\dots,2]\}$ and Hurwitz's representation

$$\begin{aligned}\rho:\langle a,\mathbf{u}_1,\mathbf{u}_2 | a^2\mathbf{u}_1\mathbf{u}_2=1\rangle &\longrightarrow \Sigma_d\\ a &\longmapsto \alpha\\ \mathbf{u}_1 &\longmapsto \gamma_1\\ \mathbf{u}_2 &\longmapsto \gamma_2\end{aligned}$$

then $\mathrm{Im}\rho$ is a transitive permutation group with $\gamma_1,\gamma_2 \in [2,\dots,2]$ and $\alpha^2\gamma_1\gamma_2 = 1$.

If $\langle\gamma_1,\gamma_2\rangle$ is transitive, by Lemma 3.3 it is imprimitive, then the branched covering is decomposable.

If $\langle\gamma_1,\gamma_2\rangle$ is not transitive and $\gamma_1\gamma_2 = 1$ then, without loss of generality, suppose $\gamma_1 = \gamma_2 = (1\ 2)(3\ 4)\dots(d-3\ d-2)(d-1\ d)$. By the relation, the options for α are, up to conjugation, either $\alpha := (2\ 3)(4\ 5)\dots(d-2\ d-1)(d\ \ 1)$ or $\alpha := (1)(2\ 3)(4\ 5)\dots(d-2\ d-1)(d)$. For the first option we have $\mathrm{Im}\rho$ equal to the group considered in the case before, then it is imprimitive. For the second one, notice that $\{1,d\}$ define a block. On the other hand, if $\gamma_1\gamma_2 \neq 1$, using $G = \langle\gamma_1,\gamma_2,\alpha|\gamma_1^2 = \gamma_2^2 = 1, \gamma_1\gamma_2 = \alpha^{-2}\rangle$ we show that $Fix(\gamma_1\gamma_2) = Fix(\gamma_1\gamma_2)^{\gamma_i} = Fix(\gamma_1\gamma_2)^{\alpha}$ for $i = 1,2$, then for all $g \in G$ we have $Fix(\gamma_1\gamma_2)^g = Fix(\gamma_1\gamma_2)$. But $Supp(\gamma_1\gamma_2) \neq \emptyset$ and G is transitive, then $Fix(\gamma_1\gamma_2) = \emptyset$. Therefore every cycle of α has length bigger than or equal to three and γ_1, γ_2 have not commun cycles. Thus, given γ_1, each transposition of γ_2 connects two transpositions of γ_1. Let $O_1,\dots,O_k$ be the orbits of the action of $\langle\gamma_1,\gamma_2\rangle$ on $\{1,\dots,d\}$, with $k > 1$. Notice that $|O_i| \geq 4$ is even, because each transposition of γ_2 connects an even number (bigger than or equal to four) of elements, and if $\langle\gamma_1,\gamma_2\rangle_i$ denotes the restriction of the action of $\langle\gamma_1,\gamma_2\rangle$ on O_i he are in the situation of Lemma 3.3 and $\langle\gamma_1,\gamma_2\rangle_i$ is imprimitive. On the other hand, considering that α makes the group G be transitive and the relation $\gamma_1\gamma_2 = \alpha^{-2}$, we conclude that α connects two orbits, O_i and O_j, only if $|O_i| = |O_j|$. Therefore all them have tha same cardinality. In particular, is a cycle of α conects O_i and O_j, whose elements are in $\gamma_1\gamma_2$ in the way $(a_{i_1}\dots a_{i_n})(a_{i_{n+1}}\dots a_{i_{2n}})$, $(b_{j_1}\dots b_{j_n})(b_{j_{n+1}}\dots b_{j_{2n}})$ respectively, whitout loss of generality, we can suppose that a cycle of α is $(a_{i_1}\ b_{j_1}\ a_{i_2}\ b_{j_2}\dots a_{i_n}\ b_{j_n})$ and thus, the blocks that we have on the orbits, given by the respective restrictions of $\langle\gamma_1,\gamma_2\rangle$ become blocks for G. Then G is imprimitive and the branched covering is decomposable. ☐

