ON THE PRESERVATION OF ELEMENTARY EQUIVALENCE AND EMBEDDING BY BOUNDED FILTERED POWERS AND STRUCTURES OF STABLE CONTINUOUS FUNCTIONS (*)

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All notation as well as definitions of the concepts mentioned in the title will be as in [M]. In that paper can also be found bibliographical references sustaining the usefulness of filtered powers and of structures of stable continuous functions. The author acknowledges helpful conversations, on the methods used here, with A.M. Sette. We recall definitions briefly.

If X is a topological space and M a set, the symbol C(X,M) will denote the set of all continuous functions from X to M, where M is given the discrete topology. Whenever it appears this symbol will have this meaning and the structures mentioned in the title, in particular, will have as domaim subsets of sets of this form. The symbol X indicates a topological space in all that follows.

If L is a first order language with equality, M an L-structure and I a set, let $\Sigma=\langle\{F_i\}_{i\in I}, \{M_i\}_{i\in I}\rangle$ be a pair of families of subsets of X and M respectively, such that each F_i is closed in X and each M_i is a substructure of M. The symbol $C(X,M,\Sigma)$ denotes the substructure of

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$C(X,M)$, consisting of all continuous $f: X \to M$ such that for all $i \in I$ $f(F_i) \subseteq M_i$. $C(X,M;\Sigma)$ is called a *filtered power* of M by X.

Let G be a group operating by homeomorphisms on X and by automorphisms on an L-structure M. That is, we have group homomorphisms $g \in G \to g_* \in Hm\ (X)$ and $g \in G \to g^* \in Aut_L(M)$. The symbol $SC(X,M;G)$ denotes the substructure of $C(X,M)$ consisting of all continuous $X \xrightarrow{f} M$ such that for all $g \in G$ and all $x \in X$ $f(g_* x) = g^* f(x)$. $SC(X,M;G)$ is called the structure of *stable continuous functions* from X to M under the action of G.

In the rest of this note it will be assumed that X is *boolean* (compact, with a basis of clopens) and that G and I are *finite*.

If X is a boolean space, $B(X)$ denotes the boolean algebra of clopens in X . To each filtered power $C(X,M;\Sigma)$ we can associate an expansion of M, $\langle M; M_1,\ldots M_n\rangle \equiv M^{\#}$ and by Stone duality, the augmented boolean algebra
$\langle B(X); I_1,\ldots,I_n\rangle \equiv B_{\#}(X)$
where I_k is the ideal in $B(X)$ determined by the closed set $F_k \subseteq X$ (see [M] or [MR] for more details). We denote by L the expansion of L appropriate to structures of type $\langle M; M_1,\ldots, M_n\rangle$. Thus $L^{\#}$ is obtained from L by adding n new unary relation symbols $P_1,\ldots,P_n$. Similarly, to each structure of stable continuous functions $SC(X,M;G)$ we can associate an expansion of M, $M^* = \langle M, (g^*)_{g \in G}\rangle$ and an expansion $B_*(X) = \langle B(X) : (g_*)_{g \in G}\rangle$ of $B(X)$. Here we make

no distinction between the homeomorphism g_* of X and the automorphism induced by g_* in B(X). L^* denotes the expansion of L , appropriate for structures of type M^*.

In all that follows X will be an arbitrary but fixed boolean space, $F_1,\ldots,F_n$ a fixed (but again arbitrary) family of closed subsets of X and G a finite group. If $M_1^{\#}$ and $M_2^{\#}$ are two expansions of L-structures M_1 and M_2 as described above, then $\Sigma_i^{\bullet} = \langle\{F_1,\ldots F_n\}, \{M_{i_1},\ldots M_{i_n}\}\rangle$, i = 1,2. We wish to present proofs of the following:

Theorem 1. If $M_1^{\#} \prec_{L^{\#}} M_2^{\#}$ ($M_1^* \equiv_{L^*} M_2^*$) then $C(X,M_1;\Sigma_1)$ is an elementary L-substructure of $C(X,M_2;\Sigma_2)$ (resp. $C(X,M_1;\Sigma_1) \equiv_L C(X,M_2;\Sigma_2)$.

Theorem 2. Same as above, with M_1^* and $SC(X,M_i;G)$ i = 1,2 in the place of $M_i^{\#}$ and $C(X,M_i,\Sigma_i)$ respectively.

