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Conformal Sub-Riemannian Geometry in Dimension 3

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0 Introduction

Conformal geometry considers scale-invariant properties of Riemannian manifolds. See [14] for a modern introduction to conformal structures via second order frames. Complex structures are related to oriented conformal structures in such a way that, in real dimension two, both structures are equivalent.

Any 3-dimensional manifold has a contact structure ([13, 15]), that is, a non- integrable distribution characterized by $\theta \wedge d\theta \neq 0$ at every point, where θ is a 1-form anihilating the distribution. Contact structures first appeared in Mechanics, but they also appeared intermingled with complex structures in [16]. Poincaré examined the boundary of domains in C^2 in an attempt to understand uniformization in the case of two complex variables. The structure on the boundary is nowadays abstracted in the concept of a CR-structure (Cauchy-Riemann structure).

It turns out that we may view the CR-structure as a conformal structure on the contact distribution. Webster defined the concept of pseudo-Hermitian structure in [19] and noted that it relates to CR-structures in the same way as Riemannian structures relate to conformal ones. In [12], the general case of a metric structure on a contact distribution is treated and in [10] the conformal geometry of this structure is analysed based on the treatment of the CR case ([3, 4, 6]).

A metric defined on a distribution is also called a sub-Riemannian structure. In dimension 3 a contact sub-Riemannian structure is equivalent to a pseudo-Hermitian structure, and its conformal geometry is equivalent to a CR structure. In this paper we make explicit the relation between sub-Riemannian, pseudo-Hermitian and CR structures in the case of dimension 3. We analyse Cartan's CR invariant by expressing it in terms of sub-Riemannian data and compute it for all homogeneous sub-Riemannian manifolds classified in [9]. See Table 1.

1 Basic structures

Let $\mathcal{D}$ be a contact distribution defined on a 3-dimensional smooth manifold M, that is, there is a 1-form θ on M such that $\ker d\theta = \mathcal{D}$ and $\theta \wedge d\theta \neq 0$. We will consider the following structures:

Definition 1.1 *a.* $(M, \mathcal{D}, J)$ is a *CR-structure* if J is a smoothly varying linear endomorphism on $\mathcal{D}$ which satisfies $J^2 = -1$.

b. $(M, \mathcal{D}, g)$ is a *sub-Riemannian structure* if g is a smoothly varying positive definite symmetric bilinear form on $\mathcal{D}$.

c. $(M, \mathcal{D}, [g])$ is a *conformal sub-Riemannian structure* if $[g]$ is a conformal class of sub-Riemannian metrics.

2 Sub-Riemannian and pseudo-Hermitian structures

Let $(M, \mathcal{D}, g)$ be a sub-Riemannian structure. The adapted coframe bundle is the bundle of positively oriented orthonormal adapted coframes θ, θ^1, θ^2 satisfying $d\theta = 2\theta^1 \wedge \theta^2$. If θ', $\theta^{1'}$, $\theta^{2'}$ is another adapted coframe, then

$$\begin{aligned} \theta' &= \theta \\ \theta^{i'} &= a^i_j \theta^j \quad \text{where } (a^i_j) \in SO(2) \end{aligned}$$

Theorem 2.1 ([11, 19]) *There exists a unique connection form ω and torsion forms τ^1, τ^2 such that*

$$
\begin{aligned}
d\theta^1 &= \theta^2 \wedge \omega + \theta \wedge \tau^1 \\
d\theta^2 &= -\theta^1 \wedge \omega + \theta \wedge \tau^2
\end{aligned}
$$

with $\tau^1 \wedge \theta^1 + \tau^2 \wedge \theta^2 = 0$.

The curvature form is

$$\Omega = d\omega$$

and we write then

$$\Omega = K\theta^1 \wedge \theta^2 + W_1\theta^1 \wedge \theta + W_2\theta^2 \wedge \theta \tag{1}$$

It will be important to collect the Bianchi identities in the following. First observe that we may choose θ^1 and θ^2 such that $\tau^1 = \tau_0\theta^1$ and $\tau^2 = -\tau_0\theta^2$. This defines a parallelism on the manifold in the case $\tau_0 \neq 0$. We have

$$
\begin{aligned}
-W_1 - \tau_{02} &= 2\tau_0\omega_1 \\
W_2 - \tau_{01} &= 2\tau_0\omega_2 \\
K_0 - W_{12} - W_1\omega_1 + W_{21} - W_2\omega_2 &= 0 \\
-\omega_{12} + \omega_{21} - (\omega_1)^2 - (\omega_2)^2 + 2\omega_0 &= K \\
-\omega_{10} + \omega_{01} - \omega_1\tau_0 - \omega_0\omega_2 &= W_1 \\
-\omega_{20} + \omega_{02} - \omega_0\omega_1 - \omega_2\tau_0 &= W_2
\end{aligned}
$$

Here we used the convention $\alpha = \alpha_1\theta^1 + \alpha_2\theta^2 + \alpha_0\theta$ for a 1-form α on M and, in particular, $df = f_1\theta^1 + f_2\theta^2 + f_0\theta$ for a function f defined on M.