Proposition 3.5. *Every primitive branched covering of degree 4 realizing the finite collection $\mathscr{D} = \{[2,2],\dots,[2,2]\}$ as branch data on the projective plane is decomposable.*

Proof. Let $(M, \phi, \mathbb{R}P^2, B_\phi, 4)$ be a primitive branched covering with branch data $\mathscr{D}$. Let $t \geq 2$ be the number of partitions in $\mathscr{D}$ then $\nu(\mathscr{D}) = 2t$, its Hurwitz's representation is given by

$$\begin{array}{rcl} \rho : \langle a, \mathrm{u}_1, \ldots, \mathrm{u}_t | a^2 \prod_{i=1}^{t} \mathrm{u}_i = 1 \rangle & \longrightarrow & \Sigma_4 \\ a & \longmapsto & \alpha \\ \mathrm{u}_i & \longmapsto & \gamma_i \end{array}$$

and the possible images for $\prod_{i=1}^{t} \mathrm{u}_i$ are, without loss of generality, either $(1)(2)(3)(4)$ or $(12)(34)$. Let $U = \langle \gamma_1, \ldots, \gamma_t \rangle < G = \mathrm{Im}\rho$. If U is transitive then $U \cong \langle (12)(34), (13)(24) \rangle$ is imprimitive, where every pair of elements is a block. Thus, if $\rho(\prod_{i=1}^{t} \mathrm{u}_i) = 1$, for every α, $G = \langle U, \alpha \rangle$ is imprimitive. On the other hand, if $\rho(\prod_{i=1}^{t} u_i) = (12)(34)$, necessarly either $\alpha = (1324)$ or $\alpha = (1423)$, whatever the case, we will have $\{1, 2\}$ as block of G. If U is not transitive then $U \cong \langle (12)(34) \rangle$, thus for guarantee the transitivity of G, either $\alpha = (13)(24), (13)$ or (1324) and whatever the case, we always have G imprimitive. $\square$

We summarize the case d even in the following theorem:

Theorem 3.6. *let d even and $\mathscr{D} = \{D_1, \ldots, D_t\}$ an admmisible data on the projective plane such that $d \leq \nu(\mathscr{D}) \equiv 0 \pmod 2$. Then $\mathscr{D}$ is realizable by an indecomposable primitive branched covering on the projective plane if, and only if, it satisfies one of the following cases: either*
(1) $d = 2$, or
(2) There is $i \in \{1, \ldots, t\}$ such that $D_i \neq [2, \ldots, 2]$, or
(3) $d > 4$ and $t > 2$. $\square$

4 Odd degree

When d is even, notice that in $(P_h, \phi, \mathbb{R}P^2, B_\phi, d)$, by (1), h is odd. Recall also that a primitive branched covering on the projective plane with only one ramification point looks like $(\mathbb{R}P^2, \phi, \mathbb{R}P^2, \{x\}, d)$ and it is always decomposable (see Proposition 2.1). Then, the remaining cases are the decomposable data $\mathscr{D} = \{D_1, \ldots, D_t\}$ such that $t > 1$.

Proposition 4.1. *Let d be odd and $\mathscr{D}$ an admissible data on the projective plane such that $d - 1 \leq \nu(\mathscr{D}) \equiv 0 \pmod 2$. If $\mathscr{D}$ contain a partition like $[d]$ then ther is an indecomposable primitive branched covering on the projective plane realizing $\mathscr{D}$ as branch data.*

