Quantum integrability of the Alday-Arutyunov-Frolov model

A. Melikyan,[a,b] **A. Pinzul,**[a,b] **V.O. Rivelles**[c] **and G. Weber**[c]

[a]*Instituto de Física, Universidade de Brasília,*
70910-900, Brasília, DF, Brasil

[b]*International Center of Condensed Matter Physics,*
C.Postal 04667, Brasilia, DF, Brazil

[c]*Instituto de Física, Universidade de São Paulo,*
C. Postal 66318, 05315-970, São Paulo, SP, Brasil

E-mail: amelik@gmail.com, apinzul@unb.br, rivelles@fma.if.usp.br, weber@fma.if.usp.br

ABSTRACT: We investigate the quantum integrability of the Alday-Arutyunov-Frolov (AAF) model by calculating the three-particle scattering amplitude at the first non-trivial order and showing that the S-matrix is factorizable at this order. We consider a more general fermionic model and find a necessary constraint to ensure its integrability at quantum level. We then show that the quantum integrability of the AAF model follows from this constraint. In the process, we also correct some missed points in earlier works.

KEYWORDS: Exact S-Matrix, Integrable Field Theories, Bethe Ansatz, Sigma Models

ARXIV EPRINT: 1106.0512

Contents

1 Introduction

The AdS/CFT correspondence [1] has provided deep insight into the intricate dynamics of both gauge and string theories, much of which can be ascribed to the uncovering of integrable structures on both sides of the duality [2–7].[1] On the string theory side, the classical integrability of the sigma-model describing the superstring on $AdS_5 \times S^5$ is relatively well understood [9]. Though, in order to better understand the AdS/CFT conjecture it is necessary to quantize the superstring theory. This, however, has not yet been achieved using the conventional methods developed in the context of quantum integrable systems.

---

[1]For a comprehensive review of the role of integrability in the context of the AdS/CFT correspondence, see [8].

It is, nevertheless, feasible and interesting to study string theory truncated to smaller subsectors which are dual to closed sectors of the gauge theory [10–14]. Even though such reduced models may lose some important properties of the full theory, such as conformal invariance, they are still expected to be classically integrable, providing simpler but nonetheless representative examples of the difficulties associated with the quantization of the superstring theory on $AdS_5 \times S^5$. One important case is the Alday-Arutyunov-Frolov (AAF) model [14]. It arises in this context as the consistent truncation of the superstring theory in the uniform gauge to the $\mathfrak{su}(1|1)$ sector. It is interesting also to note that the AAF model also appears in a subsector of type IIA superstrings on $AdS_4 \times \mathbb{CP}^3$ [15].

The AAF model is a particularly interesting example of a classically integrable model in several aspects. It is the first non-trivial purely fermionic integrable model which is highly non-linear and singular[2] when compared to the standard case of the fermionic Thirring model. Such singularities in the context of integrable systems are hard to handle. One example where such difficulties are present is the Landau-Lifshitz model [16–18], which is generated by a string in the $\mathfrak{su}(2)$ subsector. It has been shown that the complete understanding of the quantum inverse scattering method is only possible upon careful analysis of the singularities, the associated quantum operators, and the reconstruction of the correct Hilbert space. Another intriguing aspect of the AAF model is its non-linear structure. Unlike the simpler Thirring model where the interaction vertex is of fourth order in the fermionic fields, the AAF model contains also a sixth order interaction term. As we will see, this introduces further obstacles in the analysis of quantum integrability. It is the purpose of this paper to understand the interplay among the several types of interactions which in the end results in the quantum integrability of the model.

The AAF model inherits the classical integrability of the superstring theory on $AdS_5 \times S^5$, as the Lax representation of the full string sigma-model admits the same consistent truncation [14]. This classical integrability was then conjectured to hold at quantum level by [19], where the corresponding Bethe equations were derived from the knowledge of the two-particle S-matrix and the assumption of the S-matrix factorization. In this case the AAF model was regarded as a two-dimensional field theory and its S-matrix was computed by perturbative methods.

The quantum inverse scattering method is the only reliable and desirable method to account for all non-perturbative effects in an integrable model. Unfortunately, it has not yet been developed for the AAF model due to its singular and non-linear nature. In this situation, the perturbative approach is essentially the only available method to probe quantum integrability and to obtain the Bethe equations without serious technical problems. It is clear, however, that within the perturbative quantum field theoretic approach non-perturbative information may be lost. Still, this does not happen in many known integrable models, for which the quantum inverse scattering method leads to the same results as the perturbative calculations. This happens because, in such cases, the S-matrix can be found exactly to all orders. We mention here another interesting aspect of the AAF model which

---

[2]By singularity we mean the presence of derivatives in the interaction Hamiltonian which results in a very singular quantum mechanical Hamiltonian and associated quantum conserved charges.

sets it apart from all other known classical integrable models. Namely, the highly non-linear nature of its Poisson structure, which extends up to the sixth order in the fermions and its spatial derivatives. In perturbation theory this information is not essential, but it is very desirable to understand its effect within the quantum inverse scattering formalism. Thus, until the quantum inverse scattering is fully developed and understood, perturbative calculations are essentially the only working tool at our disposal.

There is, however, an alternative formulation of the AAF model, as explained in the original work [14], since it is possible to perform a field redefinition to trivialize the Poisson structure at the price of getting a complicated Hamiltonian. Nonetheless, both approaches are plagued with the usual problems found in the quantization process of continuous integrable models, which reflect the ill-defined operator product at the same point. Although there exists standard discretization techniques that can, in principle, be used to avoid this problem, they usually lead to very complicated results and, more importantly, they might not be readily applicable to the more involved string model on $AdS_5 \times S^5$. Thus, it is desirable to deal directly with the continuous AAF model, which should shed some light on possible ways to overcome the fundamental quantization difficulties of the full string model.

Yet another possibility is to find an alternative gauge choice that linearizes the equations of motion. Actually, if the string model truncated to the $\mathfrak{su}(1|1)$ sector has its reparametrization invariance fixed by means of the uniform light-cone gauge [20], it becomes a two-dimensional theory for free massive Dirac fermions. In this case the quantization is trivial and the spectrum can be easily obtained. However, it is not clear whether the classical equivalence between the AAF model in the uniform gauge and in the uniform light-cone gauge survives quantization. The reason for that lies in the fact that the conformal invariance, which is necessary for quantum gauge equivalence, is broken in the reduction to the classically closed sector. That being the case, it is an interesting question to quantize the AAF model in the uniform gauge and compare its spectrum with the one obtained from the free action in [20].

In this paper we probe the quantum integrability of the AAF model in the uniform gauge by analyzing the factorizability of its S-matrix. We proceed along the lines of our earlier work [21] and consider the three-particle scattering within the framework of quantum field theory. It is important to bear in mind that a necessary condition for a factorizable scattering is the absence of genuine three-particle interactions. Clearly, this is, *a priori*, not the case for the AAF model since it explicitly contains a three-particle interaction vertex. In fact, we show that even in the first non-trivial order, the S-matrix factorization property can be verified only if the higher order contributions are taken into account.

For our approach, it is convenient to assign canonical mass dimensions to the originally dimensionless fields of the AAF model. This naturally leads to introduction of two dimensionful coupling constants, one for each interaction vertex. In this paper we analyze a more general model, treating the coupling constants as independent, and derive a necessary constraint to ensure the *quantum* integrability of the model. This constraint effectively reduces the number of coupling constants to one, which is in complete agreement with the AAF model. Indeed, the original *classical* dimensionless AAF action contains only one parameter λ, and we show that our general constraint is consistent with this action.

The intricate mechanism behind S-matrix factorization is the same unveiled for the Landau-Lifshitz (LL) model in [21]. However, in the AAF case, its verification is not so straightforward due a substantially more complicated diagrammatic analysis. As a result, the daunting perturbative computations make it hard to understand the various cancellations necessary for the quantum integrability of the model. Moreover, some missed factors in the previous literature have been revealed and corrected in this paper. They were found in the process of proving the conditions for quantum integrability. In particular, the Lagrangian for the interacting massive Dirac fermion in two dimensions, originally derived by [14], has a missing factor of $\frac{1}{2}$ in front of the three-particle interaction term. Its absence would prevent quantum integrability, as a delicate fine-tuning between the coupling constants of the different interaction vertices is required for S-matrix factorization. Even though this missing factor does not affect the two-particle calculations performed in [19], there is a further crucial overall sign difference, which leads to the derivation of the inverse S-matrix instead of the proper one, changing all the subsequent analysis concerning excited and bound states.

Several other technical subtleties, absent in the much simpler LL case, make the computation of scattering amplitudes a much harder problem for the AAF model. First, the fact that the theory is relativistic invariant demands a quantization with respect to a false vacuum in order to render the propagator purely retarded. Nevertheless, the two poles of the propagator should be carefully taken into account, and are essential in the analysis of the continuity of the scattering amplitudes and in the cancellation of non-integrable contributions. Moreover, the presence of spinorial products requires a great care in the combinatorial analysis, making the higher order calculations quite involved.

The paper is organized as follows. In section 2 we give a very brief review of the AAF model incorporating the factor missed by [14]. In section 3 we set up the AAF model as a quantum field theory and prepare all the necessary tools for computing the two- and three-particle S-matrices. In section 4 we give another derivation of the two-particle S-matrix based on standard techniques. In section 5 we present our analysis of the three-particle S-matrix and we show its factorization at the first non-trivial order. Finally, we collect some important technical details in the appendices.

2 Overview of the Alday-Arutyunov-Frolov model

In this section we review the AAF model [14], which emerges as a result of the consistent truncation of the superstring sigma model on $AdS_5 \times S^5$ to the $\mathfrak{su}(1|1)$ sector. Let us briefly remind the reduction process.

The $\mathfrak{su}(1|1)$ sector of superstring theory is defined to be the smallest sector of the full $AdS_5 \times S^5$ theory to contain all the states dual to the operators contained in $\mathfrak{su}(1|1)$ sector of $\mathcal{N} = 4$ SYM. The latter consists, in the $\mathcal{N} = 1$ language, of gauge invariant composite operators made of products between a complex scalar Z from the scalar supermultiplet and a Weyl fermion Ψ from the gaugino supermultiplet. In order to proceed with the truncation it is necessary to single out a string scalar field to be in correspondence with the field Z from the dual gauge theory, while keeping only the time coordinate from AdS_5

non-zero. The residual bosonic symmetry algebra can then be used to decompose the original 16 complex fermions into four sectors, comprising 4 fermions each. It is possible to reduce the superstring equations of motion to one of these sectors, and furthermore set a pair of the remaining fermions consistently to zero. Next, it is tempting to put one of these in direct correspondence with the gauge theory fermion Ψ. However, this is not the case, as a consistent truncation which keeps only one fermion non-zero is forbidden by the cubic couplings arising from the Wess-Zumino term in the superstring Lagrangian. It is important to bear in mind that the $\mathfrak{su}(1|1)$ sector of superstring theory does not coincide with the $\mathfrak{su}(1|1)$ closed sector of the dual gauge theory, since, to begin with, they contain a different number of degrees of freedom.

In addition to the conditions discussed above, one must still fix the reparametrization invariance of the superstring action. A fitting way to do this corresponds to imposing the uniform gauge [13], which identifies the world-sheet time τ with the AdS_5 global time t, and fixes the only non-vanishing component $J = J_3$ of the S^5 angular momentum to be equal to the corresponding $\mathfrak{u}(1)$ charge. By imposing the uniform gauge, the two bosonic degrees of freedom remaining after the reduction to the $\mathfrak{su}(1|1)$ sector: the S^5 angle ϕ corresponding to the scalar Z and the AdS_5 global time t, are removed. In particular, by solving the constraints introduced by this gauge choice, the angle ϕ can now be expressed in terms of the fermionic coordinates. So that the remaining physical degrees of freedom are purely fermionic. Remarkably, the two complex space-time fermions can be grouped into a single two-component world-sheet Dirac spinor. Accordingly, the action for the truncated model reduces to a non-trivially interacting Lorentz invariant action of the massive Dirac fermion on the flat two-dimensional world-sheet.

The extensive details of the derivation, as well as the notations which we also follow here, can be found in the original paper [14]. Our starting point is the classically integrable AAF model Lagrangian (see the expression (5.3) of [14]):

$$\mathcal{L} = -J - \frac{iJ}{2}\left(\bar{\psi}\rho^0\partial_0\psi - \partial_0\bar{\psi}\rho^0\psi\right) + i\kappa\left(\bar{\psi}\rho^1\partial_1\psi - \partial_1\bar{\psi}\rho^1\psi\right) + J\bar{\psi}\psi +$$
$$+ \frac{iJ}{4}\left(\bar{\psi}\rho^0\partial_0\psi - \partial_0\bar{\psi}\rho^0\psi\right)\bar{\psi}\psi - \frac{i\kappa}{2}\left(\bar{\psi}\rho^1\partial_1\psi - \partial_1\bar{\psi}\rho^1\psi\right)\bar{\psi}\psi - \frac{J}{2}\left(\bar{\psi}\psi\right)^2 +$$
$$+ \frac{\kappa}{2}\epsilon^{\alpha\beta}\left(\bar{\psi}\partial_\alpha\psi\ \bar{\psi}\rho^5\partial_\beta\psi - \partial_\alpha\bar{\psi}\psi\ \partial_\beta\bar{\psi}\rho^5\psi\right) + \frac{\kappa}{8}\epsilon^{\alpha\beta}\left(\bar{\psi}\psi\right)^2\partial_\alpha\bar{\psi}\rho^5\partial_\beta\psi, \qquad (2.1)$$

where the Dirac matrices ρ^0, ρ^1 and ρ^5 are defined in appendix A in (A.3), and the Levi-Civita tensor is such that $\epsilon^{01} = \epsilon_{10} = 1$. By means of the following field redefinition:

$$\psi \to \psi + \frac{1}{4}\psi\left(\bar{\psi}\psi\right), \quad \bar{\psi} \to \bar{\psi} + \frac{1}{4}\bar{\psi}\left(\bar{\psi}\psi\right), \qquad (2.2)$$

one can simplify the Lagrangian (2.1) further, and write it in the form:

$$\mathcal{L}_{AAF} = -J - \frac{iJ}{2}\left(\bar{\psi}\rho^0\partial_0\psi - \partial_0\bar{\psi}\rho^0\psi\right) + i\kappa\left(\bar{\psi}\rho^1\partial_1\psi - \partial_1\bar{\psi}\rho^1\psi\right) + J\bar{\psi}\psi +$$
$$+ \frac{\kappa}{2}\epsilon^{\alpha\beta}\left(\bar{\psi}\partial_\alpha\psi\ \bar{\psi}\rho^5\partial_\beta\psi - \partial_\alpha\bar{\psi}\psi\ \partial_\beta\bar{\psi}\rho^5\psi\right) - \frac{\kappa}{8}\epsilon^{\alpha\beta}\left(\bar{\psi}\psi\right)^2\partial_\alpha\bar{\psi}\rho^5\partial_\beta\psi. \qquad (2.3)$$

It is important to emphasize here a significant difference in the last interacting term of our Lagrangian (2.3) when compared to the Lagrangian (5.5) of [14]. The extra factor

$\frac{1}{2}$ that appears in our Lagrangian is crucial, as we will show in the subsequent sections, for the quantum integrability of the model, and was missed in [14].[3] As a result, this missed factor had propagated to [19], where the two-particle S-matrix was obtained for the first time. Although essential for the S-matrix factorization of $n \geq 3$ particles scattering process, and consequently for the quantum integrability, this extra factor does not affect the two-particle S-matrix calculation. However, another missed point, as we will explain below, effectively changes the two-particle S-matrix of [19] to its inverse.

We stress that all these results have been obtained by analyzing the quantum integrability of the model, which imposes a strong constraint on the coupling constants, and then checking its consistency in the classical limit. This is explained in details in the subsequent sections.

3 The AAF model as quantum field theory

Due to the highly non-trivial form of the Poisson brackets, which extends up to the eighth order in the fermions and their spatial derivatives, it is not an easy task to directly quantize the theory, defined by the Lagrangian (2.3), by the standard methods of the quantum inverse scattering method. However, since we are interested in probing the quantum integrability of the AAF model, there is an alternative framework: to consider the AAF model as a quantum field theory and study the factorability of its S-matrix. In this section we set up all the necessary tools for computing the two- and three-particle S-matrices.

