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Research Paper

Representations of Smith algebras which are free over the Cartan subalgebra

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ABSTRACT

In this paper, we study the category of modules over the Smith algebra which are free of finite rank over the unital polynomial subalgebra generated by the Cartan element h and obtain families of such simple modules of arbitrary rank. In the case of rank one we obtain a full description of the isomorphism classes, a simplicity criterion, and an algorithm to produce all composition series. We show that all such modules have finite length and describe the composition factors and their multiplicity.

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1. Introduction

In [15], Smith defined a class of algebras similar to the enveloping algebra of $\mathfrak{sl}_2$, essentially by replacing the standard relation $[e, f] = h$ in $U(\mathfrak{sl}_2)$ with the relation $[e, f] = g(h)$, where g is an arbitrary polynomial in h. We will denote these algebras by $\mathcal{S}(g)$. Among other results, Smith classified the finite-dimensional simple $\mathcal{S}(g)$-modules, seen as quotients of Verma modules, and introduced an analog of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ for $\mathcal{S}(g)$.

The Smith algebras have been extensively studied and are related to down-up algebras, a class of algebras introduced by Benkart and Roby in [1], inspired by the relations satisfied by the down and up operators on a differential poset. Down-up algebras also display many similarities with enveloping algebras of three-dimensional Lie algebras, and include those Smith algebras $\mathcal{S}(g)$ with $\deg g \leqslant 1$. Later, in [4], Cassidy and Shelton introduced generalized down-up algebras, which include all the Smith algebras $\mathcal{S}(g)$.

As with the enveloping algebra of $\mathfrak{sl}_2$, every Smith algebra has a Casimir element, which generates its center and acts as a scalar on simple $\mathcal{S}(g)$-modules. The corresponding factor rings of $\mathcal{S}(g)$ by the maximal ideal of the center have been considered by Joseph [9, Lemma 3.1], where simplicity criteria were given, and by Hodges [7], as algebras of invariants of the Weyl algebra under the action of a cyclic group. Allowing $\mathcal{S}(g)$ to be defined over a ring, then some of these quotients can further be seen as invariant rings of differential operators on a multiplicity-free representation of an algebraic group under the action of its derived subgroup [14]. Another interesting connection is with the Zhu algebra of a vertex operator algebra associated to a positive definite rank-one lattice, which is shown in [5] to be isomorphic to a finite-dimensional quotient of $\mathcal{S}(g)$.

Our main interest is the representation theory of the Smith algebras $\mathcal{S}(g)$. As we mentioned, the finite-dimensional irreducible representations, the Verma modules and category $\mathcal{O}$ have already been investigated in [15] (see also [8]). In Block's classification [2] of simple $U(\mathfrak{sl}_2)$-modules, along with the weight modules one finds also Whittaker modules and other modules defined via localization, the latter being torsion free over the polynomial algebra in h. A class of modules which has recently gained a lot of attention in the context of Lie algebras is given by the modules which are free of finite rank over the enveloping algebra of a Cartan subalgebra. These have been introduced and studied in [12,13,16], and they are in a certain sense opposite to weight modules, as the action of the Cartan subalgebra is torsion free, rather than semisimple. In particular, free rank one simple $\mathfrak{sl}_{n+1}$-modules were classified in [12], and in [16] such modules over $\mathfrak{sl}_{n+1}$ were also constructed from modules over Witt algebras W_n. Similarly, such simple $\mathfrak{sp}_{2n}$-modules were classified in [13]. These are the only simple finite-dimensional algebras for which there exist modules that are free over the enveloping algebra of a Cartan subalgebra. Parabolic induction from simple $U(h)$-free modules was studied in [3].

In this paper, we investigate the category of $\mathcal{S}(g)$-modules which are free of finite rank over the unital subalgebra generated by h and obtain families of such simple modules of arbitrary rank. We dedicate particular attention to the case of rank one, where we obtain a full description of the isomorphism classes, a simplicity criterion, and an algorithm to produce all composition series, resulting in a proof that such modules have finite length and in a full description of the composition factors and their multiplicity.

Notations and conventions We work over an algebraically closed field $\Bbbk$ of characteristic zero. Since a monic polynomial in $\Bbbk[h]$ is fully determined by its set of roots, we get a bijection between the finite submultisets of $\Bbbk$ and monic polynomials in $\Bbbk[h]$. For convenience, we associate the field $\Bbbk$ to the zero polynomial. Given any $f(h) \in \Bbbk[h]$, we let R_f denote its multiset of roots. Conversely, given any finite multiset X of elements of $\Bbbk$, we let $\mathsf{poly}_X \in \Bbbk[h]$ denote the unique monic polynomial such that $\mathsf{R}_{\mathsf{poly}_X} = X$, that is, $\mathsf{poly}_X = \prod_{\lambda \in X}(h - \lambda)$. Adopting the usual convention that an empty product equals 1, we assume $\mathsf{poly}_\emptyset = 1$. Moreover, we follow the convention that $\mathsf{deg}\, 0 = -\infty$, with its usual arithmetic properties.

Given a multiset X of elements of $\Bbbk$ and $\lambda \in \Bbbk$, we denote by $X \setminus \{\lambda\}$ (respectively, $X \cup \{\lambda\}$) the multiset obtained from X by reducing (respectively, increasing) by one the multiplicity of λ in X, and proceed similarly for the difference and union of arbitrary multisets. For example, $\{1, 2, 2, 5, 5, 5\} \setminus \{3, 5\} = \{1, 2, 2, 5, 5\}$ and $\{1, 2, 2, 5, 5, 5\} \cup \{3, 5\} = \{1, 2, 2, 3, 5, 5, 5, 5\}$. The cardinality $|X|$ of the (finite) multiset X is the sum of the multiplicities of its elements. The underlying set obtained from X (by eliminating repeated elements) will be denoted by $\underline{X}$. Thus, $|\{1, 2, 2, 5, 5, 5\}| = 6$ and $\underline{\{1, 2, 2, 5, 5, 5\}} = \{1, 2, 5\}$.

For $n \in \mathbb{N} = \mathbb{Z}_{\geqslant 0}$, set $[n] = \{1, \dots, n\}$, so in particular $[0] = \emptyset$.

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2. The Smith algebra

Fix a polynomial $g(h) \in \Bbbk[h]$. The *Smith algebra* $\mathcal{S}(g)$ is the unital associative algebra over $\Bbbk$ generated by x, y, h with definition relations:

$$[h, y] = y, \quad [h, x] = -x \quad \text{and} \quad [y, x] = g(h). \tag{2.1}$$

This algebra was introduced by Smith in [15]. In case $g(h) = 0$, the generators x and y commute and the representation theory of $\mathcal{S}(0)$ assumes characteristics which often diverge from the general theory in case $g \neq 0$. In fact, $\mathcal{S}(0)$ is the enveloping algebra of a 3-dimensional solvable (non-nilpotent) Lie algebra. **Thus, henceforth we will always implicitly assume that $g \neq 0$.**

By [15, Lemma 1.4], there exists $u(h) \in \Bbbk[h]$ such that $g(h) = u(h-1) - u(h)$. Moreover, $u(h)$ is uniquely determined up to its constant term, which can be arbitrary, and $\deg(u) = \deg(g) + 1 \geqslant 1$. Fixing one such u, we denote the Smith algebra $\mathcal{S}(g)$ by $\mathcal{S}_u$ and remark that $\mathcal{S}_u = \mathcal{S}_{u+C}$, for any $C \in \Bbbk$.

Let $z_u = xy - u(h) = yx - u(h-1)$. It is easy to see that z_u is a central element in $\mathcal{S}_u$ and for this reason we call z_u the *Casimir element* associated with u. In addition, the center $\mathsf{Z}(\mathcal{S}_u)$ of $\mathcal{S}_u$ is $\Bbbk[z_u]$, the polynomial algebra in z_u (see [15, Proposition 1.5], or [11, Proposition 2.9]).

Next we show that the algebra $\mathcal{S}_u$ acts on the polynomial algebra $\Bbbk[t]$ by differential operators, with scalar central character. Denote by $A_1 = \Bbbk[t, \partial]$, where $\partial = \frac{d}{dt}$, the first Weyl algebra over $\Bbbk$, realized here as the ring of differential operators on $\Bbbk[t]$ with polynomial coefficients. Since $g \neq 0$, then $\deg(u) = \deg(g) + 1 \geqslant 1$ and hence $\mathsf{R}_{u+C} \neq \emptyset$ for any $C \in \Bbbk$. For $C \in \Bbbk$, every root $\lambda \in \mathsf{R}_{u+C}$ and submultiset $X \subseteq \mathsf{R}_{u+C} \setminus \{\lambda\}$ define the polynomials:

$$Q_X(h) = \mathsf{poly}_X \quad \text{and} \quad P_X(h) = \frac{u(h-1)+C}{Q_X(h-1)(h-(\lambda+1))}.$$

Equivalently, $P_X(h) = \xi\, \mathsf{poly}_{\mathsf{R}_{u+C} \setminus (\{\lambda\} \cup X)}(h-1)$, where ξ is the leading coefficient of $u + C$.

