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Equivalence modulo preprojectives for algebras which are a quotient of a hereditary.

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EQUIVALENCE MODULO PREPROJECTIVES FOR ALGEBRAS WHICH ARE A QUOTIENT OF A HEREDITARY

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1 Introduction.

Let us recall that an algebra Λ is equivalent to a hereditary algebra Λ' modulo preprojectives up to de level n if the categories

$$\frac{mod\,\Lambda}{add\,\underline{\underline{P}}^n(\Lambda)}$$

and

$$\frac{mod\,\Lambda'}{add\,\underline{\underline{P}}^n(\Lambda')}$$

are equivalent (*see* [M-1]).

Our results in [M-2] gave some conditions for this to happen, but they were not entirely satisfactory because we had to make some extra assumptions. For example. that the so called *-property were satisfied and, also, that Λ had no injectives in $add\,\underline{\mathrm{P}}^n$. Besides, to allow for our general statements, we also required that Λ' had no ring components of finite representation type.

In the present article. we will limit ourselves to the simpler case when Λ is a k-finite dimensional algebra (k an algebraically closed fiel), when $n = 1$ and when Λ is a quotient of a hereditary algebra. This means. of course, that Λ has the form $\frac{k\mathbf{Q}}{\mathbf{I}}$, where $\mathbf{Q}$ is a quiver with no oriented cycles and where $\mathbf{I}$ is an admissible ideal.

We will take profit of our results of [M-3] which facilitate the calculation of preprojectives of level 1.

We will use the same notations as in [M-1, M-2, M-3]. In particular, $\mathbf{V} = add\,\underline{\underline{\mathrm{P}}}^1$ will be the category of the projective and level 1-preprojective modules, and $\underline{\underline{\mathbf{I}}}$ will be the quiver of *relatively* **V**-*injective* Λ-modules.

We will consider also a one-point extension of the form

$$\tilde{\Lambda} = \begin{pmatrix} k & M_0 \\ 0 & \Lambda \end{pmatrix}.$$

In this case, we will denote, in general but not always, by $\tilde{M}$ the $\tilde{\Lambda}$-module $(0,0,M)$ defined by a Λ-module M, and this will be our natural way of considering mod-Λ as embedded in mod-$\tilde{\Lambda}$. On the other hand, $\tilde{P}_0$ will be reserved for the new projective $(k, 1, M_0)$. Also, the symbol $\tilde{\mathbf{V}}$ will denote the subcategory $add\,\underline{\underline{\mathrm{P}}}^1(\tilde{\Lambda})$ and $\tilde{\underline{\underline{\mathbf{I}}}}$ will indicate the subquiver of $\tilde{\mathbf{V}}$-injective $\tilde{\Lambda}$-modules. Besides, we will denote with τ the Auslander-Reiten translation associated to Λ and wiht $\tilde{\tau}$ the one associated to $\tilde{\Lambda}$.

This context will be used to prove some results by induction.

The main theorems in this paper are the following.

Theorem 1 *Let Λ be an algebra quotient of a hereditary algebra which is equivalent to a hereditary algebra Λ' modulo preprojectives up to the level 1. Then, there exists an algebra $\tilde{\Lambda}$, obtained from Λ by repeated one-point extension such that the following properties are satisfied.*

1) $\tilde{\Lambda}$ is equivalent to a hereditary algebra $\tilde{\Lambda}'$ modulo preprojectives up to the level 1.

2) $\tilde{\Lambda}$ has no injectives in $\tilde{\mathbf{V}} = add\underline{\underline{P}}^1(\tilde{\Lambda})$.

3) $\tilde{\Lambda}'$ has no non simple ring components of finite representation type.

Theorem 2 *Let Λ be an algebra quotient of a hereditary algebra. In order that Λ be equivalent to a hereditary algebra modulo preprojectives up to the level 1, it is necessary anf sufficient that the quiver $\underline{\mathbf{I}}$ of the relative injectives be hereditary (v. [M-2]). In this case, $\underline{\mathbf{I}}$ consists of the indecomposable images of injective Λ-modules laying outside of* **V**.