The starting point is the action defined by the Lagrangian (2.3):

$$ S = \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \, \mathscr{L}_{AAF}, \tag{3.1} $$

which is not, however, explicitly Lorentz invariant. This can be readily fixed if we rescale the world-sheet coordinate σ:

$$ \sigma \rightarrow -\frac{2\kappa}{J}\sigma, \tag{3.2} $$

leading to:

$$ S = \frac{\kappa}{\pi} \int d\tau \int_{-\frac{\pi J}{\kappa}}^0 d\sigma \left[-1 - \frac{i}{2} \left(\bar{\psi}\rho^\alpha \partial_\alpha \psi - \partial_\alpha \bar{\psi}\rho^\alpha \psi \right) + \bar{\psi}\psi - \right. $$
$$ \left. - \frac{1}{4} \epsilon^{\alpha\beta} \left(\bar{\psi}\partial_\alpha \psi \, \bar{\psi}\rho^5 \partial_\beta \psi - \partial_\alpha \bar{\psi}\psi \, \partial_\beta \bar{\psi}\rho^5 \psi \right) + \frac{1}{16} \epsilon^{\alpha\beta} \left(\bar{\psi}\psi \right)^2 \partial_\alpha \bar{\psi}\rho^5 \partial_\beta \psi \right]. \tag{3.3} $$

As defined in the Lagrangian (3.3), the kinetic term has a (-1) sign in front of it, when compared to the standard convention. There are two equivalent ways to develop the perturbation theory. One can work directly with the signs defined in (3.3), and in this case one has to be careful with the mode expansion (see the next section), since the energy is negative definite now. In other words, the particle and anti-particle operators are switched in comparison with the standard textbook convention. Alternatively, we could modify the Lagrangian and make the kinetic term, together with the energy, positive definite. To

---

[3]We thank S. Frolov for confirming this correction.

achieve this, we can make the transformation $\mathscr{L} \longrightarrow (-1)\mathscr{L}$. On the classical level this transformation does not change the dynamics of the model. The situation on the quantum level is slightly more complicated. Firstly, the propagator will acquire an additional overall minus sign, and, secondly, the poles will be switched, so that the corresponding positive and negative energy states are interchanged, compared to the first approach. Both approaches are equivalent, however, one cannot mix the two, as it appears to be the case in [19], and which had led to the inverted S-matrix.[4] Here, we choose the second path, and make the kinetic term positive by multiplying the classical Lagrangian by (-1).

Finally, we redefine the integration variable $\sigma \longrightarrow \sigma + \frac{2\pi J}{\sqrt{\lambda}}$ as in [19], neglect the constant term in (3.3), fix $\kappa = \frac{\sqrt{\lambda}}{2}$, and change the Dirac matrices basis through the similarity transformation (A.5). Then, the action becomes:

$$
S = \frac{\sqrt{\lambda}}{2\pi} \int d\tau \int_0^{\frac{2\pi J}{\sqrt{\lambda}}} d\sigma \left[\frac{i}{2} \left(\bar{\psi}\,\gamma^\alpha \partial_\alpha \psi - \partial_\alpha \bar{\psi}\,\gamma^\alpha \psi \right) - \bar{\psi}\psi + \right.
$$
$$
\left. + \frac{1}{4}\epsilon^{\alpha\beta} \left(\bar{\psi}\partial_\alpha \psi\,\bar{\psi}\,\gamma^3 \partial_\beta \psi - \partial_\alpha \bar{\psi}\psi\,\partial_\beta \bar{\psi}\,\gamma^3 \psi \right) - \frac{1}{16}\epsilon^{\alpha\beta} \left(\bar{\psi}\psi \right)^2 \partial_\alpha \bar{\psi}\,\gamma^3 \partial_\beta \psi \right]. \quad (3.4)
$$

Up to now, we have been working only with dimensionless quantities. However, for our purposes it is convenient to assign canonical mass dimensions to the fields. To start with, we perform the following coordinate transformation [19]:

$$
x^\alpha \to y^\alpha = \frac{\sqrt{\lambda}}{2\pi} x^\alpha, \quad \text{with} \quad x^0 = \tau, \quad x^1 = \sigma, \quad (3.5)
$$

under which the action (3.4) becomes:

$$
S = \int dy^0 \int_0^J dy^1 \left[i\bar{\psi}\slashed{\partial}\psi - \frac{2\pi}{\sqrt{\lambda}}\bar{\psi}\psi + \right.
$$
$$
\left. + \frac{\sqrt{\lambda}}{8\pi}\epsilon^{\alpha\beta} \left(\bar{\psi}\partial_\alpha \psi\,\bar{\psi}\,\gamma^3 \partial_\beta \psi - \partial_\alpha \bar{\psi}\psi\,\partial_\beta \bar{\psi}\,\gamma^3 \psi \right) - \frac{\sqrt{\lambda}}{32\pi}\epsilon^{\alpha\beta} \left(\bar{\psi}\psi \right)^2 \partial_\alpha \bar{\psi}\,\gamma^3 \partial_\beta \psi \right], \quad (3.6)
$$

while the dependence of the fermionic fields in the original world-sheet dimensionless co-ordinates is simply $\psi = \psi \left(\frac{\sqrt{\lambda}}{2\pi}\tau, -\frac{\lambda}{2\pi J}\sigma + J \right)$. Identifying the term multiplying the factor $\bar{\psi}\psi$ with the mass of the theory:

$$
m = \frac{2\pi}{\sqrt{\lambda}}, \quad (3.7)
$$

we can assert the mass-dimensions.

First, we note that λ, the 't Hooft coupling, is proportional to l_s^{-2}, where l_s is the string length, so that the canonical mass-dimension assigned to the coordinates: $[y] = -1$. Thus, as we demand the action to be dimensionless, we conclude from the kinetic and the mass term that: $[\psi] = \frac{1}{2}$ and $[m] = 1$. Clearly, this amounts to $[\lambda] = -2$, hence, regarding only the free terms, the mass-dimensions attributed to fields are so far consistent. However, when we turn to the interaction terms of (3.6), we see that in the two-particle interaction

---

[4]See the discussion at the end of the section 5.3 on the relation between our results and the results of [19].

term there is one additional mass-dimension, while in the three-particle one, there are two. Therefore, one must introduce in the Lagrangian two coupling constants for the two- and three-particle interaction vertices, and assign to each of them the corresponding dimension: g_2, with $[g_2] = -1$; and g_3, with $[g_3] = -2$, respectively. The action (3.6), thus, becomes:

$$S = \int dy^0 \int_0^J dy^1 \left[i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi + \frac{g_2}{4m}\epsilon^{\alpha\beta} \left(\bar{\psi}\partial_\alpha\psi \ \bar{\psi} \ \gamma^3 \partial_\beta\psi - \partial_\alpha\bar{\psi}\psi \ \partial_\beta\bar{\psi} \ \gamma^3\psi \right) - \right.$$
$$\left. - \frac{g_3}{16m}\epsilon^{\alpha\beta} \left(\bar{\psi}\psi \right)^2 \partial_\alpha\bar{\psi} \ \gamma^3 \partial_\beta\psi \right]. \tag{3.8}$$

At this stage the two coupling constants are not independent, since they are derived from the AAF model, containing only one parameter λ. However, it is interesting to relax this condition, and consider a more general model in which the two coupling constants are independent. This generalization is also convenient for the perturbative analysis, as it allows to keep track of contributions from different vertices. Then, the requirement of quantum integrability, as we will show below, relates the two coupling constants in a manner consistent with the classical dimensionless action (3.6). For the rest of the parameter space our generalized model remains a well-defined quantum field theory, though, non-integrable.

It is important to stress that our action (3.8) essentially differs from the one used by [19][5] in three aspects. First, there was only one coupling constant introduced in [19] for both interaction vertices, which is not correct, since g_2 and g_3 have different dimensions. In addition, as we have already mentioned above, there is an additional factor of $\frac{1}{2}$ in the last interaction term, which, as we will see, is crucial for the quantum integrability. And finally, our action differs from the one in [19] by an overall sign. This choice has profound consequences in the analysis of the quantum field theory defined by (3.8), as, for instance, the sign of the free Lagrangian determines the role of creation and annihilation operators in the mode expansion of the fields. In particular, we note that the sign choice of [19] in (4.3) is not consistent with their mode expansion (4.5) and (4.6). Clearly this affects the interplay between the interaction and the free Lagrangian, the more tangible effect of this change being that the model defined by (4.3) is not quantum integrable, since its S-matrix fails to factorize. Unfortunately, the analysis of [19] was not sensitive to this inaccuracy, and as a result the correct S-matrix is in fact the inverse of the one obtained in [19]. This requires further analysis of excited and bound states.

3.1 Quantization of the free theory

The next step it to canonically quantize the free theory defined by the action (3.8), in this case, the massive two-dimensional Dirac fermion, satisfying the Dirac equation:

$$i\partial\!\!\!/\psi - m\psi = 0 \tag{3.9}$$

For the free theory the highly non-local Poisson brackets lead, nevertheless, to standard equal-time anticommutation relations:

$$\left\{ \psi^a(x), \psi^b(x') \right\} = 0, \quad \left\{ \psi^{a\dagger}(x), \psi^{b\dagger}(x') \right\} = 0, \quad \left\{ \psi^a(x), \psi^{b\dagger}(x') \right\} = \delta^{ab}\delta(x - x'). \tag{3.10}$$

---

[5]See equation (4.3) of [19].

The free quantum Hamiltonian becomes:

$$H = \int dx^1 \left(-i\bar{\psi}\,\gamma^1\partial_1\psi + m\bar{\psi}\psi \right). \tag{3.11}$$

and the field expansion takes the form (see appendix A for useful definitions):

$$\psi(x) = \int \frac{dp_1}{2\pi} \left[a(p_1)u(p)e^{-ip\cdot x} + b(-p_1)v(-p)e^{ip\cdot x} \right], \tag{3.12}$$

$$\bar{\psi}(x) = \int \frac{dp_1}{2\pi} \left[a^\dagger(p_1)\bar{u}(p)e^{ip\cdot x} + b^\dagger(-p_1)\bar{v}(-p)e^{-ip\cdot x} \right], \tag{3.13}$$

where $p_0 = \omega(p)$. Inverting the relations (3.12) and (3.13), and using (3.10), we obtain the canonical anticommutation relations for the oscillators:

$$\left\{ a(k_1), a^\dagger(p_1) \right\} = 2\pi\delta(k_1 - p_1), \quad \left\{ b(-k_1), b^\dagger(-p_1) \right\} = 2\pi\delta(k_1 - p_1). \tag{3.14}$$

The Hamiltonian (3.11) reads, then:

$$H = \int \frac{dp_1}{2\pi} p_0 \left[a^\dagger(p_1)a(p_1) - b^\dagger(p_1)b(p_1) \right]. \tag{3.15}$$

The fact that the AAF model is a relativistic invariant theory poses a further obstacle: its propagator is not purely retarded. Therefore, one cannot proceed in the standard manner (see, for example, [19, 21–23]) to compute the S-matrix, where this fact was paramount to control the loop corrections, and to calculate the sum of all Feynman diagrams. Nevertheless, we can employ the same technique used in [24] to overcome this shortcoming. The idea consists in quantizing the theory not with respect to its true ground state, but to a pseudo-vacuum, which, by definition, is the state annihilated by the field operator:

$$\psi(x)|0\rangle = 0. \tag{3.16}$$

In this case, all anti-particle levels are left empty, and the S-matrix can be computed by the same methods employed for the non-relativistic theories. Finally, from this "bare" S-matrix one can obtain the Bethe equations, the solution of which enables one to fill back the Dirac sea, and thus, reconstruct the true ground state [25]. It is important to bear in mind that the filling of the Dirac sea should change drastically the spectrum and the S-matrix. Naturally, (3.16) implies:

$$a(p_1)|0\rangle = b(p_1)|0\rangle = 0 \Rightarrow H|0\rangle = 0. \tag{3.17}$$

We define the pseudo-particle states as:

$$a^\dagger(p_1)|0\rangle = |p\rangle, \quad b^\dagger(p_1)|0\rangle = |\tilde{p}\rangle, \tag{3.18}$$

Then, it is easy to verify that:

$$H|p\rangle = p_0|p\rangle, \quad H|\tilde{p}\rangle = -p_0|\tilde{p}\rangle,$$

hence, the operators $a^\dagger(p_1)$ $\left(a(p_1)\right)$ and $b^\dagger(p_1)$ $\left(b(p_1)\right)$ create (annihilate) pseudo-particles with momentum p_1 and energy $p_0 = +\omega(p)$ and $p_0 = -\omega(p)$. The physical vacuum is, therefore, obtained by exciting all the negative energy modes of the pseudo-vacuum. Finally, we obtain the purely retarded propagator:

$$D(x - x') = \langle 0|T\psi(x)\bar\psi(x')|0\rangle = (i\slashed\partial + m) \int \frac{d^2p}{4\pi^2} \frac{ie^{-ip\cdot(x-x')}}{p^2 - m^2 + 2i\varepsilon p_0}, \qquad (3.19)$$

with $d^2p = dp_0\, dp_1$.

3.2 Scattering of the pseudo-particles in the AAF model

In the following, we will be interested in the computation of the two- and three-particle S-matrix. For the sake of simplicity we will only consider the scattering between pseudo-particles with positive energy (the excitations over the pseudo-vacuum created by $a^\dagger$), since the scattering involving pseudo-particles with negative energy (created by $b^\dagger$) can be easily obtained by an analytical continuation to complex rapidities θ,

$$p_0 = m\cosh\theta \ \text{ and } \ p_1 = m\sinh\theta, \ \text{ if } \ \begin{cases} \theta = \alpha \in \mathbb{R} & \Rightarrow p_0 \in \mathbb{R}_+ \ \text{ and } \ p_1 \in \mathbb{R} \\ \theta = i\pi - \alpha, & \alpha \in \mathbb{R} \Rightarrow p_0 \in \mathbb{R}_- \ \text{ and } \ p_1 \in \mathbb{R} \end{cases}.$$

Therefore, the pseudo-particles with positive energy are appropriately described by real-valued rapidities $\theta = \alpha \in \mathbb{R}$, accordingly the ones with negative energy are naturally parametrized by imaginary rapidities $\theta = i\pi - \alpha, \alpha \in \mathbb{R}$. Since the S-matrix is a meromorphic function of the rapidities [26, 27], it simultaneously describes the scattering of both types of pseudo-particles. In this case, without any loss of generality, we can take the external states to be composed solely of pseudo-particles in the positive mass-shell:

$$|\mathbf{p}\rangle = \prod_{i=1}^n a^\dagger(p_1^i)|0\rangle, \quad \langle\mathbf{k}| = \langle 0|\prod_{i=1}^n a(k_1^i). \qquad (3.20)$$

For future convenience, we introduce the following notation for the interaction vertices, see figure 1, already writing them in normal ordering:

$$: \mathscr{L}_{Int}^{(2)} := \frac{g_2}{4m}\epsilon^{\alpha\beta} : \left(\bar\psi\partial_\alpha\psi\ \bar\psi\,\gamma^3\partial_\beta\psi - \partial_\alpha\bar\psi\psi\ \partial_\beta\bar\psi\,\gamma^3\psi\right) :$$
$$= -\mathbb{G}_{ac,bd}^{\alpha\beta}\left(\bar\psi^a\bar\psi^c\partial_\alpha\psi^b\partial_\beta\psi^d - \partial_\alpha\bar\psi^a\partial_\beta\bar\psi^c\psi^b\psi^d\right), \qquad (3.21)$$

$$: \mathscr{L}_{Int}^{(3)} := -\frac{g_3}{16m}\epsilon^{\alpha\beta} : \left(\bar\psi\psi\right)^2 \partial_\alpha\bar\psi\partial_\beta\psi := -\mathbb{H}_{ace,bdf}^{\alpha\beta}\bar\psi^a\bar\psi^c\partial_\alpha\bar\psi^e\psi^b\psi^d\partial_\beta\psi^f, \qquad (3.22)$$

where we defined the matrices

$$\mathbb{G}_{ac,bd}^{\alpha\beta} := \frac{g_2}{8m}\epsilon^{\alpha\beta}\mathbb{P}_{ac,bd}, \qquad \text{with} \quad \mathbb{P} := \mathbb{1}_2 \otimes \gamma^3 - \gamma^3 \otimes \mathbb{1}_2, \qquad (3.23)$$

$$\mathbb{H}_{ace,bdf}^{\alpha\beta} := -\frac{g_3}{16m}\epsilon^{\alpha\beta}\mathbb{Q}_{ace,bdf}, \quad \text{with} \quad \mathbb{Q} := \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \gamma^3. \qquad (3.24)$$
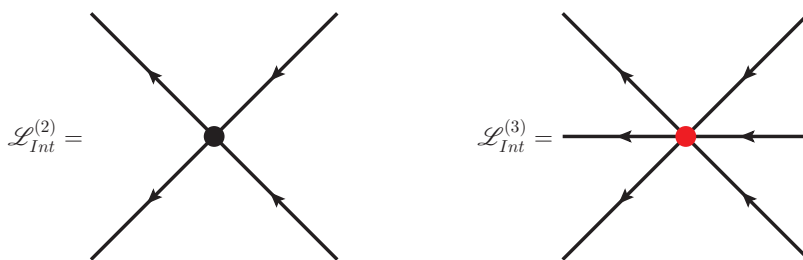
Figure 1. Interaction vertices in momentum representation.