For this purpose we will use the caracterization, due to Fraisse ([Fr]) of elementary equivalence through the existence of certain families of partial isomorphisms ("back and forth" tecniques) . All results we need are in [F], whose notation will be followed closely. In particular if M_1 and M_2 are structures, $P(M_1, M_2)$ will denote the set of partial isomorphisms from M_1 to M_2.

We have the following

Fundamental Lemma : With notational conventions as above , there is a natural injection $C(X,P(M_1,M_2)) \xrightarrow{\Delta} P(C(X,M_1)$, $C(X,M_2)) = T$ such that

a. If $S^{\#} = P(M_1^{\#}, M_2^{\#})$, define $\Delta^{\#}: C(X,S^{\#}) \to T$ by $\Delta^{\#}(f) = \Delta(f)|_{\mathrm{dom}\, \Delta(f) \cap C(X,M_1;\Sigma_1)}$. Then Im $\subseteq P(C(X,M_1;\Sigma_1), C(X,M_2;\Sigma_2)) \equiv T^{\#}$.

b. If $S^{*} = P(M_1^{*}, M_2^{*})$, let G operate on it trivially and define $\Delta^{*}: SC(X,S^{*};G) \to T$ by $\Delta^{*}(f) = \Delta(f)|_{\mathrm{dom}\, \Delta(f) \cap SC(X,M_1;G)}$. Then Im $\Delta_{*} \subseteq P(SC(X,M_1;G), SC(X,M_2;G)) \equiv T^{*}$.

c. Define, for $f,g \in C(X,P(M_1,M_2))$, $f \subseteq g$ if for all $x \in X$, $f(x) \subseteq g(x)$ (i.e., dom $f(x) \subseteq$ dom $g(x)$ and $g(x)|_{\mathrm{dom}\, f(x)} = f(x)$). Then, if $f \subseteq g$, $\Delta(f) \subseteq \Delta(g)$.

d. If $(S_n)_{n \in \omega}$ has the back and forth property in $P(M_1,M_2)$ (as in definition 1.4, p. 252 of [F]), then $(\Delta(C(X,S_n)))_{n \in \omega}$ has the same property in T.

e. If $(S_n)_{n \in \omega}$ has the back and forth property in $P(M_1^{*}, M_2^{*})$, let $S_n^{*} = S_{n|G|}$. Then $(\Delta^{*}(SC(X,S_n^{*};G)))_{n \in \omega}$ has the same property in T^{*}.

Proof. Given $f: X \to P(M_1,M_2)$ put dom $\Delta(f) = \{\alpha \in C(X,M_1): \forall x \in X \;\; \alpha(x) \in \mathrm{dom}\, f(x)\}$. Then define for $\alpha \in$ dom $\Delta(f)$

$$\Delta(f)(\alpha)(x) = f(x)(\alpha(x)).$$

For notational simplicity if $X \xrightarrow{h} A$ is constant on a clopen $u \subseteq X$ we will denote by $h|_u$ both the restriction of h to u and its value on u.

Fact 1 : *$\Delta(f)(\alpha)$ is a continuous*: choose, given $\alpha \in$ dom $\Delta(f)$, a partition $u_1, \ldots, u_s$ of X such that both $f|_{u_j}$ and $\alpha|_{u_j}$ are constant. Then for $m \in M_2$, $[\Delta(f)(\alpha)]^{-1}(m) = \bigcup\{u_j : f|_{u_j}(\alpha|_{u_j}) = m\}$, which is clopen in X.

It is straightforward to verify that Δ is injective. Notice now that if $f(x) \in S$ for all $x \in X$ and $\alpha \in C(X,M_1;\Sigma_1)$, then $(f)(\alpha) \in C(X,M_2;\Sigma_2)$: if $x \in F_k$, then $\alpha(x) \in M_{1k}$ and so $\Delta(f)(\alpha)\ (x) = f(x)(\alpha(x)) \in M_{2k}$. Similarly if $f \in C(X,S^*,G)$ and $\alpha \in \mathrm{dom}\ (f) \cap SC(X,M_1;G)$ then $\Delta(f)(\alpha) \in SC(X,M_2;G)$: given $g \in G$ and $x \in X$