We draw some consequences from the Bianchi identities. If $\tau_0 = 0$ then $W_1 = W_2 = 0$. On the other hand, in the case that $\tau_0 \neq 0$ and K, τ_0, W_1, W_2 are constant, we have the relations

$$\omega_1 = -W_1/2\tau_0, \quad \omega_2 = W_2/2\tau_0 \quad \text{and} \quad \omega_0 = \frac{1}{2}(K + W_1^2 + W_2^2). \tag{2}$$

There are two possible cases: $W_1 = -W_2$ and $W_1 = W_2$. If $W_1 \neq 0$, then $\omega_0 = \tau_0$ and $\omega_0 = -\tau_0$ in each case, respectively.

We will show how the sub-Riemannian structure is related to the pseudo-Hermitian structure defined by Webster in [19]. In fact, in dimension 3, the structures are the same. A pseudo-Hermitian structure is defined fixing a distribution $\mathcal{D}$, a complex operator J in $\mathcal{D}$ and fixing a form $\theta = \theta$ with kernel $\mathcal{D}$. The equations of Webster's connection are obtained by defining

$$\begin{aligned} \theta^1 &= \theta^1 + i\theta^2 \\ \tau^1 &= \tau^1 + i\tau^2 \\ \theta^1_1 &= -i\omega \end{aligned}$$

With this notation, we easily see that we have

$$\begin{aligned} d\theta &= i\theta^1 \wedge \theta^{\bar{1}} \\ d\theta^1 &= \theta^1 \wedge \theta^1_1 + \theta \wedge \tau^1 \end{aligned}$$

with conditions

$$\begin{aligned} \theta^1_1 + \theta^{\bar{1}}_{\bar{1}} &= 0 \\ \tau^1 \wedge \theta^{\bar{1}} &= 0 \end{aligned}$$

Here $\theta^{\bar{1}}_{\bar{1}} = \bar{\theta}^1_1$.

The curvature of this connection is

(3) $$\Omega^1_1 = d\theta^1_1 = R\theta \wedge \theta^{\bar{1}} + W\theta^1 \wedge \theta - \bar{W}\theta^{\bar{1}} \wedge \theta.$$

Comparing the pseudo-Hermitian curvature (3) with the sub-Riemannian one (1), we obtain

(4) $$R = \frac{1}{2}K$$

(5) $$W = -\frac{1}{2}W_2 - \frac{i}{2}W_1$$

3 Sub-conformal and CR-structures

In this section we describe the bundles which are associated to the CR-structure and to the sub-conformal structure. Note that in dimension 3 both structures coincide. See [6] for details on the CR-case and [10] for the sub-conformal case.

Let $(M, \mathcal{D}, [g])$ be a sub-conformal structure. We let E' be the line-bundle of all sub-Riemannian metrics in the conformal class $[g]$. Given a sub-Riemannian metric, that is, a section of this bundle, there exists a canonical contact form θ such that

$$d\theta = 2\theta^1 \wedge \theta^2 + h_i\theta^i \wedge \theta$$

where θ^i is an orthonormal coframe on $\mathcal{D}$. We could also consider the line bundle E of all contact forms associated to the sub-Riemannian metrics in this sense. It is clear then that there exists a fiber bundle isomorphism between those two bundles. To explicit this isomorphism, consider a trivialization of E', that is, a choice of a sub-Riemannian metric g and the corresponding trivialization of E, that is, the contact form θ. Then the bundle map is defined as $\lambda g \to \lambda\theta$. We will identify E with E' in the following considerations.

We will construct a bundle Y of 1-forms over the bundle E. We begin by defining the tautological form ω. Given a point e in E, consider a coframe θ^i as above. At e we consider the pull-back of θ^i and all forms defined by

$$\omega^i = \sqrt{\lambda} a^i_j \theta^j + v^i\omega \quad \text{where } (a^i_j) \in SO(2);$$

finally we define the form ϕ by imposing the equation

$$d\omega = 2\omega^1 \wedge \omega^2 + \omega \wedge \phi.$$

Observe that for each choice of ω^i, ϕ is then any form in the familly

$$\phi = -\frac{d\lambda}{\lambda} + 2(v^1\omega^2 - v^2\omega^1) + s\omega$$

The bundle of all forms ω, ω^i, ϕ is denoted by Y. This is a G-structure with G the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ v^i & u^i_j & 0 \\ s & -2v^k u^k_j & 1 \end{pmatrix}$$

where $(u^i_j) \in SO(2)$.