Proof. Suppose $\mathscr{D} = \{D_1, \ldots, D_t\}$ with $D_t = [d]$. For $i = 1, \ldots, t-1$ choose $\gamma_i \in \Sigma_d$ with cyclic structure given by D_i. If $\prod_{i=1}^{t-1} \gamma_i \neq 1_d$, then its cyclic structure defines a new partition $D = [d_1, \ldots, d_s]$ of d such that $\nu(D) = d - s \equiv \sum_{i=1}^{t-1} \nu(D_i) = \nu(\mathscr{D}) - \nu(D_t) \equiv d - 1 \equiv 0 \pmod 2$, therefore s is odd. If $s = 1$, define $\gamma_t = (\prod_{i=1}^{t-1} \gamma_i)^{-1}$, $\alpha = (1\ 1^{\gamma_t})$ and the representation

$$\begin{array}{rcl} \rho : \langle a, \{\mathrm{u_i}\}_{i=1}^{t} | a^2 \prod_{i=1}^{t} \mathrm{u_i} = 1 \rangle & \longrightarrow & \Sigma_d \\ a & \longmapsto & \alpha \\ \mathrm{u_i} & \longmapsto & \gamma_i \end{array}$$

where $\mathrm{Im}\rho$ is a primitive permutation group, because by the structure of α, every block containing the element 1 contain also $1^\alpha = 1^{\gamma_t}$ and since γ_t is a d-cycle, this block contain everybody, therefore the block is trivial. If $s > 1$, we apply the Corolrio Z for $\prod_{i=1}^{t-1} \gamma_i$ and therefore there is a d-cycle γ_t such that $\prod_{i=1}^{t} \gamma_i$ is a d-cycle and $\langle \prod_{i=1}^{t-1} \gamma_i, \gamma_t \rangle$ is primitive. Moreover, since $\prod_{i=1}^{t} \gamma_i$ is an odd length cycle, there is $\alpha \in \Sigma_d$ such that $\alpha^2 = \prod_{i=1}^{t} \gamma_i$. We define the following representation:

$$\begin{array}{rcl} \rho : \langle a, \{\mathrm{u_i}\}_{i=1}^{t} | a^2 \prod_{i=1}^{t} \mathrm{u_i} = 1 \rangle & \longrightarrow & \Sigma_d \\ a & \longmapsto & \alpha^{-1} \\ \mathrm{u_i} & \longmapsto & \gamma_i \end{array}$$

with $\mathrm{Im}\rho$ primitive because it contains $\langle \prod_{i=1}^{t-1} \gamma_i, \gamma_t \rangle$. Whatever tha case, Theorema YO guarantee that the primitive branched covering associated to each one of the representations above is indecomposable.

If $\prod_{i=1}^{t-1} \gamma_i = 1_d$ and there is some γ_i with a cycle with length bigger than or equal to 3, we change γ_i by γ_i^{-1}. If each γ_i is a product of cycles of length less than or equal to 2 we change the symbol of a transposition by a symbol in another cycle. thus, without change the cyclic structure of the permutations, the new product $\prod_{i=1}^{t} \gamma_i$ is different of the identity and we are in the case before. □

By the proposition above, it remains to analyze the cases where every partition in $\mathscr{D}$ is different of $[d]$.

4.1 A special case

Let $d \in \mathbf{Z}^+$ be an odd integer and $\mathscr{D} = \{D_1, D_2\}$ a decomposable data on the projective plane such that $D_i \neq [d]$, for $i = 1, 2$, and $\nu(\mathscr{D}) = d - 1$.

Since $\mathscr{D}$ is decomposable, there exist a non-trivial factorization of d, say $d = u.w$, and a non-trivial factorization of $\mathscr{D}$, say $\mathscr{D} = \mathscr{U}.\mathscr{W}$.

Conjecture. *Let $\lambda \in D_1$, $\beta \in D_2$. If $\mathscr{U} = \{[u]\}$ then a transitive permutation group like $\langle \alpha, \beta, \lambda | \alpha^2 \lambda \beta = 1 \rangle$ imprimitivo.*

If $\mathscr{U} = \{[u]\}$ then there exist $u+1$ collections of partitions of w, they are $W_0, W_1, \ldots, W_u$, such that, without loss of generality,

$$D_1 = [u.W_0], \quad D_2 = [W_1, \ldots, W_u]$$

with $W_0 \neq [w]$. Then each component of D_1 is divisible by u and each component of D_2 is less than or equal to w.