Lemma 2.2. *There exists a morphism of algebra $\varphi_{C,\lambda,X} \colon S_u \to A_1$ such that*

$$x \mapsto \partial Q_X(\theta - 1), \quad y \mapsto t P_X(\theta + 1) \quad \text{and} \quad h \mapsto \theta,$$

where $\theta = t\partial + \lambda + 1$. Moreover, z_u is mapped to C.

Proof. Define actions of x, y and h on $\Bbbk[t]$ as in the statement above. Since $\theta t = t(\theta+1)$ and $\theta\partial = \partial(\theta - 1)$, it follows that

$$(tP_X(\theta+1))\,(\partial Q_X(\theta-1)) = t\partial P_X(\theta) Q_X(\theta-1) = ((\theta-1)-\lambda) P_X(\theta) Q_X(\theta-1),$$
$$(\partial Q_X(\theta-1))\,(tP_X(\theta+1)) = \partial t Q_X(\theta) P_X(\theta+1) = (\theta-\lambda) P_X(\theta+1) Q_X(\theta).$$

Then, from the equality $u(h) + C = (h-\lambda)P_X(h+1)Q_X(h)$, it follows that $[y, x]$ acts on $\Bbbk[t]$ as $((\theta-1)-\lambda)P_X(\theta)Q_X(\theta-1) - (\theta-\lambda)P_X(\theta+1)Q_X(\theta) = u(\theta-1) + C - u(\theta) - C = g(\theta)$, which is the action of $g(h)$. Similarly, the relations $[h, y] = y$ and $[h, x] = -x$ are also preserved by the action, thus inducing an $\mathcal{S}_u$-module structure by differential operators on $\Bbbk[t]$, and hence the given morphism of algebras. It is straightforward to show that $\varphi_{C,\lambda,X}(z_u) = C$. $\square$

Example 2.3. Let $g(h) = h$, so that $\mathcal{S}(h) \simeq U(\mathfrak{sl}_2)$, the universal enveloping algebra of $\mathfrak{sl}_2$. Then we can take $u(h) = -\frac{1}{2}h(h+1)$, $C = 0$, $\lambda = 0$ and $X = \{-1\}$, so that

$Q_X(h) = h + 1$ and $P_X(h) = -\frac{1}{2}$. We obtain an action of $\mathfrak{sl}_2$ on $\Bbbk[t]$ where x acts by $\partial(t\partial + 1)$, y acts by $-\frac{1}{2}t$ and h acts by $t\partial + 1$.

Concretely,

$$x \cdot t^k = k(k+1)t^{k-1}, \quad y \cdot t^k = -\frac{1}{2}t^{k+1} \quad \text{and} \quad h \cdot t^k = (k+1)t^k, \quad \text{for all } k \geqslant 0.$$

Using the fact that the action of x lowers the degree in t, annihilating only the constant polynomials, and the action of y raises it, a straightforward argument shows that this is an irreducible representation of $\mathfrak{sl}_2$.

As a consequence of the previous lemma, and using exponential modules for the Weyl algebra (compare [6] for the case of $\mathfrak{sl}_2$), we can construct a class of non-weight representations of $\mathcal{S}_u$ as follows.

Definition 2.4. (Exponential modules) Let $p \in \Bbbk[t]$ be a polynomial and consider the A_1-module $\Bbbk[t]e^p$. Given $C \in \Bbbk$, $\lambda \in \mathsf{R}_{u+C}$ and $X \subseteq \mathsf{R}_{u+C} \setminus \{\lambda\}$ a submultiset, define $\mathcal{E}(p, C, \lambda, X)$ to be the $\mathcal{S}_u$-module induced from the A_1-module $\Bbbk[t]e^p$ via the map $\varphi_{C,\lambda,X}$ from Lemma 2.2.

Theorem 2.5. *Assume that* $\deg p \geqslant 1$*. Then* $\mathcal{E}(p, C, \lambda, X)$ *is a* $\Bbbk[h]$*-free module of rank* $\deg p$*. Furthermore, if there is no* $\mu \in \mathsf{R}_{u+C} \setminus \{\lambda\}$ *such that* $\mu - \lambda \in \mathbb{Z}_{\geqslant 1}$ *then* $\mathcal{E}(p, C, \lambda, \mathsf{R}_{u+C} \setminus \{\lambda\})$ *is simple.*

Proof. Let $n = \deg p$. We claim that $B = \{e^p, te^p, \dots, t^{n-1}e^p\}$ is a $\Bbbk[h]$-basis of $\mathcal{E}(p, C, \lambda, X)$.

From the relation $(h - (\lambda + 1 + s)) \cdot t^s e^p = t^{s+1}p'e^p$ we can show, by induction on s, that $t^s e^p \in \Bbbk[h]B$ for all $s \in \mathbb{N}$, so we conclude that B generates $\Bbbk[t]e^p$ as a $\Bbbk[h]$-module. Now notice that $h \cdot qe^p = (t(q' + qp') + (\lambda+1)q)e^p$, for all $q \in \Bbbk[t]$. In particular, $h \cdot qe^p = \hat{q}e^p$, where $\deg \hat{q} = \deg p + \deg q$. Thus, we can conclude that $r(h) \cdot qe^p = \hat{q}e^p$, for some $\hat{q} \in \Bbbk[h]$ such that

$$\deg \hat{q} = (\deg r)(\deg p) + \deg q.$$

Suppose, by contradiction, that $\sum_{i=0}^{n-1} r_i(h) \cdot t^i e^p = 0$, for some $r_i \in \Bbbk[h]$, not all zero. If there is a unique i such that $r_i \neq 0$, then $0 = r_i(h) \cdot t^i e^p = \hat{q}e^p$ with $\deg \hat{q} = i + n \deg r_i \geqslant 0$. Thus $\hat{q} \neq 0$, which contradicts the equality $\hat{q}e^p = 0$. So assume that at least two of the r_i are nonzero. Then there are $0 \leqslant i < j \leqslant n-1$ such that $r_i, r_j \neq 0$ and $(\deg r_i)n + i = (\deg r_j)n + j$. Hence $(\deg r_i - \deg r_j)n = j - i \in [n-1]$. As $[n-1]$ contains no multiples of n, this is impossible. Therefore B is $\Bbbk[h]$-linearly independent and the claim is proved.

Now consider the case $X = \mathsf{R}_{u+C} \setminus \{\lambda\}$. In this case, y acts as multiplication by βt, for some $\beta \in \Bbbk^\times$. Replacing y with y/β, we can, and will, assume that $\beta = 1$, for simplicity, so that y acts as multiplication by t.

Let $V \subseteq \mathcal{E}(p, C, \lambda, X)$ be a nonzero submodule. We claim that $t^i e^p \in V$, for some $i \in \mathbb{N}$. First, notice that $(h - (\lambda+1) - yp'(y))qe^p = tq'e^p$, for all $q \in \Bbbk[t]$. In particular, $(h-(\lambda+1)-yp'(y))t^j e^p = jt^j e^p$, for all $j \in \mathbb{N}$, so $\left\{t^j e^p \mid j \in \mathbb{N}\right\}$ is a basis of $\mathcal{E}(p, C, \lambda, X)$ of eigenvectors for the action of $(h-(\lambda+1)-yp'(y))$. Thus, this operator has a diagonal action on $\mathcal{E}(p, C, \lambda, X)$ and hence also on V. Since the eigenspaces are one-dimensional, V must contain some eigenvector, say $t^i e^p$, for some $i \in \mathbb{N}$.