In the next section we set up our inductive context and obtain ways of relating our equivalences modulo preprojectives from $\tilde{\Lambda}$ to Λ . The most important result in it is the fact that equivalence modulo preprojectives up to the level 1 with a hereditary algebra implies the *-property (*see* Prop. 2).

In Section 3 we establish a situation which allows to go up from Λ to a "bigger" algebra $\tilde{\Lambda}$ and this is used in Section 4 to prove our theorems.

2 Conditions for going down and applications.

Let us consider a basic artin algebra $\tilde{\Lambda}$, which is indecomposable as a ring, that is equivalent to a hereditary algebra $\tilde{\Lambda}'$ modulo preprojectives up to the level 1 and let $\tilde{\alpha}$ denote the equivalence $\frac{mod\,\tilde{\Lambda}}{\tilde{\mathbf{V}}} \to \frac{mod\,\tilde{\Lambda}'}{\tilde{\mathbf{V}}'}$ (where $\tilde{\mathbf{V}}'$ denotes *add* $\underline{P}^1(\tilde{\Lambda}')$). We will require also that $\tilde{\Lambda}$ is not simple.

Let us assume further that there are simple injective $\tilde{\Lambda}$-modules outside of $\mathbf{V}$. Of course, the mere existence of simple injectives always happens if $\tilde{\Lambda}$ is a quotient of a hereditary.

Hence, we want to examine at this moment if this assumption introduces any significant restrictions. In the first place let us consider the case when $\tilde{I}_0$ is a simple injective which is projective. Then $\tilde{\Lambda}$ would split off a ring component isomorphic to k, but this possibility cannot happen under our hypotheses.

Let us consider next the case when $\tilde{I}_0$ is a simple injective which is a preprojective of level 1. If this is so, since there are no irreducible maps starting at $\tilde{I}_0$, the middle term of the Auslander-Reiten sequence which ends at this module must be an injective. Therefore, since $\tilde{I}_0$ is in $\underline{P}_1(\tilde{\Lambda})$, this middle term must be an indecomposable projective injective, P^*, and $\tau\,\tilde{I}_0 =: X$, the radical of P^*, must be simple too. Then we see that if P' is any other projective indecomposable, $\tilde{\Lambda}(P^*, P') = 0$, so that $\tilde{\Lambda}$ has the triangular form above with $\tilde{P}_0 = P^*$ and $M_0 = X$. But then, as it is easy to show, $\frac{mod\,\tilde{\Lambda}}{\tilde{\mathbf{V}}}$ is equivalent to $\frac{mod\,\Lambda}{\mathbf{V}}$, and this shows that our situation reduces to that of an algebra Λ whose ordinary quiver has one vertex less.

Let us pick then a simple injective $\tilde{\Lambda}$-module $\tilde{I}_0$ that is not in $\tilde{\mathbf{V}}$. Using, respectively, D and $*$ to denote the ordinary duality and the usual duality between projectives, we introduce the projectives

$$\tilde{P}_0 = (D\tilde{I}_o)^* \qquad \tilde{P'}_0 = (D\tilde{I}'_0)^*$$

where $\tilde{I}'_0 = \tilde{\alpha}(\tilde{I}_0)$.

As we can easily see, we have then that

$$\tilde{\Lambda}(\tilde{P}_0, \tilde{P}) = 0 \qquad \tilde{\Lambda}'(\tilde{P'}_0, \tilde{P}') = 0$$

for any other indecomposable projective $\tilde{\Lambda}$-module $\tilde{P}$ and, respectively, for any other indecomposable projective $\tilde{\Lambda}'$-module $\tilde{P}'$. This means that each algebra may be considered as a one-point extension as follows.

$$\tilde{\Lambda} = \begin{pmatrix} k & M_0 \\ 0 & \Lambda \end{pmatrix} \qquad \tilde{\Lambda}' = \begin{pmatrix} k & M'_0 \\ 0 & \Lambda' \end{pmatrix}.$$

Before going on, we introduce the following lemmas that will be needed later. The first one is a result given in [G-1, Cor. 3.5], the second, an interesting fact, is a sort of analogue of the former, which applies to one-point extensions, as does the third one.