$$f(g_* x)(\alpha(g_* x)) = f(x)(\alpha(g_* x)) = g^* f(x)(\alpha(x)).$$

Thus a and b will be verified, as well as the assertion as to the range of Δ in the statement of the Lemma, as soon as we ascertain that $\Delta(f)$ is a partial isomorphism. If R is a n-ary relation symbol in L and $\alpha_1, \ldots \alpha_n$ are in dom $\Delta(f)$ then $C(X,M_1) \models R(\alpha_1,\ldots,\alpha_n) \Longleftrightarrow \forall x \in X\ M_1\ \ R(\alpha_1(x),\ldots,\alpha_n(x)) \Longleftrightarrow \forall x \in X\ M_2 \models R(f(x)(\alpha(x)),\ldots,f(x)(\alpha(x)))$. Function and constant symbols may be handled analogously. The verification of <u>c</u>. is straightforward For <u>d</u>., suppose we are given $f \in C(X,S_n)$ and $\alpha \in C(X,M_2,\Sigma_2)$. Let $u_1,\ldots,u_s$ be a partition of X such that $f|_{u_j}$ and $\alpha|_{u_j}$ are constant. For each $1 \le j \le s$, there is $g_j \in S_{n-1}$ such that $g_j \supseteq f|_{u_j}$ and $\alpha|_{u_j} \in \mathrm{Im}\ g_j$, $g_j(\beta_j) = \alpha|_{u_j}$. Define $g: X \to S_{n-1}$ such that $g|_{u_j} = g_j\ \ 1 \le j \le s$ and $\beta : X \to M_1$ by $\beta|_{u_j} = \beta_j$. Then, since $g \supseteq f$, $\Delta(g) \supseteq \Delta(f)$ and we have $\alpha \in \mathrm{Im}\ \Delta(g)$. The case in which $\alpha \in C(X,M_1;\Sigma_1)$ is similarly disposed of. Thus <u>d</u>. is verified. For <u>e</u>. we need

<u>Fact 2</u> : If $\alpha \in SC(X,M;G)$ we can find a partition $u_1,\ldots,u_s$ of X into invariant clopens (i.e. $g_* u_j = u_j\ \forall g\ \ G$) such that $\alpha(u_j) \le |G|$ = order of G.

<u>Proof</u> : On M we have an equivalence relation where $m_1 \sim m_2$ iff $m_2 = g^* m_1$ for some $g \in G$. Let $[m_1],\ldots,[m_s]$ be a

partition of Im α into disjoint equivalence classes, and put

$$u_j = \{x \in X : \alpha(x) \in [m_j]\}, \quad 1 \le j \le s.$$

It is easy to verify that $u_1, \ldots, u_s$ satisfy the conclusions of Fact 2. Now given $f \in SC(X, S_n^* ; G)$ and $\alpha \in SC(X, M_2; G)$ we can find a partition $v_1, \ldots, v_t$ of into <u>invariant</u> clopens in such a way that $f|_{V_k}$ is constant and $\alpha(V_k) \le |G|$. To obtain such a partition it is sufficient to consider the common refinement of the partition obtained in Fact 2 and that determined by requiring that f be constant on each of its members. Note that since the action of G on S^* is trivial, the set of points of X where f is constant is clopen invariant. We can find for each $1 \le k \le t$, $h_k \in S_{n-1}^*$ such that Im h_k $\supseteq \alpha(V_k)$. Define $h : X \to S_{(n-1)|G|}$ by $h|_{V_k} = h_k$. Since each V_k is invariant, $h \in SC(X, S_{n-1}^*; G)$ and $h \supseteq f$. Define $\alpha : X \to M_1$ by the following rule : If $x \in V_k$, $1 \le k \le t$, $\alpha(x) = h_k^{-1}(\alpha(x)$. Observe that once it is shown that β is in $SC(X, M_1; G)$ the proof is finished : it is obvious that $\alpha \in$ Im $\Delta^*(h)$, $\Delta^*(h)(\beta) = \alpha$. We have for $m \in M_1$
$\beta^{-1}(m) = \bigsqcup_{1 \le k \le t} \{x \in X : \alpha(x) = h_k(m)\}$, clearly clopen. Thus β is continuous. On the other hand given $x \in X$, there is $1 \le k \le t$, such that $x \in V_k$. Then $\beta(g_* x) = h_k^{-1}(\alpha(g_* x)) = h_k^{-1}(g^* \alpha(x)) = g^* h_k^{-1}(\alpha(x) = g^* \beta(x)$. In the above computations we have used the fact that V_k is invariant. This completes the proof of the Lemma. Δ

One can immediately conclude.

