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A Hopf-Galois correspondence for free algebras

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Abstract

A Galois correspondence is exhibited between right coideals subalgebras of a finite-dimensional pointed Hopf algebra acting homogeneously and faithfully on a free associative algebra and free subalgebras containing the invariants of this action.

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Introduction

A Galois correspondence for subalgebras of a free algebra has been constructed by Kharchenko for group actions by automorphisms and for Lie algebra actions

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by derivations. More precisely, Kharchenko proved in [5] that there exists a one-to-one correspondence between the subgroups of a finite group of homogeneous automorphisms of a free algebra and its free subalgebras containing the subalgebra of invariants of the group action. An analogous correspondence for the case of a restricted finite-dimensional Lie algebra of derivations of a free algebra over a field of positive characteristic was presented in [6]. Both correspondences can be obtained from Galois correspondences established for prime algebras on which groups of automorphisms or Lie algebras of derivations act in an X-outer fashion. (For automorphisms see [12], and for derivations see [6]. We also refer to [7, Section 6.1] for proofs of these results.)

Group actions by automorphisms and Lie algebra actions by derivations are examples of pointed Hopf algebra actions on associative algebras. In this note we show that Kharchenko's Galois correspondences for free algebras can be framed in the context of Hopf algebra actions.

In Section 1 we show how a generalization of the correspondences for X-outer actions on prime algebras, obtained by Masuoka, Westreich, and Yanai ([13,14,8]), can be used in order to exhibit a Galois correspondence between free subalgebras of a free algebra and right coideal subalgebras of a finite-dimensional pointed Hopf algebra acting homogeneously and faithfully on it (Theorem 1.2). The specialization to free algebras depends on two facts which are proved in the following two sections. The first, dealt with in Section 2, is the fact that faithful, homogeneous actions of finite-dimensional pointed Hopf algebras on free algebras are X-outer. The second, proved in Section 3, is the fact that subalgebras of invariants of free algebras acted homogeneously upon by Hopf algebras are always free. We end, in Section 4, by giving a proof of Theorem 1.2 based on the results of the previous sections.

1 A Galois correspondence

In this section we shall describe a Galois correspondence between free subalgebras of a free algebra and right coideal subalgebras of a Hopf algebra acting on it. We shall use the notation of [10] throughout.

Let us start by recalling that a subalgebra U of a prime algebra R is said to be *rationally complete* if $Ir \subseteq U$, for a nonzero ideal I of U and an element r of R, implies $r \in U$. (Such subalgebras are called *ideal-cancellable* in [12].)

If H is a finite-dimensional pointed Hopf algebra acting on a prime algebra R, this action can be extended to an action of H on Q, the symmetric Martindale ring of quotients of R (see [11]); so we can consider the smash product $Q\#H$. For nonempty subsets V, W of $Q\#H$, let $V^W = \{v \in V :$

$vw = wv$, for all $w \in W\}$. The action of H on R is said to be *X-outer* if $(Q\#H)^R = K$, the extended centroid of R. (This is the original definition given by Milinski in [9]).

We shall make use of the following result.

Theorem 1.1 ([8, Theorem 3.5]) *Let k be a field and let H be a finite-dimensional pointed Hopf algebra over k acting on a prime k-algebra R such that the action is X-outer. Let Q denote the symmetric Martindale ring of quotients of R and let K denote the center of Q. Let $\mathcal{S}$ be the set of all right H-comodule subalgebras Λ of $K\#H$ containing K, and let $\mathcal{F}$ be the set of all rationally complete subalgebras U of R containing R^H. Then the map*

$$\Phi : \mathcal{F} \longrightarrow \mathcal{S}$$
$$U \longmapsto (K\#H)^U$$

is bijective with inverse given by

$$\Psi : \mathcal{S} \longrightarrow \mathcal{F}$$
$$\Lambda \longmapsto R^\Lambda.$$

It should be remarked that, for a right coideal Λ of H, R^Λ coincides with the usual subalgebra of invariants of R under the action of Λ, that is $R^\Lambda = \{r \in R : h \cdot r = \varepsilon(h)r$, for all $h \in \Lambda\}$ (see [15]).