Lemma 1 *Let P be an indecomposable injective projective, non simple, Λ-module whose socle is S and whose radical is R. Then the following is the Auslander-Reiten sequence beginning at R.*

$$0 \to R \xrightarrow{\begin{pmatrix} i \\ f \end{pmatrix}} P \oplus R/S \xrightarrow{(p,g)} P/S \to 0$$

Lemma 2 *Let us assume that $\tilde{\Lambda}$ is an algebra quotient of a hereditary, that $\tilde{P}_0$ is a projective, co-local $\tilde{\Lambda}$-module whose top is a simple injective, and let us denote its radical by $\tilde{M}_0$. Then, the Auslander-Reiten sequence starting at $\tilde{M}_0$ has the following form, where the undefined modules are explicitely given in the proof.*

$$0 \to \tilde{M}_0 \xrightarrow{\begin{pmatrix} i \\ p \end{pmatrix}} \tilde{P}_0 \oplus \tilde{E} \xrightarrow{(p',i')} \tilde{F} \to 0.$$

PROOF.The assumptions allow us to use our standard notations for $\tilde{\Lambda}$ as a one-point extension of Λ. Hence, $\tilde{M}_0$ has the form $(0,0,M_0)$ and $\tilde{P}_0$ is given by $(k,1,M_0)$. If M_0 is Λ-injective, we fall in the case treated in Lemma 1. Assuming this is not so, we will show the statement is true for $\tilde{E} = (0,0,E)$, and for $\tilde{F} = (k, i, E/soc M_0)$ where

$$0 \to M_0 \xrightarrow{i} E \to Q \to 0$$

is the Λ-Auslander-Reiten sequence going out from M_0.

We observe next that, since the algebra is a quotient of a hereditary and since M_0 is co-local, $\Lambda(M_0, M_0) \cong k$.

We are going to show directly that the map in the statement of the lemma starting at $(0,0,M_0)$ is minimal left almost split. But, before that, let us show that if a module of the form $(k^n, \begin{pmatrix} f \\ x \end{pmatrix}, M_0^m \oplus X)$, where X does not have any direct component isomorphic to M_0, is indecomposable, then the module has one of the following forms.

$$(k,1,M_0) = \tilde{P}_0,$$
$$(0,0,M_0),$$
$$(k^n, x, X).$$

In order to obtain this result it is enough to reduce f to its cannonical form, that is to a matrix with just one non-zero block (at the upper left corner) which would be equal to an identity matrix. It is clear then that, given a morphism $(0,x)$ from $(0,0,M_0)$ to an indecomposable, if the latter is of our first form, $(0,x)$ factors through it; if it is of the third kind, $(0,x)$ factors through $(0,0,E)$, and, finally, if this indecomposable is $(0,0,M_0)$, $(0,x)$ is split monic unless $x=0$, in which case it factors through any of the two. □

Lemma 3 *Let us assume that $\tilde{\Lambda}$ is a one-point extension as above. If $X \xrightarrow{f} Y$ is an irreducible map in mod Λ, then f does not factor through an indecomposable $\tilde{\Lambda}$-module of the form $\tilde{\tau}^{-1}(0,0,L)$.*

PROOF. Let us suppose that a factorization, as the following, does exist, where the middle term is $\tilde{\tau}^{-1}(0,0,L)$.

$$\underbrace{(0,0,X) \xrightarrow{(0,g)} (k^l,r,Z) \xrightarrow{(0,h)} (0,0,Y)}_{f}$$

Then, since this implies that h factors through cok r, we obtain factorizations $f = hg = t.\mathrm{cok}\, r.g$. Hence, in case L is an injective Λ-module, r is a surjection and f is equal to 0. On the other hand, if L is not injective, $\frac{Z}{Im\, r} =: \overline{Z}$ is the indecomposable $\tau^{-1}L$ (see [M-3]) and, using those factorizations and the fact that f is Λ-irreducible, it follows easily that $\overline{Z}$ is a direct summand of Z, a contradiction because the indecomposable $\tilde{\tau}^{-1}(0,0,L)$ whould then split off a summand $(0,0,\overline{Z})$. □

Corollary 1 *With the hypotheses and notations above, if f is irreducible, then it does not factor through $\tilde{\mathbf{V}}$.*

The following propositions will be the support of all our induction arguments in this article and they justify the name to this section.