Let $i \in \mathbb{N}$ be minimum such that $t^i e^p \in V$. Using induction on the number of elements of X, one can prove that

$$Q_X(h)t^j e^p = \left(\prod_{\mu \in X} (\lambda + 1 + j - \mu) + tq_j \right) t^j e^p,$$

for $j \in \mathbb{N}$ and for some $q_j \in \Bbbk[t]$. Since

$$\begin{aligned} V \ni x \cdot t^i e^p &= Q_X(\theta)\partial t^i e^p = Q_X(h)(it^{i-1} + t^i p')e^p \\ &= i \left(\prod_{\mu \in X} (\lambda + i - \mu) + tq_{i-1} \right) t^{i-1} e^p + Q_X(h)p'(y)t^i e^p \\ &= i \prod_{\mu \in X} (\lambda + i - \mu) t^{i-1} e^p + \left(iq_{i-1}(y) + Q_X(h)p'(y)\right) t^i e^p, \end{aligned}$$

we deduce that $i \prod_{\mu \in X}(\lambda + i - \mu)t^{i-1}e^p \in V$. By the minimality of i, we conclude that $i \prod_{\mu \in X}(\lambda + i - \mu) = 0$ and from the hypothesis that $\mu - \lambda \notin \mathbb{Z}_{\geqslant 1}$ for all $\mu \in X$, it must be that $i = 0$. Therefore $V = \mathcal{E}(p, C, \lambda, X)$ and the simplicity of $\mathcal{E}(p, C, \lambda, \mathsf{R}_{u+C} \setminus \{\lambda\})$ is established. □

3. The category $\mathfrak{U}$ of $\Bbbk[h]$-free $\mathcal{S}_u$-modules

Denote by $\mathfrak{U}$ the category of $\mathcal{S}_u$-modules that are free of finite rank over the subalgebra $\Bbbk[h]$. In this section we describe a skeleton of the category $\mathfrak{U}_1$, the full subcategory of $\mathfrak{U}$ consisting of modules that are free of rank one over $\Bbbk[h]$. We show that any module in $\mathfrak{U}_1$ is of finite length, give an algorithm to determine all of its composition series, and give an explicit classification of the simple objects in $\mathfrak{U}_1$.

Let $M \in \mathfrak{U}$ have rank n, so we may assume that $M = \Bbbk[h]^n$ as a $\Bbbk[h]$-module. Let $1_1, \dots, 1_n \in \Bbbk[h]^n$ be its canonical basis. We have

$$y(h^k \cdot 1_i) = (h-1)^k y 1_i \quad \text{and} \quad x(h^k \cdot 1_i) = (h+1)^k x \cdot 1_i, \quad \text{for } i \in [n] \text{ and } k \in \mathbb{N}.$$

Therefore,

$$yf(h) \cdot 1_i = f(h-1)y \cdot 1_i \quad \text{and} \quad xf(h) \cdot 1_i = f(h+1)x \cdot 1_i, \quad \text{for } i \in [n] \text{ and } f(h) \in \Bbbk[h]. \tag{3.1}$$

In particular, the action of $\mathcal{S}_u$ on M is uniquely defined by a choice

$$y \cdot 1_i =: p_i = (p_{i,1}, p_{i,2}, \dots, p_{i,n}) \in \Bbbk[h]^n, \tag{3.2}$$

$$x \cdot 1_i =: q_i = (q_{i,1}, q_{i,2}, \dots, q_{i,n}) \in \Bbbk[h]^n, \tag{3.3}$$

for all $i \in [n]$. By considering that $[y, x] = g(h) = u(h-1) - u(h)$, we deduce that the $p_{i,j}$ and the $q_{i,j}$ must satisfy the relations

$$g(h)1_i = \sum_{\ell=1}^{n} \left(\sum_{j=1}^{n} q_{i,j}(h-1)p_{j,\ell}(h) - p_{i,j}(h+1)q_{j,\ell}(h) \right) 1_\ell,$$

for all $i \in [n]$. Writing $Q = (q_{i,j}), P = (p_{i,j}) \in M_n(\Bbbk[h])$, we see that the above is equivalent to the following matrix equation over $\Bbbk[h]$:

$$Q(h-1)P(h) - P(h+1)Q(h) = g(h)I, \tag{3.4}$$

where $I \in M_n(\Bbbk[h])$ is the identity matrix. In fact, it is easy to see that (3.2) and (3.3) define a $\mathcal{S}_u$-module structure on $M = \Bbbk[h]^n$ extending the action of $\Bbbk[h]$ by multiplication if and only if (3.4) holds.

Now, suppose that M has a central character $\chi_M : \Bbbk[z_u] \to \Bbbk$, so that $zm = \chi_M(z)m$, for all $z \in \mathsf{Z}(\mathcal{S}_u) = \Bbbk[z_u]$ and all $m \in M$. Set $C = \chi_M(z_u)$. Then we have $xy1_i = (z_u + u)1_i = (u + C)1_i$, which becomes

$$(u(h) + C)I = P(h+1)Q(h), \tag{3.5}$$

in matrix form. Then (3.4) implies that

$$(u(h-1) + C)I = Q(h-1)P(h), \tag{3.6}$$

which translates to $yx1_i = (u(h-1)+C)1_i$. Conversely, notice that (3.5) and (3.6) imply (3.4) and moreover that M has a central character χ_M with $\chi_M(z_u) = C$.

Example 3.7. Let $C \in \Bbbk$, $\lambda \in \mathsf{R}_{u+C}$ and $X \subseteq \mathsf{R}_{u+C} \setminus \{\lambda\}$. Let $p = \sum_{j=0}^{n} \alpha_j t^j \in \Bbbk[t]$ of degree $n \geqslant 1$ and $\mathcal{E} = \mathcal{E}(p, C, \lambda, X)$. By Theorem 2.5, $\mathcal{E}$ is $\Bbbk[h]$-free with basis $\{e^p, \dots, t^{n-1}e^p\}$. Hence there is an isomorphism of $\Bbbk[h]$-modules $\Bbbk[h]^n \to \mathcal{E}$ such that

$$1_i \mapsto t^{i-1}e^p, \quad i \in [n].$$

Via this isomorphism, $\Bbbk[h]^n$ inherits from $\mathcal{E}$ a structure of $\mathcal{S}_u$-module.

Recall, from Definition 2.4 and Lemma 2.2, that h acts on $\mathcal{E}$ as $\theta = t\partial + \lambda + 1 = \partial t + \lambda$, x acts as $\partial Q_X(\theta - 1) = Q_X(\theta)\partial$ and y acts as $tP_X(\theta + 1) = P_X(\theta)t$. Set $v_i = t^{i-1}e^p$, for $i \in [n]$. Then we have, for $i \geqslant 2$,

$$x \cdot v_i = Q_X(\theta)\partial t^{i-1}e^p = Q_X(\theta)(\partial t)t^{i-2}e^p = Q_X(\theta)(\theta - \lambda)v_{i-1} = Q_X(h)(h-\lambda)\cdot v_{i-1}.$$

We conclude that $q_{i,j} = Q_X(h)(h-\lambda)\delta_{i-1,j}$, for all $i, j \in [n]$ with $i \geqslant 2$, where $\delta_{k,\ell}$ is the Kronecker delta.

Now we take $i = 1$:

$$x \cdot v_1 = Q_X(\theta)\partial\, e^p = Q_X(\theta)p' e^p = Q_X(\theta)\sum_{j=1}^{n} j\alpha_j t^{j-1}e^p = Q_X(h)\cdot\sum_{j=1}^{n} j\alpha_j v_j.$$

We conclude that $q_{1,j} = Q_X(h) j\alpha_j$, for all $j \in [n]$. Therefore, we obtain

$$Q(h) = Q_X(h)\begin{bmatrix} \alpha_1 & 2\alpha_2 & 3\alpha_3 & \cdots & (n-1)\alpha_{n-1} & n\alpha_n \\ (h-\lambda) & 0 & 0 & \cdots & 0 & 0 \\ 0 & (h-\lambda) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & (h-\lambda) & 0 \end{bmatrix}.$$

Similarly, we obtain

$$\begin{aligned} P(h) &= P_X(h)\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \frac{(h-(\lambda+1))}{n\alpha_n} & -\frac{\alpha_1}{n\alpha_n} & -\frac{2\alpha_2}{n\alpha_n} & \cdots & -\frac{(n-2)\alpha_{n-2}}{n\alpha_n} & -\frac{(n-1)\alpha_{n-1}}{n\alpha_n} \end{bmatrix} \\ &= P_X(h)\left(\operatorname{Comp}\left(\frac{tp' - (h-(\lambda+1))}{n\alpha_n}\right)\right)^t, \end{aligned}$$

where $\operatorname{Comp}(f(t))$ denotes the companion matrix of $f(t) \in (\Bbbk[h])[t]$, as a polynomial in t.

3.1. The category $\mathfrak{U}_1$

Now we will focus on the category $\mathfrak{U}_1$ of $\mathcal{S}_u$-modules which are free of rank 1 over $\Bbbk[h]$. In the following, we will identify $M \in \mathfrak{U}_1$ with $\Bbbk[h]$, the (left) regular $\Bbbk[h]$-module. We set $p_M = y \cdot 1$ and $q_M = x \cdot 1$. Whenever there is no ambiguity, we will simply denote these elements of $\Bbbk[h]$ by p and q, respectively.