<u>Theorem 1 b</u> : If $M_1^{\#} \equiv_{L^*} M_2^{\#}$, then $C(X, M_1; \Sigma_1) \equiv_L C(X, M_2; \Sigma_2)$.

and

Theorem 2 b : If $M_1^* \underset{L^*}{\equiv} M_2^*$, then $SC(X,M_1;G) \underset{L}{\equiv} SC(X,M_2;G)$.

The proofs come from the fundamental Lemma plus Theorem 1.13, p.257 of [F] in the case that *L is of finite type.* In case L is not of finite type, observe that any sentence in L is a sentence in a finite reduct of L. Thus if it is true in $C(X,M_1;\Sigma_1)$ it will be true in $C(X,M_2;\Sigma_2)$ by the result above. A similar argument may, of course, be applied to obtain Theorem 2b in full generality. *We have, therefore, Theorems 1b and 2b for all languages L of any type.*

In the case that $M_1^{\#} <_{L^{\#}} M_2^{\#}$ (or $M_1^* <_{L^*} M_2^*$) we have $C(X,M_1;\Sigma_1) \equiv E_1 \subseteq C(X,M_2;\Sigma_2) \equiv E_2$, cannonically. To show that one is an elementary substructure of the other, we will prove.

Fact 3 : For any reduct of L of finite type, L', including the unary predicate symbols $P_1,\ldots,P_n$, for any $t \in \omega$ and for any $\{\alpha_1,\ldots,\alpha_t\} \subseteq E_1$, $\langle E_1,\alpha_1,\ldots,\alpha_t\rangle \underset{L'(\underline{\alpha}_1,\ldots\underline{\alpha}_t)}{\equiv} \langle E_2, \alpha_1,\ldots,\alpha_t\rangle$.

Proof: Let $\{m_1,\ldots,m_s\} = \bigsqcup_{j=1}^{t} \mathrm{Im}\,\alpha_j$. Since $M_1^{\#} <_{L}^{\#} M_2^{\#}$, we have $\langle M_1^{\#},m_1,\ldots,m_s\rangle \underset{L'(\underline{m}_1,\ldots,\underline{m}_s)}{\equiv} \langle M_2^{\#}, m_1,\ldots,m_s\rangle$. Let $\langle M_i^{\#}, \overrightarrow{m}\rangle$ denote $\langle M_i^{\#}, m_1,\ldots,m_s\rangle$, i=1,2. Observe that in Fraissé's result we can assume that the constants belong to the domain of all partial isomorphisms involved in the back and forth arguments. Observe also that if $f \in C(X,P(\langle M_1^{\#}, \overrightarrow{m}\rangle, M_2^{\#}, \overrightarrow{m}\rangle))$ then $\Delta^{\#}(f)(\alpha_j) = \alpha_j$ $1 \le j \le t$. By the fundamental Lemma, we may conclude (again using Theorem 1.13 in [F]) that $\langle E_1, \overrightarrow{\alpha}\rangle \underset{L'(\overrightarrow{\alpha})}{\equiv} \langle E_2, \overrightarrow{\alpha}\rangle$ which proves Fact 3. Δ

From Fact 3 comes imediately, that if $M_1'' <_{L''} M_2''$, then $E_1 <_L E_2$. And Theorem 1 is proven in its entirety. Arguments entirely analous to those described above will prove Theorem 2.

Theorems 1 and 2 generalise the corresponding statements for bounded boolean powers in [B] (Theorem 6.5(i) and (ii)), and even in the case of boolean powers provide a new proof of the aforementioned result. In closing we would like to point out some related questions :

1) If $\langle B(X), I_1,\ldots,I_n\rangle \equiv \langle B(Y), I_1',\ldots I_n'\rangle$ is it true that $C(X,M;\Sigma) \equiv C(Y,M;\Sigma')$?

2) Same as above for structures of stable continuous functions, that is If $B_*(X) \equiv B_*(Y)$ is it true that $SC(X,M;G) \equiv SC(Y,M;G)$?

3) Same as 1) and 2) above, for elementary embedding.

4) If $\{M_i\}_{i\in I}$ is an inductive system of L-structures and $M = \varinjlim M_i$ is its inductive limit, under what cirscunstances $M_i <_L M_i'$, implies $\varinjlim M_i = M <_L \varinjlim M_i'$?

When M is finite, questions 1), 2) and 3) have affirmative answers (see[M]).

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