Proposition 1 *Let us assume that $\tilde{\Lambda}$ is a one-point extension algebra which is equivalent to the hereditary algebra $\tilde{\Lambda}'$ modulo preprojectives up to the level 1 (as above). Then Λ is also equivalent to the hereditary algebra Λ' modulo preprojectives up to the level 1.*

PROOF. Let us denote again by $\tilde{\alpha}$ the equivalence map. The other equivalence in our statement may be obtained as a restsriction of $\tilde{\alpha}$, and we will denote it by α.

It is clear that $\frac{mod\,\tilde{\Lambda}}{\tilde{\mathbf{V}}}$ is equivalent to Comod(add $\underline{\underline{\tilde{\mathbf{I}}}}/\tilde{\mathbf{V}}$) or to Comod(add $\underline{\underline{\tilde{\mathbf{I}}}}_0/\tilde{\mathbf{V}}$), where $\underline{\underline{\tilde{\mathbf{I}}}}_0$ is the family of indecomposables in $\underline{\underline{\tilde{\mathbf{I}}}}$ except the simple injective $\tilde{I}_0$. And similarly for Λ'.

Hence the proposition will be proved if we can show that any morphism f in add $\underline{\underline{\mathbf{I}}}$ which is 0 modulo $\tilde{\mathbf{V}}$ is also 0 modulo $\mathbf{V}$. Using underlining to denote the class of a morphism modulo $\mathbf{V}$, let us assume the f is 0 modulo $\check{\mathbf{V}}$ but $\underline{f} \neq 0$. Now, modulo $\mathbf{V}$, all these morphisms are linear combinations of composites of irreducibles and we can make an argument by induction using our Cor. 1. Since maps in add $\tilde{\underline{\underline{\mathbf{I}}}}$ (in case of indecomposable codomains) are 0 modulo $\check{\mathbf{V}}$ or are epimorphisms (see [M-1]), our induction step is very easily carried on. □

Proposition 2 *Let us assume that $\tilde{\Lambda}$ is an algebra, quotient of a hereditary, which is equivalent to the hereditary algebra $\tilde{\Lambda}'$ modulo preprojectives up to the level* 1 *(as above). Then, $\tilde{\underline{\underline{\mathbf{I}}}}$ is hereditary.*

PROOF. We can proceed by an induction argument on the number of simple modules using the preceding proposition. Keeping our notations, we have then that Λ is equivalent modulo preprojectives up to the level 1 to a hereditary algebra Λ' and that, hence, $\underline{\underline{\mathbf{I}}}$ is hereditary.

Let us recall (*see* [M-1] and [M-2, Def.1] that this means that, as a subquiver of Γ_Λ, $\underline{\underline{\mathbf{I}}}$ is open to the right in $\Gamma_\Lambda \setminus \mathbf{V}$, does not have any oriented cycles, contains the left boundary of $\mathbf{V}$ and has the property that the $\underline{\underline{\mathbf{I}}}$-component of the minimal left almost split map goint out from one of its elements is an epimorphism (*see* [M-2, proof of Prop. 2.1]). What we have to do is to show that then $\tilde{\underline{\underline{\mathbf{I}}}}$ satisfies the same properties.