In the case of CR-structures we also form the line bundle E of contact forms and denote by $\omega = \omega$ the tautological form. Y will be the G-structure of all coframes satisfying the equation

$$d\omega = i\omega^1 \wedge \omega^{\bar{1}} + \omega \wedge \phi.$$

where $\omega^1 = \omega^1 + i\omega^2$, $\phi = \phi$.

In dimension 3, a sub-conformal structure is equivalent to a CR-structure. In complex notation write the group G as the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ v^1 & u & 0 & 0 \\ v^{\bar 1} & 0 & \bar u & 0 \\ s & iuv^{\bar 1} & -i\bar u v^1 & 1 \end{pmatrix}$$

where $u \in U(1)$.

We use now the convention $\alpha^{\bar 1} = \bar\alpha^1$ and $\alpha_1 = \alpha^{\bar 1}$ for α^1 a complex valued 1-form.

Theorem 3.1 **([3, 4, 6])** *On Y there exists a unique parallelism given by the forms ω, ω^i, ϕ, ϕ^i_j, ϕ^i, ψ such that the following equations are satisfied*

$$\begin{aligned} d\omega &= i\omega^1 \wedge \omega^{\bar 1} + \omega \wedge \phi \\ d\omega^1 &= \omega^1 \wedge \phi^1_1 + \omega \wedge \phi^1 \\ d\phi &= i\omega_{\bar 1} \wedge \phi^{\bar 1} + i\phi_{\bar 1} \wedge \omega^{\bar 1} + \omega \wedge \psi \\ d\phi^1_1 &= i\omega_1 \wedge \phi^1 - 2i\phi_1 \wedge \omega^1 - \frac{1}{2}\psi \wedge \omega \\ d\phi^1 &= \phi \wedge \phi^1 + \phi^1 \wedge \phi^1_1 - \frac{1}{2}\psi \wedge \omega^1 + Q\omega^{\bar 1} \wedge \omega \end{aligned}$$

with the condition $\phi - \phi^1_1 - \phi^{\bar 1}_{\bar 1} = 0$.

We also have one more equation:

$$d\psi = -\psi \wedge \phi + 2i\phi^1 \wedge \phi_1 + \rho \wedge \omega$$

It is not difficult to see that $\rho = 2Q_{\bar 1}\theta^1 + 2Q_1\theta^{\bar 1}$. See Cheng ([5]).

The Cartan connection on Y is the $\mathfrak{su}(2,1)$ valued form

$$\pi = \begin{pmatrix} -\frac{1}{3}(\phi^1_1 + \phi) & \omega^1 & 2\omega \\ -i\phi_1 & \frac{1}{3}(2\phi^1_1 - \phi) & 2i\omega_1 \\ -\frac{1}{4}\psi & \frac{1}{2}\phi^1 & \frac{1}{3}(\phi + \phi^{\bar 1}_{\bar 1}) \end{pmatrix}$$

and the curvature form for this connection is

$$\Pi = \begin{pmatrix} 0 & 0 & 0 \\ -i\bar Q\omega^1 \wedge \omega & 0 & 0 \\ -\frac{1}{4}\rho \wedge \omega & \frac{1}{2}Q\omega^{\bar 1} \wedge \omega & 0 \end{pmatrix}$$

(see ([6]).

In the case of a conformal sub-Riemannian structure in arbitrary dimension we have the following theorem proved in [10].

Theorem 3.2 ([10]) *There exists a unique parallelism of Y given by the forms ω, ω^i, ϕ, ω^i_j, ϕ^i, ψ satisfying the equations*