Notice that, by Lemma 2.2, the Casimir element z_u acts on any exponential module by a scalar. Next, we show that this property holds for all modules in $\mathfrak{U}_1$.

Lemma 3.8. *Let $M \in \mathfrak{U}_1$. Then M admits a central character χ_M that satisfies $\chi_M(z_u) = p(h+1)q(h) - u(h) \in \Bbbk$.*

Proof. By equation (3.4), we must have

$$q(h-1)p(h) - p(h+1)q(h) = g(h) = u(h-1) - u(h).$$

Let $f(h) = q(h)p(h+1)$. Then we have $f(h-1) - f(h) = g(h)$. By [12, Lemma 4], the solution of such an equation is unique up to the constant term. Therefore, $f(h) = u(h) + C$, for some $C \in \Bbbk$. In particular $C = p(h+1)q(h) - u(h)$ and

$$z_u \cdot 1 = (xy - u(h)) \cdot 1 = x \cdot p - u(h) \overset{(3.1)}{=} p(h+1)q(h) - u(h) = C,$$

thus proving the lemma. □

Let $M \in \mathfrak{U}_1$ and $C = \chi_M(z_u)$. Let $\xi_C \in \Bbbk^\times$ be the leading coefficient of $u(h)+C$. Since $u(h) + C = p(h+1)q(h)$, it follows that there is a multiset partition $\mathsf{R}_{u+C} = X \coprod Y$, where $X = \mathsf{R}_q$ and $Y = \mathsf{R}_{p(h+1)} = \mathsf{R}_p - 1$. Hence,

$$q(h) = \xi_q \,\mathsf{poly}_X = \xi_q \prod_{\alpha \in X} (h - \alpha) \quad \text{and} \quad p(h) = \xi_p \,\mathsf{poly}_{Y+1} = \xi_p \prod_{\alpha \in Y} (h - (\alpha + 1)),$$

with $\xi_q, \xi_p \in \Bbbk^\times$ the leading coefficients of q and p, respectively, so that $\xi_q\xi_p = \xi_C$. Thus, $M = \Bbbk[h]$ is described by C, X and ξ_q, and we will denote it by $A_C(X, \xi_q)$.

Given $\lambda \in \Bbbk^\times$, let φ_λ be the algebra automorphism of $\mathcal{S}_u$ defined by

$$\varphi_\lambda(x) = \lambda x, \quad \varphi_\lambda(y) = \lambda^{-1} y \quad \text{and} \quad \varphi_\lambda(h) = h.$$

For any $M \in \mathcal{S}_u{-}\mathsf{mod}$, define $\mathsf{F}_\lambda M \in \mathcal{S}_u{-}\mathsf{mod}$ to be the module M with $\mathcal{S}_u$-action twisted by φ_λ, i.e., $s \cdot m = \varphi_\lambda(s)m$, for all $s \in \mathcal{S}_u$, $m \in \mathsf{F}_\lambda M$. This defines a family of functors

$$\mathsf{F}_\lambda : \mathcal{S}_u{-}\mathsf{mod} \longrightarrow \mathcal{S}_u{-}\mathsf{mod},$$

for all $\lambda \in \Bbbk^\times$. It is easy to see that $\mathsf{F}_\lambda \mathsf{F}_\mu = \mathsf{F}_{\lambda\mu}$, for all $\lambda, \mu \in \Bbbk^\times$. In particular, the F_λ define category autoequivalences.

Notice now that $\mathsf{F}_\lambda A_C(X, \xi_q) = A_C(X, \lambda\xi_q)$, for all $\lambda \in \Bbbk^\times$, so in particular $A_C(X, \xi_q) = \mathsf{F}_{\xi_q} A_C(X, 1)$. Thus, it suffices to study the modules of the form $A_C(X, 1)$, which we simply denote by $A_C(X)$.

We summarize the above construction.

Definition 3.9. Let $C \in \Bbbk$ and let X be an arbitrary submultiset of the multiset R_{u+C} of roots of $u(h) + C$. Let $Y = \mathsf{R}_{u+C} \setminus X$, the multiset complement of X in R_{u+C}. Let $q(h) = \mathsf{poly}_X = \prod_{\alpha \in X} (h - \alpha)$ and $p(h) = \frac{u(h-1)+C}{q(h-1)} \in \Bbbk[h]$. Then $A_C(X) = \Bbbk[h]$ is the regular $\Bbbk[h]$-module, with action extended to $\mathcal{S}_u$ by

$$xf(h) = f(h+1)q(h) \quad \text{and} \quad yf(h) = f(h-1)p(h), \quad \text{for all } f(h) \in \Bbbk[h].$$

We have proved the following lemma.

Lemma 3.10. *Let $M \in \mathcal{S}_u-\mathsf{mod}$. Then $M \in \mathfrak{U}_1$ if and only if then there exist $\lambda \in \Bbbk^\times$, $C \in \Bbbk$ and a submultiset X of R_{u+C} such that $M \simeq \mathsf{F}_\lambda A_C(X)$.*

Lemma 3.11. *Let $C, C' \in \Bbbk$, X and X' be submultisets of R_{u+C} and $\mathsf{R}_{u+C'}$, respectively, and $\lambda, \lambda' \in \Bbbk^\times$. Then $\mathsf{F}_\lambda A_C(X) \simeq \mathsf{F}_{\lambda'} A_{C'}(X')$ if and only if $C = C'$, $\lambda = \lambda'$ and $X = X'$.*

Proof. Assume that $M = \mathsf{F}_\lambda A_C(X) \simeq \mathsf{F}_{\lambda'} A_{C'}(X') = M'$. Then the central characters must be the same, so $C = C'$. Moreover, any isomorphism of $\mathcal{S}_u$-modules is in particular an isomorphism of $\Bbbk[h]$-modules, and hence given by multiplication by a nonzero scalar. Thus it can be assumed that the identity map is an isomorphism between the given $\mathcal{S}_u$-modules. Then, by checking the action of x, we deduce that the isomorphism maps λq_M to $\lambda' q_{M'}$. Hence these polynomials have the same roots and the same leading coefficient, and it follows that $X = X'$ and $\lambda = \lambda'$. □

From Lemmas 3.10 and 3.11 we obtain a classification of the objects in $\mathfrak{U}_1$.

Corollary 3.12. *The following family is a skeleton of the category $\mathfrak{U}_1$:*

$$\left\{ \mathsf{F}_\lambda A_C(X) \mid C \in \Bbbk,\, \lambda \in \Bbbk^\times \text{ and } X \subseteq \mathsf{R}_{u+C} \text{ (a submultiset)} \right\}.$$

3.2. The exponential modules in $\mathfrak{U}_1$

From Theorem 2.5 we know that the exponential modules in $\mathfrak{U}_1$ are precisely those of the form $\mathcal{E}(p, C, \lambda, X)$ with $\deg p = 1$. We will see that the latter exhaust all isomorphism classes in $\mathfrak{U}_1$, except for the isomorphism classes of $A_C(\mathsf{R}_{u+C}, \alpha)$, with $C, \alpha \in \Bbbk$ and $\alpha \neq 0$. Using the symmetry of the Weyl algebra A_1, we define *dual* exponential modules $\mathcal{E}(p, C, \lambda, X)^\vee$ which will cover the remaining isomorphism classes in $\mathfrak{U}_1$.

Fix $p(t) = \alpha t + \beta$, with $\alpha, \beta \in \Bbbk$ and $\alpha \neq 0$. We know that $\mathcal{E}(p, C, \lambda, X) \simeq A_C(\widetilde{X}, \xi)$, for some submultiset $\widetilde{X} \subseteq \mathsf{R}_{n+C}$ and $\xi \in \Bbbk^\times$. These are determined by $x \cdot 1 = \xi \, \mathsf{poly}_{\widetilde{X}}(h)$ in $A_C(\widetilde{X}, \xi)$. Since $\mathrm{End}_{\mathcal{S}_u}(A_C(\widetilde{X}, \xi)) = \Bbbk 1$, where 1 stands for the identity on $A_C(\widetilde{X}, \xi)$, we can assume that the isomorphism $A_C(\widetilde{X}, \xi) \to \mathcal{E}(p, C, \lambda, X)$ takes the $\Bbbk[h]$-generators $1 \in A_C(\widetilde{X}, \xi)$ to $e^p \in \mathcal{E}(p, C, \lambda, X)$. Then from $\xi \, \mathsf{poly}_{\widetilde{X}}(h) = x \cdot 1$ we obtain

$$\xi \, \mathsf{poly}_{\widetilde{X}}(h) \cdot e^p = x \cdot e^p = Q_X(\theta) \partial e^p = \alpha \, \mathsf{poly}_X(h) \cdot e^p$$

As $\mathcal{E}(p, C, \lambda, X)$ is a free $\Bbbk[h]$-module on $\{e^p\}$, it follows that $\xi \, \mathsf{poly}_{\widetilde{X}}(h) = \alpha \, \mathsf{poly}_X(h)$, so $\xi = \alpha$ and $\widetilde{X} = X$.