It is clear from [M-1] that $\tilde{\underline{\underline{\mathbf{I}}}}$ satisfies the first two conditions. Also the third condition follows because we are working here only with preprojectives up to the level 1. We are going to apply the properties of cok_{M_0}, the cokernel functor introduced in [M-3, §4], especially the fact that if M_0 is not Λ-injective it applies the $\tilde{\Lambda}$-Auslander-Reiten sequence beginning at, say, $(0, 0, N)$, into the Λ-Auslander-Reiten sequence beginning at N. Also, a $\tilde{\Lambda}$-module $\tilde{Q}$ is preprojective of level 1 if and only if the cokernel functor applies it into a Λ-preprojective of level 1 or into some $\tau^{-1}N$, where N is an indecomposable component of M_0.

Let $\tilde{X}$ be a $\tilde{\Lambda}$-indecomposable not in $\check{\mathbf{V}}$ but linked by an arrow to an indecomposable $\tilde{Q}$ of $\check{\mathbf{V}}$. Let us show that then $\tilde{X}$ belongs to $\tilde{\underline{\underline{\mathbf{I}}}}$. Obviously, we can assume that $\tilde{Q}$ is a preprojective of level 1, and it is easy to reduce the proof to the case where $\tilde{\tau}\tilde{X}$, which is of the form $(0, 0, Y)$, belongs to $\underline{\underline{P}}_1(\tilde{\Lambda})$. Hence, Y is also a preprojective of level 1 and it follows that X, the image of $\tilde{X}$ under cok_{M_0}, is outside of $\mathbf{V}$ and is linked by an arrow to Q (image of $\tilde{Q}$), an indecomposable of $\mathbf{V}$. Hence, X is in $\underline{\underline{\mathbf{I}}}$. Now, if X is $\mathbf{V}$-simple, all its successors in Γ_Λ belong to $\underline{\underline{P}}_1$, which implies that all successors of $\tilde{X}$ are in $\underline{\underline{P}}_1$ too and. hence, that $\tilde{X}$ is also $\check{\mathbf{V}}$-simple. On the other hand, if X is not simple, it is linked by an arrow to a Λ-projective. It follows that we have also that $\tilde{X}$ is linked by an arrow to a projective. We have again that $\tilde{X} = (0, 0, X)$ and that it belongs to $\tilde{\underline{\underline{\mathbf{I}}}}$.

In order to show the last property (the one that guaranties the validity of our so-called *-property) we proceed by induction observing that, if our property is satisfied by all members of $\tilde{\underline{\mathbf{I}}}$ that follow (in the ordering defined by the arrows) some points $\tilde{I}_1, ..., \tilde{I}_t$ and if g is an epimorphism with domain equal to the direct sum of those, then g is not zero modulo $\mathbf{V}$, and hence not 0 modulo $\check{\mathbf{V}}$ (see the proof of Prop. 1). Let us suppose then that $\tilde{I} \xrightarrow{f} X$ is the $\tilde{\underline{\mathbf{I}}}$-component of the left minimal almost split map going out from $\tilde{I} \in \tilde{\underline{\mathbf{I}}}$, that f is a monomorphism and that $\tilde{I}$ is followed by elements of $\tilde{\underline{\mathbf{I}}}$ that all satisfy our property. The, if $g = cokf$, it follows that gf is 0 and, since f is an epimorphism modulo $\check{\mathbf{V}}$, g is 0 modulo $\check{\mathbf{V}}$. This is a contradiction that completes the proof. □

3 Conditions for going up and applications.

We begin with the following lemma.

Lemma 4 *Let $\tilde{\Lambda}$ be an algebra quotient of a hereditary and such that its relative injectives quiver $\tilde{\underline{\mathbf{I}}}$* (case n=1) *is hereditary (in particular, $\tilde{\Lambda}$ may be equivalent modulo preprojectives up to the level 1 to a hereditary algebra). Then, if $\tilde{I}$ is an injective $\tilde{\Lambda}$-module sitting in $\underline{\underline{P}}_1(\tilde{\Lambda})$, either $\tilde{I}$ is simple or all its predecessors in Γ_Λ are in $\check{\mathbf{V}}$. Moreover, if P is any immediate predecessor of $\tilde{I}$ (in the Auslander-Reiten quiver) which is projective, then P is a uniserial all of whose submodules (except, perhaps, its socle) are projective.*