4 Two-particle scattering

The two-particle S-matrix for the AAF model has been first obtained in [19].[6] Here we give another derivation, based on a general technique which, in principle, can be applied to a large class of integrable models (see, for example, [21, 23]).

The fact that the propagator is now purely retarded implies that the two-particle S-matrix is given by the sum of bubble diagrams (and its spinorial twists), as depicted in figure 2. Hence, we need only consider the contribution of the two-particle interaction term, $\mathscr{L}_{Int}^{(2)}$. Without any loss in generality, we assume the incoming momenta[7] to be ordered: $p_1^1 > p_1^2$ and on-shell.

The two-particle S-matrix is then determined from the relation:

$$\langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle = S(p^1, p^2) \delta_-^{(2)}(p^1, p^2; k^1, k^2), \tag{4.1}$$

where:

$$\delta_\pm^{(2)}(p^1, p^2; k^1, k^2) = 4\pi^2 \left[\delta(k_1^1 - p_1^1)\delta(k_1^2 - p_1^2) \pm \delta(k_1^1 - p_1^2)\delta(k_1^2 - p_1^1) \right], \tag{4.2}$$

and the scattering amplitude is given by:

$$\begin{aligned}
\langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle &= \langle k^1 k^2 | e^{i \int \mathscr{L}_{Int}^{(2)} d^2 x} | p^1 p^2 \rangle \\
&= \langle k^1 k^2 | p^1 p^2 \rangle + i \langle k^1 k^2 | \int \mathscr{L}_{Int}^{(2)} d^2 x \, | p^1 p^2 \rangle - \\
&\quad - \frac{1}{2} \langle k^1 k^2 | T \left(\int \mathscr{L}_{Int}^{(2)} d^2 x \right)^2 | p^1 p^2 \rangle + \cdots .
\end{aligned} \tag{4.3}$$

The non-scattering term is easily computed and yields:

$$\langle k^1 k^2 | p^1 p^2 \rangle = \delta_-^{(2)}(p^1, p^2; k^1, k^2). \tag{4.4}$$

---

[6]See, however, the above discussion on the sign difference.

[7]In the following, we will always write the space-time momenta as p_μ^i, where $\mu = 0, 1$ is the Lorentz index and $i = 1, \ldots, n$, the particle label.
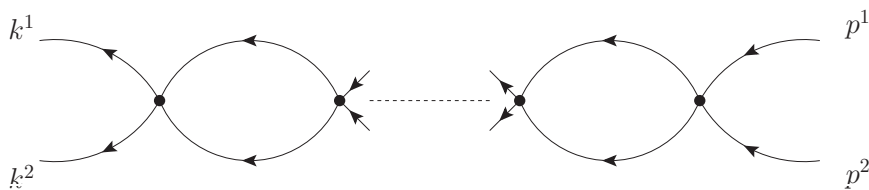
Figure 2. Bubble diagram for two-pseudo-particle scattering.

At the tree level, we need to evaluate:

$$\langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle \Big|_{g_2} = i \langle k^1 k^2 | \int \mathscr{L}_{Int}^{(2)} \, d^2 x \, | p^1 p^2 \rangle$$

$$= -i \mathbb{G}_{ac,bd}^{\alpha\beta} \langle k^1 k^2 | \int d^2 x \left(\bar{\psi}^a \bar{\psi}^c \partial_\alpha \psi^b \partial_\beta \psi^d - \partial_\alpha \bar{\psi}^a \partial_\beta \bar{\psi}^c \psi^b \psi^d \right) | p^1 p^2 \rangle. \quad (4.5)$$

We can compute all the integrals to obtain the off-shell[8] tree level term:

$$\langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle \Big|_{g_2} = -\frac{ig_2}{8m} \left\{ k^2 \times k^1 \left(\bar{U}_{12}^k + \bar{U}_{21}^k \right) \mathbb{P} \left(U_{12}^p - U_{21}^p \right) \right. \quad (4.6)$$

$$\left. + p^2 \times p^1 \left(-\bar{U}_{12}^k + \bar{U}_{21}^k \right) \mathbb{P} \left(U_{12}^p + U_{21}^p \right) \right\} 4\pi^2 \delta^{(2)} \left(k^1 + k^2 - p^1 - p^2 \right).$$

where we introduced the shorthand notation for the bi-spinors:

$$\bar{U}_{ij}^k = \bar{u}(k^i) \otimes \bar{u}(k^j), \quad U_{ij}^p = u(p^i) \otimes u(p^j),$$

and denoted: $\epsilon^{\alpha\beta} p_\alpha^i p_\beta^j = p^i \times p^j$.

Taking the outcoming pseudo-particles to be on-shell as well, we can use the identity:

$$4\pi^2 \delta^{(2)} \left(k^1 + k^2 - p^1 - p^2 \right) = \left| \frac{p_0^1 \, p_0^2}{p^2 \times p^1} \right| \delta_+^{(2)} (p^1, p^2; k^1, k^2), \quad (4.7)$$

together with the fact that for our ordering of incoming momenta $p^2 \times p^1 > 0$ to write:

$$\langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle \Big|_{g_2} = -\frac{ig_2}{2m} p_0^1 \, p_0^2 \, \bar{U}_{21}^p \, \mathbb{P} \, U_{12}^p \, \delta_-^{(2)} (p^1, p^2; k^1, k^2). \quad (4.8)$$

Finally, we compute the spinorial product

$$\bar{U}_{21}^p \, \mathbb{P} \, U_{12}^p = \frac{p^2 \times p^1}{p_0^1 \, p_0^2}, \quad (4.9)$$

to obtain the on-shell tree-level amplitude:

$$\langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle \Big|_{g_2} = -\frac{ig_2}{2m} \left(p^2 \times p^1 \right) \delta_-^{(2)} (p^1, p^2; k^1, k^2). \quad (4.10)$$

---

[8]To be more precise, the outcoming pseudo-particles are off-shell, though the assumption of on-shell incoming pseudo-particles is not used in the derivation of (4.6), thus justifying calling it off-shell.

The one-loop amplitude computation is rather more involved as it contains more terms to take into account. Namely, we have to evaluate:

$$
\begin{aligned}
\langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle \big|_{g_2^2} &= -\frac{1}{2} \langle k^1 k^2 | T \left(\int \mathscr{L}_{Int}^{(2)} \, d^2 x \right)^2 | p^1 p^2 \rangle \\
&= -\frac{1}{2} \mathbb{G}_{ac,bd}^{\alpha\beta} \mathbb{G}_{eg,fh}^{\gamma\delta} \langle k^1 k^2 | T \int d^2 x \, d^2 y \, \left[\bar{\psi}^a \bar{\psi}^c \partial_\alpha \psi^b \partial_\beta \psi^d \bar{\phi}^e \bar{\phi}^g \partial_\gamma \phi^f \partial_\delta \phi^h - \right. \\
&\quad - \bar{\psi}^a \bar{\psi}^c \partial_\alpha \psi^b \partial_\beta \psi^d \partial_\gamma \bar{\phi}^e \partial_\delta \bar{\phi}^g \phi^f \phi^h - \partial_\alpha \bar{\psi}^a \partial_\beta \bar{\psi}^c \psi^b \psi^d \bar{\phi}^e \bar{\phi}^g \partial_\gamma \phi^f \partial_\delta \phi^h + \\
&\quad \left. + \partial_\alpha \bar{\psi}^a \partial_\beta \bar{\psi}^c \psi^b \psi^d \partial_\gamma \bar{\phi}^e \partial_\delta \bar{\phi}^g \phi^f \phi^h \right] | p^1 p^2 \rangle,
\end{aligned}
\tag{4.11}
$$

where to avoid cluttering, we denoted $\psi \equiv \psi(x)$ and $\phi \equiv \psi(y)$. Expanding the T-product, we realize that only the terms with two contractions of the type

$$
\overparen{\psi \phi} \ \overparen{\psi \phi}, \quad \text{or} \quad \overparen{\bar{\psi} \phi} \ \overparen{\bar{\psi} \phi}
$$

contribute. The other terms vanish identically, either because the inner product

$$
\langle k^1 k^2 | f(a, a^\dagger) | p^1 p^2 \rangle = 0 \quad \text{or} \quad \langle k^1 k^2 | g(\psi, \bar{\psi}) | p^1 p^2 \rangle \propto \theta(x^0 - y^0) \theta(y^0 - x^0) = 0,
$$

with $f(a, a^\dagger)$ and $g(\psi, \bar{\psi})$ being arbitrary functions. Hence,

$$
\begin{aligned}
\langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle \big|_{g_2^2} &= 2 \left(\frac{g_2}{8m} \right)^2 \left[\left(k^2 \times k^1 \right) \left(p^2 \times p^1 \right) \left(\bar{U}_{12}^k + \bar{U}_{21}^k \right) \mathbb{P} \, I_0(p^1, p^2) \, \mathbb{P} \left(U_{12}^p + U_{21}^p \right) - \right. \\
&\quad - \left(p^2 \times p^1 \right) \left(\bar{U}_{12}^k - \bar{U}_{21}^k \right) \mathbb{P} \, I_1(p^1, p^2) \, \mathbb{P} \left(U_{12}^p + U_{21}^p \right) - \\
&\quad - \left(k^2 \times k^1 \right) \left(\bar{U}_{12}^k + \bar{U}_{21}^k \right) \mathbb{P} \, I_1(p^1, p^2) \, \mathbb{P} \left(U_{12}^p - U_{21}^p \right) + \\
&\quad \left. + \left(\bar{U}_{12}^k - \bar{U}_{21}^k \right) \mathbb{P} \, I_2(p^1, p^2) \, \mathbb{P} \left(U_{12}^p - U_{21}^p \right) \right] 4\pi^2 \delta^{(2)} \left(k^1 + k^2 - p^1 - p^2 \right),
\end{aligned}
\tag{4.12}
$$

where $I_i(p^1, p^2)$, $i = 0, 1, 2$ are defined in appendix B by equations (B.2), (B.3) and (B.4).

The completeness relations (A.13), together with our ordering for the incoming momenta and the identity for the on-shell momenta:

$$
\frac{\left(p^2 \times p^1 \right) \left(p_0^1 + p_0^2 \right)}{\left(p_1^1 - p_1^2 \right) \left(p^1 + p^2 \right)^2} = \frac{1}{2},
\tag{4.13}
$$

lead to the following central relation between the one-loop amplitude and the off-shell[9] tree-level amplitude (4.6)

$$
\langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle \big|_{g_2^2} = 2 \left(p^2 \times p^1 \right) \left(\frac{-i g_2}{8m} \right) \langle k^1 k^2 | \hat{S} | p^1 p^2 \rangle \big|_{g_2}.
\tag{4.14}
$$

Since the outcoming pseudo-particles are off-shell, equation (4.14) suggests that we can regard the one-loop scattering amplitude as the interaction vertex in momentum representation, with the incoming momenta p^1 and p^2 on-shell, multiplied by some function of this

---

[9]Here we actually mean that only the outcoming pseudo-particles are off-shell.

pair of momenta and the coupling constant. Therefore the n-loop scattering amplitude corresponds to the product of n of these modified vertices:

$$\langle k^1 k^2|\hat{S}|p^1 p^2\rangle\Big|_{g_2^{n+1}} = \left[2\left(p^2 \times p^1\right)\left(\frac{-ig_2}{8m}\right)\right]^n \langle k^1 k^2|\hat{S}|p^1 p^2\rangle\Big|_{g_2}$$

$$\overset{\text{on-shell}}{=} 2\left(-\frac{ig_2}{4m}\left(p^2 \times p^1\right)\right)^{n+1} \delta_-(p^1, p^2; k^1, k^2), \qquad (4.15)$$

where in the last step, we took the outcoming pseudo-particles k^1 and k^2 on-shell.

We are now in the position to obtain the full scattering amplitude:

$$\langle k^1 k^2|\hat{S}|p^1 p^2\rangle = \langle k^1 k^2|p^1 p^2\rangle + \sum_{n=1}^{\infty}\langle k^1 k^2|\hat{S}|p^1 p^2\rangle\Big|_{g_2^n}$$

$$= \frac{1 - \frac{ig_2}{4m}p^2 \times p^1}{1 + \frac{ig_2}{4m}p^2 \times p^1}\delta_-(p^1, p^2; k^1, k^2), \qquad (4.16)$$

from which, by comparison with (4.1), we read off the S-matrix for the scattering of two-particle,

$$S(p^1, p^2) = \frac{1 - \frac{ig_2}{4m}p^2 \times p^1}{1 + \frac{ig_2}{4m}p^2 \times p^1}. \qquad (4.17)$$

It is crucial to notice that our S-matrix is the inverse of the one derived by [19].

5 Three-particle scattering

In this section we analyze the S-matrix factorization, which reflects the quantum integrability of the model. The first step in this program is to consider the S-matrix for the scattering of three pseudo-particles and confirm that it can be properly written as the product of three S-matrices for two-particle scattering (4.17). Before going into the details of the actual three-particle scattering amplitudes, just as we did for the two-particle case in the previous section, we take a closer look at the factorizable expression for the three-particle S-matrix, namely,

$$S(p^1, p^2, p^3) = S(p^1, p^2)S(p^1, p^3)S(p^2, p^3)$$

$$= \frac{1 - \frac{ig_2}{4m}p^2 \times p^1}{1 + \frac{ig_2}{4m}p^2 \times p^1} \cdot \frac{1 - \frac{ig_2}{4m}p^3 \times p^1}{1 + \frac{ig_2}{4m}p^3 \times p^1} \cdot \frac{1 - \frac{ig_2}{4m}p^3 \times p^2}{1 + \frac{ig_2}{4m}p^3 \times p^2}$$

$$= 1 + 2\sum_{n=1}^{3}\left[\left(-\frac{ig_2}{4m}\right)\left(p^2 \times p^1 + p^3 \times p^1 + p^3 \times p^2\right)\right]^n +$$

$$+ 2\left(\frac{ig_2}{4m}\right)^3\left(p^2 \times p^1 + p^3 \times p^1\right)\left(p^2 \times p^1 + p^3 \times p^2\right)\left(p^3 \times p^1 + p^3 \times p^2\right) +$$

$$+ O(g_2^4). \qquad (5.1)$$

In the following, we will compute the three-particle scattering amplitude and show that this necessary condition (5.1) is satisfied, up to the first non-trivial order in g_2 and g_3, provided a very precise relation between g_2 and g_3.

5.1 Diagrammatic calculations

In this case, we must consider the full interaction Lagrangian $\mathscr{L}_{Int} = \mathscr{L}_{Int}^{(2)} + \mathscr{L}_{Int}^{(3)}$, as both the initial and final states involve three pseudo-particles:

$$|\mathbf{p}\rangle = |p^1 p^2 p^3\rangle = a^\dagger\left(p_1^1\right) a^\dagger\left(p_1^2\right) a^\dagger\left(p_1^3\right)|0\rangle \quad \text{and} \quad \langle\mathbf{k}| = \langle k^1 k^2 k^3| = \langle 0|a\left(k_1^3\right) a\left(k_1^2\right) a\left(k_1^1\right).$$

Similarly, we will assume without any loss of generality that the incoming pseudo-particles are on-shell and have their momenta ordered: $p_1^1 > p_1^2 > p_1^3$. The analyticity in the coupling constants of the three-particle scattering amplitude implies:

$$\langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle = \langle\mathbf{k}|\mathbf{p}\rangle + \langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_g + \langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_{g^2} + \cdots, \tag{5.2}$$

where g stands either for g_1 or g_2. The non-scattering term is easily computed:

$$\langle\mathbf{k}|\mathbf{p}\rangle = 3!(2\pi)^3 \mathcal{A}_p\left[\delta(k^1 - p^1)\delta(k^2 - p^2)\delta(k^3 - p^3)\right]. \tag{5.3}$$

Here, we introduced the antisymmetrization operator, or simply, antisymmetrizator, defined by:

$$\mathcal{A}_q[f(\mathbf{q})] := \frac{1}{3!}\sum_A \text{sign}(A)f(A[\mathbf{q}]), \tag{5.4}$$

with the sum taken over all possible permutations of $(1,2,3)$ and the vector $A[\mathbf{q}] := \left(q^{A_1}, q^{A_2}, q^{A_3}\right)$.