If t is the number of components of D_1 then $d-t+1$ is the number of components of D_2. Notice that $Fix\beta \neq \emptyset$, otherwise each component of D_2 is bigger than or equal to 2 then $2(d-t+1) \leq d$ and $d \leq 2(t-1)$, impossible because D_1 has t non-trivial components whose sum is d. Thus $2(d-t+1-|Fix\beta|) + |Fix\beta| \leq d$ then

$$d - 2t + 2 \leq |Fix\beta|$$

$$|Supp\beta| \leq 2(t-1)$$

Then β has at most $s \leq t-1$ non-trivial cycles and moreover, it moves at most $2(t-1)$ elements. Let us suppose

$$\lambda = \lambda_1 \ldots \lambda_t$$

$$\beta = \beta_1 \ldots \beta_s.$$

Proposition 4.2. *With the hypothesis and the notation above, if $\langle \lambda, \beta \rangle$ is transitive, the conjecture is true.*

Proof. First of all we will show that under these hypothesis $\lambda\beta$ is a d-cycle. Let us suppose that β_i is a t_i-cycle for $i = 1, \ldots, s$, then

$$\sum_{i=1}^{s} t_i - s + 1 = t \tag{5}$$

because $\nu(\mathscr{D}) = (d-t) + (\sum_{i=1}^{s}(t_i - 1)) = d - t + t - 1 = d - 1$. Then without loss of generality we can define β_1 as the bridge of the first t_1 cycles of λ, this is, β_1 is the cycle formed by the firs element of the first t_1 cycles of λ. Then we consider the permutation $\lambda\beta_1$ with $t - t_1 + 1$ cycles and we define β_2 as the bridge between the first t_2 cyclies of $\lambda\beta_1$, with $\{\text{elements of } \beta_2\} \subset \{1, \ldots, d\} - \{\text{elements of } \beta_1\}$. Now, we consider the permutation $\lambda\beta_1\beta_2$ with $t - t_1 - t_2 - 2$ cycles and we define β_3 in the same way of the cases before. Thus, in the end of the process notice that $\lambda\beta$ defined in this way is a permutation with $t - \sum_{i=1}^{s} t_i + s$ cycles, and by (5) $\lambda\beta$ is a d-cycle.

β_1 is a non-trivial t_1-cycle of β, with $t_1 \leq t$, then β_1 is defined taken one and only one element of t_1 different cycles of λ. Thus, without loss of generality

$$\lambda\beta_1 = (\{\lambda_1\} \dots \{\lambda_{t_1}\})\ \lambda_{t_1+1} \dots \lambda_t \tag{6}$$

where $\{\lambda_i\}$ denotes the sequence of elements in λ_i, for $i = 1, \dots, t_1$. □

Let us suppose $D_1 = [d_{1_1}, \dots, d_{1_r}]$ with $1 < r \leq w$, $\lambda \in D_1$, $\beta \in D_2$ and $\alpha \in \Sigma_d$ such that $\langle \alpha, beta, \lambda | \alpha^2 \lambda \beta = 1 \rangle$ is a transitive permutation group. Then $\lambda = \lambda_1 \dots \lambda_{d_1}$, where λ_i is a d_{1_i}-cycle, $i = 1, \dots, r$ and we can consider $\beta = \beta_1 \dots \beta_u$, where $\beta_i \in W_i$ with the symbols in β_i different of the symbols in β_j, whenever $i \neq j$ for $i = 1, \dots, u$. Since the number of components of D_2 is $d - r + 1$ then $\mathrm{Fix}\beta \neq \emptyset$, otherwise $d \geq 2(d - r + 1)$ then $d \leq 2(w - 1)$ but $d = u(w - 1) + u$. Moreover $|Fix\beta| \geq w(u - 2) + 2$ therefore $|Supp\beta| \leq 2(w - 1)$.

