Bayesian reference analysis for the Generalized Gamma distribution

Pedro Luiz Ramos, and Francisco Louzada

***Abstract*—In this paper, we present a Bayesian reference analysis for the generalized gamma distribution by using a reference prior, which has important properties such as one-to-one invariance under reparametrization, consistent marginalization, consistent sampling and leads to a proper posterior density. A simulation study is performed in order to verify the efficiency of our proposed methodology.**

***Index Terms*—Bayesian Inference, Generalized Gamma Distribution, Objective Prior, Reference Prior.**

I. Introduction

The generalized gamma (GG) distribution [1] has proven to be very flexible in characterizing the fading phenomenon occurring in wireless communications [2], [3], [4], [5]. A random variable X follows a GG distribution if its probability density function (PDF) is given by

$$f(x|\phi,\mu,\alpha)=\frac{\alpha}{\Gamma(\phi)}\mu^{\alpha\phi}x^{\alpha\phi-1}\exp\left(-(\mu x)^{\alpha}\right), \qquad (1)$$

where $x>0$, $\Gamma(\phi)=\int_0^\infty e^{-x}x^{\phi-1}dx$ is the gamma function, $\alpha>0$ and $\phi>0$ are the shape parameters and $\mu>0$ is a scale parameter. The GG distribution has relevant PDF with various sub-models, such as the Weibull, gamma distribution, Log-Normal, Nakagami-m, half-normal, Rayleigh, Maxwell-Boltzmann and chi distributions.

The GG distribution is best known in the wireless communications scenario as $\alpha-\mu$ distribution since it is a generalization of the Nakagami-μ distribution [5]. Yacoub [6] presented an important review of the $\alpha-\mu$ distribution and its applications in this area. The author argued that the proposed distribution "has as its base a fading model. Thence, its parameters are directly associated with the physical properties of the propagation medium".

Although the maximum likelihood estimators (MLEs) for the parameters of the GG distribution have been discussed earlier [7], the MLEs may not be a good choice, since in many cases the asymptotic confidence intervals are not achieved even for samples greater than 400 [8]. In order to overcome this problem, a Bayesian inference can be considered. Ramos et al. [9] discussed an objective Bayesian analysis for the GG distribution and proposed a geometric mean Jeffreys/Reference prior to performing inference. However, the Jeffreys prior may not be a good choice in the multiparametric case [10] as well as its subsequently geometric mean with other priors.

In this letter, we derived a reference prior for the GG distribution, which leads to a proper posterior density with interesting properties, such as one-to-one invariance, consistent marginalization and consistent sampling properties [10]. These results are important to perform inference for the parameters of the GG distribution, especially for small and moderate sample sizes.

P.L. Ramos and F. Louzada are in the Institute of Mathematical Science and Computing at University of São Paulo, São Carlos, Brazil.

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II. Bayesian Reference Analysis

The main objective of reference Bayesian analysis introduced by Bernardo [11] with further developments (see Bernardo [10] and the references therein) is to specify a prior distribution where the dominant information in the posterior distribution is provided by the data. To achieve such prior the authors maximize the expected Kullback-Leibler divergence between the posterior distribution and the prior.

Bernardo [10] reviewed different procedures to derive reference priors considering ordered parameters. The necessary steps to obtain reference priors can be seen in Proposition A.1.

Theorem II.1. *Let $\boldsymbol{\theta}=(\phi,\mu,\alpha)$ be the vector of ordered parameters. Then, the $\boldsymbol{\theta}$-reference prior is given by*

$$\pi_R(\phi,\mu,\alpha)\propto\frac{1}{\alpha\mu}\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi+\phi^2\psi'(\phi)-1}}. \qquad (2)$$

where $\psi(k)=\frac{\partial}{\partial k}\log\Gamma(k)=\frac{\Gamma'(k)}{\Gamma(k)}$ is the digamma function and $\psi'(k)=\frac{\partial}{\partial k}\psi(k)$ is the trigamma function.