We specialize to the case when R is free. It is well-known that a free algebra of rank greater than 1 over a field is symmetrically closed, that is, its symmetric Martindale ring of quotients coincides with it (see, *e. g.* [5, Lemma]). Therefore, if $R = k\langle X\rangle$, with $|X| > 1$, then $Q = k\langle X\rangle, K = k$, and $K\#H$ is naturally isomorphic to H. Of course, in this case, $\mathcal{S}$ is just the set of all right coideal subalgebras of H.

Recall that an algebra R is graded if there are subspaces R_n, $n \in \mathbf{N}$, of R such that $R = \bigoplus_{n\in\mathbf{N}} R_n$, $R_iR_j \subseteq R_{i+j}$, and $1 \in R_0$. We say that an action of a Hopf algebra H on a graded algebra R is *homogeneous* if $H \cdot R_n \subseteq R_n$ for all $n \in \mathbf{N}$.

A free algebra $k\langle X\rangle$ can be graded by assigning arbitrary positive degrees to the elements of X and then extending this definition in a natural way to all monomials of $k\langle X\rangle$. For a given positive integer n the homogeneous component of degree n is then defined to be the linear span of all the monomials of degree n; degree 0 is assigned to the elements of k. This induces a degree function on $k\langle X\rangle$ defined on an element $f \in k\langle X\rangle$ as being the highest degree occurring among the monomials in the support of f. All gradings of $k\langle X\rangle$ considered in this paper will be of this kind. An action of a Hopf algebra H on a free algebra $k\langle X\rangle$ with a fixed grading is homogeneous, by the definition above, if and only if either $h\cdot x = 0$ or $h\cdot x$ is homogeneous of the same degree as x, for all $h \in H$ and all $x \in X$. When $k\langle X\rangle$ is given the usual grading, where each $x \in X$ has degree 1, the action of H is homogeneous if and only if, for each $h \in H$ and each $x \in X$, there exist $\alpha_{xy} \in k$, finitely many of which nonzero, such that $h \cdot x = \sum_{y\in X} \alpha_{xy} y$. Such actions are also called *linear*.

An action of a Hopf algebra H on an algebra R is said to be *faithful* if R is faithful as H-module. Our main aim in this note is to present a proof of the following result.

Theorem 1.2 *Let k be a field, let X be a set with $|X| > 1$, and let $R = k\langle X\rangle$ be the free associative algebra on X over k. Let H be a finite-dimensional pointed Hopf algebra which acts homogeneously and faithfully on R. Let $\mathcal{S}$ be the set of all right coideal subalgebras Λ of H, and let $\mathcal{F}$ be the set of all free subalgebras of R containing the subalgebra of invariants R^H of R under H. Then the map*

$$\begin{aligned} \Phi : \mathcal{F} &\longrightarrow \mathcal{S} \\ U &\longmapsto H^U = \{h \in H : hf = fh \text{ in } R\#H, \text{ for all } f \in U\} \end{aligned}$$

is bijective with inverse given by

$$\begin{aligned} \Psi : \mathcal{S} &\longrightarrow \mathcal{F} \\ \Lambda &\longmapsto R^\Lambda = \{f \in R : h\cdot f = \varepsilon(h) f, \text{ for all } h \in \Lambda\}. \end{aligned}$$

We remark that for a subalgebra U of R, the set ${}^U H = \{h \in H : h \cdot f = \varepsilon(h) f, \text{ for all } f \in U\}$, a candidate for the image of U under Φ in Theorem 1.2, although always containing H^U, does not necessarily coincide with it. We refer the reader to [14, Example 0.4] for an example in which $H^U \subsetneq {}^U H$. Of course, in the case of automorphisms or derivations, these two sets do coincide.

A proof of Theorem 1.2 will be given in Section 4.

2 Outerness of homogeneous actions

In this section we show that a faithful and homogeneous action of a finite-dimensional pointed Hopf algebra on a free algebra is always X-outer.

In what follows ν will denote the degree function associated to a fixed grading on the free algebra $k\langle X\rangle$.

We start with the following property of free algebras.