For the tree-level amplitude we need to evaluate:

$$\langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_g = i\langle\mathbf{k}|T\int d^2 x \left(\mathscr{L}_{Int}^{(2)} + \mathscr{L}_{Int}^{(3)}\right)|\mathbf{p}\rangle, \tag{5.5}$$

leading to the off-shel tree-level amplitude,

$$\langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_g = (3!)^2 \mathcal{A}_{p,k}\left[\frac{ig_2}{8m}\left(k^2 \times k^1 + p^2 \times p^1\right)\bar{U}_{12}^k \,\mathbb{P}\, U_{21}^p \, 2\pi\delta(k^3 - p^3) - \right.$$
$$\left. - \frac{ig_3}{16m}\left(k^3 \times p^3\right)\bar{U}_{123}^k \,\mathbb{Q}\, U_{123}^p\right] 4\pi^2\delta^{(2)}(\mathbf{k} - \mathbf{p}), \tag{5.6}$$

where we introduced the almost self-evident notation for the tri-spinors:

$$\bar{U}_{ijl}^k = \bar{u}(k^i) \otimes \bar{u}(k^j) \otimes \bar{u}(k^l), \quad U_{ijl}^p = u(p^i) \otimes u(p^j) \otimes u(p^l).$$

Proceeding in the same way as with the two-particle case, we can impose the mass-shell condition on the outcoming pseudo-particles in the first term from (5.6), so that we can apply the identity (4.7) and use the fact that for our ordering of initial momenta $p^i \times p^j > 0$ if $i > j$ together with (4.9) to obtain:

$$\langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_g = -\frac{ig_2}{2m}\left[p^2 \times p^1 + p^3 \times p^1 + p^3 \times p^2\right]\langle\mathbf{k}|\mathbf{p}\rangle -$$
$$- \frac{ig_3}{16m}(3!)^2 \mathcal{A}_{p,k}\left[k^3 \times p^3 \,\bar{U}_{123}^k \,\mathbb{Q}\, U_{123}^p\right] 4\pi^2\delta^{(2)}(\mathbf{k} - \mathbf{p}). \tag{5.7}$$

It is crucial to realize that there is no identity similar to (4.7) involving the three momenta. Hence, it is not possible to compute the spinorial product within the second term in (5.6), so as to reduce it to an expression proportional to $\langle \mathbf{k} | \mathbf{p} \rangle$. Remembering that for an integrable model the S-matrix is expected to be of the form:

$$\langle \mathbf{k} | \hat{S} | \mathbf{p} \rangle = S(\mathbf{p}) \langle \mathbf{k} | \mathbf{p} \rangle, \tag{5.8}$$

it is then clear that not only one cannot derive the expression (5.1) in the tree-level approximation, but it is not even possible to write the S-matrix in the form (5.8). Even though this seems to be a formidable obstacle to prove quantum integrability, it is not so. Indeed, the same situation arises in a similar calculation for the Landau-Lifshitz model [21], where it was found that such troublesome terms[10] are cancelled out by certain contributions coming from higher order (in g) scattering amplitudes.

We consider here the one-loop scattering amplitude to track down the contribution to cancel out the second term in (5.6) and, thus, render the tree-level S-matrix factorizable.

$$
\begin{aligned}
\langle \mathbf{k} | \hat{S} | \mathbf{p} \rangle \big|_{g^2} &= -\frac{1}{2} \langle \mathbf{k} | T \left[\int d^2 x \, \left(\mathscr{L}_{Int}^{(2)} + \mathscr{L}_{Int}^{(3)} \right) \right]^2 | \mathbf{p} \rangle \\
&= -\frac{1}{2} \langle \mathbf{k} | T \int d^2 x \, d^2 y \, \Big[\mathscr{L}_{Int}^{(2)}(x) \mathscr{L}_{Int}^{(2)}(y) + \mathscr{L}_{Int}^{(2)}(x) \mathscr{L}_{Int}^{(3)}(y) + \mathscr{L}_{Int}^{(3)}(x) \mathscr{L}_{Int}^{(2)}(y) + \\
&\quad + \mathscr{L}_{Int}^{(3)}(x) \mathscr{L}_{Int}^{(3)}(y) \Big] | \mathbf{p} \rangle \\
&\equiv V_1^{(2)}(\mathbf{k}, \mathbf{p}) + V_2^{(2)}(\mathbf{k}, \mathbf{p}) + V_3^{(2)}(\mathbf{k}, \mathbf{p}) + V_4^{(2)}(\mathbf{k}, \mathbf{p}). \tag{5.9}
\end{aligned}
$$

As noted in [21], this cancellation can only happen between the diagrams of the same order in $\hbar$.[11] Clearly, all tree-level diagrams are of the order $\hbar^0$, hence, at this stage, we can focus only on the first term from the expansion (5.9), $V_1^{(2)}(\mathbf{k}, \mathbf{p})$, as it is the only term which contains diagrams of the zeroth order in $\hbar$. The corresponding Feynman diagrams are depicted in figures 3 and 4.

The analysis of the time-ordered product expansion goes along the same lines as in the two-particle case, since only the two-particle interaction term comes into play in $V_1^{(2)}$. However, as there are now three pseudo-particles in the initial and final states, the contribution of the terms with only one contraction becomes also non-zero. Therefore, one must take into account the contributions coming from the terms containing only the following contractions:

$$\overbrace{\psi \bar{\phi}}, \; \overbrace{\bar{\psi} \phi}, \; \overbrace{\psi \bar{\phi}} \; \overbrace{\psi \bar{\phi}}, \text{ or } \overbrace{\bar{\psi} \phi} \; \overbrace{\bar{\psi} \phi}.$$

---

[10]Such as the second term in (5.6).

[11]We consider here the loop expansion which corresponds to an expansion in powers of $\hbar$. Thus, it is clear that diagrams with a different number of loops cannot cancel each other. Nevertheless, we stress that there are two different coupling constants g_2 and g_3, the dimensions of which are such that the second diagram of figure 1 and the first diagram of figure 3 are of the same order, and may cancel each other (we show below that it is indeed the case). Therefore, one must consider all possible diagrams for a given order in $\hbar^n$.
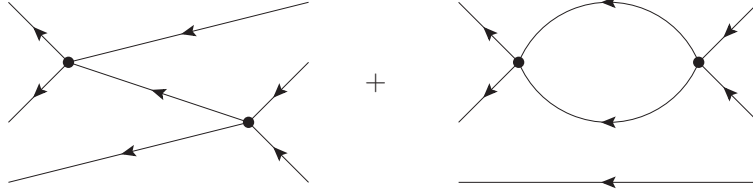
Figure 3. Feynman diagrams for the one-loop scattering amplitude $V_1^{(2)}(\mathbf{k}, \mathbf{p})$. Here, the first graph corresponds to the principal value contribution coming from the one-contraction terms, while the second corresponds to the sum of the delta contribution from the one-contraction term with the contribution coming from the term with two contractions.
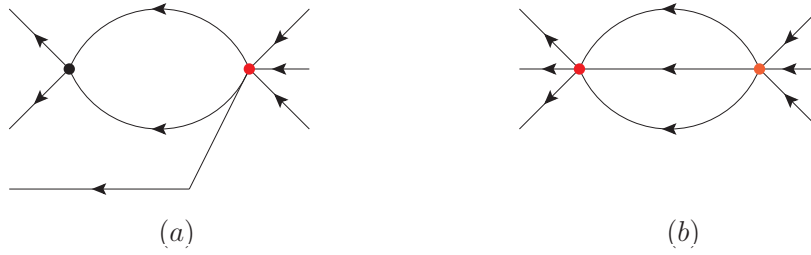

Figure 4. Feynman diagrams corresponding to the one-loop amplitudes: (a) $V_2^{(2)}(\mathbf{k}, \mathbf{p})$ and $V_3^{(2)}(\mathbf{k}, \mathbf{p})$, (b) $V_4^{(2)}(\mathbf{k}, \mathbf{p})$.

The contribution of the two-contraction terms can be written in the form:

$$V_{1(2C)}^{(2)}(\mathbf{k}, \mathbf{p}) = \frac{1}{2}\left(-\frac{3!\,g_2}{4m}\right)^2 \mathcal{A}_{p,k}\left\{\bar{U}_{12}^k \,\mathbb{P}\left[\left(p^2 \times p^1\right)\left(k^2 \times k^1\right)I_0(p^1, p^2) + \left(k^2 \times k^1 - \right.\right.\right. \tag{5.10}$$
$$\left.\left.\left. - p^2 \times p^1\right)I_1(p^1, p^2) - I_2(p^1, p^2)\right]\mathbb{P}U_{21}^p \, 8\pi^3\delta^{(2)}(k^1 + k^2 - p^1 - p^2)\delta(k^3 - p^3)\right\},$$

with the integrals $I_0(p^1, p^2)$, $I_1(p^1, p^2)$ and $I_2(p^1, p^2)$ defined in appendix B. Again, by using the completeness relations (A.13) and imposing the mass-shell condition, we can employ the identity (4.13) in conjunction with our ordering for the incoming momenta to conclude that

$$V_{1(2C)}^{(2)}(\mathbf{k}, \mathbf{p}) = 2\left(-\frac{ig_2}{4m}\right)^2 \left[(p^2 \times p^1)^2 + (p^3 \times p^1)^2 + (p^3 \times p^2)^2\right]\langle\mathbf{k}|\mathbf{p}\rangle. \tag{5.11}$$

On the other hand, the evaluation of the one-contraction terms is considerably more complex, and can be written in the form:

$$V_{1(1C)}^{(2)}(\mathbf{k}, \mathbf{p}) = \left(-\frac{3!\,g_2}{4m}\right)^2 \mathbb{P}_{ac,bd}\,\mathbb{P}_{eg,fh}\,\mathcal{A}_{p,k}\bigg\{\left[-\left(p^1 \times p^2\right)\left(k^2 \times k^3\right)I + \epsilon^{\alpha\beta}\left(\left(p^1 \times p^2\right)\right.\right. +$$
$$\left.\left. + \left(k^2 \times k^3\right)k_\beta^1\right)I_\alpha - \epsilon^{\alpha\beta}\epsilon^{\gamma\delta}k_\beta^1 p_\delta^3 I_{\alpha\gamma}\right]^{fa}$$
$$\times \bar{u}^c(k_1^1)\bar{u}^e(k_1^2)\bar{u}^g(k_1^3)u^b(p_1^1)u^d(p_1^2)u^h(p_1^3)\bigg\}, \tag{5.12}$$

where the integrals I, I_α and $I_{\alpha\gamma}$ are defined in appendix B by equations (B.5), (B.6) and (B.7), respectively. One crucial feature of the aforementioned integrals is that they all are proportional to a sum of the delta term and the principal value (p.v.) term:

$$\underbrace{\frac{\slashed{\Delta} + m}{4\omega(\Delta)}\, 2\pi\Big[\delta\left(\Delta_0 - \omega(\Delta)\right) - \delta\left(\Delta_0 + \omega(\Delta)\right)\Big]}_{\text{delta term}} + \underbrace{\frac{i\left(\slashed{\Delta} + m\right)}{\Delta^2 - m^2}}_{\text{p.v. term}}, \tag{5.13}$$

where we have defined $\Delta \equiv p^1 + p^2 - k^1$.

This split of the one-contraction contribution plays an important role in the subsequent analysis. As we will see, the contribution of the delta term from (5.13) to (5.12) will combine with (5.11) to yield the factorizable S-matrix at one-loop, while the one coming from the p.v. term in (5.13) will cancel the contribution of the three-particle interaction Lagrangian at tree-level, which prevented S matrix factorability. Upon substitution of the integrals I, I_α and $I_{\alpha\gamma}$, the expression (5.12) greatly simplifies,

$$V_{1(1C)}^{(2)}(\mathbf{k}, \mathbf{p}) = \left(-\frac{3!\, ig_2}{4m}\right)^2 \mathcal{A}_{p,k}\bigg\{\left(p^2 \times p^1 + \Delta \times k^1\right)\left(k^3 \times k^2 + \Delta \times p^3\right)\bigg[\frac{i}{\Delta^2 - m^2} + \tag{5.14}$$

$$+ \frac{2\pi}{4\omega(\Delta)}\Big(\delta\left(\Delta_0 - \omega(\Delta)\right) - \delta\left(\Delta_0 + \omega(\Delta)\right)\Big)\bigg]\bar{U}_{123}^k \mathbb{M}(\Delta) U_{213}^p\bigg\} 4\pi^2 \delta^{(2)}(\mathbf{k} - \mathbf{p}),$$

where we introduced

$$\mathbb{M}(q) := \gamma^3 \otimes (\slashed{q} + m) \otimes \gamma^3 - \gamma^3 \otimes \gamma^3(\slashed{q} + m) \otimes \mathbb{1}_2 - \mathbb{1}_2 \otimes (\slashed{q} + m)\gamma^3 \otimes \gamma^3$$
$$+ \mathbb{1}_2 \otimes \gamma^3(\slashed{q} + m)\gamma^3 \otimes \mathbb{1}_2. \tag{5.15}$$

The contribution of the delta term is easier to evaluate, as the delta functions implement the mass-shell conditions for the pseudo-particles with positive and negative energies, respectively. Since we assumed the scattering pseudo-particles to have positive energy, the condition $\Delta_0 + \omega(\Delta)$ cannot be satisfied for any $\mathbf{p}$ and $\mathbf{k}$, and therefore, we can disregard the second delta function above. We can conveniently rewrite the positive-energy delta function as follows:

$$\delta\Big(\Delta_0 - \omega(\Delta)\Big) = \frac{p_0^1\, p_0^2}{|p^2 \times p^1|}\Big[\delta(k_1^1 - p_1^1) + \delta(k_1^1 - p_1^2)\Big], \tag{5.16}$$

so that the delta function for overall energy-momentum conservation can be further simplified, allowing us to use the identity (4.7) and our ordering of initial momenta to trivially compute the spinorial products, along the same lines as with the two-particle case. After long but straightforward calculations the delta term contribution reduces to:

$$4\left(-\frac{ig_2}{4m}\right)^2\Big[(p^2 \times p^1)(p^3 \times p^1) + (p^2 \times p^1)(p^3 \times p^2) + (p^3 \times p^2)(p^3 \times p^1)\Big]\langle\mathbf{k}|\mathbf{p}\rangle. \tag{5.17}$$

Equations (5.11) and (5.17) can be easily combined, yielding:

$$
\begin{aligned}
V_1^{(2)}(\mathbf{k}, \mathbf{p}) =& 2\left(-\frac{ig_2}{4m}\right)^2 \left[(p^2 \times p^1) + (p^3 \times p^2) + (p^3 \times p^1)\right]^2 \langle \mathbf{k}|\mathbf{p}\rangle + \\
& + i\left(-\frac{3!\, ig_2}{4m}\right)^2 \mathcal{A}_{k,p} \left\{ \frac{\left[p^2 \times p^1 + \Delta \times k^1\right]\left[k^3 \times k^2 + \Delta \times p^3\right]}{\Delta^2 - m^2} \bar{U}_{123}^k \mathbb{M}(\Delta) U_{213}^p \right\} \cdot \\
& \cdot 4\pi^2 \delta^{(2)}(\mathbf{k} - \mathbf{p}).
\end{aligned} \tag{5.18}
$$

Clearly, the first term in (5.18) amounts to the complete contribution to a factorizable S-matrix at one-loop (5.1). It is worth pointing out that, according to the scheme proposed in [21], the remaining terms appearing in the one-loop scattering amplitude (5.9), $V_i^{(2)}(\mathbf{k}, \mathbf{p})$, $i = 2, 3, 4$, must be cancelled out by higher order contributions so as to have a factorizable S-matrix.

5.2 Continuity of the scattering amplitudes

Before moving onto the proof of the S-matrix factorization at first order, we pause to address one important subtlety that arises during the evaluation of the integrals I, I_α and $I_{\alpha\gamma}$ (see (5.12)), and which is intimately related to the formal proof of the S-matrix factorization for quantum integrable models [26, 28]. Let us consider one of these integrals (see appendix B, equations (B.5), (B.6) and (B.7)):

$$
I = \iint d^2x\, d^2y\, e^{-ix \cdot \Delta - iy \cdot \tilde{\Delta}} \int \frac{d^2q}{4\pi^2} e^{iq \cdot (x-y)} D(q) \tag{5.19}
$$
$$
= (\slashed{\Delta} + m) \left\{ \frac{2\pi}{4\omega(\Delta)} \left[\delta\left(\Delta_0 - \omega(\Delta)\right) - \delta\left(\Delta_0 + \omega(\Delta)\right)\right] + \frac{i}{\Delta^2 - m^2} \right\} 4\pi^2 \delta^{(2)}(\mathbf{k} - \mathbf{p}).
$$

Strictly speaking, the result in (5.19) is valid only for the $\Delta_0 - \omega(\Delta) \neq 0$ case. Indeed, this is the case for the standard situation of one pole on a real line.[12] However, a new feature in this model is that one needs to carefully take into account both poles of the propagator in (5.19).[13] Let us briefly explain how the computation is done (full details are given in the appendix B.2). The contour of the integration over q_0 is split into integrations over the segments $(-\infty, -\omega(q) - \epsilon)$, $(-\omega(q) + \epsilon, \omega(q) - \epsilon)$, and $(\omega(q) + \epsilon, +\infty)$, which is what we called the p.v. term, and the integrations over the two semi-circles around the two poles at $\pm\omega(q)$, which result in the δ-function terms in (5.19). A very careful analysis of the p.v. term shows that a typical integral to be evaluated has the form:

$$
\int_0^\infty dx \sin(x\eta)\, \text{si}(x\epsilon) = \left\{ \begin{array}{l} -\frac{\pi}{2\eta},\ \eta^2 > \epsilon^2 \\ 0,\ \eta^2 < \epsilon^2 \end{array} \right. , \tag{5.20}
$$

where $\eta \equiv \Delta_0 - \omega(\Delta)$, and $\text{si}(x)$ is the sine integral (see, for example, (6.252) of [29]). The result in (5.19) is valid for $\eta \neq 0$, and, therefore, $\eta^2 > \epsilon^2$, since we should consider the limit

---

[12]This is often formally written as $\frac{1}{x \pm i0} = \mp i\pi\delta(x) + \text{p.v.}\left(\frac{1}{x}\right)$.