$$
\begin{aligned}
&d\omega = h_{ij}\omega^i \wedge \omega^j + \omega \wedge \phi \\
&d\phi = 2h_{ij}\phi^i \wedge \omega^j + b_{ij}\omega^i \wedge \omega^j + \omega \wedge \psi \\
&d\omega^i_j + \omega^i_k \wedge \omega^k_j + h_{kj}\omega^i \wedge \phi^j - h_{ki}\omega^j \wedge \phi^k - \frac{1}{2}b_{kj}\omega^k \wedge \omega^i + \frac{1}{2}b_{ki}\omega^k \wedge \omega^j \\
&\quad -h_{kj}\phi^j \wedge \omega^k + h_{ki}\phi^j \wedge \omega^k + h_{ij}\omega^k \wedge \phi^k = \\
&\quad S^i_{jkm}\omega^k \wedge \omega^m + (V^i_{jk}\omega^k + W^i_{jk}\phi^k) \wedge \omega \\
&d\phi^i - \frac{1}{2}\phi \wedge \phi^i - \phi^j \wedge \omega^i_j + \frac{1}{2}\psi \wedge \omega^i = \\
&\quad \frac{1}{2}(V^i_{jk} + V^i_{kj})\omega^k \wedge \omega^j + W^i_{jk}\phi^k \wedge \omega^j + P^i_j\omega^j \wedge \omega + R^i_j\phi^j \wedge \omega + U\psi \wedge \omega \\
&d\psi + \psi \wedge \phi - 2h_{ij}\phi^i \wedge^j + (2h_{ij}R^i_k + 4b_{jk})\phi^k \wedge \omega^j + 2h_{ij}U^i\psi \wedge \omega^j = \\
&\quad \rho \wedge \omega + Q_{ij}\omega^i \wedge \omega^j
\end{aligned}
$$

with conditions

$$
\begin{aligned}
\omega^i_j &= -\omega^j_i \\
(b_{ij}) &\in \mathfrak{g} \\
(S^i_{jim}) &\in (\mathfrak{g}')^\perp \\
V^i_{ji} &= 0 \\
P^i_i &= 0
\end{aligned}
$$

where $\mathfrak{g} = \ker(ad_H) \cap so(2n)$, $(\mathfrak{g}')^\perp = im(ad_H) \cap sim(2n)$, *for* $H = (h_{ij})$, ρ

is a 1-form, and the conditions

$$\begin{aligned}
h_{ij} &= -h_{ji} \\
b_{ij} &= -b_{ji} \\
b_{ijk} &= -b_{jik} \\
b_{ijk} + b_{jki} + b_{kij} &= 0 \\
S^i_{jkm} = -S^j_{ikm} &= -S^i_{jmk} \\
S^i_{jkm} + S^i_{kmj} + S^i_{mjk} &= 0 \\
V^i_{jk} &= -V^j_{ik} \\
W^i_{jk} &= -W^j_{ik}
\end{aligned}$$

(sim($2n$) denotes the vector space of $2n \times 2n$ real symmetric matrices).

The parallelism defined by Chern for the CR-structure in [6] corresponds to the case in the above Theorem where $H = J$ and $(\omega^i_j) \in \mathfrak{g}$.

4 Cartan's invariant

Cartan's invariant is Q and it is defined on Y. A change in coframes

$$\begin{aligned}
\theta' &= \lambda\theta \\
\theta^{1'} &= \sqrt{\lambda} u \theta^1 + v^1 \theta
\end{aligned}$$

transforms Cartan's invariant

$$Q = Q'(\lambda)^2 u^2 \tag{6}$$

Definition 4.1 We say a CR-structure is *umbilic* at $p \in M$ if $Q(p) = 0$. Otherwise, we say the structure is *non-umbilic* at p.

To obtain an invariant defined on the base manifold M, we first observe that, if the point is umbilical, then the invariant $Q = 0$ is well defined on the base manifold at p. If the point is not umbilical, we shall find a parallelism on M at p by imposing $Q'(p) = i$. This condition fixes a unique coframe θ, θ^1 at p. Invariants A, B and C are thus obtained on M from the equation

$$d\theta^1 = A\theta^1 \wedge \theta^{\bar{1}} + B\theta \wedge \theta^1 + C\theta \wedge \theta^{\bar{1}} \tag{7}$$

Remark 4.1 a. If a CR-structure is umbilical on a neighbourhood of a point, then the CR-structure is locally CR-equivalent to S^3 near that point.

b. If two nowhere umbilical CR-structures have the same constant invariants A, B and C, then they are locally CR-equivalent.

5 Cartan's invariant via sub-Riemannian invariants

It is now our goal to write Cartan's invariant in terms of the sub-Riemannian data. Following Webster [19] and a correction in [1], we embbed the sub-Riemannian structure into the sub-conformal structure by fixing a section

$$(8)\qquad \begin{aligned} \phi_1^1 &= \theta_1^1 + \frac{1}{4} R\theta \\ \phi^1 &= \tau^1 + \frac{i}{4} R\theta^1 + E\theta \\ \psi &= G\theta + i(\bar{E}\theta^1 - E\theta^{\bar{1}}) \end{aligned}$$

where $E = \frac{R_1}{6} + \frac{2iW}{3}$

Recall the last equation from Theorem 3.1

$$(9)\qquad d\phi^1 - \phi \wedge \phi^1 - \phi^1 \wedge \phi_1^1 + \frac{1}{2}\psi \wedge \omega^1 = Q\omega^{\bar{1}} \wedge \omega$$

and substitute relations (8) into (9) to get

$$(10)\qquad Q = \tau_0 \theta_{\bar{1}0}^{\bar{1}} - \tau_0 \theta_{10}^1 - \frac{i}{2} R\tau_0 + E\theta_{1\bar{1}}^1$$