Lemma 2.1 *Let $f \in k\langle X\rangle$. If for every $x \in X$ there exists a homogeneous element $h_x \in k\langle X\rangle$ of degree $\nu(x)$ such that $\nu(xf - fh_x) \leq \nu(x)$ then $f \in k$.*

Proof. First, we prove by induction on the degree of f that if $x \in X$ and $h \in k\langle X\rangle$ is homogeneous of degree $\nu(x)$ satisfying $xf = fh$, then $f \in k[x]$. If $f \in k$, then the result is trivially true. Suppose the assumption is true for polynomials of degree less than n, with $n > 0$, and let $f \in k\langle X\rangle$ be of degree n such that $xf = fh$. Decomposing f as $f = \lambda + f'$, with $\lambda \in k$ and f' having zero independent term, we get $0 = xf - fh = \lambda(x - h) + xf' - f'h$, which implies $\lambda(x - h) = 0$ and $xf' = f'h$, since a monomial occurring with nonzero coefficient in $xf' - f'h$ would have degree greater than $\nu(x)$. Now, the relation $xf' = f'h$ and the fact that f' has zero independent term imply that $f' = xf''$, where $f'' \in k\langle X\rangle$ is of degree $n - \nu(x)$. We have $xxf'' = xf' = f'h = xf''h$, hence $xf'' = f''h$. By the induction hypothesis, we have $f'' \in k[x]$. Therefore, $f' = xf'' \in k[x]$ and so, $f = \lambda + f' \in k[x]$.

Now, in order to prove the lemma, let $f = \lambda + f'$, where $\lambda \in k$ and f' has zero independent term, and suppose that for every $x \in X$ there exists a homogeneous element $h_x \in k\langle X\rangle$ of degree $\nu(x)$ such that $\nu(xf - fh_x) \leq \nu(x)$. Then $xf - fh_x = \lambda x - \lambda h_x + xf' - f'h_x$. Since each monomial occurring with nonzero coefficient in $xf' - f'h_x$ would have degree greater than $\nu(x)$, we must have $xf' - f'h_x = 0$, for every $x \in X$. By what we have seen above, $f' \in \bigcap_{x\in X} k[x] = k$. Hence $f' = 0$, and so, $f = \lambda \in k$. $\square$

Theorem 2.2 *Let k be a field. Then every faithful and homogeneous action of a finite-dimensional pointed Hopf k-algebra H on the free algebra $k\langle X\rangle$ is X-outer, that is, $(k\langle X\rangle \# H)^{k\langle X\rangle} = k$.*

Proof. Let $\{H_n\}$ be the coradical filtration of H. By the Taft-Wilson Theorem (see, *e.g.* [10, Theorem 5.4.1]), there exist subspaces W_m of H, with $H_m = H_{m-1} \oplus W_m$, and, for each $m \in \mathbb{N}$, a basis Y_m of W_m such that if $h \in Y_m$ then

$$\Delta h = \tau_h \otimes h + w, \tag{1}$$

where $\tau_h \in Y_0 = G(H)$ (here $G(H)$ stands for the set of grouplike elements of H) and $w \in H \otimes H_{m-1}$. Let $\xi \in k\langle X\rangle \# H_n$. Then

$$\xi = \sum_{i=0}^{n} \sum_{h \in Y_i} f_h h, \tag{2}$$

where $f_h \in k\langle X\rangle$, all but a finite number being zero. Hence, for each $x \in X$, we have

$$x\xi - \xi x = \sum_{h\in Y_n} \Big(xf_h - f_h(\tau_h \cdot x)\Big)h + y,$$

for some $y \in k\langle X\rangle \# H_{n-1}$. Therefore, the component of $x\xi - \xi x$ in the subspace $k\langle X\rangle \# W_n$ is