[13]We remind that although the relativistic propagator is a retarded propagator by the choice of the false vacuum (3.16), one still needs to take into account both poles of the propagator when computing the integrals.

$\epsilon \to 0$. Let us note that the $\eta = 0$ point corresponds exactly to the integrability condition, namely, to the condition that the set of initial momenta is equal to the set of final momenta. In other words, the result in (5.19) is valid for the set of momenta which are not at the integrability point. Let us now turn to the case for which $\eta = 0$. This corresponds to the integrability point, and the integral (5.20) is equal to zero. Therefore, one in principle will obtain a different result from the one in (5.19). This is somewhat puzzling, as one would expect continuity of the scattering amplitude in the external momenta, and this issue should not have come up. It is clear, however, that even though each separate term, as the one above, should not be in principle continuous, the continuity in the external momenta should be restored when the contribution of all diagrams in each order in $\hbar$ is taken into account. The artificial singularity, that arises here in the point $\eta = 0$, appears because we split "by hand" the scattering amplitude into several terms corresponding to the integrals (B.5), (B.6) and (B.7).

This special point makes the analysis technically much more complicated, due to the enormous number of permutations. Indeed, by considering one particular integrability condition, corresponding to one fixed choice of the initial and final set of momenta, one needs to explicitly write down all the terms in (5.12), consider separately the subset corresponding to $\eta = 0$, and the subset for which $\eta \neq 0$. This is quite difficult to deal with, and instead we exclude this special case by the following argument. To be more precise in our analysis, we should, strictly speaking, consider localized wave-packet distributions, corresponding to the scattering particles. In this case the special point $\eta = 0$ is never reached, and in fact it should not even be taken into account. We then obtain the result in (5.19), which is used below to show the factorization of the S-matrix. Let us note that the formal proof of S-matrix factorizability for the quantum integrable systems requires consideration of such localized wave-packet distributions (see for details [27, 28]). Thus, our consideration is in complete agreement with the formal proof of the S-matrix factorization, and, therefore, we will use the result in (5.19), as well as similar results for the integrals in (B.5), (B.6) and (B.7) in appendix B.

Finally, we note that this difficulty does not arise in the calculation for simpler models, such as, the Landau-Lifshitz model [21]. This is because the propagator of the Landau-Lifshitz model has only one pole, and the special case $\eta = 0$ does not contribute, in other words, the p.v. is simply equal to zero. In contrast, in the AAF model there are two poles, and the $\eta = 0$ case produces a non-zero p.v. contribution. However, as we have explained above by utilizing the localized wave-packet distributions, the p.v. should be a continuous function of the external momenta, and $\eta = 0$ case plays no role in further analysis.

5.3   S-matrix factorization

In this section, we prove S-matrix factorization at first non-trivial order by showing that the second term in (5.7) and (5.18) cancel each other. The idea is to rewrite them in terms of rapidities, so that it is possible to compute the spinorial products without resorting to an identity, such as (4.7), and then work out the antisymmetrizations. Consider the rapidities:

$$p_0^i = m \cosh \theta_i, \ p_1^i = m \sinh \theta_i, \ k_0^i = m \cosh \eta_i, \ k_1^i = m \sinh \eta_i, \ i = 1, 2, 3.$$

The non-integrable contribution at tree-level is easily evaluated:

$$-\frac{ig_3}{16m}(3!)^2 \mathcal{A}_{p,k}\left[k^3 \times p^3\, \bar{U}^k_{123}\, \mathbb{Q}\, U^p_{123}\right] = -ig_3 m \cosh\left(\sum_{i=1}^3 \frac{\eta_i - \theta_i}{2}\right) F\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right), \qquad (5.21)$$

where we introduced the rapidity-dependent function:

$$F\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right) = \frac{\sinh\left(\frac{\eta_1-\eta_2}{2}\right)\sinh\left(\frac{\eta_1-\eta_3}{2}\right)\sinh\left(\frac{\eta_2-\eta_3}{2}\right)\sinh\left(\frac{\theta_1-\theta_2}{2}\right)\sinh\left(\frac{\theta_1-\theta_3}{2}\right)\sinh\left(\frac{\theta_2-\theta_3}{2}\right)}{\sqrt{\cosh\eta_1\cosh\eta_2\cosh\eta_3\cosh\theta_1\cosh\theta_2\cosh\theta_3}}.$$
$$(5.22)$$

The next step is to reduce the second term from (5.18) to the opposite of the tree-level contribution (5.21), though in this case the calculations are considerably more involved. First, we recast the argument of the antisymmetrizer in terms of rapidities:

$$\frac{\left[p^2 \times p^1 + \Delta \times k^1\right]\left[k^3 \times k^2 + \Delta \times p^3\right]}{\Delta^2 - m^2}\,\bar{U}^k_{123}\mathbb{M}(\Delta)U^p_{213} = -\frac{8m^3 G\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)}{\sqrt{\prod_{i=1}^3 \cosh\eta_i\cosh\theta_i}}, \quad (5.23)$$

where

$$G\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right) = \cosh\left(\frac{\eta_2-\eta_3}{2}\right)\cosh\left(\frac{\theta_1-\theta_2}{2}\right)\cosh\left(\frac{\eta_2-\theta_3}{2}\right)\coth\left(\frac{\eta_1-\theta_1}{2}\right)\cdot$$
$$\cdot \sinh\left(\frac{\eta_1-\theta_2}{2}\right)\sinh\left(\frac{\eta_3-\theta_3}{2}\right)\sinh\left(\frac{\eta_2-\eta_3-\theta_2+\theta_3}{2}\right). \qquad (5.24)$$

Then, instead of directly antisymmetrizing it, it is more profitable if we rewrite $G\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)$ so as to minimize the number of hyperbolic functions depending only on the difference of one η and one θ, obtaining, thus the following decomposition:

$$G\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right) = \frac{1}{4}\sum_{i=1}^8 A_i\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right), \qquad (5.25)$$

with the factors $A_i\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)$ defined in appendix C by (C.1)–(C.8). It is clear then that we can apply the antisymmetrizer at each $A_i\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)$ independently. Remarkably,

$$\mathcal{A}_{\theta,\eta}\left[A_i\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)\right] = 0, \quad \text{for } i = 2, 4, 5, 7 \quad \text{and} \quad \mathcal{A}_{\theta,\eta}\left[A_6\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)\right] = \mathcal{A}_{\theta,\eta}\left[A_8\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)\right]. \quad (5.26)$$

We give the long explicit expressions for the non-zero terms in the appendix C. The expression for $G\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)$ is considerably simplified upon antisymmetrization

$$\mathcal{A}_{\theta,\eta}\left[G\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)\right] = \frac{1}{4}\mathcal{A}_{\theta,\eta}\left[A_1\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right) + A_3\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right) + 2A_6\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)\right], \qquad (5.27)$$

but not yet reduced to the opposite of (5.21).

In addition, there is still the condition of overall energy and momentum conservation, which has not been so far imposed on (5.27). In fact, as we restrict the two-momenta to this submanifold, by implementing the overall delta function $\delta^{(2)}\left(\mathbf{k} - \mathbf{p}\right)$, we have shown

that both $\mathcal{A}_{\theta,\eta}\left[A_3\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]$ and $\mathcal{A}_{\theta,\eta}\left[A_6\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]$ vanish identically. The most straightforward way to do this is to consider light-cone momenta:

$$p_\pm^i = e^{\pm\theta} \quad \text{and} \quad k_\pm^i = e^{\pm\eta} \quad (i = 1, 2, 3),$$

for which the mass-shell condition is convenient recast as:

$$p_-^i = \frac{1}{p_+^i} \quad \text{and} \quad k_-^i = \frac{1}{k_+^i}.$$

This provides a suitable way not only to write $\mathcal{A}_{\theta,\eta}\left[A_3\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]$ and $\mathcal{A}_{\theta,\eta}\left[A_6\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]$, as they depend only on hyperbolic functions, but to implement the conservation of energy and momentum without introducing non-linear relations amongst the two-momenta. Then, after very long and tedious calculations one finds:

$$\mathcal{A}_{\theta,\eta}\left[A_3\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]\delta^{(2)}\left(\mathbf{k}-\mathbf{p}\right) = 0 \quad \text{and} \quad \mathcal{A}_{\theta,\eta}\left[A_6\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]\delta^{(2)}\left(\mathbf{k}-\mathbf{p}\right) = 0. \tag{5.28}$$

Thus, we obtain:

$$\begin{aligned} (3!)^2\,\mathcal{A}_{\theta,\eta}\left[G\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]\delta^{(2)}\left(\mathbf{k}-\mathbf{p}\right) &= \frac{1}{4}\,(3!)^2\,\mathcal{A}_{\theta,\eta}\left[A_1\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)-A_3\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]\delta^{(2)}\left(\mathbf{k}-\mathbf{p}\right) \\ &= 2\cosh\left(\sum_{i=1}^3 \frac{\eta_i-\theta_i}{2}\right)\sinh\left(\frac{\eta_1-\eta_2}{2}\right)\sinh\left(\frac{\eta_1-\eta_3}{2}\right)\cdot \\ &\quad \cdot \sinh\left(\frac{\eta_2-\eta_3}{2}\right)\sinh\left(\frac{\theta_1-\theta_2}{2}\right)\sinh\left(\frac{\theta_1-\theta_3}{2}\right)\cdot \\ &\quad \cdot \sinh\left(\frac{\theta_2-\theta_3}{2}\right)\delta^{(2)}\left(\mathbf{k}-\mathbf{p}\right). \end{aligned} \tag{5.29}$$

And finally,

$$\begin{aligned} i\left(-\frac{3!\,ig_2}{4m}\right)^2\mathcal{A}_{k,p}\left\{\frac{\left[p^2\times p^1 + \Delta\times k^1\right]\left[k^3\times k^2 + \Delta\times p^3\right]}{\Delta^2 - m^2}\bar{U}_{123}^k\mathsf{M}(\Delta)U_{213}^p\right\}4\pi^2\delta^{(2)}\left(\mathbf{k}-\mathbf{p}\right) = \\ = ig_2^2 m\cosh\left(\sum_{i=1}^3\frac{\eta_i-\theta_i}{2}\right)F\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)4\pi^2\delta^{(2)}\left(\mathbf{k}-\mathbf{p}\right). \end{aligned} \tag{5.30}$$

Noting also that there should be a factor of $4\pi\delta^{(2)}\left(\mathbf{k}-\mathbf{p}\right)$ multiplying the tree-level non-integrable term (5.21), we can easily conclude that the term which prevented S-matrix factorization at tree level (5.21) is indeed cancelled by the p.v. contribution coming from the one-contraction diagrams at one-loop (5.30), provided the following constraint on the coupling constants holds:

$$g_2^2 = g_3. \tag{5.31}$$

This is one of our central results, which provides the necessary condition for the quantum integrability of the AAF model. Note that this condition is in complete agreement with the mass dimensions assigned to the coupling constants g_2 and g_3, and the classical Lagrangian (3.8), as discussed in section 3. Indeed, with (5.31), one can easily go back from

the Lagrangian (3.8) with coupling constants to its dimensionless counterpart (3.6) with the correct coefficients.

Therefore, if we denote $g = g_2$, the scattering amplitude for the three-particle scattering becomes

$$\langle \mathbf{k} | \hat{S} | \mathbf{p} \rangle = \left\{ 1 + 2 \sum_{n=1}^{2} \left[\left(-\frac{ig}{4m} \right) \left(p^2 \times p^1 + p^3 \times p^1 + p^3 \times p^2 \right) \right]^n \right\} \langle \mathbf{k} | \mathbf{p} \rangle + O(g^3), \quad (5.32)$$

where we included the one-loop amplitudes $V_i^{(2)}(\mathbf{k}, \mathbf{p})$, $i = 2, 3, 4$ in $O(g^3)$, for

$$V_2^{(2)}(\mathbf{k}, \mathbf{p}) \sim V_3^{(2)}(\mathbf{k}, \mathbf{p}) \sim g_2 g_3 \sim g^3 \quad \text{and} \quad V_4^{(2)}(\mathbf{k}, \mathbf{p}) \sim g_3^2 \sim g^4.$$

The first term in (5.32) corresponds to the factorizable[14] three-particle S-matrix at second order in g. Consequently, in order to have S-matrix factorization even at tree-level, it is mandatory to consider higher loop amplitudes, as they yield the counterterms for lower order non-integrable terms.

It is not difficult to see that this remarkable scheme of cancellations would have been completely ruined had we not corrected the factor of $\frac{1}{2}$ missed by [14] and the overall sign difference in comparison with by [19], as they were paramount for the intricate fine-tuning between the interaction terms required for S-matrix factorization. Rectifying the signs in the action (3.8) had profound consequences in the analysis thereafter, the most notable being that instead of deriving the S-matrix proposed by [19] we obtained its inverse. Remarkably, our two-particle S-matrix (4.17), written in terms of rapidities:

$$S(\theta_1, \theta_2) = \frac{1 - \frac{img_2}{4} \sinh(\theta_1 - \theta_2)}{1 + \frac{img_2}{4} \sinh(\theta_1 - \theta_2)}, \quad (5.33)$$

is very similar to the S-matrix for two-particle scattering of the massive Thirring model:

$$S_{\text{Thirring}}(\theta, \theta') = \frac{1 - \frac{ig}{2} \tanh \frac{\theta - \theta'}{2}}{1 + \frac{ig}{2} \tanh \frac{\theta - \theta'}{2}}. \quad (5.34)$$

We can now write down the correct Bethe equations for the AAF model:

$$e^{iJm \sinh \theta_i} = \prod_{k \neq i} \frac{1 + \frac{img_2}{4} \sinh(\theta_i - \theta_k)}{1 - \frac{img_2}{4} \sinh(\theta_i - \theta_k)}, \quad (5.35)$$

which is the first step in analyzing the bound and negative energy states, and obtaining the physical S-matrix and excitations for both repulsive and attractive cases [19, 30, 31].

We conclude the paper by making a comparison with the analysis and the results of [19]. Towards this end, we start from the original AAF action (3.3) and consider the transformation:[15]

$$\tau \to -\tau, \quad \sigma \to -\sigma, \quad \rho^\alpha \to -\rho^\alpha, \quad (5.36)$$

---

[14]Compare (5.32) with (5.1), which we derived only from the knowledge of the two-particle S-matrix.

[15]Here we also neglect the constant term, fix $\kappa = \frac{\sqrt{\lambda}}{2}$ and change the Dirac matrices basis through (A.5).

Then the action becomes:

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d\tau \int_0^{\frac{2\pi J}{\sqrt{\lambda}}} d\sigma \left[\frac{i}{2} \left(\bar{\chi} \, \gamma^\alpha \partial_\alpha \chi - \partial_\alpha \bar{\chi} \, \gamma^\alpha \chi \right) - \bar{\chi} \chi - \right.$$
$$\left. - \frac{1}{4} \epsilon^{\alpha\beta} \left(\bar{\chi} \partial_\alpha \chi \, \bar{\chi} \, \gamma^3 \partial_\beta \chi - \partial_\alpha \bar{\chi} \chi \, \partial_\beta \bar{\chi} \, \gamma^3 \chi \right) - \frac{1}{16} \epsilon^{\alpha\beta} \left(\bar{\chi} \chi \right)^2 \partial_\alpha \bar{\chi} \, \gamma^3 \partial_\beta \chi \right], \qquad (5.37)$$

where we have denoted $\chi(\tau, \sigma) \equiv \psi(-\tau, -\sigma)$. This action differs from our action (3.4) by the sign of the quartic term, and this is essentially the action considered by [19], written in terms of the field $\chi(\tau, \sigma)$. Thus, the two actions are different already on the classical level. Careful analysis shows that the S-matrix of [19] corresponds to the scattering of the χ-particles, rather than the original ψ-particles. It is easy to see from the mode expansions for both fields, that going from $\chi(\tau, \sigma)$ to $\psi(\tau, \sigma)$ corresponds to interchanging the particle and anti-particle operators $a^\dagger \longleftrightarrow b^\dagger$, as the transformation (5.36) involves time inversion. Hence, it is not surprising that our S-matrix (5.33) is the inverted result of [19], and to make a connection between the two results we must change the coupling constant $g_2 \to -g_2$. Indeed, our general result (5.33) explicitly shows this.