5 Final comments

We would like first to observe that the case where we have two branching points is in fact the main case. Using some algebraic argument in terms of the representations we can show that a positive solution for this case implies that the problem has a solution for all cases.

In few examples we can show that the problem has a posite solution. For example, by brute force the case of degree 9 we can give a complete classification of the coverings in terms of the braching date. We observed that the only cases (which are 4) where there is a branch data such date it can not be realized by an indecomposable is when the excess is 8 which is $d - 1$. Many other examples gives support to possibility that if the excess is greater than $d - 1$ then there is always a realization by an indecomposable.

The table below shows all possible branch data which can be realized by decomposable, which are the ones we are interested. We can show that the only cases where we cann't find a realization by an indecomposable, are theones given item 8, 19, 20 and 26. These cases have excess equal to $8 = d - 1$.

	$4 \leq r_1 + r_2 \leq 10$	$\mathscr{D}$	$\nu(\mathscr{D})$	$\mathscr{U}$	$\nu(\mathscr{U})$	α	β	$\alpha\beta$
1	2+2	{[6,3], [6,3]}	14	{[3], [3]} / {[2,1], [2,1]}	4/2	(1 2 3 4 5 6)(7 8 9)	(1 7 3)(2 4 5 6 8 9)	(1 4 6 7 9 3 5 8 2)
2	2+4	{[6,3], [6,1,1,1]}	12	{[2,1], [2,1]}	2	(1 2 3 4 5 6)(7)(8)(9)	(1 7 8)(2 3 4 5 6 9)	(1 3 5 9 2 4 6 7 8)
3	2+4	{[6,3], [4,2,2,1]}	12	{[2,1], [2,1]}	2	(1 2 3 4)(5 6)(7 8)(9)	(1 5 7)(2 9 3 4 6 9)	(1 9 3 6 7 2 4 5 8)
4	2+4	{[6,3], [3,3,2,1]}	12	{[3]}	2	(1 2 3)(4 5 6)(7 8)(9)	(1 4 7)(2 9 3 5 6 8)	(1 9 3 4 6 7 2 5 8)
5	2+4	{[6,3], [3,2,2,2]}	12	{[2,1], [2,1]}	2	(1 2 3)(4 5)(6 7)(8 9)	(1 4 6)(2 8 3 5 7 9)	(1 8 2 5 6 9 3 4 7)
6	2+6	{[6,3], [3,2,1,1,1,1]}	10	{[3]}	2	(1 2 3)(4 5)(6)(7)(8)(9)	(1 4 6)(2 7 8 9 3 5)	(1 7 8 9 3 4 2 5 6)
7	2+6	{[6,3], [2,2,2,1,1,1]}	10	{[2,1], [2,1]} / {[3]}	2	(1 2)(3)(4 5)(6)(7 8)(9)	(1 3 4)(2 6 7 9 5 8)	(1 6 7 2 3 4 8 9 5)
8	2+8	{[6,3], [2,1,1,1,1,1,1]}	8	{[3]}	2	X	X	X
9	3+3	{[1,2,6], [1,2,6]}	12	{[1,2], [1,2]}	2	(1 2 3 4 5 6)(7 8)(9)	(1)(2 7)(3 9 4 5 6 8)	(1 7 3 5 8 2 9 4 6)
10	3+3	{[1,2,6], [2,3,4]}	12	{[1,2], [2,1]}	2	(1 2 3 4 5 6)(7 8)(9)	(1 7)(2 9 4)(3 5 6 8)	(1 9 4 6 7 3 2 5 8)
