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The 2-Decomposition Conjecture for a new class of graphs☆

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Abstract

The 2-Decomposition Conjecture, equivalent to the 3-Decomposition Conjecture stated in 2011 by Hoffmann-Ostenhof, claims that every connected graph G with vertices of degree 2 and 3, and satisfying that $G - E(C)$ is disconnected for every cycle C, admits a decomposition into a spanning tree and a matching. In this work we show that the 2-Decomposition Conjecture holds for graphs whose vertices of degree 3 induce a collection of cacti in which each vertex belongs to a cycle.

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1. Introduction

The terminology used in this is work is standard and we refer the reader to [4] for missing definitions. All graphs considered in this work are finite and have no loops (but may contain parallel edges). As usual, we say that a graph is cubic (resp. subcubic) if all its vertices have degree 3 (resp. at most three).

A *Homeomorphically Irreducible Spanning Tree (HIST)* is a spanning tree without vertices of degree 2. The problem of deciding whether a graph contains a HIST [3] is NP-complete, even for subcubic graphs [7]. This topic has

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been studied by many researchers [3, 5, 6, 8, 18], and it is related to the topic addressed in this paper, as we shall explain. Let G be a cubic graph, T be a spanning tree of G, and $G' = G - E(T)$. Each component of G' is either a path or a cycle, so every cubic graph can be decomposed into a spanning tree, a collection of cycles, and a collection of paths. Moreover, T is a HIST if and only if G' is a collection of cycles. Thus the problem of deciding whether a cubic graph G contains a HIST is equivalent to the problem of deciding whether G can be decomposed into a spanning tree and a collection of cycles. Not all cubic graphs can be decomposed into a spanning tree and a collection of cycles; necessary conditions for the existence of such decomposition have been shown by Hoffmann-Ostenhof, Noguchi, and Ozeki [12]. However, a related but more relaxed decomposition of cubic graphs has been conjectured by Hoffmann-Ostenhof [9].

Conjecture 1.1 (Hoffmann-Ostenhof, 2011 [9]). *Every connected cubic graph can be decomposed into a spanning tree, a collection of cycles, and a matching (possibly empty).*

Conjecture 1.1 is known as the 3-*Decomposition Conjecture*. Clearly, if a cubic graph G contains a HIST, then Conjecture 1.1 holds for G. This conjecture has attracted the attention of many researchers, and although the general problem remains open, its claim has been proven for some subclasses of graphs. Liu and Li [15] verified it for cubic traceable graphs. Ozeki and Ye [16] verified it for 3-connected planar cubic graphs and 3-connected cubic graphs on the projective plane. Later, Hoffmann-Ostenhof, Kaiser and Ozeki [11] extended the result of Ozeki and Ye by verifying Conjecture 1.1 for all planar cubic graphs. Recently Xie, Zhou and Zhou [17] verified Conjecture 1.1 for cubic graphs containing a 2-factor of three cycles and, independently, Hong, Liu and Yu [13] and Aboomahigir, Ahanjideh and Akbari [1] verified the conjecture for claw-free cubic graphs.

In addition, some weaker forms of Conjecture 1.1 have been verified. Akbari, Jensen, and Siggers [2] proved that every cubic graph can be decomposed into a spanning forest, a collection of cycles, and a matching. Li and Cui [14] showed that every cubic graph can be decomposed into a spanning tree, one cycle, and a collection of paths with length at most 2.

A cycle C in a connected graph G is *separating* if $G - E(C)$ is disconnected. Let $\mathcal{G}$ be the class of connected graphs in which every cycle is separating and each vertex has degree 2 or 3. It is known that Conjecture 1.1 is equivalent to the following conjecture, known as the 2-*Decomposition Conjecture* (see [11, Proposition 14]).

Conjecture 1.2 (Hoffmann-Ostenhof, 2016 [10]). *Every graph in $\mathcal{G}$ can be decomposed into a spanning tree and a matching.*

Conjecture 1.2 is fairly new. At the best of our knowledge, the only work addressing directly Conjecture 1.2 is the one conducted by Hoffmann-Ostenhof, Kaiser and Ozeki [11], where the authors verify Conjecture 1.2 for the planar case.