To simplify the exposition we will suppose from now on that K, τ_0, W_1, W_2, are constants. If $\tau_0 = 0$ then $Q = 0$. Otherwise, substituting the sub-Riemannian data (4), (5) and (2) into (10), we get

$$Q = i\{\frac{3}{4}\tau_0 K + \frac{1}{6\tau_0}(W_1^2 + W_2^2)\}$$

As we observed before, to obtain the parallelism in the non-umbilical case we set $Q' = i$ in (6) to get

$$\lambda^2 u^2 = \frac{3}{4}\tau_0 K + \frac{1}{6\tau_0}(W_1^2 + W_2^2)$$

Now there are two cases to consider, whether the right hand side of the equation above is positive or negative. We have respectively $u = 1$ and

$u = i$ and find $C = \frac{u\tau_0}{\bar{u}|\lambda|}$ from (7). Similarly, $A = \frac{u}{4\tau_0\sqrt{|\lambda}}(W_2 + iW_1)$ and $B = -\frac{i}{2|\lambda|}(K + 2W_1^2)$. It is better to work with the following expression as Cartan's invariant

$$C = \frac{u}{\bar{u}}\frac{1}{C^2} = \frac{3}{4}\frac{K}{\tau_0} + \frac{1}{3}\frac{W_1^2}{\tau_0^3},$$

as it allows to detect some umbilical cases with non-vanishing torsion. We next compute the value of this invariant for the examples of homogeneous sub-Riemannian manifolds classified in [9]. The final results are summarized in Table 1.

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type	G	A	B	τ_0	K	W_1	W_2	C
1	H^3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0	0	0	0	0
2a	$\widetilde{Euc_2^+}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$\frac{a}{2}$	0	0	$\frac{3}{2}$
3a	$Poinc_2^+$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$-\frac{a}{2}$	0	0	$-\frac{3}{2}$
4ad	$\widetilde{SU(1,1)}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$	$\frac{a-d}{4}$	$-\frac{a+d}{2}$	0	0	$-\frac{3}{2}\frac{1+s^2}{1-s^2}$
5ad	$\widetilde{SU(1,1)}'$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$	$\frac{a+d}{4}$	$\frac{a-d}{2}$	0	0	$\frac{3}{2}\frac{1-s^2}{1+s^2}$
6ad	$SU(2)$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$	$\frac{a-d}{4}$	$\frac{a+d}{2}$	0	0	$\frac{3}{2}\frac{1+s^2}{1-s^2}$
7ab	$\Sigma_+(b)$	$\begin{pmatrix} 0 & 1 & -b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$a(\frac{1}{2}-b^2)$	$a^{\frac{3}{2}}b\frac{\sqrt{2}}{4}$	$-a^{\frac{3}{2}}b\frac{\sqrt{2}}{4}$	$\frac{3}{2}-\frac{b^2}{3}$
8ab	$\Sigma_-(b)$	$\begin{pmatrix} 0 & -1 & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$-a(\frac{1}{2}+b^2)$	$a^{\frac{3}{2}}b\frac{\sqrt{2}}{4}$	$a^{\frac{3}{2}}b\frac{\sqrt{2}}{4}$	$-\frac{3}{2}-\frac{b^2}{3}$

Table 1: 3-dimensional sub-homogeneous spaces: all of them are Lie groups G; $\{X_1, X_2, Y = [X_1, X_2]\}$ is a basis of the Lie algebra of G, $\{X_1, X_2\}$ is a basis of the distribution and A is the matrix of ad_Y restricted to the distribution; B is the matrix of the inner product on the distribution; τ_0, K, W_1 and W_2 are sub-Riemannian invariants; a, b and d are positive parameters; we may assume $a \geq d$ for types (4) and (6); each one of types (1), (2), (3), (7b) and (8b) gives rise to a unique homogeneous conformal sub-Riemannian manifold (or, what is the same, homogeneous CR-manifold), but each one of types (4), (5) and (6) gives rise to a one-parameter family of homogeneous conformal sub-Riemannian manifolds indexed by $s = (d/a)^{1/2}$, and $s \in (0, 1]$ for types (4) and (6) and $s > 0$ for type (5); C is Cartan's CR invariant.

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