11	3+3	{[2,3,4], [2,3,4]}	12	{[1,2], [1,2]}	2	(1 2 3 4)(5 6 7)(8 9)	(1 5)(2 4 8)(3 6 7 9)	(1 8 3 2 6 9 4 5 7)
12	3+3	{[3 3 3], [3,3,3]}	12	{[3], [3]}	4	(1 2 3)(4 5 6)(7 8 9)	(1 4 7)(2 3 5)(6 8 9)	(1 3 4 2 5 8 6 7 9)
13	3+5	{[1,2,6], [4,2,1,1,1]}	10	{[1,2], [2,1]}	2	(1 2 3 4 5 6)(7 8)(9)	(1)(2)(3)(4 7)(5 9 6 8)	(1 2 3 7 5 8 4 9 6)
14	3+5	{[1,2,6], [2,2,2,2,1]}	10	{[1,2], [1,2]}	2	(1 2 3 4 5 6)(7 8)(9)	(1)(2 7)(3 9)(4 6)(5 8)	(1 7 5 4 8 2 9 3 6)
15	3+5	{[2,3,4], [4,2,1,1,1]}	10	{[1,2], [1,2]}	2	(1 2 3 4)(5 6 7)(8 9)	(1)(2)(3)(5 4)(6 8 7 9)	(1 2 3 5 8 6 9 7 4)
16	3+5	{[2,3,4], [2,2,2,2,1]}	10	{[2,1], [2,1]}	2	(1 2)(3 4)(5 6)(7 8)(9)	(1 3)(2 5 7)(4 9 6 8)	(1 5 8 2 3 9 6 7 4)
17	3+5	{[3,3,3], [3,2,2,1,1]}	10	{[3]}	2	(1 2 3)(4 5 6)(7 8 9)	(1)(2)(4 7)(5 3)(6 8 9)	(1 2 5 8 6 7 9 4 3)
18	3+5	{[3,3,3], [3,3,1,1,1]}	10	{[3]}	2	(1 2 3)(4 5 6)(7 8 9)	(1)(2)(4)(5 7 3)(6 8 9)	(1 2 5 8 6 4 7 9 3)
19	3+7	{[3,3,3], [3,1,1,1,1,1,1]}	8	{[3]}	2	X	X	X
20	3+7	[3,3,3], [2,2,1,1,1,1,1]	8	{[3]}	2	X	X	X
21	4+6	{[6,1,1,1], [2,2,2,1,1,1]}	8	{[2,1], [2,1]}	2	(1 2 3 4 5 6)(7)(8)(9)	(1)(2)(3)(4 7)(5 8)(6 9)	(1 2 3 7 4 8 5 9 6)
22	4+6	{[4,2,2,1], [2,2,2,1,1,1]}	8	{[2,1], [2,1]}	2	(1 2)(3 4)(5 6)(7)(8)(9)	(1)(3 2)(5 4)(6 7 8 9)	(1 3 5 7 8 9 6 4 2)
23	4+6	{[3,2,2,2], [2,2,2,1,1,1]}	8	{[1,2], [1,2]}	2	(1 2)(3 4)(5 6)(7)(8)(9)	(1 3)(2 5)(4 7)(6 8 9)	(1 5 8 9 6 2 3 7 4)
24	5+5	{[4,2,1,1,1], [4,2,1,1,1]}	8	{[2,1], [2,1]}	2	(1 2 3 4)(5 6)(7)(8)(9)	(1)(2)(3)(5 7)(6 8 9 4)	(1 2 3 6 7 5 8 9 4)
25	5+5	{[4,2,1,1,1], [2,2,2,2,1]}	8	{[2,1], [2,1]}	2	(1 2 3 4)(5 6)(7)(8)(9)	(1)(2 5)(3 7)(4 8)(6 9)	(1 5 9 6 2 7 3 8 4)
26	5+5	{[2,2,2,2,1], [2,2,2,2,1]}	8	{[2,1], [2,1]}	2	X	X	X
27	4+4	{[6,1,1,1], [6,1,1,1]}	10	{[2,1], [2,1]}	2	(1 2 3 4 5 6)(7)(8)(9)	(1)(2)(3)(4 7 8 9 5 6)	(1 2 3 7 8 9 5 4 6)
28	4+4	{[6,1,1,1], [4,2,2,1]}	10	{[2,1], [2,1]}	2	(1 2 3 4)(5 6)(7 8)(9)	(1)(2)(3)(4 5 7 9 6 8)	(1 2 3 5 8 9 6 7 4)
29	4+4	{[6,1,1,1], [3,2,2,2]}	10	{[2,1], [2,1]}	2	(1 2 3 4 5 6)(7)(8)(9)	(1 7)(2 8)(3 9)(4 5 6)	(1 8 2 9 3 5 4 6 7)
30	4+4	{[4,2,2,1], [4,2,2,1]}	10	{[2,1], [2,1]}	2	(1 2 3 4)(5 6)(7 8)(9)	(1)(2 5)(3 7)(4 9 6 8)	(1 5 8 3 9 6 2 7 4)
31	4+4	{[4,2,2,1], [3,2,2,2]}	10	{[2,1], [2,1]}	2	(1 2 3 4)(5 6)(7 8)(9)	(1 5)(2 7)(3 9)(4 6 8)	(1 7 4 5 8 2 9 3 6)
32	4+4	{[3,2,2,2], [3,2,2,2]}	10	{[1,2], [1,2]}	2	(1 2)(3 4)(5 6)(7 8 9)	(1 3)(2 5)(4 7)(6 8 9)	(1 5 8 6 2 3 7 9 4)