Proof.We use our induction context and our standard notations. If $\tilde{I}$ is a Λ-module, we have the two possibilities that it is a simple or that its Auslander-Reiten translate is projective. In the first case we have our statement holds and, in the second case, $\tilde{I} = (0,0,I)$ means that $\Lambda(M_0, I) = 0$, implying that $\tilde{\tau}\tilde{I} = (0,0,\tau I)$ (*see* [M-3]), which is $\tilde{\Lambda}$-projective. Also, the projective predecessors correspond to projective predecessors in the Auslander-Reiten quiver of Λ.

Let us consider now the case in which $\tilde{I} = (\Lambda(M_0, I), e, I)$, with I an injective Λ-module. Since $\tilde{I}$ is in $P_1(\tilde{\Lambda})$, it follows from [M-3, §4] that I must be also projective. We claim that the Auslander-Reiten sequence ending at $\tilde{I}$ has the following form.

$$0 \to (0,0,X) \to (0,0,I) \oplus (k,x,X) \to \tilde{I} \to 0$$

where x is given by the map from M_0 onto the socle of I. In order to prove this, we observe first that there must be an indecomposable component of the middle term of the form (k,x,Y), with the length of Y not greater than the length of X. Then the corresponding irreducible starting at $(0,0,X)$ must be monic, and, hence, $X = Y$ and our assertion follows. On the other hand, since (k,x,X) is linked to a preprojective of level 1, and since it is not simple, it must be en $\tilde{\mathbf{V}}$. If it were projective, it would be $(k, x, X) = (k, 1, M_0)$, implying that M_0, the image of x has to be simple.

If, on the other hand, (k, x, X) is preprojective of level 1, then (by the properties of the cokernel functor (*see* [M-3]), X must be projective.

Now the argument continues by induction. Let us assume that we arrived to an Auslander-Reiten sequence of the form

$$0 \to (0, 0, X_i) \to (0, 0, X_{i-1}) \oplus (k, x, X_i) \to (k, x, X_{i-1}) \to 0$$

where $X_i = rad\, X_{i-1}$ are projective. Let $X_{i+1} = rad\, X_i = \tau(k, x, X_i)$ and where x is given by the epimorphism from M_0 to the socle of I. Then it is very easy to check that our argument above may be repeated, showing that either $X_{i+1} = M_0$, a simple module, or it is a projective. This completes de proof. □

Proposition 3 *Let Λ be an algebra quotient of a hereditary whose quiver of relative injectives* (case n=1) *is hereditary (in particular, Λ can be assumed to be equivalent to a hereditary algebra modulo preprojectives up to the level* 1*). Let us consider an indecomposable, co-local Λ-modulo, M_0, that is a quotient of an element in $\underline{\mathbf{I}}$ and that is a component of the radical of a projective. Let us assume further that M_0 is not in $\underline{\underline{P}}_1(\Lambda)$. Then, for the one-point extension*

$$\tilde{\Lambda} = \begin{pmatrix} k & M_0 \\ 0 & \Lambda \end{pmatrix},$$

in order that $\tilde{\underline{\mathbf{I}}}$ be hereditary it is necessary and sufficient *that the following two conditions are satisfied, where J is any indecomposable injective Λ-modulo not in $\mathbf{V}$.*

1) *Every irreducible map $J \to L$ where L is an indecomposable injective Λ-module, induces an epimorphism from $\Lambda(M_0, J)$ to $\Lambda(M_0, L)$.*
2) *For every irreducible morphism $J \xrightarrow{f} L$, where L is a non injective indecomposable Λ-module, $T(M_0, J) \subset ker\, f$.*

Also, it happens then that properties 1) *and* 2) *are satisfied for the same module M_0 as a module* $(0, 0, M_0)$ *in mod* $\tilde{\Lambda}$.

(The notation $T(X, Y)$ is used for the *trace* of X in Y – that is the sum of all images of morphisms from X to Y.)