Proof. See Appendix A. □

The joint posterior distribution for ϕ,μ and α, using the reference prior distribution (2) is given by

$$\pi_R(\boldsymbol{\theta}|\boldsymbol{x})=\frac{\pi(\phi)}{d(\boldsymbol{x})}\frac{\alpha^{n-1}}{\Gamma(\phi)^n}\mu^{n\alpha\phi-1}\prod_{i=1}^{n}x_i^{\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n}x_i^{\alpha}\right\}, \qquad (3)$$

where

$$d(\boldsymbol{x})=\int_{\mathcal{A}}\frac{\alpha^{n-1}\pi(\phi)}{\Gamma(\phi)^n}\mu^{n\alpha\phi-1}\prod_{i=1}^{n}x_i^{\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n}x_i^{\alpha}\right\}d\boldsymbol{\theta},$$

$\pi(\phi)=\sqrt{\dfrac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi+\phi^2\psi'(\phi)-1}}$ and $\mathcal{A}=\{(0,\infty)\times(0,\infty)\times(0,\infty)\}$ is the parameter space for $\boldsymbol{\theta}$.

Theorem II.2. *The posterior (3) is a proper posterior distribution, i.e., $d(\boldsymbol{x})<\infty$.*

Proof. See Appendix B. □

It is important to point out that using Proposition A.1 we can obtain six different reference priors. However, the reference prior (2) is the only one that returned a proper posterior.

Due to the consistent marginalization property of the reference prior the reference marginal posterior distribution of ϕ and α is

$$\pi_R(\phi,\alpha|\boldsymbol{x}) \propto \alpha^{n-2}\frac{\pi(\phi)\Gamma(n\phi)}{\Gamma(\phi)^n}\left(\frac{\sqrt[n]{\prod_{i=1}^n x_i^\alpha}}{\sum_{i=1}^n x_i^\alpha}\right)^{n\phi} .$$

The conditional posterior distributions for α, μ and ϕ are given as follows:

$$\pi_R(\alpha|\phi,\boldsymbol{x}) \propto \alpha^{n-2}\left(\frac{\sqrt[n]{\prod_{i=1}^n x_i^\alpha}}{\sum_{i=1}^n x_i^\alpha}\right)^{n\phi},$$

$$\pi_R(\phi|,\alpha,\boldsymbol{x}) \propto \frac{\pi(\phi)\Gamma(n\phi)}{\Gamma(\phi)^n}\left(\frac{\sqrt[n]{\prod_{i=1}^n x_i^\alpha}}{\sum_{i=1}^n x_i^\alpha}\right)^{n\phi}, \quad (4)$$

$$\pi_R(\mu|\phi,\alpha,\boldsymbol{x}) \sim \mathrm{GG}\left(n\phi, \left(\sum_{i=1}^n x_i^\alpha\right)^{\frac{1}{\alpha}}, \alpha\right).$$

where $\mathrm{GG}(\phi,\mu,\alpha)$ is the PDF of the GG distribution (1).

The conditional posterior distributions (4) are useful to achieve the convergence of the Markov chain Monte Carlo (MCMC) methods. Since the conditional distributions of α and ϕ do not have closed forms, the Metropolis-Hastings algorithm was used to obtain the subsequent quantities.

A. Comparison between the Jeffreys/Reference prior and the proposed reference prior

Ramos et al. [9] proved that the Jeffreys prior led to an improper posterior in the case of the GG distribution and such posterior should not be. Moreover, it was presented a reference prior where (μ,ϕ,α) are the ordered parameters that also led to an improper posterior. Although both priors led to improper posteriors, the obtained geometric mean returned a proper posterior. The proposed Jeffreys/Reference prior is given by

$$\pi_{JR}(\alpha,\mu,\phi) \propto \frac{\pi_{JR}(\phi)}{\mu\sqrt{\alpha}},$$

where

$$\pi_{JR}(\phi) \propto \sqrt[4]{\phi^2\psi'(\phi)^3 - \psi'(\phi)^2 - \psi'(\phi) - \frac{\psi(\phi)^2\left(\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1\right)}{2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2 + 1}} .$$

However, the Jeffreys prior may not be adequate in many situations and can lead to marginalization paradoxes and strong inconsistencies (see Bernardo [10, pg. 41] and the references therein), while the reference priors return consistent results. Such problems may remain if we consider the geometric mean with the Jeffreys prior. On the other hand, the proposed (ϕ,μ,α)-reference prior provided a posterior distribution with interesting properties, such as one-to-one invariance, consistent marginalization, and consistent sampling properties. It is worth mentioning that is not easy to find a reference prior that returns a proper posterior as many reference priors can be obtained depending on the ordered parameters or the selected compact spaces.