Combining the preceding considerations with Lemma 3.11, we obtain a characterization of the exponential modules for $\mathcal{S}_u$ of rank 1.

Lemma 3.13. *Let $p, \widetilde{p} \in \Bbbk[h]$ with $\mathsf{deg}\, p = 1 = \mathsf{deg}\, \widetilde{p}$, $C, \widetilde{C} \in \Bbbk$, $\lambda \in \mathsf{R}_{u+C}$, $\widetilde{\lambda} \in \mathsf{R}_{u+\widetilde{C}}$ and $X \subseteq \mathsf{R}_{u+C} \setminus \{\lambda\}$, $\widetilde{X} \subseteq \mathsf{R}_{u+\widetilde{C}} \setminus \{\widetilde{\lambda}\}$ submultisets. The following hold:*

(a) $\mathcal{E}(p, C, \lambda, X) \simeq A_C(X, \alpha)$, *where* $p'(t) = \alpha \in \Bbbk^\times$;
(b) $\mathcal{E}(p, C, \lambda, X) \simeq \mathcal{E}(\widetilde{p}, \widetilde{C}, \widetilde{\lambda}, \widetilde{X})$ *if and only if* $p' = \widetilde{p}'$, $C = \widetilde{C}$ *and* $X = \widetilde{X}$.

In particular, all modules $A_C(X, \alpha)$ are exponential modules, except for $X = \mathsf{R}_{u+C}$. In order to be able to include these latter ones, we define the modules $\mathcal{E}(p, C, \lambda, X)^\vee$ using the symmetry of A_1.

Concretely, let $\tau \colon A_1 \to A_1$ be the automorphism defined by $t \mapsto \partial$ and $\partial \mapsto -t$. Then the algebra morphism $\widetilde{\varphi}_{C,\lambda,X} \colon S_u \to A_1$ defined by $\widetilde{\varphi}_{C,\lambda,X} = \tau \circ \varphi_{C,\lambda,X}$ induces an $\mathcal{S}_u$-module structure on the A_1-module $\Bbbk[t]e^p$, denoted by $\mathcal{E}(p, C, \lambda, X)^\vee$. So h acts as $\widetilde{\theta} = \tau(\theta) = -\partial t + \lambda + 1$, x acts as $-Q_X(\widetilde{\theta})t$ and y acts as $P_X(\widetilde{\theta})\partial$.

As before, we obtain a characterization of the modules $\mathcal{E}(p, C, \lambda, X)^\vee$ in case $\mathsf{deg}\, p = 1$.

Lemma 3.14. *Let $p \in \Bbbk[h]$ with $\mathsf{deg}\, p = 1$, $C \in \Bbbk$, $\lambda \in \mathsf{R}_{u+C}$ and $X \subseteq \mathsf{R}_{u+C} \setminus \{\lambda\}$ a submultiset. Then*

$$\mathcal{E}(p, C, \lambda, X)^\vee \simeq A_C(X \cup \{\lambda\}, \alpha^{-1}), \quad \text{where } \alpha = p'(t) \in \Bbbk^\times.$$

In particular, $A_C(\mathsf{R}_{u+C}, \alpha^{-1}) \simeq \mathcal{E}(p, C, \lambda, \mathsf{R}_{u+C} \setminus \{\lambda\})^\vee$.

3.3. The submodule structure of $A_C(X)$

Next, we study the simplicity of the modules $A_C(X)$ and, moreover, we will produce an algorithm to describe the composition series for these modules. We will find that $A_C(X)$ always has finite length as an $\mathcal{S}_u$-module.

Unless otherwise noted, throughout this subsection, $C \in \Bbbk$, X denotes an arbitrary submultiset of R_{u+C} and the polynomials $p, q \in \Bbbk[h]$ are as in Definition 3.9. In particular, q is monic, $X = \mathsf{R}_q$ and $Y = \mathsf{R}_{u+C} \setminus X = \mathsf{R}_{p(h+1)}$. Define

$$\mathcal{L}_C(X) = \{t(h) \in \Bbbk[h] \mid t(h) \text{ is monic, } t(h) \mid t(h-1)p(h) \text{ and } t(h) \mid t(h+1)q(h)\} \bigcup \{0\}.$$

We think of $\mathcal{L}_C(X)$ as a poset, under the polynomial divisibility relation.

Lemma 3.15. *There is an order reversing bijection between $\mathcal{L}_C(X)$ and the lattice of submodules of $A_C(X)$. Under this correspondence, $t \in \mathcal{L}_C(X)$ is mapped to $t\Bbbk[h] \subseteq A_C(X)$. Moreover, if $t \neq 0$ then $t\Bbbk[h] \simeq A_C(\mathsf{R}_{\overline{q}})$, where $\overline{q} = \frac{t(h+1)q(h)}{t(h)}$.*

Proof. Let M be an $\mathcal{S}_u$-submodule of $A_C(X)$. Then M is an ideal of $\Bbbk[h]$, by restriction, so it follows that $M = t(h)\Bbbk[h]$, for some $t(h) \in \Bbbk[h]$. If $t \neq 0$, then we can assume that t is monic, in which case M determines t. Since M is stable under the action of x, we have

$$t(h+1)q(h) = xt(h) \in M = t(h)\Bbbk[h],$$

thus $t(h)$ divides $t(h+1)q(h)$. Similarly, looking at the action of y, we deduce that $t(h)$ divides $t(h-1)p(h)$.

Conversely, let $0 \neq t \in \mathcal{L}_C(X)$ and set $\overline{q} = \frac{t(h+1)q(h)}{t(h)}$, $\overline{p} = \frac{t(h-1)p(h)}{t(h)} \in \Bbbk[h]$. Notice that $\overline{q}(h)\overline{p}(h+1) = q(h)p(h+1) = u(h) + C$, so that $\mathsf{R}_{\overline{q}}$ and $\mathsf{R}_{\overline{p}(h+1)}$ define a partition of R_{u+C}. What's more,

$$\begin{aligned} xt(h)f(h) &= t(h+1)f(h+1)q(h) = t(h)f(h+1)\overline{q} \quad \text{and} \\ yt(h)f(h) &= t(h-1)f(h-1)p(h) = t(h)f(h-1)\overline{p}, \end{aligned}$$

for all $f(h) \in \Bbbk[h]$. Hence $t\Bbbk[h]$ is a submodule of $A_C(X)$ isomorphic to $A_C(\mathsf{R}_{\overline{q}})$. The order reversing property is clear. $\square$

As all nonzero submodules of $A_C(X)$ are of the form $A_C(X')$, for some submultiset X' of R_{u+C}, in order to find simplicity criteria and composition series for $A_C(X)$, it suffices to determine all of the maximal submodules of $A_C(X)$. By the previous result, this is tantamount to finding all minimal elements of $\mathcal{L}_C(X) \setminus \{1\}$, i.e. all $t \in \mathcal{L}_C(X)$ with no proper nontrivial factors in $\mathcal{L}_C(X)$.

It will be convenient to introduce a partial order relation on $\Bbbk$, given by $\alpha \preccurlyeq \beta \iff \beta - \alpha = n1_{\Bbbk}$, for some $n \in \mathbb{N}$ (recall that $\mathsf{char}(\Bbbk) = 0$, so $\preccurlyeq$ is indeed antisymmetric).