PROOF.Let us show first that the conditions are necessary. Let us assume that $\tilde{\mathbf{I}}$ is hereditary and let us consider the $\tilde{\Lambda}$-injective associated to the Λ-injective J. It is $\tilde{J} = (\Lambda(M_0, J), e, J)$, and similarly in the case of L. We remark that the $\tilde{\Lambda}$-irreducible maps starting from $\tilde{J}$ are obtained as components of the natural epimorphism $\tilde{J} \to \tilde{J}/soc\,\tilde{J}$, and the same is true for Λ. Hence, in the case of condition 1, that map must induce an epimorphism from $\tilde{J}$ onto $\tilde{L} = (\Lambda(M_0, L), e, L)$, so that condition 1 has to be satisfied. Similarly, when

L is not injective, the map must be onto $\check{L} = (0, 0, L)$, which implies condition 2.

We now show that our conditions are sufficient. As we know, the vertices of $\underline{\underline{I}}$ correspond to vertices of $\check{\underline{\underline{I}}}$ in the following way. a non-injective, I corresponds to a non injective $(0, 0, I)$, and an injective J to an injective $(\Lambda(M_0, J), e, J)$. Hence, it is clear that the two conditions in the statement mean that this correspondence extends to a monomorphism of quivers from $\underline{\underline{I}}$ to $\check{\underline{\underline{I}}}$. Another, possible vertices of $\check{\underline{\underline{I}}}$ are the new injective $(k, 0, 0)$, the components of $rad\,\check{P}_0$ and the immediate succesors of them in $\Gamma_{\check{\Lambda}}$.

Since $rad\,\check{P}_0 = M_0$, we see, by Lemma 2, that its successors are given by the components of the left, minimal, almost-split map $M_0 \to E$. The hypothesis means that. if M_0 is projective, so is $(0, 0, M_0)$, and all its successors are in $\check{\mathbf{V}}$ too. On the other hand, if M_0 is not projective, it is in $\underline{\underline{I}}$, and so are its succesors outside of $\mathbf{V}$.

The only thing left to be checked now is that if M_0 has a successor Q lying in $\underline{\underline{P}}_1(\Lambda)$, then $(0, 0, Q)$ lies in $\underline{\underline{P}}_1(\check{\Lambda})$. For this, we have to show that $\Lambda(M_0, \tau(Q)) = 0$. We observe that, if this is not the case, there would be a non zero endomorphism of $\tau(Q)$ of the form: $\tau(Q) \to M_0 \to \tau(Q)$. We will show that τQ is local and, hence, since our algebra is quotient of a hereditary, this would imply a contradiction.

As a matter of fact. the hypothesis implies that τI must be projective. On the other hand. as it is easy to see, the irreducible map between τQ and I has to be epic and. since in our case, the trace of a member of $\underline{\underline{I}}$ in a member of $\mathbf{V}$ lies inside the radical of it, it follows that the irreducible from τI to τQ is an epimorphism too. Therefore, as we claimed, τQ is local. This completes the proof that $\check{\underline{\underline{I}}}$ is hereditary.

As for the last assertion, it is direct and we left it to the reader. □

Remark. Let us observe that, in the context of the preceeding proposition, $\check{\underline{\underline{I}}}$ is equal to $\underline{\underline{I}}$ only in the case that the new injective $(k, 0, 0) =: \check{J}_0$ is in $\check{\mathbf{V}}$, or, more precisely, when it is in $\underline{\underline{P}}_1(\check{\Lambda})$. Since all predecessors of $\check{J}_0$ must be injectives. but one of them has to be projective, it follows easily that the Auslander-Reiten sequence ending up at $\check{J}_0$ is

$$0 \to (0, 0, M_0) \to (k, 1, M_0) \to (k, 0, 0) \to 0.$$

But, then. it follows that M_0 is a simple, injective Λ-module, which is impossible under the hypothesis of the proposition.