It is important to emphasize, that our quantum integrability condition (5.31) is invariant under the transformation $g_2 \to -g_2$, and consequently, both models (3.8) and (5.37) are integrable. Finally, we stress that even though this action arises from string theory, our generalized model with two independent coupling constants is an interesting case to investigate on its own, where both repulsive and attractive cases should be considered separately [30, 31].

6 Conclusion

In this paper we have considered the quantum integrability of the AAF model, and showed the S-matrix factorizability in the first non-trivial order. As we explain in the main text, the AAF model requires introducing two dimensional coupling constants, and one of our main results is a necessary relation between these coupling constants in order to guarantee the quantum integrability. With this quantum constraint we were able to reveal and correct several missed factors in the previous works, as well as to derive the correct S-matrix. The latter is the inverse of the one found in [19]. This also changes the analysis of bound and negative energy states. In the process, the mechanism behind the cancellations of non-integrable parts is understood at the perturbative level.

As discussed in the introduction, the AAF model is a very interesting fermionic integrable model, which is considerably more complex than its simpler fermionic Thirring model counterpart. While for the latter we have a number of techniques to understand its quantization, for the AAF model this is not the case due to its complexity. Even though the quantum inverse scattering method is a relatively straightforward procedure, in the AAF model the main difficulty lies in the type of the interaction Hamiltonian, namely, in the singular behavior of the quantum mechanical Hamiltonian and all other conserved charges. Let us note, that in the Landau-Lifshitz model one encounters exactly the same type of singularity, and the development of the quantum inverse scattering method required

considerable effort and careful analysis of operator products and their regularization, as well as the construction of the correct Hilbert space [16–18, 32]. It has also been shown that the self-adjointness of the operators is essentially equivalent to the S-matrix factorization. It is desirable to perform a similar analysis for the AAF model, and we plan to do it in the future. The lattice version of the AAF model should be also understood in the process. It would be also interesting to compare the two extended Hilbert spaces for the two models. Indeed, we do not expect that the constructions of the self-adjoint operators and corresponding extensions should coincide, as they carry a different number of degrees of freedom. Moreover, one should understand whether the two extensions can be accommodated in some larger space to fit both fermionic and bosonic degrees of freedom. This will be a very important step in understanding the correct Hilbert space for the entire string on $AdS_5 \times S^5$, as we have emphasized in [16–18].

While the AAF model has some similarity to the Landau-Lifshitz model, there are a few distinctions that make the AAF model a more intriguing theory. In particular, the Poisson brackets structure in the AAF model is highly non-linear, due to the presence of the time derivatives in the forth and sixth order of the interaction vertices, extending up to the eighth order in fermionic fields, which makes it quite difficult to develop the standard quantum inverse scattering method. While there are examples of such models, for instance the *anisotropic* Landau-Lifshitz model where the standard commutation relations between the fields should be modified in the quantum theory, resulting in non-linear *Sklyanin algebra*, it is hard to deal with such theories. Besides that, there is a deep relation between the algebraic structure and regularization of the singular Yang-Baxter relations. In addition, the non-linearity in the commutation relations in the AAF model already appears in the classical theory. This is somewhat unusual, and may lead to the loss of some non-perturbative effects in the perturbative analysis. For other known models this does not happen, and the S-matrix perturbative calculations have produced consistent results. However, strictly speaking, the perturbative S-matrix calculations are not reliable, and the full picture can be understood only within the framework of the quantum inverse scattering method. Therefore, developing the latter, together with the careful analysis of bound and negative states, as well as construction of the physical S-matrix and excitations, should be the main focus in the future investigations.

Acknowledgments

A   Two-dimensional Dirac equation

Consider the two-dimensional Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0, \tag{A.1}$$

with the Dirac matrices γ^μ, $\mu = 0, 1$, belonging to the SO(1, 1) Clifford algebra:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \eta^{\mu\nu} \mathbb{1}_2 \,, \tag{A.2}$$

where the two-dimensional Minkowsky metric is $\eta = \mathrm{diag}\,(1, -1)$ and the symbol $\mathbb{1}_2$ stands for the 2×2 unit matrix. In the main text, we consider the following faithful representations of (A.2):

$$\rho^0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho^5 = \rho^0 \rho^1, \tag{A.3}$$

and

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^3 = \gamma^0 \gamma^1, \tag{A.4}$$

which are related by the similarity transformation bellow:

$$\gamma^\mu = M \rho^\mu M^{-1}, \quad M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}. \tag{A.5}$$

A.1 Plane-wave solutions

Substituting the plane-wave solution:

$$\psi_a(x) = e^{-ip \cdot x} u_a(p), \quad \text{with} \quad p \cdot x = \eta^{\mu\nu} p_\mu x_\nu. \tag{A.6}$$

into (A.1) leads to the equation of motion for the spinor:

$$(\not{p} - m)\, u_a(p) = 0, \tag{A.7}$$

which has non-trivial solutions, if and only if,

$$\det (\not{p} - m) = 0 \Rightarrow m^2 - p^2 = 0,$$

i.e., if and only if, the momentum p is on mass-shell, solving this condition for the energy, we get:

$$p_0 = \sqrt{p_1^2 + m^2} =: \omega(p). \tag{A.8}$$

The normalized solutions of (A.7), with positive and negative energy, are:

$$u(p) = \begin{pmatrix} \sqrt{\frac{\omega(p) - p_1}{2\omega(p)}} \\ \sqrt{\frac{\omega(p) + p_1}{2\omega(p)}} \end{pmatrix}, \quad v(p) = \begin{pmatrix} \sqrt{\frac{\omega(p) + p_1}{2\omega(p)}} \\ -\sqrt{\frac{\omega(p) - p_1}{2\omega(p)}} \end{pmatrix}. \tag{A.9}$$

Therefore, the plane-wave solutions for the Dirac equation in two dimensions are:

$$\begin{aligned} \psi_+(x) &= e^{-ip \cdot x}\, u(p), \quad \text{with} \quad (\not{p} - m)\, u(p) = 0, \\ \psi_-(x) &= e^{ip \cdot x}\, v(-p), \quad \text{with} \quad (\not{p} + m)\, v(-p) = 0. \end{aligned} \tag{A.10}$$

A.2 Completeness and orthonormality relations

The solutions $u(p)$ and $v(p)$ (A.9) satisfy the orthonormality relations:

$$
\begin{aligned}
u^\dagger(p)u(p) = 1, \ v^\dagger(p)v(p) = 1, \\
u^\dagger(p)v(p) = 0, \ v^\dagger(p)u(p) = 0,
\end{aligned}
\tag{A.11}
$$

as well as,

$$
\begin{aligned}
\bar{u}(p)u(p) = \tfrac{m}{\omega(p)}, \ \bar{v}(p)v(p) = -\tfrac{m}{\omega(p)}, \\
\bar{u}(p)v(p) = \tfrac{p_1}{\omega(p)}, \ \bar{v}(p)u(p) = \tfrac{p_1}{\omega(p)},
\end{aligned}
\tag{A.12}
$$

where the conjugate spinors are defined in the usual way: $\bar{u} = u^\dagger\gamma^0$ and $\bar{v} = v^\dagger\gamma^0$.

Finally, the completeness relations for the spinors $u(p)$ e $v(p)$ are:

$$
u(p)\bar{u}(p) = \frac{\slashed{p} + m}{2\omega(p)}, \quad v(-p)\bar{v}(-p) = \frac{\slashed{p} - m}{2\omega(p)}.
\tag{A.13}
$$

B Table of useful integrals

Let $D(q)$ be the propagator in momentum space:

$$
D(q) = \frac{i\left(\slashed{q} + m\right)}{q^2 - m^2 + 2i\epsilon q_0}.
\tag{B.1}
$$

The relevant momentum space integrals for two-particle scattering are:

$$
I_0(p^1, p^2) = \int \frac{d^2q}{4\pi^2}\, D(q) \otimes D(p^1 + p^2 - q)
\tag{B.2}
$$

$$
= \frac{p_0^1 + p_0^2}{4|p_1^1 - p_1^2|(p^1 + p^2)^2} \left[(\slashed{p}^1 + m) \otimes (\slashed{p}^2 + m) + (\slashed{p}^2 + m) \otimes (\slashed{p}^1 + m) \right] + D_0,
$$

$$
I_1(p^1, p^2) = \int \frac{d^2q}{4\pi^2}\, \left[(p^1 + p^2 - q) \times q \right] D(q) \otimes D(p^1 + p^2 - q)
\tag{B.3}
$$

$$
= \frac{p^2 \times p^1 (p_0^1 + p_0^2)}{4|p_1^1 - p_1^2|(p^1 + p^2)^2} \left[(\slashed{p}^1 + m) \otimes (\slashed{p}^2 + m) - (\slashed{p}^2 + m) \otimes (\slashed{p}^1 + m) \right] + D_1,
$$

$$
I_2(p^1, p^2) = \int \frac{d^2q}{4\pi^2}\, \left[(p^1 + p^2 - q) \times q \right]^2 D(q) \otimes D(p^1 + p^2 - q)
\tag{B.4}
$$

$$
= \frac{\left(p^2 \times p^1\right)^2 (p_0^1 + p_0^2)}{4|p_1^1 - p_1^2|(p^1 + p^2)^2} \left[(\slashed{p}^1 + m) \otimes (\slashed{p}^2 + m) + (\slashed{p}^2 + m) \otimes (\slashed{p}^1 + m) \right] + D_2,
$$

where D_i, $i = 0, 1, 2$ stand for their respective divergent parts.

For three-particle scattering the following integrals are also needed:

$$I = \iint d^2x\, d^2y\, e^{-ix\cdot\Delta - iy\cdot\tilde{\Delta}} \int \frac{d^2q}{4\pi^2}\, e^{iq\cdot(x-y)} D(q) \tag{B.5}$$

$$= (\slashed{\Delta} + m)\left\{ \frac{2\pi}{4\omega(\Delta)}\left[\delta\left(\Delta_0 - \omega(\Delta)\right) - \delta\left(\Delta_0 + \omega(\Delta)\right) \right] + \frac{i}{\Delta^2 - m^2} \right\} 4\pi^2 \delta^{(2)}(\mathbf{k} - \mathbf{p}),$$

$$I_\alpha = \iint d^2x\, d^2y\, e^{-ix\cdot\Delta - iy\cdot\tilde{\Delta}} \int \frac{d^2q}{4\pi^2}\, e^{iq\cdot(x-y)} q_\alpha D(q) \tag{B.6}$$

$$= \Delta_\alpha (\slashed{\Delta} + m)\left\{ \frac{2\pi}{4\omega(\Delta)}\left[\delta\left(\Delta_0 - \omega(\Delta)\right) - \delta\left(\Delta_0 + \omega(\Delta)\right) \right] + \frac{i}{\Delta^2 - m^2} \right\} 4\pi^2 \delta^{(2)}(\mathbf{k} - \mathbf{p}),$$

$$I_{\alpha\gamma} = \iint d^2x\, d^2y\, e^{-ix\cdot\Delta - iy\cdot\tilde{\Delta}} \int \frac{d^2q}{4\pi^2}\, e^{iq\cdot(x-y)} q_\alpha q_\gamma D(q) \tag{B.7}$$

$$= \Delta_\alpha \Delta_\gamma (\slashed{\Delta} + m)\left\{ \frac{2\pi}{4\omega(\Delta)}\left[\delta\left(\Delta_0 - \omega(\Delta)\right) - \delta\left(\Delta_0 + \omega(\Delta)\right) \right] + \frac{i}{\Delta^2 - m^2} \right\} 4\pi^2 \delta^{(2)}(\mathbf{k} - \mathbf{p}),$$

where $\Delta = p^1 + p^2 - k^1$ and $\tilde{\Delta} = p^3 - k^2 - k^3$.

B.1 Integrals for two-particle scattering: computational details

In this small section, we quickly outline the main steps involved in the evaluation of (B.2) and comment on some of the nuances of the result. We note that the presence of factors such as $(p^1 + p^2 - q) \times q$ in the other two integrals, i.e., (B.3) and (B.4), do not introduce any serious technical complication, despite increasing the superficial degree of divergence. Before going into the details of this calculation, we would like to stress that despite the apparent symmetry with respect to the momenta p^1 and p^2 in (B.8), which manifests in the dependence of the integral only on the sum $p^1 + p^2$, this is not the case. Indeed, as we already discussed in the beginning of the section 4, one must choose a particular ordering for the incoming momenta p_1^1 and p_1^2.[16] Therefore, after integration over q_0, the position of the poles will depend on this ordering, and, as a consequence, the final answer may depend not only on the sum $p^1 + p^2$. In fact, this is the case, as the explicit calculations below show.

We can easily perform the integration over q_0 of the integral

$$I_0(p^1, p^2) = \int \frac{d^2q}{4\pi^2} \frac{i(\slashed{q} + m)}{q^2 - m^2 + 2i\epsilon q_0} \otimes \frac{i(\slashed{p}^1 + \slashed{p}^2 - \slashed{q} + m)}{(p^1 + p^2 - q)^2 - m^2 + 2i\epsilon(p_0^1 + p_0^2 - q_0)}, \tag{B.8}$$

by closing the integration contour in the lower complex half-plane, so that it encloses two of the four simple poles of (B.8), as depicted in figure 5. The remaining integration over q_1 can then be reduced to the following form:

$$I_0(p^1, p^2) = \frac{1}{4i\pi (p^1 + p^2)^2} \int dq_1 \frac{f(q_1)}{(q_1 - p_1^1 - i\eta)(q_1 - p_1^1 + i\eta)}, \tag{B.9}$$

where $f(q_1)$ is a polynomial in q_1 of degree two with some coefficients, but still symmetric in the external momenta. It is important to notice that η is a function of ϵ and the external

---
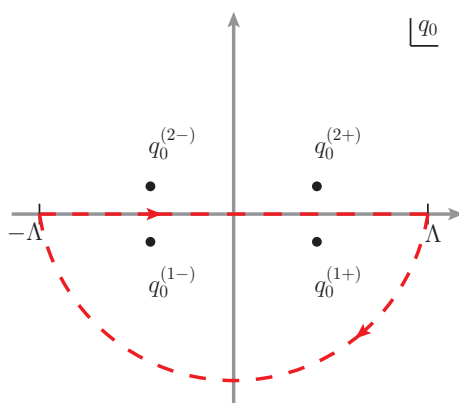
[16]In this paper we choose $p_1^1 > p_1^2$.

Figure 5. Pole prescription and contour of integration for computing the integral over q_0 in (B.8).

momenta, which has the following form:

$$\eta = \frac{4\epsilon p_0^1 p_0^2 \left(p_0^1 + p_0^2\right)}{\left(p_1^1 - p_1^2\right)\left(p^1 + p^2\right)^2}.$$

From this expression it is clear that in order to perform the remaining integration over q_1 one must choose a concrete ordering of the incoming momenta, which fixes the positions of the poles, and ensures that η is a well-behaved function. This in turn breaks the symmetry between p^1 and p^2, as it is clear from the denominator of (B.9).

Noting that

$$\int \frac{dq \left(aq^2 + bq + c\right)}{\left(q - p_1^1 - i\eta\right)\left(q - p_1^2 + i\eta\right)} = \left[a\left(P_-^2 + P_+^2\right) + bP_+ + c\right]\int \frac{dx}{\left(x - P_- - i\eta\right)\left(x + P_- + i\eta\right)} +$$
$$+ a\int dx, \tag{B.10}$$

where we wrote $f(q) = aq^2 + bq + c$ and introduced

$$P_+ := \frac{p_1^1 + p_1^2}{2} \quad \text{and} \quad P_- := \frac{p_1^1 - p_1^2}{2},$$

we can finally use

$$\int \frac{dx}{\left(x - P_- - i\eta\right)\left(x + P_- + i\eta\right)} = \frac{i\pi}{|P_-|} \tag{B.11}$$

to obtain the result (B.2), with

$$D_0 = 2a \lim_{\Lambda \to \infty} \Lambda \quad \text{where} \quad a = -\left(p_0^1 + p_0^2\right)\left(\gamma^0 \otimes \gamma^0 + \gamma^1 \otimes \gamma^1\right) + \left(p_1^1 + p_1^2\right)\left(\gamma^0 \otimes \gamma^1 + \gamma^1 \otimes \gamma^0\right). \tag{B.12}$$

B.2 Integrals for three-particle scattering: computational details

The computation of the integrals (B.5)–(B.7) is considerably more involved and exhibits some interesting features, which are essential for demonstrating the S-matrix factorization
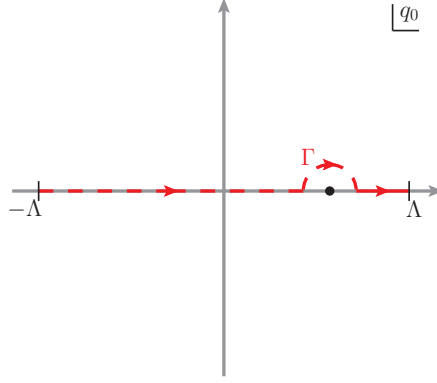
Figure 6. Integration path Γ for the first term in (B.15).

property. For the reader's convenience, we outline here the main steps for evaluating (B.5). We note that the other two integrals, namely (B.6) and (B.7), can be computed by exactly the same method, and the extra factors of momentum introduce no serious technical complication.