References

[1] K. Barański, On realizability of branched coverings of the sphere, *Topology Appl.*, **116** 3, 279-291 (2001)

[2] N. A. V. Bedoya, Revestimentos remificados e o problema da decomponibilidade, Tese de doutorado IME-USP-Junho-2008.

[3] N. A. V. Bedoya, Decomposability problem on branched coverings, preprint (2008).

[4] S. Bogataya, S. Bogatyĭ and H. Zieschang. On compositions of open mappings .*Mat. Sb.*, **193**, 3-20 (2002)

[5] S. A. Bogatyi, D. L. Gonçalves, E. A. Kudryavtseva and H. Zieschang, Realization of primitive branched coverings over closed surfaces.*Advances in topological quantum field theory*, **179**, 297-316 (2004)

[6] S. Bogatyi, D. L. Gonçalves, E. Kudryavtseva and H. Zieschang. Realization of primitive branched coverings over closed surfaces following the Hurwitz approach.*Cent. Eur. J. Math.*, **1** (2003)

[7] K. Borsuk and R. Molski, On a class of continuous mappings, *Fund. Math.*, **45**, 84-98 (1957)

[8] J. D.Dixon and B. Mortimer, Permutation groups, *Graduate Texts in Mathematics*, **163**, (1996)

[9] A. L. Edmonds, R. S. Kulkarni and R. E. Stong, Realizability of branched coverings of surfaces,*Trans. Amer. Math. Soc.*, **282**, (1984)

[10] C. L. Ezell, Branch point structure of covering maps onto nonorientable surfaces, *Trans. Amer. Math. Soc.*, **243**, 123-133 (1978)

[11] A. Hurwitz, Über Riemaniannische Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.*, **39**, 1-60 (1891)

[12] D. H. Husemoller, Ramified coverings of Riemann surfaces, *Duke Math. J.*, **29**, 167-174 (1962)

[13] E. Pervova and C. Petronio, On the existence of branched coverings between surfaces with prescribed branch data. I. *Algebr. Geom. Topol.*, **6**, 1957-1985 (2006)

[14] E. Pervova and C. Petronio, Realizability and exceptionality of candidate surface branched covers: methods and results, **arXiv.org:0709.0150**, (2007)

[15] M. A. Pascali and C. Petronio, Surface branched covers and geometric 2-orbifolds, **arXiv.org:0709.2026**, (2007)