Therefore, $\check{\underline{\underline{I}}}$ is, up to isomorphism, always obtained from $\underline{\underline{I}}$ by adding one vertex, a sink, corresponding to the simple injective $\check{J}_0 = (k, 0, 0)$, and additional arrows from some points of $\underline{\underline{I}}$ to it. These arrows start preciesly at those points representing injectives $\check{J} = (\Lambda(M_0, J), e, J)$ such that $T(M_0, J) = soc\,J$.

If the process of going up from Λ is iterated (taking always "the same" M_0), we obtain algebras $\Lambda, \tilde{\Lambda} =: \Lambda_1, \tilde{\Lambda}_1 =: \Lambda_2, ...$ and the injectives J linked by arrows to the new simple injective correspond always to "the same" vertices of the relative injectives quiver. In other words, thinking of this quiver, the process consists, each time, in adding arrows, starting at some particular vertices and going to a new vertex.

4 Proofs of the theorems.

4.1 Proof of Theorem 1.

We show first that our iterating process can lead us from our given algebra to another algebra that has no injectives in $\underline{\underline{P}}^1$. We do it by induction, showing that we can always obtain a one-point extension with one bijective less. In fact, given a bijective P and calling M_0 its radical, we can apply Prop. 3 to build a one-point extension $\tilde{\Lambda}$. For this algebra, the projective corresponding to P is $(0, 0, P)$, but the corresponding injective is $(\Lambda(M_0, P), e, P) \neq (0, 0, P)$. Hence, $\tilde{\Lambda}$ has one bijective less.

Now, let us assume that our algebra has no injective projectives, and let us see a similar way of obtaining another algebra with no injectives in $\underline{\underline{P}}_1$. Using again our standard notation, let us suppose that J is an injective in $\underline{\underline{P}}_1$. As we know, J cannot be simple for, otherwise, our algebra would have a bijective module. Then, aplying Lemma 4, let P be a projective predecessor of J and let M_0 be its radical. By Prop. 3, if $\tilde{\Lambda}$ is the one-point extension defined by M_0, we get another algebra with $\underline{\mathbf{I}}$ hereditary and where the injective corresponding to J, $\tilde{J}$ cannot be in $\tilde{\mathbf{V}}$ (if it were preprojective of level 1, J would have to be projective). Hence the number of injectives in $\tilde{\mathbf{V}}$ diminishes by 1.

Now, to complete the proof of our theorem, we show that our process may take us to an algebra such that the quiver of the relative injectives has no components whose underlying graph is Dynkin, except for components reduced to isolated points. It is enough to observe that we only have to take for M_0 any non injective non simple element in $\underline{\mathbf{I}}$ and repeat the construction until we get more than 3 arrows starting at some given vertex. This should be done for each component of $\underline{\mathbf{I}}$ that might have a Dynkin underlying graph not reduced to a point.

Hence, we have arrived to an algebra $\tilde{\Lambda}$ which is equivalent to a hereditary algebra $\tilde{\Lambda}'$ modulo preprojectives up to the level 1, that has no injectives in $\tilde{\mathbf{V}}$ and such that $\tilde{\Lambda}'$ has no components of finite representation type. The proof of Theorem 1 is complete. □

4.2 Proof of Theorem 2.

We know from Prop. 2 that the requirement that $\underline{\mathbf{I}}$ be hereditary is necessary for the equivalence to a hereditary algebra modulo preprojectives up to the level 1.

In order to show that it is also sufficient, we begin with our given algebra and perform the same process as in the proof of Theorem 1. Eventually we arrive to $\tilde{\Lambda}$ such that $\tilde{\underline{\mathbf{I}}}$ is hereditary, and there are no injectives in $\tilde{\mathbf{V}}$. Then by the theorem in [M-2], $\tilde{\Lambda}$ is equivalent to a hereditary algebra modulo preprojectives up to the level 1. Finally, repeated applications of our Prop. 1 lead us to conclude that our original algebra Λ is also equivalent to a hereditary algebra modulo preprojectives up to the level 1. The proof of Theorem 2 is complete. □

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