Introducing light-cone-like coordinates: $x = x_+ + x_-$ and $y = x_+ - x_-$, (B.5) becomes

$$I = 4 \iint d^2x_+ \, d^2x_- \, e^{-ix_+ \cdot (\Delta + \tilde{\Delta})} e^{-ix_- \cdot (\Delta - \tilde{\Delta})} \int \frac{d^2q}{4\pi^2} \, e^{2iq \cdot x_-} \frac{i \, (\slashed{q} + m)}{q^2 - m^2 + 2i\epsilon q_0}. \quad \text{(B.13)}$$

We can then integrate over x_+, to obtain the overall energy-momentum conservation delta function $\delta^{(2)}\left(\Delta + \tilde{\Delta}\right) = \delta^{(2)}\left(\mathbf{k} - \mathbf{p}\right)$. Rescaling, $x_- \to -\frac{1}{2}x$ and using the exact decomposition for the propagator:

$$\frac{1}{q^2 - m^2 + 2i\epsilon q_0} = \frac{1}{2\omega(q)} \left[\frac{1}{q_0 - \omega(q) + i\epsilon} - \frac{1}{q_0 + \omega(q) + i\epsilon} \right], \quad \text{(B.14)}$$

we obtain:

$$I = \int d^2x \, e^{ix \cdot \Delta} \int \frac{d^2q}{4\pi^2} \, \frac{i \, (\slashed{q} + m) \, e^{-ix \cdot q}}{2\omega(q)} \left[\frac{1}{q_0 - \omega(q) + i\epsilon} - \frac{1}{q_0 + \omega(q) + i\epsilon} \right] 4\pi^2 \delta^{(2)}\left(\mathbf{k} - \mathbf{p}\right). \quad \text{(B.15)}$$

For computing the integral over q_0 of the first term in (B.15), we introduce the integration path:

$$\Gamma = \begin{cases} q_0\,, & q_0 \in \left[-\Lambda, \omega(q) - \epsilon\right) \bigcup \left(\omega(q) + \epsilon, \Lambda\right] \\ q_0 - \omega(q) = \epsilon e^{-i\theta}\,, & \theta \in [0, \pi] \quad (C_\epsilon) \end{cases}, \quad \text{(B.16)}$$

depicted in figure 6. Then,

$$\int \frac{dq_0}{2\pi} \frac{(\slashed{q} + m) \, e^{-ix^0 q_0}}{q_0 - \omega(q) + i\epsilon} = \lim_{\substack{\Lambda \to \infty \\ \epsilon \to 0}} \left\{ \left[\int_{-\Lambda}^{\omega(q) - \epsilon} + \int_{\omega(q) + \epsilon}^{\Lambda} \right] \frac{dq_0}{2\pi} \frac{(\slashed{q} + m) \, e^{-ix^0 q_0}}{q_0 - \omega(q)} + \right.$$

$$\left. + \frac{1}{2\pi} \left[(\slashed{q} + m) \, e^{-ix^0 q_0} \right] \Big|_{q_0 = \omega(q)} \int_{C_\epsilon} \frac{dq_0}{q_0 - \omega(q)} \right\}$$

$$= \fint \frac{dq_0}{2\pi} \frac{(\slashed{q} + m) \, e^{-ix^0 q_0}}{q_0 - \omega(q)} - \frac{i}{2} \left[(\slashed{q} + m) \, e^{-ix^0 q_0} \right] \Big|_{q_0 = \omega(q)}, \quad \text{(B.17)}$$

which is nothing but the sum of a principal value term with a delta term that gives rise to the decomposition (5.13). The second term of (B.15) can be computed in the same vein. Hence,

$$I = \int d^2x \, e^{ix \cdot \Delta} \int \frac{dq_1}{2\pi} \frac{ie^{-ix^1 q_1}}{2\omega(q)} \left\{ -\frac{i}{2} \left[(\slashed{q} + m) \, e^{-ix^0 q_0} \Big|_{q_0 = \omega(q)} - (\slashed{q} + m) \, e^{-ix^0 q_0} \Big|_{q_0 = -\omega(q)} \right] + \right.$$
$$\left. + \fint \frac{dq_0}{2\pi} \, (\slashed{q} + m) \, e^{-ix^0 q_0} \left[\frac{1}{q_0 - \omega(q)} - \frac{1}{q_0 + \omega(q)} \right] \right\} 4\pi^2 \delta^{(2)}(\mathbf{k} - \mathbf{p}). \tag{B.18}$$

As stated above, the terms in the first line of (B.18) will contribute to the delta terms in the decomposition (5.13). In order to conclude this, it is only necessary to group the exponentials together, and realize that one can safely exchange the order of integrations to obtain a delta function from the integration over x. Namely,

$$\int d^2x \, e^{ix \cdot \Delta} \int \frac{dq_1}{2\pi} \frac{ie^{-ix^1 q_1}}{2\omega(q)} \left\{ -\frac{i}{2} \left[(\slashed{q} + m) \, e^{-ix^0 q_0} \Big|_{q_0 = \omega(q)} - (\slashed{q} + m) \, e^{-ix^0 q_0} \Big|_{q_0 = -\omega(q)} \right] \right\} =$$
$$= \frac{\slashed{\Delta} + m}{4\omega(\Delta)} 2\pi \left[\delta \left(\Delta_0 - \omega(\Delta) \right) - \delta \left(\Delta_0 + \omega(\Delta) \right) \right]. \tag{B.19}$$

Deriving the p.v. contribution demands more work. First, let us denote the principal value integral from (B.18) simply as $\fint$ and note that:

$$(\slashed{q} + m) \left[\frac{1}{q_0 - \omega(q)} - \frac{1}{q_0 + \omega(q)} \right] = \gamma^0 \omega(q) \left[\frac{1}{q_0 - \omega(q)} + \frac{1}{q_0 + \omega(q)} \right] +$$
$$+ (q_1 \gamma^1 + m) \left[\frac{1}{q_0 - \omega(q)} - \frac{1}{q_0 + \omega(q)} \right]. \tag{B.20}$$

Hence, if we use the identities:

$$\fint \frac{dq_0}{2\pi} \, e^{-ix^0 q_0} \left[\frac{1}{q_0 - \omega(q)} - \frac{1}{q_0 + \omega(q)} \right] = -\frac{2}{\pi} \sin \left(\omega(q) x^0 \right) \int_\epsilon^\infty \frac{dz}{z} \sin \left(x^0 z \right), \tag{B.21}$$

$$\fint \frac{dq_0}{2\pi} \, e^{-ix^0 q_0} \left[\frac{1}{q_0 - \omega(q)} + \frac{1}{q_0 + \omega(q)} \right] = -\frac{2i}{\pi} \cos \left(\omega(q) x^0 \right) \int_\epsilon^\infty \frac{dz}{z} \sin \left(x^0 z \right), \tag{B.22}$$

where we left implicit the $\epsilon \to 0$ limit, we can write:

$$\fint = \int d^2x \, e^{ix \cdot \Delta} \int \frac{dq_1}{2\pi} \frac{ie^{-ix^1 q_1}}{2\omega(q)} \left\{ -\frac{2i}{\pi} \gamma^0 \omega(q) \cos \left(\omega(q) x^0 \right) - \frac{2}{\pi} (q_1 \gamma^1 + m) \sin \left(\omega(q) x^0 \right) \right\} \cdot$$
$$\cdot \int_\epsilon^\infty \frac{dz}{z} \sin \left(x^0 z \right) 4\pi \delta^{(2)} \left(\mathbf{k} - \mathbf{p} \right)$$
$$= \frac{i}{8\pi^2} \int d^2x \int dq_1 \frac{e^{ix^1 (\Delta_1 - q_1)}}{\omega(q)} \left[\int_{-\infty}^{-\epsilon} + \int_\epsilon^\infty \right] \frac{dz}{z} \left\{ \left[\omega(q) \gamma^0 - q_1 \gamma^1 - m \right] e^{ix^0 [\Delta_0 + \omega(q) - z]} + \right.$$
$$\left. + \left[\omega(q) \gamma^0 + q_1 \gamma^1 + m \right] e^{ix^0 [\Delta_0 - \omega(q) - z]} \right\} 4\pi^2 \delta^{(2)} \left(\mathbf{k} - \mathbf{p} \right). \tag{B.23}$$

Next, we exchange the order of integrations so as to evaluate the integrals over x^1 and x^0 first. The integration over x^1 factorizes and clearly yields a delta function, which can

then be used to integrate over q_1, casting $q_1 = \Delta_1$. The situation involving the integration over x^0 is, however, more delicate. Here the limit $\epsilon \to 0$ plays a paramount role, as it removes the point $z = 0$ from the integration domain, thus, removing the $\sim \frac{1}{z}$ singularity, and we can safely proceed as before. Namely, integrate over x^0 to obtain a delta function, which in turn allows us to perform the integration over z. Concluding, thus, that:

$$\begin{aligned} I &= \frac{i}{2\omega(\Delta)} \left\{ \frac{\omega(\Delta)\gamma^0 - \Delta_1\gamma^1 - m}{\Delta_0 + \omega(\Delta)} + \frac{\omega(\Delta)\gamma^0 + \Delta_1\gamma^1 + m}{\Delta_0 - \omega(\Delta)} \right\} 4\pi^2\delta^{(2)}\left(\mathbf{k} - \mathbf{p}\right) \\ &= \frac{i\left(\slashed{\Delta} + m\right)}{\Delta^2 - m^2} 4\pi^2\delta^{(2)}\left(\mathbf{k} - \mathbf{p}\right). \end{aligned} \tag{B.24}$$

Finally, by adding (B.19) and (B.24) together, we obtain the desired result, (B.5).

We stress, however, that a careful analysis of whether the factor $\Delta_0 \pm \omega(\Delta)$ vanishes is of great importance, as it may imply divergent or discontinuous scattering amplitudes (see section 5.2 for the discussion of this subtlety). The plus case is easier to understand, since we deal only with pseudo-particles with positive energy, and thus, if we impose the mass-shell condition,[17] one easily sees that such a polynomial has no real roots. This is obviously not the case for the on-shell polynomial coming from $\Delta_0 - \omega(\Delta)$, which can be solved, say for k^1, giving $k^1 = p^1$ or $k^1 = p^2$. Although we can still perform the integrations over x^0 and z as we did above, even if $\Delta_0 - \omega(\Delta) = 0$, the result is obviously not the same as (B.24). In fact, it changes in such a way as to avoid any divergencies coming from the vanishing of the denominator. There remains only the question about the continuity of the forthcoming scattering amplitudes as we approach the integrability point. But as discussed in the main text, this issue can be dismissed by considering localized wave-packet distributions, because in this case the condition $\Delta_0 - \omega(\Delta) = 0$ is never satisfied.

C Tree-level factorizability computational details

In this appendix we collect all the additional formulae needed for computing the cancellation of the spurious contributions that prevented S-matrix factorization. The factors $A_i\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)$ which provide the decomposition (5.25) of $G\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right)$ are:

$$\begin{aligned} A_1\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right) = &\cosh\left(\frac{\eta_2 - \eta_3}{2}\right)\cosh\left(\frac{\theta_1 - \theta_2}{2}\right)\cosh\left(\frac{\theta_2 - \theta_3}{2}\right)\operatorname{csch}\left(\frac{\eta_1 - \theta_1}{2}\right)\sinh\left(\frac{\eta_2 - \eta_3}{2}\right) \cdot \\ &\cdot \sinh\left(\frac{2\eta_1 - \theta_1 - \theta_2}{2}\right)\sinh\left(\frac{\eta_2 + \eta_3 - 2\theta_3}{2}\right), \end{aligned} \tag{C.1}$$

$$\begin{aligned} A_2\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right) = &-\cosh\left(\frac{\eta_2 - \eta_3}{2}\right)\cosh\left(\frac{\theta_1 - \theta_2}{2}\right)\cosh\left(\frac{\theta_2 - \theta_3}{2}\right)\operatorname{csch}\left(\frac{\eta_1 - \theta_1}{2}\right) \cdot \\ &\cdot \sinh^2\left(\frac{\eta_2 - \eta_3}{2}\right)\sinh\left(\frac{2\eta_1 - \theta_1 - \theta_2}{2}\right), \end{aligned} \tag{C.2}$$

$$\begin{aligned} A_3\left(\boldsymbol{\eta}, \boldsymbol{\theta}\right) = &\cosh\left(\frac{\eta_2 - \eta_3}{2}\right)\cosh\left(\frac{\theta_1 - \theta_2}{2}\right)\cosh\left(\frac{\theta_2 - \theta_3}{2}\right)\operatorname{csch}\left(\frac{\eta_1 - \theta_1}{2}\right)\sinh\left(\frac{\eta_2 - \eta_3}{2}\right) \cdot \\ &\cdot \sinh\left(\frac{\theta_1 - \theta_2}{2}\right)\sinh\left(\frac{\eta_2 + \eta_3 - 2\theta_3}{2}\right), \end{aligned} \tag{C.3}$$

---

[17]Remember that $\Delta = p^1 + p^2 - k^1$.

$$A_4\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)=-\cosh\left(\frac{\eta_2-\eta_3}{2}\right)\cosh\left(\frac{\theta_1-\theta_2}{2}\right)\cosh\left(\frac{\theta_2-\theta_3}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_1}{2}\right)\cdot$$
$$\cdot\sinh^2\left(\frac{\eta_2-\eta_3}{2}\right)\sinh\left(\frac{\theta_1-\theta_2}{2}\right),\tag{C.4}$$

$$A_5\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)=-\cosh^2\left(\frac{\eta_2-\eta_3}{2}\right)\cosh\left(\frac{\theta_1-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_1}{2}\right)\sinh\left(\frac{\theta_2-\theta_3}{2}\right)\cdot$$
$$\cdot\sinh\left(\frac{2\eta_1-\theta_1-\theta_2}{2}\right)\sinh\left(\frac{\eta_2+\eta_3-2\theta_3}{2}\right),\tag{C.5}$$

$$A_6\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)=\cosh^2\left(\frac{\eta_2-\eta_3}{2}\right)\cosh\left(\frac{\theta_1-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_1}{2}\right)\sinh\left(\frac{\eta_2-\eta_3}{2}\right)\cdot$$
$$\cdot\sinh\left(\frac{\theta_2-\theta_3}{2}\right)\sinh\left(\frac{2\eta_1-\theta_1-\theta_2}{2}\right),\tag{C.6}$$

$$A_7\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)=-\cosh^2\left(\frac{\eta_2-\eta_3}{2}\right)\cosh\left(\frac{\theta_1-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_1}{2}\right)\sinh\left(\frac{\theta_1-\theta_2}{2}\right)\cdot$$
$$\cdot\sinh\left(\frac{\theta_2-\theta_3}{2}\right)\sinh\left(\frac{\eta_2+\eta_3-2\theta_3}{2}\right),\tag{C.7}$$

$$A_8\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)=\cosh^2\left(\frac{\eta_2-\eta_3}{2}\right)\cosh\left(\frac{\theta_1-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_1}{2}\right)\sinh\left(\frac{\eta_2-\eta_3}{2}\right)\cdot$$
$$\cdot\sinh\left(\frac{\theta_1-\theta_2}{2}\right)\sinh\left(\frac{\theta_2-\theta_3}{2}\right).\tag{C.8}$$