Let $t \in \mathcal{L}_C(X)$ and assume that $t \neq 0, 1$. Then R_t is a nonempty finite multiset with cardinality equal to $\deg(t) \geqslant 1$, and thus it decomposes as a finite union of maximal chains (the connected components of the Hasse diagram of the poset R_t):

$$\begin{aligned} s_1 &: \alpha_1^1 \preccurlyeq \cdots \preccurlyeq \alpha_{k_1}^1; \\ &\vdots \\ s_\ell &: \alpha_1^\ell \preccurlyeq \cdots \preccurlyeq \alpha_{k_\ell}^\ell. \end{aligned}$$

Set

$$t_{s_i}(h) = \prod_{j=1}^{k_i}(h - \alpha_j^i), \qquad \overline{t_{s_i}}(h) = t(h)\left(t_{s_i}(h)\right)^{-1},$$

so that $t(h) = t_{s_1}(h) \cdots t_{s_\ell}(h) = t_{s_i}(h)\overline{t_{s_i}}(h)$. Thus $t_{s_i}(h)\overline{t_{s_i}}(h) \mid t_{s_i}(h-1)\overline{t_{s_i}}(h-1)p(h)$ and, since $\mathsf{gcd}\left(t_{s_i}(h), \overline{t_{s_i}}(h)\right) = 1$, the latter is equivalent to

$$t_{s_i}(h) \mid t_{s_i}(h-1)\overline{t_{s_i}}(h-1)p(h) \quad \text{and} \quad \overline{t_{s_i}}(h) \mid t_{s_i}(h-1)\overline{t_{s_i}}(h-1)p(h). \tag{3.16}$$

Moreover, as $\mathsf{gcd}\left(t_{s_i}(h), \overline{t_{s_i}}(h-1)\right) = 1 = \mathsf{gcd}\left(t_{s_i}(h-1), \overline{t_{s_i}}(h)\right)$, (3.16) above is equivalent to

$$t_{s_i}(h) \mid t_{s_i}(h-1)p(h) \quad \text{and} \quad \overline{t_{s_i}}(h) \mid \overline{t_{s_i}}(h-1)p(h).$$

Replacing $p(h)$ with $q(h)$ we deduce that $t \in \mathcal{L}_C(X) \iff t_{s_1}, \ldots, t_{s_\ell} \in \mathcal{L}_C(X)$, and $t\Bbbk[h] = \bigcap_{i=1}^{\ell} t_{s_i}\Bbbk[h]$. Thus, we may assume that the roots of $t(h)$ form a chain

$$\alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_k,$$

with $k \geqslant 1$. Then, from $t(h) \mid t(h-1)p(h)$ we deduce that α_1 is a root of p, i.e. $\alpha_1 - 1 \in \mathsf{R}_{p(h+1)} = Y$. Similarly, from $t(h) \mid t(h+1)q(h)$ we deduce that $\alpha_k \in \mathsf{R}_q = X$. We call such a multiset a (p,q)-*chain*. Note that $\alpha_k - (\alpha_1 - 1) = \alpha_k - \alpha_1 + 1 \in \mathbb{Z}_{\geqslant 1}$.

Lemma 3.17. *If $\alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_k$ is a (p,q)-chain, then $s(h) = \prod_{j=0}^{m}(h-(\alpha_1+j)) \in \mathcal{L}_C(X)$, where $m = \alpha_k - \alpha_1$.*

Proof. We have

$$(h-\alpha_1)s(h-1) = \prod_{j=0}^{m+1}(h-(\alpha_1+j)) = s(h)(h-(\alpha_1+m+1)),$$

$$(h-(\alpha_1+m))s(h+1) = \prod_{j=-1}^{m}(h-(\alpha_1+j)) = s(h)(h-(\alpha_1-1)).$$

Since $h-\alpha_1 \mid p(h)$ and $h-\alpha_k = h-(\alpha_1+m) \mid q(h)$, it follows that

$$s(h) \mid s(h-1)p(h) \quad \text{and} \quad s(h) \mid s(h+1)q(h). \quad \square$$

Corollary 3.18. *Let $C \in \Bbbk$ and X be a submultiset of R_{u+C}, with $\mathsf{R}_{u+C} = X \coprod Y$. Consider the $\mathcal{S}_u$-module $A_C(X) \in \mathfrak{U}_1$, as given in Definition 3.9. Then $A_C(X)$ is simple if and only if $(X-Y) \cap \mathbb{Z}_{\geqslant 1} = \emptyset$, i.e. if and only if there are no $\alpha \in Y$ and $\beta \in X$ such that $\beta - \alpha \in \mathbb{Z}_{\geqslant 1}$.*

Proof. Assume that $A_C(X)$ is simple and assume, by contradiction, that there exist $\alpha \in Y$ and $\beta \in X$ such that $\beta-\alpha = m+1$, for some $m \in \mathbb{N}$. Let $s(h) = \prod_{j=0}^{m}(h-(\alpha_1+j))$, with $\alpha_1 = \alpha + 1$. Then $\deg(s) = m+1 \geqslant 1$ and $\alpha_1 \preccurlyeq \alpha_1 + 1 \preccurlyeq \cdots \preccurlyeq \alpha_1 + m = \beta$ is a (p,q)-chain, so $s(h) \in \mathcal{L}_C(X)$ by Lemma 3.17. Hence, $s(h)\Bbbk[h]$ is a proper nontrivial submodule of $A_C(X)$, a contradiction.

Conversely, suppose that $(X-Y) \cap \mathbb{Z}_{\geqslant 1} = \emptyset$ and let $S \subseteq A_C(X)$ be a submodule. Then $S = s(h)\Bbbk[h]$ for some $s(h) \in \mathcal{L}_C(X)$. If $\deg(s) \geqslant 1$, then the multiset R_s is finite and nonempty. Thus, by the preceding considerations, there is a divisor $t(h)$ of $s(h)$ with $\deg(t) \geqslant 1$ such that R_t is a (p,q)-chain, say $\alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_k$, with $\alpha_1 - 1 \in Y$ and $\alpha_k \in X$. Thence, $\alpha_k - (\alpha_1 - 1) = \alpha_k - \alpha_1 + 1 \in \mathbb{Z}_{\geqslant 1} \cap (X-Y) = \emptyset$, a contradiction. Thus, either $s(h) = 1$ or $s(h) = 0$, proving that $A_C(X)$ is simple. $\square$

Let us return to the classification of the minimal elements $t \in \mathcal{L}_C(X) \setminus \{1\}$. We already know that $t = 0$ is minimal if and only if $(X - Y) \cap \mathbb{Z}_{\geqslant 1} = \emptyset$, so let's assume that $\mathsf{deg}\,(t) \geqslant 1$. From our previous considerations, we know that R_t is a (p, q)-chain, say $\alpha_1 \preccurlyeq \cdots \preccurlyeq \alpha_k$, with $k \geqslant 1$.

Suppose there is $1 \leqslant i < k$ such that $\alpha_{i+1} - \alpha_i \geqslant 2$. Then, writing $t(h) = t_1(h)t_2(h)$ with $t_1(h) = \prod_{j \leqslant i}(h - \alpha_j)$ and $t_2(h) = \prod_{j > i}(h - \alpha_j)$, the argument we have used before in (3.16) also shows that $t \in \mathcal{L}_C(X) \iff t_1, t_2 \in \mathcal{L}_C(X)$, and $t\Bbbk[h] = t_1\Bbbk[h] \cap t_2\Bbbk[h]$. Thus, we may further assume that $\alpha_{i+1} - \alpha_i \in \{0, 1\}$, for all $1 \leqslant i < k$. We call such a chain a *gapless chain*. What's more, taking $m = \alpha_k - \alpha_1$ and $s(h) = \prod_{j=0}^{m}(h - (\alpha_1 + j))$, we see that $s(h)$ divides $t(h)$ (because the chain is gapless) and $s(h) \in \mathcal{L}_C(X)$, by Lemma 3.17. As $\mathsf{deg}\,(s) = m + 1 \geqslant 1$, it follows from the minimality of t that $t = s$; whence, t is separable. In other words, the roots of t are all distinct and form a gapless (p, q)-chain $\alpha_1 \precneqq \cdots \precneqq \alpha_k$.

Proposition 3.19. *Let $C \in \Bbbk$, X, Y and $\mathcal{L}_C(X)$ be as above. Then t is minimal in $\mathcal{L}_C(X) \setminus \{1\}$ if and only if either one of the following conditions hold:*

- *(a) $t = 0$ and $(X - Y) \cap \mathbb{Z}_{\geqslant 1} = \emptyset$ (i.e., $A_C(X)$ is simple);*
- *(b) R_t is a finite gapless (p, q)-chain with no repeated elements, say $\alpha_1 \precneqq \cdots \precneqq \alpha_k$, with $\mathsf{deg}\,(t) = k$, such that:*
  - *(i) there is no $i < k$ with $\alpha_i \in X$;*
  - *(ii) there is no $i > 1$ with $\alpha_i - 1 \in Y$.*

Proof. The direct implication is clear from the preceding discussion. Conversely, condition (a) clearly implies the minimality of $t = 0$. So assume that the roots of t form the chain $\alpha_1 \precneqq \cdots \precneqq \alpha_k$, satisfying the conditions in (b). In particular, $t \neq 0$, as R_t is finite.