$$\sum_{h\in Y_n} \Big(xf_h - f_h(\tau_h \cdot x)\Big)h. \tag{3}$$

We prove first that $(k\langle X\rangle\#H)^{k\langle X\rangle} \subseteq H$, that is, that if $\xi \in (k\langle X\rangle\#H)^{k\langle X\rangle}$ has a decomposition as in (2) then each f_h lies in k. Since $x\xi - \xi x = 0$, for each $x \in X$, we have, in particular, that (3) is zero. Then, for each $h \in Y_n$ and $x \in X$, we must have $xf_h - f_h(\tau_h \cdot x) = 0$. By Lemma 2.1, f_h is in k. Let $0 \le m < n$ and assume, by induction, that, for $m < i \le n$, we have $f_h \in k$, for each $h \in Y_i$. Decompose $\xi = \xi' + \xi''$, where $\xi' = \sum_{j=0}^{m} \sum_{h\in Y_j} f_h h$ and $\xi'' = \sum_{i=m+1}^{n} \sum_{h\in Y_i} f_h h$. By our induction hypothesis, $\xi'' \in k\#H = H$. Since the action of H on $k\langle X\rangle$ is homogeneous, for each $x \in X$, $x\xi'' - \xi''x$ belongs to $V_x\#H$, where V_x denotes the subspace of $k\langle X\rangle$ generated by all elements of degree $\nu(x)$. Then, $0 = x\xi - \xi x = x\xi' + x\xi'' - \xi'x - \xi''x$, that is, $x\xi' - \xi'x = \xi''x - x\xi''$ lies in the subspace $V_x\#H$. Since $\xi' \in k\langle X\rangle\#H_m$, we can exhibit the component of $x\xi' - \xi'x$ in the subspace $k\langle X\rangle\#W_m$, as it was done in (3), obtaining $\sum_{h\in Y_m}\big(xf_h - f_h(\tau_h \cdot x)\big)h$, and this component must belong to $V_x\#H$. Therefore $xf_h - f_h(\tau_h \cdot x)$ has degree less than or equal to $\nu(x)$, for each $x \in X$ and each $h \in Y_m$. By Lemma 2.1, $f_h \in k$, for each $h \in Y_m$. By induction, we conclude that $(k\langle X\rangle\#H)^{k\langle X\rangle} \subseteq H$.

Now, let $h \in (k\langle X\rangle\#H)^{k\langle X\rangle}$. Then, for each $f \in k\langle X\rangle$,

$$fh = hf = \sum_{(h)} (h_{(1)} \cdot f)h_{(2)}. \tag{4}$$

Since H is finite-dimensional, there exists a nonzero left integral t in H, that is, a nonzero element $t \in H$ is such that $ht = \varepsilon(h)t$, for all $h \in H$. Multiplication of (4) by t on the right yields $fht = \sum_{(h)}(h_{(1)} \cdot f)h_{(2)}t$, that is, $\varepsilon(h)ft = (h \cdot f)t$. Therefore, $\varepsilon(h)f = h \cdot f$. Since the action of H on $k\langle X\rangle$ is faithful, it follows that $h = \varepsilon(h)1 \in k$. □

3 Invariants of homogeneous actions

In this section we show that the subalgebra of invariants of a free algebra acted homogeneously upon by a Hopf algebra is free. In fact, we show that this can be proved for a slightly larger class of algebras.

Let k be a field and let R be a k-algebra with a filtration ν, that is, a map $\nu : R \longrightarrow \mathbb{N}_{-\infty}$, where $\mathbb{N}_{-\infty} = \mathbb{N} \cup \{-\infty\}$, such that $\nu(0) = -\infty, \nu(1) = 0$

and $\nu(a) \geq 0$ if $a \neq 0$, $\nu(a-b) \leq \max\{\nu(a), \nu(b)\}$, and $\nu(ab) \leq \nu(a) + \nu(b)$, for all $a, b \in R$. Let H be a Hopf algebra which acts on R. The action of H on R is said to be *compatible* with the filtration ν if

(i) $\nu(h \cdot a) \leq \nu(a)$, for all $h \in H$, $a \in R$, and
(ii) every $a \in R$ has a decomposition

$$a = a^+ + a^-,$$

with $\nu(a^+) = \nu(a)$ and $\nu(a^-) < \nu(a)$, such that, for all $h \in H$, $\nu(\varepsilon(h)a^+ - h \cdot a^+) = \nu(a^+)$, if $h \cdot a^+ \neq \varepsilon(h)a^+$.

Such a decomposition of an element $a \in R$ is called a *ν-homogeneous decomposition* of a.