Throughout this paper, given a graph G, we denote by $V_k(G)$ the set of vertices of G with degree k. We recall that a graph is a *cactus* if it is connected and every edge is contained in at most one cycle. We say that a cactus G is *thick* if every vertex in G belongs to a cycle. In this work, we verify Conjecture 1.2 for the graphs $G \in \mathcal{G}$ whose subgraph induced by $V_3(G)$ is a collection of thick cacti.

Theorem 1.3. *Every graph $G \in \mathcal{G}$ for which $G - V_2(G)$ is a collection of thick cacti can be decomposed into a spanning tree and a matching.*

We observe that Theorem 1.3 is not implied by any of the results regarding Conjecture 1.1. For every subclass $\mathcal{S}$ of cubic graphs where Conjecture 1.1 is known to hold, namely cubic traceable graphs, planar cubic graphs, 3-connected cubic graphs on the projective plane, cubic graphs containing a 2-factor of three cycles and, claw-free cubic graphs, one may obtain a cubic graph $G \notin \mathcal{S}$ from a graph in $\mathcal{G}$ satisfying the hypothesis of Theorem 1.3 such that the 3-Decomposition Conjecture holds for G.

2. Proof of Theorem 1.3

Let $\mathcal{H} \subset \mathcal{G}$ be the set of all simple graphs H in which $V_2(H)$ is a stable set and every vertex in $V_3(H)$ has precisely one neighbor in $V_2(H)$. Thus, $H - V_2(H)$ is 2-regular, as we state in the next remark.

Remark 2.1. *If H is a graph in $\mathcal{H}$, then each component of $H - V_2(H)$ is a cycle.*

Let H be a graph in $\mathcal{H}$. We refer to the cycles in $H - V_2(H)$ as the *basic cycles* of H. Note that the vertices of basic cycles of H define a partition of $V_3(H)$. Let $u \in V_2(H)$, and note that the neighbors x and y of u belong to basic cycles, say C and C', of H. If $C = C'$, then we say that the path $P = xuy$ is a *2-chord* of C. In this case, x and y are called the *ends* of P and u the *inner* vertex. If $C \neq C'$, then we say that u is a *connector*. In this case, we say that *u joins C* and C'. Moreover, we say that two connectors are *parallel* if they join the same pair of basic cycles; and a collection of connectors $\mathcal{C}$ of H is called *simple* if it contains no pair of parallel connectors.

We define the *basic cycles graph* (BC-*graph*, for short) of $H \in \mathcal{H}$, which we denote by $\tilde{H}$, as the graph whose vertices are the basic cycles of G and in which two vertices C, C' are adjacent whenever the graph G has a connector joining C and C'. Note that $\tilde{H}$ is connected because H is connected. Also, note that this definition ignores parallel connectors in the sense that a pair of parallel connectors yields only one edge in $\tilde{H}$. Given a collection $\mathcal{C}$ of connectors of H, we define the *underlying* BC-*graph* of $\mathcal{C}$, denoted by $\tilde{H}_{\mathcal{C}}$, as the spanning subgraph of $\tilde{H}$ in which two vertices C and C' are adjacent whenever there is a connector in $\mathcal{C}$ joining C and C'.

We refer to a decomposition of a graph G into a spanning forest F and a matching M as a 2-*decomposition* of G and we denote it by the ordered pair (F, M). Note that if a graph $G \in \mathcal{G}$ admits a 2-decomposition (F, M), then Conjecture 1.2 holds for G since we can complete F to a tree using edges of M. Given a 2-decomposition $\mathcal{D} = (F, M)$ of a graph $G \in \mathcal{G}$, we say that a vertex $u \in V(G)$ is a *full vertex* in $\mathcal{D}$ if every edge of G incident to the vertex u belongs to F.

The main result of this paper (Theorem 1.3) is a consequence of the following result.

Proposition 2.2. *Let $\mathcal{C}$ be a simple collection of connectors of a graph $H \in \mathcal{H}$. If $\tilde{H}_{\mathcal{C}}$ is a forest, then H admits a 2-decomposition $\mathcal{D} = (F, M)$ such that each $u \in \mathcal{C}$ either is a full vertex in $\mathcal{D}$ or is adjacent to a full vertex in $\mathcal{D}$.*

Before we present the proof of Proposition 2.2, we show how it implies Theorem 1.3.

Proof of Theorem 1.3. The proof follows by induction on $|E(G)|$. Let $G \in \mathcal{G}$ so that $G - V_2(G)$ is a collection of thick cacti. The statement clearly holds for $|E(G)| \leq 3$, so we may assume that $|E(G)| \geq 4$.