[16] F. J. Ritt, Prime and composite polynomials, *Trans. Amer. Math. Soc.*, **23**, 51-66 (1922)

TRABALHOS DO DEPARTAMENTO DE MATEMÁTICA

TÍTULOS PUBLICADOS

2007.1 GREBENEV, V. N., GRISHKOV, A. N. and OBERLACK, M. Lie algebra methods in Statistical Theory of Turbulence. 29p.

2007.2 FUTORNY, V. and SERGEICHUK, V. Change of the congruence canonical form of 2 x 2 and 3 x 3 matrices under perturbations. 18p.

2007.3 KASHUBA, I. and PATERA, J. Discrete and continuous exponential transforms of simple Lie groups of rank two. 24p.

2007.4 FUTORNY, V. and SERGEICHUK, V. Miniversal deformations of matrices of bilinear forms. 34p.

2007.5 GONÇALVES, D. L., HAYAT, C., MELLO, M.H.P.L. and ZIESCHANG, H. Spin-structures of bundles on surfaces and the fundamental group. 22p.

2007.6 GOODAIRE, E. G. and MILIES, C.P. Polynomial and group identities in alternative loop algebras.7p.

2007.7 GOODAIRE, E. G. and MILIES, C.P. Group identities on symmetric units in alternative loop algebras. 8p.

2007.8 ALEXANDRINO, M. M. Singular holonomy of singular Riemannian foliations with sections. 16p.

2007.9 ASPERTI, A. C. and VALÉRIO, B. C. Ruled Weingarten surfaces in a 3-dimensional space form. 6p.

2007.10 ASPERTI, A. C. and VILHENA, J. A. M. Spacelike surfaces in $\mathbb{L}^4$ with prescribed Gauss map and nonzero mean curvature. 22p.

2007.11 GOODAIRE, E. G. and MILIES, C. P. Involutions of RA loops. 13p.

2007.12 FERRAZ, R. A., MILIES, C.P., DEL RIO, Á. and SIMÓN, J. J. Group codes from subgroups. 9p.

2007.13 ALEXANDRINO, M. M. and TÖBEN, D. Equifocality of a singular Riemannian foliation. 10p.

2007.14 WEISS, M. A. A polynomial algorithm for deciding 3-sat. 23p.

2008.1 GRISHKOV, A. N., ZAVARNITSINE, A.V. and GIULIANI, M. L. M. Classification of subalgebras of the Cayley algebra O (q) over a finite field. 17p.

2008.2 GRISHKOV, A. and LOGINOV, E. Some generalizations of groups with triality. 13p.

2008.3 GRISHKOV, A. N. and GUERREIRO, M. On simple Lie algebras of dimension seven over fields of characteristic 2. 16p.

2008.4 GRISHKOV, A. and NAGY, G. P. Algebraic bol loops. 12p.

2008.5 GRISHKOV, A. and CHESTAKOV, I., Commutative Moufang loops and alternative algebras. 15p.

2008.6 CARRION V., H. D. Entire functions on banach spaces with u property. 10p.

2008.7 BONDARENKO, DOKUCHAEV, M.A., V. M., GUBARENI, N. M., KHIBINA, M. A. and KIRICHENKO, V. V. Representations of primitive posets. 32p.

2008.8 DOKUCHAEV, M. A., KIRICHENKO, V. V., NOVIKOV, B. V. and PLAKHOTNYK, M. V. Finite metric spaces and Latin squares. 23p.

2008.9 BEDOYA, N. A. V. and GONÇALVES, D. L. Decomposability of branched coverings on the projective plane. 13p

Nota: Os títulos publicados nos Relatórios Técnicos dos anos de 1980 a 2006 estão à disposição no Departamento de Matemática do IME-USP.
Cidade Universitária "Armando de Salles Oliveira"
Rua do Matão, 1010 - Cidade Universitária
Caixa Postal 66281 - CEP 05315-970