While condition (i) above looks natural, the rather technical condition (ii) is in fact very frequent. For instance, let R be a graded algebra, say $R = \bigoplus_{n \in \mathbb{N}} R_n$, and let H be a Hopf algebra acting homogeneously on R. Let ν be the filtration on R induced by the grading, that is, given $a \in R$, $a \neq 0$, if $a = \sum_{n \in \mathbb{N}} a_n$, with $a_n \in R_n$, let $\nu(a) = \max\{n : a_n \neq 0\}$. Then condition (i) holds trivially. Now given $a \in R$, $a \neq 0$, with homogeneous decomposition $a = a_0 + \cdots + a_d$, where $d = \nu(a)$, letting $a^+ = a_d$ and $a^- = a_0 + \cdots + a_{d-1}$, yields a ν-homogeneous decomposition of a.

We are ready to present a proof of our main result in this section, which is just an adaptation of the proofs of [5, Proposition 1] and [6, Proposition 4]. In the proof we shall make use of Cohn's weak algorithm (see [2, Chapter 2]).

Theorem 3.1 *Let k be a field and let R be a k-algebra with a filtration ν such that $\nu(\alpha) = 0$ for all $\alpha \in k$. Let H be a Hopf algebra over k which acts on R. Suppose that the action of H on R is compatible with ν. Let Λ be a right coideal of H. Then the subalgebra of invariants R^Λ of R under Λ satisfies the weak algorithm with respect to ν if R satisfies the weak algorithm with respect to ν.*

Proof. Let $A = \{a_1, \ldots, a_m\}$ be a right ν-dependent subset of R^Λ with $\nu(a_1) \leq \cdots \leq \nu(a_m)$. So A is a right ν-dependent subset of R. Choose the smallest n such that $\{a_1, \ldots, a_n\}$ is right ν-dependent in R. Since R satisfies the weak algorithm with respect to ν, some a_i (in fact, a_n) is right ν-dependent on $a_1, \ldots, a_{i-1}$. Thus, there exist $b_i \in R$ such that

$$\nu\Big(a_i - \sum_{j<i} a_j b_j\Big) < \nu(a_i)$$

and $\max_{j<i}\{\nu(a_j) + \nu(b_j)\} \leq \nu(a_i)$.

Let $J = \{j : \nu(a_j) + \nu(b_j) < \nu(a_i)\}$ and define the subset $\{c_j : j = 1, \ldots, i-1\}$ of R by

$$c_j = \begin{cases} 0 & \text{if } j \in J \\ b_j & \text{if } j \in \{1, \ldots, i-1\} \setminus J. \end{cases}$$

Then

$$\begin{aligned} \nu\Big(a_i - \sum_{j<i} a_j c_j\Big) &= \nu\Big(a_i - \sum_{j<i} a_j b_j + \sum_{j\in J} a_j b_j\Big) \\ &\le \max\Big\{\nu\Big(a_i - \sum_{j<i} a_j b_j\Big), \nu\Big(\sum_{j\in J} a_j b_j\Big)\Big\} \\ &\le \max\Big\{\nu\Big(a_i - \sum_{j<i} a_j b_j\Big), \max_{j\in J}\{\nu(a_j b_j)\}\Big\} \\ &\le \max\Big\{\nu\Big(a_i - \sum_{j<i} a_j b_j\Big), \max_{j\in J}\{\nu(a_j) + \nu(b_j)\}\Big\} \\ &< \nu(a_i) \end{aligned}$$

And $\max_{j<i}\{\nu(a_j) + \nu(c_j)\} = \max_{j<i}\{\nu(a_j) + \nu(b_j)\} \le \nu(a_i)$. In fact, by the definition of J, we have $\max_{j<i}\{\nu(a_j) + \nu(c_j)\} = \nu(a_i)$.

A further reduction is possible. For each j, let $c_j = c_j^+ + c_j^-$ be a ν-homogeneous decomposition of c_j. Then, we have

$$\begin{aligned} \nu\Big(a_i - \sum_{j<i} a_j c_j^+\Big) &= \nu\Big(a_i - \sum_{j<i} a_j c_j + \sum_{j<i} a_j c_j^-\Big) \\ &\le \max\Big\{\nu\Big(a_i - \sum_{j<i} a_j c_j\Big), \nu\Big(\sum_{j<i} a_j c_j^-\Big)\Big\} \\ &\le \max\Big\{\nu\Big(a_i - \sum_{j<i} a_j c_j\Big), \max_{j<i}\{\nu(a_j) + \nu(c_j^-)\}\Big\} \\ &< \nu(a_i), \end{aligned}$$

while $\max_{j<i}\{\nu(a_j) + \nu(c_j^+)\} \le \nu(a_i)$.