III. Numerical Analysis

In this section, a simulation study is carried out for the proposed model by computing the mean relative errors (MRE) and the mean square errors (MSE) given by $\mathrm{MRE}_i = \frac{1}{N}\sum_{j=1}^N \frac{\hat\theta_{i,j}}{\theta_i}, \mathrm{MSE}_i = \frac{1}{N}\sum_{j=1}^N(\hat\theta_{i,j}-\theta_i)^2$, for $i = 1,2,3$, where $N = 5,000$ is the number of estimates obtained through the Bayes estimator. Considering this approach a good estimation procedure should return the MREs closer to one with smaller MSEs. The results were computed with the software R [12] using 2017 as seed to generate the pseudo-random samples from the GG distribution. This procedure was performed considering $\boldsymbol{\theta} = ((0.5,0.5,3),(0.4,1.5,5))$ and $n = (50,60,\ldots,300)$. The 95% coverage probability was also evaluated using the credibility intervals. For a large number of experiments, using a 95% confidence level, the frequencies of the intervals that covered the true values of $\boldsymbol{\theta}$ should be closer to 0.95.

For each simulated sample, 31,000 iterations were performed using MCMC methods. The first 1,000 observations were discarded as initial values. The thin considered was 30 to reduce the autocorrelation among the chains. Three chains of size 1,000 were obtained for each simulated sample at the end of the procedure. The convergence of the chains were confirmed by the Geweke criterion [13] with a 95% confidence level. The posterior mode estimates were computed due to the asymmetry of the marginal posterior distributions, yielding 2,000 estimates for ϕ,μ and α. Although we have presented the results for $\boldsymbol{\theta} = ((0.4,1.5,5),(0.5,0.5,3))$, the following results were similar for other choices of ϕ,μ and α. Figures 1 and 2 show the MREs, MSEs, the coverage probability with a 95% confidence level considering our proposed approach.

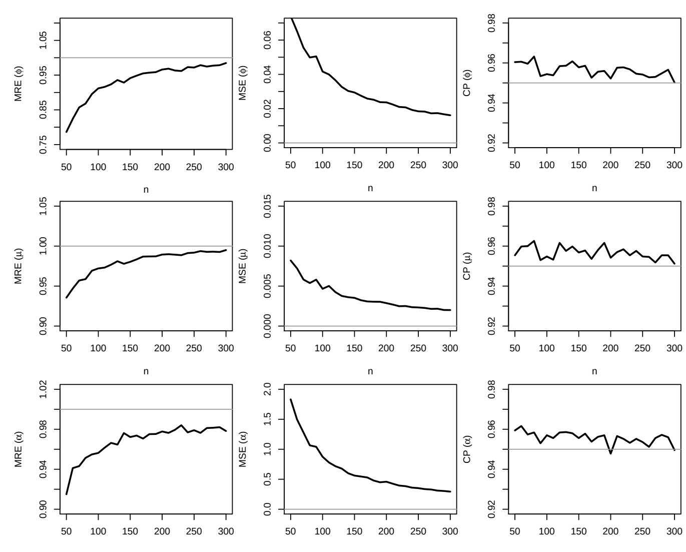

Fig. 1. MREs, RMSEs for the parameters considering $\phi = 0.5, \mu = 0.5$ and $\alpha = 4$ for $N = 5,000$ simulated samples and $n = (50,60,\ldots,300)$.

The results show that the MSEs decrease as n increases and also, as expected, the values of MREs tend to one, i.e. the estimators are asymptotically unbiased for the parameters. In addition, the coverage probability for all parameters tends to 0.95 even for small values of n.