Bellow we give the explicit expressions for the non-vanishing action of the antisymmetrizer on the factors $A_i\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)$:

$$(3!)^2\,\mathcal{A}_{\theta,\eta}\left[A_1\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]=\frac{1}{4}\left\{\operatorname{csch}\left[\frac{\eta_1-\theta_1}{2}\right]\operatorname{csch}\left[\frac{\eta_1-\theta_2}{2}\right]\operatorname{csch}\left[\frac{\eta_1-\theta_3}{2}\right]\sinh(\eta_2-\eta_3)\cdot\right.$$
$$\cdot\left[3\sinh\left(\frac{2\eta_1-\eta_2-\eta_3}{2}\right)-\sinh\left(\frac{4\eta_1+\eta_2+\eta_3-2\theta_1-2\theta_2-2\theta_3}{2}\right)-\right.$$
$$-\sinh\left(\frac{4\eta_1-\eta_2-\eta_3-2\theta_1}{2}\right)-\sinh\left(\frac{4\eta_1-\eta_2-\eta_3-2\theta_2}{2}\right)-$$
$$-\sinh\left(\frac{4\eta_1-\eta_2-\eta_3-2\theta_3}{2}\right)+\sinh\left(\frac{2\eta_1+\eta_2+\eta_3-2\theta_1-2\theta_2}{2}\right)+$$
$$+\sinh\left(\frac{2\eta_1+\eta_2+\eta_3-2\theta_1-2\theta_3}{2}\right)+\sinh\left(\frac{2\eta_1+\eta_2+\eta_3-2\theta_2-2\theta_3}{2}\right)+$$
$$+\sinh\left(\frac{2\eta_1-\eta_2-\eta_3+2\theta_1-2\theta_2}{2}\right)+\sinh\left(\frac{2\eta_1-\eta_2-\eta_3-2\theta_1+2\theta_2}{2}\right)+$$
$$+\sinh\left(\frac{2\eta_1-\eta_2-\eta_3+2\theta_1-2\theta_3}{2}\right)+\sinh\left(\frac{2\eta_1-\eta_2-\eta_3-2\theta_1+2\theta_3}{2}\right)+$$
$$\left.+\sinh\left(\frac{2\eta_1-\eta_2-\eta_3+2\theta_2-2\theta_3}{2}\right)+\sinh\left(\frac{2\eta_1-\eta_2-\eta_3-2\theta_2+2\theta_3}{2}\right)\right]+$$
$$+\operatorname{csch}\left[\frac{\eta_2-\theta_1}{2}\right]\operatorname{csch}\left[\frac{\eta_2-\theta_2}{2}\right]\operatorname{csch}\left[\frac{\eta_2-\theta_3}{2}\right]\sinh(\eta_1-\eta_3)\cdot$$
$$\cdot\left[3\sinh\left(\frac{\eta_1-2\eta_2+\eta_3}{2}\right)+\sinh\left(\frac{\eta_1+4\eta_2+\eta_3-2\theta_1-2\theta_2-2\theta_3}{2}\right)-\right.$$

$$-\sinh\left(\frac{\eta_1-4\eta_2+\eta_3-2\theta_1}{2}\right)-\sinh\left(\frac{\eta_1-4\eta_2+\eta_3-2\theta_2}{2}\right)-$$

$$-\sinh\left(\frac{\eta_1-4\eta_2+\eta_3-2\theta_3}{2}\right)-\sinh\left(\frac{\eta_1+2\eta_2+\eta_3-2\theta_1-2\theta_2}{2}\right)-$$

$$-\sinh\left(\frac{\eta_1+2\eta_2+\eta_3-2\theta_1-2\theta_3}{2}\right)-\sinh\left(\frac{\eta_1+2\eta_2+\eta_3-2\theta_2-2\theta_3}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1-2\eta_2+\eta_3+2\theta_1-2\theta_2}{2}\right)+\sinh\left(\frac{\eta_1-2\eta_2+\eta_3-2\theta_1+2\theta_2}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1-2\eta_2+\eta_3+2\theta_1-2\theta_3}{2}\right)+\sinh\left(\frac{\eta_1-2\eta_2+\eta_3-2\theta_1+2\theta_3}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1-2\eta_2+\eta_3+2\theta_2-2\theta_3}{2}\right)+\sinh\left(\frac{\eta_1-2\eta_2+\eta_3-2\theta_2+2\theta_3}{2}\right)\Bigg]-$$

$$-\operatorname{csch}\left[\frac{\eta_3-\theta_1}{2}\right]\operatorname{csch}\left[\frac{\eta_3-\theta_2}{2}\right]\operatorname{csch}\left[\frac{\eta_3-\theta_3}{2}\right]\sinh(\eta_1-\eta_2)\cdot$$

$$\cdot\Bigg[3\sinh\left(\frac{\eta_1+\eta_2-2\eta_3}{2}\right)+\sinh\left(\frac{\eta_1+\eta_2+4\eta_3-2\theta_1-2\theta_2-2\theta_3}{2}\right)-$$

$$-\sinh\left(\frac{\eta_1+\eta_2-4\eta_3+2\theta_1}{2}\right)-\sinh\left(\frac{\eta_1+\eta_2-4\eta_3+2\theta_2}{2}\right)-$$

$$-\sinh\left(\frac{4\eta_1+\eta_2-4\eta_3+2\theta_3}{2}\right)-\sinh\left(\frac{\eta_1+\eta_2+2\eta_3-2\theta_1-2\theta_2}{2}\right)-$$

$$-\sinh\left(\frac{\eta_1+\eta_2+2\eta_3-2\theta_1-2\theta_3}{2}\right)-\sinh\left(\frac{\eta_1+\eta_2+2\eta_3-2\theta_2-2\theta_3}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1+\eta_2-2\eta_3+2\theta_1-2\theta_2}{2}\right)+\sinh\left(\frac{\eta_1+\eta_2-2\eta_3-2\theta_1+2\theta_2}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1+\eta_2-2\eta_3+2\theta_1-2\theta_3}{2}\right)+\sinh\left(\frac{\eta_1+\eta_2-2\eta_3-2\theta_1+2\theta_3}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1+\eta_2-2\eta_3+2\theta_2-2\theta_3}{2}\right)+\sinh\left(\frac{\eta_1+\eta_2-2\eta_3-2\theta_2+2\theta_3}{2}\right)\Bigg]\Bigg]\Bigg\}\cdot$$

$$\cdot\sinh\left(\frac{\theta_1-\theta_2}{2}\right)\sinh\left(\frac{\theta_1-\theta_3}{2}\right)\sinh\left(\frac{\theta_2-\theta_3}{2}\right),\qquad\text{(C.9)}$$

$$(3!)^2\,\mathcal{A}_{\theta,\eta}\left[A_3\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]=\frac{1}{4}\Bigg\{-\operatorname{csch}\left(\frac{\eta_3-\theta_1}{2}\right)\operatorname{csch}\left(\frac{\eta_3-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_3-\theta_3}{2}\right)\sinh(\eta_1-\eta_2)\cdot$$

$$\cdot\Bigg[\sinh\left(\frac{\eta_1+\eta_2-2\eta_3}{2}\right)+\sinh\left(\frac{\eta_1+\eta_2-2\theta_1}{2}\right)+\sinh\left(\frac{\eta_1+\eta_2-2\theta_2}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1+\eta_2-2\theta_3}{2}\right)-\sinh\left(\frac{\eta_1+\eta_2+2\eta_3-2\theta_1-2\theta_2}{2}\right)-$$

$$-\sinh\left(\frac{\eta_1+\eta_2+2\eta_3-2\theta_1-2\theta_3}{2}\right)-\sinh\left(\frac{\eta_1+\eta_2+2\eta_3-2\theta_2-2\theta_3}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1+\eta_2-2\theta_1-2\theta_2+2\theta_3}{2}\right)+\sinh\left(\frac{\eta_1+\eta_2-2\theta_1+2\theta_2-2\theta_3}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1+\eta_2+2\theta_1-2\theta_2-2\theta_3}{2}\right)\Bigg]+$$

$$+ \operatorname{csch}\left(\frac{\eta_2-\theta_1}{2}\right)\operatorname{csch}\left(\frac{\eta_2-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_2-\theta_3}{2}\right)\sinh(\eta_1-\eta_3)\cdot$$

$$\cdot\left[\sinh\left(\frac{\eta_1-2\eta_2+\eta_3}{2}\right)+\sinh\left(\frac{\eta_1+\eta_3-2\theta_1}{2}\right)+\sinh\left(\frac{\eta_1+\eta_3-2\theta_2}{2}\right)+\right.$$

$$+\sinh\left(\frac{\eta_1+\eta_3-2\theta_3}{2}\right)-\sinh\left(\frac{\eta_1+2\eta_2+\eta_3-2\theta_1-2\theta_2}{2}\right)-$$

$$-\sinh\left(\frac{\eta_1+2\eta_2+\eta_3-2\theta_1-2\theta_3}{2}\right)-\sinh\left(\frac{\eta_1+2\eta_2+\eta_3-2\theta_2-2\theta_3}{2}\right)+$$

$$+\sinh\left(\frac{\eta_1+\eta_3-2\theta_1-2\theta_2+2\theta_3}{2}\right)+\sinh\left(\frac{\eta_1+\eta_3-2\theta_1+2\theta_2-2\theta_3}{2}\right)+$$

$$\left.+\sinh\left(\frac{\eta_1+\eta_3+2\theta_1-2\theta_2-2\theta_3}{2}\right)\right]+$$

$$+\operatorname{csch}\left(\frac{\eta_1-\theta_1}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_3}{2}\right)\sinh(\eta_2-\eta_3)\cdot$$

$$\cdot\left[\sinh\left(\frac{2\eta_1-\eta_2-\eta_3}{2}\right)-\sinh\left(\frac{\eta_2+\eta_3-2\theta_1}{2}\right)-\sinh\left(\frac{\eta_2+\eta_3-2\theta_2}{2}\right)-\right.$$

$$-\sinh\left(\frac{\eta_2+\eta_3-2\theta_3}{2}\right)+\sinh\left(\frac{2\eta_1+\eta_2+\eta_3-2\theta_1-2\theta_2}{2}\right)+$$

$$+\sinh\left(\frac{2\eta_1+\eta_2+\eta_3-2\theta_1-2\theta_3}{2}\right)+\sinh\left(\frac{2\eta_1+\eta_2+\eta_3-2\theta_2-2\theta_3}{2}\right)-$$

$$-\sinh\left(\frac{\eta_2+\eta_3-2\theta_1-2\theta_2+2\theta_3}{2}\right)-\sinh\left(\frac{\eta_2+\eta_3-2\theta_1+2\theta_2-2\theta_3}{2}\right)-$$

$$\left.\left.-\sinh\left(\frac{\eta_2+\eta_3+2\theta_1-2\theta_2-2\theta_3}{2}\right)\right]\right\}\cdot$$

$$\cdot\sinh\left(\frac{\theta_1-\theta_2}{2}\right)\sinh\left(\frac{\theta_1-\theta_3}{2}\right)\sinh\left(\frac{\theta_2-\theta_3}{2}\right), \qquad (\text{C.10})$$

$$(3!)^2\,\mathcal{A}_{\theta,\eta}\left[A_6\left(\boldsymbol{\eta},\boldsymbol{\theta}\right)\right]=\frac{1}{4}\left\{\operatorname{csch}\left(\frac{\eta_2-\eta_3}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_1}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_1-\theta_3}{2}\right)\cdot\right.$$

$$\cdot\sinh^2\left(\eta_2-\eta_3\right)\left[\sinh(\eta_1-\eta_2)+\sinh(\eta_1-\eta_3)\right]-$$

$$-\operatorname{csch}\left(\frac{\eta_1-\eta_3}{2}\right)\operatorname{csch}\left(\frac{\eta_2-\theta_1}{2}\right)\operatorname{csch}\left(\frac{\eta_2-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_2-\theta_3}{2}\right)\cdot$$

$$\cdot\sinh^2\left(\eta_1-\eta_3\right)\left[\sinh(\eta_2-\eta_1)+\sinh(\eta_2-\eta_3)\right]+$$

$$+\operatorname{csch}\left(\frac{\eta_1-\eta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_3-\theta_1}{2}\right)\operatorname{csch}\left(\frac{\eta_3-\theta_2}{2}\right)\operatorname{csch}\left(\frac{\eta_3-\theta_3}{2}\right)\cdot$$

$$\left.\cdot\sinh^2\left(\eta_1-\eta_2\right)\left[\sinh(\eta_3-\eta_1)+\sinh(\eta_3-\eta_2)\right]\right\}\cdot$$

$$\cdot\sinh\left(\frac{\theta_1-\theta_2}{2}\right)\sinh\left(\frac{\theta_1-\theta_3}{2}\right)\sinh\left(\frac{\theta_2-\theta_3}{2}\right). \qquad (\text{C.11})$$

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] [`hep-th/9711200`] [SPIRES].

[2] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for N = 4 super Yang-Mills*, *JHEP* **03** (2003) 013 [`hep-th/0212208`] [SPIRES].

[3] N. Beisert, *The Complete One-Loop Dilatation Operator of N = 4 Super Yang-Mills Theory*, *Nucl. Phys.* **B 676** (2004) 3 [`hep-th/0307015`] [SPIRES].

[4] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical / quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [`hep-th/0402207`] [SPIRES].

[5] N. Gromov and V. Kazakov, *Asymptotic Bethe ansatz from string σ-model on $S^3 \times R$*, *Nucl. Phys.* **B 780** (2007) 143 [`hep-th/0605026`] [SPIRES].

[6] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [`hep-th/0406256`] [SPIRES].

[7] G. Arutyunov and M. Staudacher, *Matching higher conserved charges for strings and spins*, *JHEP* **03** (2004) 004 [`hep-th/0310182`] [SPIRES].

[8] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, `arXiv:1012.3982` [SPIRES].

[9] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [`hep-th/0305116`] [SPIRES].

[10] N. Beisert, *The dilatation operator of N = 4 super Yang-Mills theory and integrability*, *Phys. Rept.* **405** (2005) 1 [`hep-th/0407277`] [SPIRES].

[11] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [`hep-th/0412188`] [SPIRES].

[12] L.F. Alday, G. Arutyunov and A.A. Tseytlin, *On integrability of classical superstrings in $AdS_5 \times S^5$*, *JHEP* **07** (2005) 002 [`hep-th/0502240`] [SPIRES].

[13] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$*, *JHEP* **02** (2005) 059 [`hep-th/0411089`] [SPIRES].

[14] L.F. Alday, G. Arutyunov and S. Frolov, *New integrable system of 2dim fermions from strings on $AdS_5 \times S^5$*, *JHEP* **01** (2006) 078 [`hep-th/0508140`] [SPIRES].

[15] M. Dukalski and S.J. van Tongeren, *On fermionic reductions of the $AdS_4 \times CP^3$ superstring*, *Phys. Rev.* **D 80** (2009) 046005 [`arXiv:0906.0706`] [SPIRES].

[16] A. Melikyan and A. Pinzul, *On quantum integrability of the Landau-Lifshitz model*, *J. Math. Phys.* **50** (2009) 103518 [`arXiv:0812.0188`] [SPIRES].

[17] A. Melikyan and A. Pinzul, *Thermodynamics of the quantum* SU(1,1) *Landau-Lifshitz model*, *J. Stat. Mech.* (2010) P06007 [`arXiv:1001.5450`] [SPIRES].

[18] A. Melikyan, A. Pinzul and G. Weber, *Higher charges and regularized quantum trace identities in* SU(1,1) *Landau-Lifshitz model*, J. Math. Phys. **51** (2010) 123501 [`arXiv:1008.1054`] [SPIRES].

[19] T. Klose and K. Zarembo, *Bethe ansatz in stringy σ-models*, *J. Stat. Mech.* (2006) P05006 [`hep-th/0603039`] [SPIRES].

[20] G. Arutyunov and S. Frolov, *Uniform light-cone gauge for strings in $AdS_5 \times S^5$: Solving SU(1|1) sector*, *JHEP* **01** (2006) 055 [hep-th/0510208] [SPIRES].

[21] A. Melikyan, A. Pinzul, V.O. Rivelles and G. Weber, *On S-matrix factorization of the Landau-Lifshitz model*, *JHEP* **10** (2008) 002 [arXiv:0808.2489] [SPIRES].

[22] H.B. Thacker, *Exact Integrability in Quantum Field Theory and Statistical Systems*, *Rev. Mod. Phys.* **53** (1981) 253 [SPIRES].

[23] A.K. Das, A. Melikyan and V.O. Rivelles, *The S-matrix of the Faddeev-Reshetikhin Model, Diagonalizability and PT Symmetry*, *JHEP* **09** (2007) 104 [arXiv:0707.0511] [SPIRES].

[24] H. Bergknoff and H.B. Thacker, *Method for Solving the Massive Thirring Model*, *Phys. Rev. Lett.* **42** (1979) 135 [SPIRES].

[25] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge U.K. (1997).

[26] A.B. Zamolodchikov and A.B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, *Annals Phys.* **120** (1979) 253 [SPIRES].

[27] P. Dorey, *Exact S matrices*, hep-th/9810026 [SPIRES].

[28] S.J. Parke, *Absence of particle production and factorization of the S-matrix in $1 + 1$ dimensional models*, *Nucl. Phys.* **B 174** (1980) 166 [SPIRES].

[29] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 5th edition, Academic Press, New York U.S.A. (1994)

[30] V.E. Korepin, *Direct calculation of the S matrix in the massive Thirring model*, *Theor. Math. Phys.* **41** (1979) 953 [SPIRES].

[31] V.E. Korepin, *New effects in the massive Thirring model: Repulsive case*, *Commun. Math. Phys.* **76** (1980) 165 [SPIRES].

[32] E.K. Sklyanin, *Quantization of the continuous Heisenberg ferromagnet*, *Lett. Math. Phys.* **15** (1988) 357.