Now, for all $h \in \Lambda$,

$$\begin{aligned} h \cdot \Big(a_i - \sum_j a_j c_j^+\Big) &= h \cdot a_i - \sum_j h \cdot (a_j c_j^+) \\ &= h \cdot a_i - \sum_j \sum_{(h)} (h_{(1)} \cdot a_j)(h_{(2)} \cdot c_j^+) \\ &= \varepsilon(h) a_i - \sum_j \sum_{(h)} \varepsilon(h_{(1)}) a_j (h_{(2)} \cdot c_j^+) \\ &= \varepsilon(h) a_i - \sum_j a_j \Big(\Big(\sum_{(h)} \varepsilon(h_{(1)}) h_{(2)}\Big) \cdot c_j^+\Big) \\ &= \varepsilon(h) a_i - \sum_j a_j (h \cdot c_j^+). \end{aligned}$$

So, we have

$$\begin{aligned} &\nu\Big(\sum_j a_j(\varepsilon(h)c_j^+ - h \cdot c_j^+)\Big) \\ &\quad = \nu\Big(\Big(\varepsilon(h)a_i - \sum_j a_j(h \cdot c_j^+)\Big) - \varepsilon(h)\Big(a_i - \sum_j a_j c_j^+\Big)\Big) \\ &\quad \le \max\Big\{\nu\Big(h \cdot \Big(a_i - \sum_j a_j c_j^+\Big)\Big), \nu\Big(\varepsilon(h)\Big(a - \sum_j a_j c_j^+\Big)\Big)\Big\} \\ &\quad < \nu(a) = \max_{j<i}\{\nu(a_j) + \nu(c_j^+)\}. \end{aligned}$$

If there exist j and $h \in \Lambda$ such that $\varepsilon(h)c_j^+ - h \cdot c_j^+ \neq 0$, then $\nu(\varepsilon(h)c_j^+ - h \cdot c_j^+) = \nu(c_j^+)$, because the action of H on R is compatible with ν. And, so, we would be able to write

$$\nu\Big(\sum_j a_j(\varepsilon(h)c_j^+ - h \cdot c_j^+)\Big) < \max_{j<i}\{\nu(a_j) + \nu(\varepsilon(h)c_j^+ - h \cdot c_j^+)\}.$$

Now, the above would be a right ν-dependence relation on $\{a_1, \ldots, a_{i-1}\}$, but this contradicts the choice of n. So, for all j and $h \in \Lambda$, $h \cdot c_j^+ = \varepsilon(h)c_j^+$. It follows that $\{c_j^+ : j = 1, \ldots, i-1\} \subseteq R^\Lambda$. And we have shown that R^Λ satisfies the weak algorithm with respect to ν. □

By [2, Proposition 2.4.2], a free algebra $k\langle X\rangle$ with any given grading satisfies the weak algorithm with respect to the filtration induced by that grading. Moreover, any subalgebra of it satisfying the weak algorithm with respect to this filtration is free. Therefore, the above theorem specializes to free algebras in the following way.

Corollary 3.2 *Let R be a free associative algebra and let H be a Hopf algebra* *acting homogeneously on R. Let Λ be a right coideal of H. Then R^{Λ} is a free subalgebra.*[4] □

We are now able to rescue the following results.

Corollary 3.3 ([3,5]) *The subalgebra of invariants of a group of homogeneous automorphisms of a free associative algebra is free.* □

Corollary 3.4 ([4,6]) *The subalgebra of constants of a Lie algebra of homogeneous derivations of a free associative algebra is free.* □

Finite group gradings can also be realized as actions of Hopf algebras. Thus Corollary 3.2 has also the following consequence.