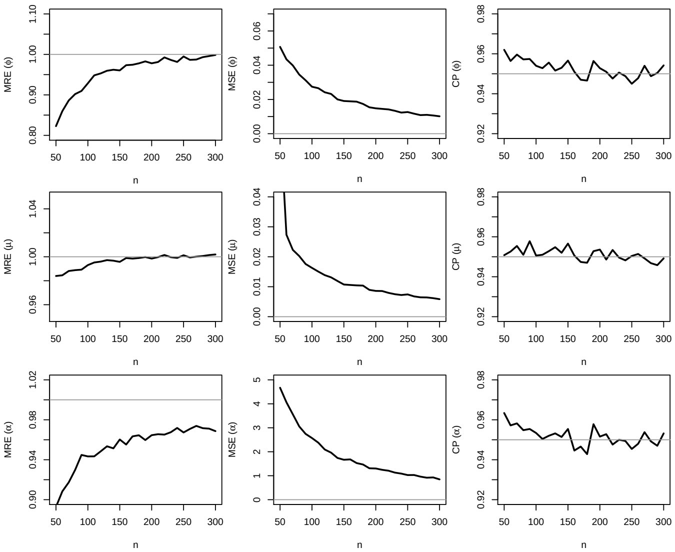

Fig. 2. MREs, RMSEs for the parameters considering $\phi = 0.4, \mu = 1.5$ and $\alpha = 5$ for $N = 5,000$ simulated samples and $n = (50, 60, \ldots, 300)$.

IV. Conclusions

In this work, we have introduced a new reference prior for the generalized gamma distribution. The proposed reference prior provided a posterior distribution with interesting properties, such as one-to-one invariance, consistent marginalization and consistent sampling properties. In addition, the proposed prior returns a proper posterior distribution and has better properties than the Jeffreys/Reference prior. Numerical results have shown that the MCMC using the proposed posterior returns good estimates for the parameters as well as desirable credibility intervals.

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Appendix A
Proof of Theorem II.1

The following proposition will be applied to obtain the reference prior for the GG distribution.

Proposition A.1. *Bernardo [10, p. 40, Theorem 14] Consider that $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ is a vector with ordered parameters and $p(\boldsymbol{\theta}|\boldsymbol{x})$ is the posterior distribution that has an asymptotically normal distribution with dispersion matrix $V(\hat{\boldsymbol{\theta}}_n)/n$, where $\hat{\boldsymbol{\theta}}_n$ is a consistent estimator of $\boldsymbol{\theta}$ and $H(\boldsymbol{\theta}) = V^{-1}(\boldsymbol{\theta})$. Moreover, V_j is the upper $j \times j$ submatrix of V, $H_j = V_j$ and $h_{j,j}(\boldsymbol{\theta})$ is the lower right element of H_j. Then, if the parameter space of θ_j is independent of $\boldsymbol{\theta}_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_m)$, for $j = 1, \ldots, m$, and $h_{j,j}(\boldsymbol{\theta})$ are factorized in the form $h_{j,j}^{\frac{1}{2}}(\boldsymbol{\theta}) = f_j(\theta_j) g_j(\boldsymbol{\theta}_{-j})$, $\quad j = 1, \ldots, m$. Then the reference prior for the ordered parameters $\boldsymbol{\theta}$ is given by $\pi_R(\boldsymbol{\theta}) = \pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) \times \cdots \times \pi(\theta_2|\theta_1)\pi(\theta_1)$, where $\pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) = f_j(\theta_j)$, for $j = 1, \ldots, m$ and there is no need for compact approximations, even if the conditional priors are not proper.*

Firstly, the parameter space of θ_j is independent of $\boldsymbol{\theta}_{-j}$ for $j = 1, 2, 3$. The Fisher information matrix can be seen in Ramos et al [9] and after some algebraic manipulations we have

$$h_{1,1}^{\frac{1}{2}}(\boldsymbol{\theta}) = \sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi + \phi^2\psi'(\phi) - 1}} = f_1(\phi) g_1(\mu) g_1(\phi)$$

where $f_1(\phi) = \sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi + \phi^2\psi'(\phi) - 1}}$, $g_1(\mu)$=1 and $g_1(\alpha) = 1$.