Let $s \in \mathcal{L}_C(X) \setminus \{1\}$ be a divisor of t. As $t \neq 0$, also $s \neq 0$ and thus $\mathsf{deg}\,(s) \geqslant 1$. Then, $\mathsf{R}_s \subseteq \mathsf{R}_t$ and the roots of s are of the form $\alpha_{i_1} \precneqq \cdots \precneqq \alpha_{i_\ell}$, with $1 \leqslant i_1 < \cdots < i_\ell \leqslant k$. The fact that $s \in \mathcal{L}_C(X)$ implies that R_s is a (p, q)-chain and (b) forces $i_1 = 1$ and $i_\ell = k$. If $s \neq t$, then there is some $j < \ell$ such that $\alpha_{i_{j+1}} - \alpha_{i_j} \geqslant 2$. Hence, by the argument preceding Proposition 3.19, $(h - \alpha_{i_1}) \cdots (h - \alpha_{i_j}) \in \mathcal{L}_C(X)$. In particular, $\alpha_{i_j} \in X$, forcing $i_j = k = i_\ell$, which contradicts $j < \ell$. The contradiction implies that $s = t$, proving the minimality of t. $\square$

Let $C \in \Bbbk$, X, $p, q \in \Bbbk[h]$ and $\mathcal{L}_C(X)$ be as above. Write $X_0 = X$, $q_0 = q$ and $p_0 = p$. The previous proposition provides a method to construct any decreasing chain of submodules

$$A_C(X_0) \supset t_1\Bbbk[h] \simeq A_C(X_1) \supset t_2t_1\Bbbk[h] \simeq A_C(X_2) \supset \cdots, \tag{3.20}$$

where t_i is a minimal element of $\mathcal{L}_C(X_{i-1}) \setminus \{1\}$. As long as $t_i \neq 0$, we can proceed with $q_i = \frac{t_i(h+1)q_{i-1}(h)}{t_i(h)}$, $p_i = \frac{t_i(h-1)p_{i-1}(h)}{t_i(h)}$ and $X_i = \mathsf{R}_{q_i}$, for $i \geqslant 1$. The minimality of t_i

implies that $A_C(X_{i-1})/A_C(X_i)$ is simple, for all $i \geqslant 1$. Then, to prove that all objects in $\mathfrak{U}_1$ have finite length, it is enough to show that, after a finite number of steps, the minimal element obtained is $t_\ell = 0$.

3.4. Composition series for $A_C(X)$

Recall the order $\preccurlyeq$ defined on $\Bbbk$. Given a multiset $Z \subseteq \Bbbk$ and $\beta \in \Bbbk$, denote by $Z_{\precnsim \beta}$ the submultiset $\{\alpha \in Z \mid \alpha \precnsim \beta\}$. Let $X \subseteq \mathsf{R}_{u+C}$ be a submultiset and take $\beta \in X$. If $(\mathsf{R}_{u+C} \setminus X)_{\precnsim \beta} \neq \emptyset$, we denote by $X \star \beta$ the submultiset of R_{u+C} defined by

$$X \star \beta = \{\hat{\beta}\} \cup X \setminus \{\beta\},$$

where $\hat{\beta} \in (\mathsf{R}_{u+C} \setminus X)_{\precnsim \beta}$ is uniquely defined by imposing the minimum distance from β, i.e., $\beta - \hat{\beta} = \min\{\beta - \alpha \mid \alpha \in (\mathsf{R}_{u+C} \setminus X)_{\precnsim \beta}\}$.

Let $t \in \mathcal{L}_C(X) \setminus \{1\}$ be minimal and suppose that $t \neq 0$. Set $q = \mathsf{poly}_X$. Then R_t is a finite gapless (p, q)-chain with no repeated elements, say $\alpha_1 \precnsim \cdots \precnsim \alpha_k$ satisfying Proposition 3.19(b). Set $\overline{q} = \frac{t(h+1)q(h)}{t(h)}$. It follows that $\mathsf{R}_{\overline{q}} = (X \setminus \{\alpha_k\}) \cup \{\alpha_1 - 1\}$. Furthermore, by Proposition 3.19(b)(ii),

$$k = \alpha_k - (\alpha_1 - 1) = \min\{\alpha_k - \beta \mid \beta \in (\mathsf{R}_{u+C} \setminus X)_{\precnsim \alpha_k}\}.$$

Thus, $\mathsf{R}_{\overline{q}} = X \star \alpha_k$. Moreover, with this notation, the chain of submodules (3.20) can be written as

$$A_C(X) \supset t_1 \Bbbk[h] \simeq A_C(X \star \beta_1) \supset t_2 t_1 \Bbbk[h] \simeq A_C(X \star \beta_1 \star \beta_2) \supset \cdots, \tag{3.21}$$

where β_i is the maximal element of the gapless (p_{i-1}, q_{i-1})-chain corresponding to t_i, a minimal element of $\mathcal{L}_C(X \star \beta_1 \star \cdots \star \beta_{i-1}) \setminus \{1\}$ which we are assuming to be nonzero.

Now, for any submultiset $Z \subseteq \mathsf{R}_{u+C}$, define

$$\ell(Z) = \sum_{\beta \in Z} |(\mathsf{R}_{u+C} \setminus Z)_{\precnsim \beta}| \geqslant 0,$$

where $|\cdot|$ denotes the number of elements of a multiset. Notice that $\ell(Z \star \beta) \leqslant \ell(Z) - 1$, whenever $Z \star \beta$ is defined. Finally, recall also that, by Corollary 3.18, $A_C(Z)$ is simple if and only if $\ell(Z) = 0$. Therefore, the chain (3.21) has maximal length bounded above by $\ell(X)$ and $\ell(X \star \beta_1 \star \cdots \star \beta_m) = 0$, for some $m \leqslant \ell(X)$. The last nonzero term of the chain (3.21) will be the simple submodule $A_C(X \star \beta_1 \star \cdots \star \beta_m)$.

From the discussion above we obtain our desired result.

Proposition 3.22. *Let $A_C(X) \in \mathfrak{U}_1$. Then $A_C(X)$ has finite length, bounded above by $\ell(X) + 1$.*

The method described above using Proposition 3.19 and the iterative construction in (3.21) gives all possible composition series for $A_C(X)$. Nevertheless, we will see that, regardless of the choices made, the final multiset $X \star \beta_1 \star \cdots \star \beta_m$ obtained, with $\ell(X \star \beta_1 \star \cdots \star \beta_m) = 0$, will always be the same. We give an algebraic proof of this result, using the notion of socle of a module M, denoted by $\mathrm{soc}(M)$, this being the sum of its simple submodules, or equivalently, its unique maximal semisimple submodule.

Corollary 3.23. *Let $A_C(X) \in \mathfrak{U}_1$. Then $\mathrm{soc}\,(A_C(X)) = A_C(X^\star)$, where $X^\star = X \star \beta_1 \star \cdots \star \beta_m$ is obtained iteratively by the method described above, terminating with $\ell(X^\star) = 0$. In particular, $X^\star$ depends only on X.*

Proof. We have seen that there exist $0 \leqslant m \leqslant \ell(X)$ and $\beta_1, \ldots, \beta_m \in \mathsf{R}_{u+C}$ such that $A_C(X \star \beta_1 \star \cdots \star \beta_m)$ is a submodule of $A_C(X)$ with $\ell(X \star \beta_1 \star \cdots \star \beta_m) = 0$, hence simple and thence contained in $\mathrm{soc}\,(A_C(X))$.

The $\Bbbk[h]$-module $A_C(X)$ is just the regular module $\Bbbk[h]$, which contains no nontrivial direct sums of submodules. It follows that the same must hold for $A_C(X)$ as an $\mathcal{S}_u$-module. Thus, its socle, being nonzero and semisimple, must be simple and equal to $A_C(X \star \beta_1 \star \cdots \star \beta_m)$. So $A_C(X \star \beta_1 \star \cdots \star \beta_m)$ is the unique simple submodule of $A_C(X)$, and the last nonzero term in all composition series for $A_C(X)$. Now, by Lemma 3.11, the uniqueness of the multiset $X \star \beta_1 \star \cdots \star \beta_m$ follows. □

Remark 3.24. Let $\mathsf{R}_{u+C} = R_1 \coprod \cdots \coprod R_k$ be the decomposition of R_{u+C} in to its maximal chains with respect to $\preccurlyeq$. Then $X^\star$ is the unique submultiset of R_{u+C} with $\ell(X^\star) = 0$ and $|R_i \cap X^\star| = |R_i \cap X|$, for all $i \in [k]$.

Next, we will describe the remaining composition factors of $A_C(X)$ and their multiplicities, obtaining as a corollary an exact formula for the length of $A_C(X)$. Since $A_C(X)/\mathrm{soc}\,(A_C(X))$ is finite dimensional, $A_C(X^\star)$ occurs with multiplicity one and all the other composition factors, if any, will be finite dimensional.