Corollary 3.5 *Let $R = k\langle X\rangle$ be a free associative algebra and suppose that R is graded by a finite group G such that, for all $x \in X$ and $g \in G$, the g-homogeneous component of x is a k-linear combination of elements of X. If e denotes the identity of G, then R_e is a free subalgebra.* □

Finally, Kharchenko asks in [5,6] whether the homogeneous hypothesis in Corollaries 3.3 and 3.4 can be removed. We could reformulate this question in the Hopf algebra context in the following way.

Is the subalgebra of invariants of a free associative algebra under the action of a finite-dimensional Hopf algebra free?

The finite-dimensional hypothesis is necessary in order to exclude Bergman's counter-example [1] (see also [7, 6.1.12]).

4 Proof of Theorem 1.2

We are ready to prove Theorem 1.2.

Let $R = k\langle X\rangle$ be a free associative algebra on a set X over a field k and let H be a finite-dimensional pointed Hopf algebra which acts homogeneously and faithfully on R. By Theorem 2.2, the action of H on R is X-outer. Applying Theorem 1.1 to R, we obtain a bijection between the set $\mathcal{S}$ of all right coideal subalgebras of H and the set $\mathcal{F}$ of all rationally complete subalgebras of R containing R^H given by the functions Φ and Ψ. Let $\mathcal{F}'$ denote the set of all

[4] This has been announced for $\Lambda = H$ by Kharchenko in *Transactions of the XX All Union Algebraic Conference*, Krasnoyarsk, 1993, p. 350, but a written proof has not been published. The authors thank I. Shestakov for having provided this information.

free subalgebras of R containing R^H. We shall prove that $\mathcal{F} = \mathcal{F}'$, that is, that a subalgebra of R containing R^H is free if and only if it is rationally complete. That $\mathcal{F}'$ is contained in $\mathcal{F}$ follows from [5, Lemma] and the remark on [12, p. 313]. On the other hand, it follows from Theorem 2.2 that $\mathcal{F} = \Psi(\mathcal{S})$; by Corollary 3.2, this last set is contained in $\mathcal{F}'$. This proofs Theorem 1.2.

References

[1] G. M. Bergman, A derivation on a free algebra whose kernel is a nonfree subalgebra, unpublished, 1981.

[2] P. M. Cohn, Free Rings and Their Relations, 2nd. Ed., Academic Press, London, 1985.

[3] D. Lane, Free Algebras of Rank Two and Their Automorphisms, Ph.D. Thesis, London, 1976.

[4] T. de W. Jooste, Primitive derivations in free associative algebras, Math. Z. 164 (1978), 15–23.

[5] V. K. Kharchenko, Algebras of invariants of free algebras, Algebra i Logika 17 (1978), 478–487 (English translation in Algebra and Logic 17 (1979), 208–258.)

[6] V. K. Kharchenko, Constants of derivations of prime rings, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), 435–461. (English translation in Izv. Math. 18 (1982), 381–401.)

[7] V. K. Kharchenko, Automorphisms and Derivations of Associative Rings, Kluwer, Dordrecht, 1991.

[8] A. Masuoka and T. Yanai, Hopf module duality applied to X-outer Galois theory, J. Algebra 265 (2003) 229–246.

[9] A. Milinski, Actions of pointed Hopf algebras on prime algebras, Comm. Algebra 23 (1995), 313–333.

[10] S. Montgomery, Hopf Algebras and Their Actions on Rings, American Mathematical Society, Providence, RI, 1993.

[11] S. Montgomery, Bi-invertible actions of Hopf algebras, Israel J. Math. 83 (1993), 45–71.

[12] S. Montgomery and D. S. Passman, Outer Galois theory of prime rings, Rocky Mountain J. Math. 14 (1984), 305–318.

[13] S. Westreich, A Galois-type correspondence theory for actions of finite-dimensional pointed Hopf algebras on prime algebras, J. Algebra 219 (1999), 606–624.

[14] S. Westreich and T. Yanai, More about a Galois-type correspondence theory, J. Algebra 246 (2001), 629–640.

[15] T. Yanai, Correspondence theory of Kharchenko and X-outer actions of pointed Hopf algebras, Comm. Algebra 25 (1997), 1713–1740.

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