$$h_{2,2}^{\frac{1}{2}}(\boldsymbol{\theta}) = \frac{\alpha}{\mu}\sqrt{\frac{\phi + \phi^2\psi(\phi) - 1}{1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2}} = g_2(\alpha) f_2(\mu) g_2(\phi)$$

where $g_2(\phi) = \sqrt{\frac{\phi + \phi^2\psi(\phi) - 1}{1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2}}$, $g_2(\alpha) = \alpha$, $f_2(\mu) = \frac{1}{\mu}$.

$$h_{3,3}^{\frac{1}{2}}(\boldsymbol{\theta}) = \frac{\sqrt{1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2}}{\alpha} = f_3(\alpha) g_3(\mu) g_3(\phi)$$

where $g_3(\phi) = \sqrt{1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2}$, $g_3(\mu) = 1$ and $f_3(\alpha) = \frac{1}{\alpha}$.

From the Proposition A.1, for the ordered parameters (ϕ, μ, α) the conditional reference priors are

$$\pi(\alpha|\mu, \phi) \propto f_3(\alpha) \propto \frac{1}{\alpha}, \quad \pi(\mu|\phi) \propto f_2(\mu) \propto \frac{1}{\mu},$$

$$\pi(\phi) \propto f_1(\phi) \propto \sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi + \phi^2\psi'(\phi) - 1}}.$$

Therefore, the joint $\boldsymbol{\theta}$-reference prior is

$$\pi_R(\phi,\mu,\alpha) \propto \pi(\alpha|\mu,\phi)\pi(\mu|\phi)\pi(\phi) \propto \frac{1}{\alpha\mu}\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi+\phi^2\psi'(\phi)-1}}.$$

Appendix B
Proof of Theorem II.2

Since $\frac{\alpha^{n-1}\pi(\phi)}{\Gamma(\phi)^n}\mu^{n\alpha\phi-1}\prod_{i=1}^n x_i^{\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} \geq 0$ we have

$$\begin{aligned} d(\boldsymbol{x}) &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{\alpha^{n-1}\pi(\phi)}{\Gamma(\phi)^n}\mu^{n\alpha\phi-1}\prod_{i=1}^n x_i^{\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\phi d\alpha \\ &= \int_0^\infty\int_0^\infty \frac{\alpha^{n-2}\pi(\phi)}{\Gamma(\phi)^n}\prod_{i=1}^n x_i^{\alpha\phi}\frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}d\phi d\alpha \\ &= s_1+s_2+s_3+s_4, \end{aligned}$$

where

$$\begin{aligned} s_1 &= \int_0^1\int_0^1 \frac{\alpha^{n-2}\pi(\phi)}{\Gamma(\phi)^n}\prod_{i=1}^n x_i^{\alpha\phi}\frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}d\phi d\alpha \\ &\propto \int_0^1\int_0^1 \alpha^{n-2}\times 1\times\phi^{n-\frac{3}{2}}\left(\frac{\sqrt[n]{\prod_{i=1}^n x_i^\alpha}}{\sum_{i=1}^n x_i^\alpha}\right)^{n\phi}d\phi d\alpha \\ &= \int_0^1 \alpha^{n-2}\int_0^1 \phi^{n-\frac{3}{2}}e^{-n\phi\,\mathrm{q}(\alpha)}d\phi d\alpha \\ &= \int_0^1 \alpha^{n-2}\frac{\gamma(n-\frac{1}{2},n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n-\frac{1}{2}}}d\alpha \\ &\propto \int_0^1 \alpha^{n-2}\frac{1}{1^{n-\frac{1}{2}}}\times 1\,d\alpha < \int_0^1 \alpha^{n-\frac{3}{2}}d\alpha < \infty. \end{aligned}$$

$$\begin{aligned} s_2 &= \int_1^\infty\int_0^1 \frac{\alpha^{n-2}\pi(\phi)}{\Gamma(\phi)^n}\prod_{i=1}^n x_i^{\alpha\phi}\frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}d\phi d\alpha \\ &\propto \int_1^\infty\int_0^1 \alpha^{n-2}\times 1\times\phi^{n-\frac{3}{2}}\left(\frac{\sqrt[n]{\prod_{i=1}^n x_i^\alpha}}{\sum_{i=1}^n x_i^\alpha}\right)^{n\phi}d\phi d\alpha \\ &= \int_1^\infty \alpha^{n-2}\int_0^1 \phi^{n-\frac{3}{2}}e^{-n\phi\,\mathrm{q}(\alpha)}d\phi d\alpha \\ &= \int_1^\infty \alpha^{n-2}\frac{\gamma(n-\frac{1}{2},n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n-\frac{1}{2}}}d\alpha \\ &\propto \int_1^\infty \alpha^{n-2}\frac{1}{\alpha^{n-\frac{1}{2}}}\times 1\,d\alpha < \int_1^\infty \alpha^{-\frac{3}{2}}d\alpha < \infty, \end{aligned}$$