We summarize the classification of simple $\mathcal{S}_u$-modules of finite dimension given by Smith in [15] (see also [10]). Let $\lambda \in \Bbbk$, and $\Bbbk_\lambda = \Bbbk v_\lambda$ be the one-dimensional $\Bbbk[h]$-module where h acts by λ. Let $\mathfrak{b} \subseteq \mathcal{S}_u$ be the unital subalgebra generated by h and y. Then $\Bbbk_\lambda$ becomes a $\mathfrak{b}$-module by defining $y v_\lambda = 0$. The Verma module of highest weight λ for $\mathcal{S}_u$ is defined by

$$V(\lambda) = \mathcal{S}_u \otimes_{\mathfrak{b}} \Bbbk_\lambda \simeq \Bbbk[x].$$

Theorem 3.25. *[15] Let $\lambda \in \Bbbk$, then $V(\lambda)$ has a unique maximal submodule and hence a unique simple subquotient, denoted by $L(\lambda)$. Furthermore, any simple $\mathcal{S}_u$-module of dimension j is isomorphic to*

$$L(\lambda) = V(\lambda)/x^j V(\lambda),$$

for some $\lambda \in \Bbbk$, *where* j *is the minimal positive integer such that* $u(\lambda) - u(\lambda - j) = 0$.

Lemma 3.26. *Let* $A_C(X) \in \mathfrak{U}_1$ *and assume that* $A_C(X)$ *is not simple. Let* $0 \neq t \in \mathcal{L}_C(X) \setminus \{1\}$ *be minimal. Then* $A_C(X)/t\Bbbk[h] \simeq L(\beta)$, *where* β *is the maximal element of the gapless* (p, q)*-chain corresponding to* t.

Proof. Let $t(h) = (h-(\hat{\beta}+1))(h-(\hat{\beta}+2)) \cdots (h-\beta)$, with $\hat{\beta} \preccurlyeq \beta$. Then $N = A_C(X)/t\Bbbk[h]$ has dimension equal to $\beta - \hat{\beta}$. Define $w \neq 0$ to be the class of $t(h)/(h-\beta)$ in N. A straightforward computation using the fact that $h - (\hat{\beta} + 1)$ divides p shows that

$$yw = 0, \quad hx^k w = (\beta - k)x^k w, \quad \text{and} \quad yx^{k+1}w = (u(\beta - (k+1)) - u(\beta))x^k w,$$

for all $k \in \mathbb{N}$. Let k_0 be the minimal positive integer such that $x^{k_0}w = 0$. Then we have

$$0 = yx^{k_0}w = (u(\beta - k_0) - u(\beta))x^{k_0-1}w.$$

As N is simple, it follows that $N = \mathsf{span}_{\Bbbk}\left\{x^k w \mid k = 0, \ldots, k_0 - 1\right\}$, a simple $\mathcal{S}_u$-module of highest weight β. Thus, $N \simeq L(\beta)$. Since $x^{k_0-1}w \neq 0$, this implies that $u(\beta - k_0) - u(\beta) = 0$, and we have $\beta - \hat{\beta} = \dim_{\Bbbk} N = k_0$. □

Remark 3.27. As a converse to the previous result, any simple finite-dimensional $\mathcal{S}_u$-module $L(\lambda)$ can be seen as a quotient of $A_C(X) \in \mathfrak{U}_1$, for some $C \in \Bbbk$ and some $X \subseteq \mathsf{R}_{u+C}$. Indeed, suppose that $\dim_{\Bbbk} L(\lambda) = j \geqslant 1$ and set $C = -u(\lambda)$. Then, by Theorem 3.25, $\lambda, \lambda - j \in \mathsf{R}_{u+C}$ and $A_C(\{\lambda\})$ is well defined. Moreover, $\lambda - j + 1 \preccurlyeq \cdots \preccurlyeq \lambda$ forms a (p, q)-chain for $X = \{\lambda\}$, so $t_\lambda(h) = \prod_{i=0}^{j-1}(h - (\lambda - i)) \in \mathcal{L}_C(\{\lambda\})$, by Lemma 3.17. Finally, the minimality of j given in Theorem 3.25 ensures, by Proposition 3.19, that $t_\lambda\Bbbk[h]$ is a maximal submodule of $A_C(\{\lambda\})$, isomorphic to $A_C(\{\lambda - j\})$, and $A_C(\{\lambda\})/t_\lambda\Bbbk[h] \simeq L(\lambda)$, by Lemma 3.26.

Now, for every submultiset Z of R_{u+C}, define the map $\varphi_Z \colon \underline{\mathsf{R}_{u+C}} \to \mathbb{N}$ by

$$\varphi_Z(\beta) = \min\left\{|(\mathsf{R}_{u+C} \setminus Z)_{\preccurlyeq\beta}|, |Z_{\succcurlyeq\beta}|\right\},$$

where $Z_{\succcurlyeq\beta} = \{\alpha \in Z \mid \beta \preccurlyeq \alpha\}$ and $\underline{\mathsf{R}_{u+C}}$ is the underlying set obtained from R_{u+C}.

We are ready to describe the composition factors of $A_C(X)$ and their multiplicities. It turns out that this is best phrased using the Grothendieck group $K_0(\mathcal{S}_u)$, which is the free abelian group on the isomorphism classes of finitely generated $\mathcal{S}_u$-modules, modulo the short exact sequences.

Theorem 3.28. *Consider the Grothendieck group* $K_0(\mathcal{S}_u) = \{[M] \mid M \in \mathcal{S}_u\text{−}\mathsf{mod}\}$. *Let* $A_C(X) \in \mathfrak{U}_1$. *Then*

$$[A_C(X)] = [A_C(X^\star)] + \sum_{\beta \in \underline{\mathsf{R}_{u+C}}} \varphi_X(\beta)[L(\beta)] \in K_0(\mathcal{S}_u),$$

where $A_C(X^\star) = \operatorname{soc}(A_C(X))$.

Proof. We prove it by induction on $\ell(X)$.

If $\ell(X) = 0$, then clearly φ_X is the constant null map, $X = X^\star$ and the claim is proved. Suppose that $\ell(X) > 0$ and assume the claim to be true for any submultiset $Z \subseteq \mathsf{R}_{u+C}$ such that $\ell(Z) < \ell(X)$. There exists a minimal $t \in \mathcal{L}_C(X) \setminus \{1\}$ and $t \neq 0$, since $\ell(X) > 0$. Let β be the maximal element of the gapless (p, q)-chain corresponding to t (see Proposition 3.19). Then, by Lemmas 3.15 and 3.26, we have an exact sequence

$$0 \to A_C(X \star \beta) \to A_C(X) \to L(\beta) \to 0.$$

Then $[A_C(X)] = [A_C(X \star \beta)] + [L(\beta)] \in K_0(\mathcal{S}_u)$. Since $\ell(X \star \beta) \leqslant \ell(X) - 1$, it follows by the induction hypothesis that

$$[A_C(X)] = [L(\beta)] + [A_C(X^\star)] + \sum_{\alpha \in \underline{\mathsf{R}_{u+C}}} \varphi_{X\star\beta}(\alpha)[L(\alpha)] \in K_0(\mathcal{S}_u),$$

where $A_C(X^\star) = \operatorname{soc}(A_C(X \star \beta)) = \operatorname{soc}(A_C(X))$. So it is sufficient to prove that

$$\varphi_X(\beta) = \varphi_{X\star\beta}(\beta) + 1 \quad \text{and} \quad \varphi_X(\alpha) = \varphi_{X\star\beta}(\alpha), \quad \text{for all } \alpha \neq \beta. \tag{3.29}$$

Computing φ_X and $\varphi_{X\star\beta}$, we obtain:

$$|(\mathsf{R}_{u+C} \setminus X \star \beta)_{\preccurlyeq\alpha}| = \begin{cases} |(\mathsf{R}_{u+C} \setminus X)_{\preccurlyeq\alpha}|, & \alpha \neq \beta; \\ |(\mathsf{R}_{u+C} \setminus X)_{\preccurlyeq\beta}| - 1, & \alpha = \beta; \end{cases}$$

$$|(X \star \beta)_{\succcurlyeq\alpha}| = \begin{cases} |X_{\succcurlyeq\alpha}|, & \alpha \neq \beta; \\ |X_{\succcurlyeq\beta}| - 1, & \alpha = \beta. \end{cases}$$

Thus, (3.29) is satisfied and the theorem is proved. □

Corollary 3.30. *The module* $A_C(X) \in \mathfrak{U}_1$ *has length* $1 + \sum_{\beta \in \underline{\mathsf{R}_{u+C}}} \varphi_X(\beta)$.

Data availability

No data was used for the research described in the article.

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