$$\begin{aligned} s_3 &= \int_0^1\int_1^\infty \frac{\alpha^{n-2}\pi(\phi)}{\Gamma(\phi)^n}\prod_{i=1}^n x_i^{\alpha\phi}\frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}d\phi d\alpha, \\ &\propto \int_0^1\int_1^\infty \alpha^{n-2}\times\frac{1}{\phi^{\frac{3}{2}}}\times n^{n\phi}\phi^{\frac{n-1}{2}}\left(\frac{\sqrt[n]{\prod_{i=1}^n x_i^\alpha}}{\sum_{i=1}^n x_i^\alpha}\right)^{n\phi}d\phi d\alpha \end{aligned}$$

$$\begin{aligned} s_3 &= \int_0^1 \alpha^{n-2}\int_1^\infty \phi^{\frac{n-2}{2}-1}e^{-n\phi\,\mathrm{p}(\alpha)}d\phi d\alpha \\ &= \int_0^1 a^{n-2}\frac{\Gamma(\frac{n-2}{2},n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n-2}{2}}}d\alpha \\ &\propto \int_0^1 \alpha^{n-2}\frac{1}{(\alpha^2)^{\frac{n-2}{2}}}\times 1\,d\alpha = \int_0^1 \alpha^0 d\alpha < \infty, \end{aligned}$$

and

$$\begin{aligned} s_4 &= \int_1^\infty\int_1^\infty \frac{\alpha^{n-2}\pi(\phi)}{\Gamma(\phi)^n}\prod_{i=1}^n x_i^{\alpha\phi}\frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}d\phi d\alpha \\ &\propto \int_0^1\int_1^\infty \alpha^{n-2}\times\frac{1}{\phi^{\frac{3}{2}}}\times n^{n\phi}\phi^{\frac{n-1}{2}}\left(\frac{\sqrt[n]{\prod_{i=1}^n x_i^\alpha}}{\sum_{i=1}^n x_i^\alpha}\right)^{n\phi}d\phi d\alpha \\ &= \int_0^1 \alpha^{n-2}\int_1^\infty \phi^{\frac{n-2}{2}-1}e^{-n\phi\,\mathrm{p}(\alpha)}d\phi d\alpha \\ &= \int_1^\infty a^{n-2}\frac{\Gamma(\frac{n-2}{2},n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n-2}{2}}}d\alpha \\ &\propto \int_1^\infty \alpha^{n-2}\frac{1}{\alpha^{\frac{n-2}{2}}}\times\alpha^{\frac{n-2}{2}-1}e^{-n\log\left(\frac{x_m}{\sqrt[n]{\prod_{i=1}^n x_i}}\right)\alpha}d\alpha \\ &= \int_1^\infty \alpha^{-3}e^{-L\alpha}d\alpha < \infty \end{aligned}$$

where $x_m = \max(x_1,\ldots,x_n)$, $L = n\log\left(\frac{x_m}{\sqrt[n]{\prod_{i=1}^n x_i}}\right) > 0$, $\mathrm{p}(\alpha) = \log\left(\frac{\frac{1}{n}\sum_{i=1}^n x_i^\alpha}{\sqrt[n]{\prod_{i=1}^n x_i^\alpha}}\right)$, $\mathrm{q}(\alpha) = \log\left(\frac{\sum_{i=1}^n x_i^\alpha}{\sqrt[n]{\prod_{i=1}^n x_i^\alpha}}\right)$ and $\gamma(y,x) = \int_0^x w^{y-1}e^{-w}dw$.

Therefore, we have $d(\boldsymbol{x}) = s_1+s_2+s_3+s_4 < \infty$.