First, suppose that G is not a simple graph. Then G contains loops or parallel edges. If there is a loop, then it is a cycle that is not separating, which is a contradiction. If there are three copies of an edge, then the cycle containing any two of these copies is not a separating cycle, again a contradiction. Thus, we may assume that G has precisely two copies, say e and e', of an edge xy. If $d(x) = d(y) = 2$, then $(\{e\}, \{e'\})$ is a decomposition as desired. If $d(x) = 2$ and $d(y) = 3$, then y has degree 1 in $G - V_2(G)$, and hence $G - V_2(G)$ is not a collection of thick cacti, a contradiction. Thus, by symmetry, we may assume that $d(x) = d(y) = 3$. Let u (resp. v) be the neighbor of x (resp. y) distinct from y (resp. x). Since e and e' form a cycle, say C, the graph $G - E(C)$ is disconnected (and thus, $u \neq v$). Note that $G' = G - x - y \cup uv$ is a graph in $\mathcal{G}$ and $G' - V_2(G)$ is a collection of thick cacti. Since $|E(G')| < |E(G)|$, by the induction hypothesis G' admits a decomposition into a spanning tree T' and a matching M'. We may assume that $uv \in E(T')$ because uv is a cut edge of G'. Therefore $((T' - uv) \cup \{ux, e, yv\}, M' \cup \{e'\})$ is a 2-decomposition of G, as desired. Therefore, from now on, we assume that G is a simple graph.

In what follows, we obtain a graph in $\mathcal{H}$ and a simple collection of connectors satisfying the hypothesis of the Proposition 2.2. Let H be the graph obtained from G by the following operations.

(1) replacing every path $P_j = ux_1x_2 \cdots x_kv$ for which $k \geq 1$, $u, v \in V_3(G)$, and $x_i \in V_2(G)$ for $1 \leq i \leq k$, by a path uw_jv, where w_j is a new vertex (if $k = 1$ we just rename x_1); and
(2) subdividing once every edge of $G - V_2(G)$ that does not belong to a cycle of $G - V_2(G)$.

Let $\mathcal{C}$ be the set of vertices added due to the subdivisions in the step (2). It is straightforward that $H \in \mathcal{H}$. Furthermore, since $G - V_2(G)$ is a collection of thick cacti, $\mathcal{C}$ is a simple collection of connectors of H whose underlying BC-graph $\tilde{H}_{\mathcal{C}}$ is a forest. By Proposition 2.2, the graph H admits a 2-decomposition $\mathcal{D} = (F, M)$ such that, for every $u \in \mathcal{C}$, the vertex u is a full vertex in $\mathcal{D}$ or it is adjacent to a full vertex in $\mathcal{D}$.

Now, from (F, M) we obtain a 2-decomposition (F^*, M^*) of G. Let xy be an edge in M. If $d_H(x) = d_H(y) = 3$, then we put xy in M^*. Thus, we may assume, that $d_H(x) = 2$. If x was added in (1), then, without loss of generality, there is a path $P_j = ux_1x_2 \cdots x_ky$ (with $k \geq 1$, $u, y \in V_3(G)$, $x_i \in V_2(G)$ for $1 \leq i \leq k$) that was replaced by the path uw_jy, where $w_j = x$. In this case, we put x_ky in M^*. If x was added in (2), then $x \in \mathcal{C}$. Since x is not a full vertex in $\mathcal{D}$, the vertex x must be adjacent to a full vertex in $\mathcal{D}$, say z, and so we put yz in M^*. Since M is a transversal of the cycles of H, i.e., M contains an edge in each cycle of H, by construction M^* is also a transversal of the cycles of G, which implies that $F^* = G - E(M^*)$ is a forest.

Finally, to obtain a decomposition of G into a spanning tree and a matching from (F^*, M^*), one may find a minimal subset $S \subset M^*$ such that $F^* \cup S$ is connected. □

For our next result, we define the following convenient notation to refer to the successor of a vertex in a cycle. Given a cycle $C = w_1w_2 \cdots w_pw_1$ and a vertex $w_i \in V(C)$, for every $i \in \{1, \ldots, p\}$, we define $w_i^+ = w_{i+1}$, where $w_{p+1} = w_1$. Now we prove Proposition 2.2 by proving the following stronger statement.

Proposition 2.3. *Let $\mathcal{C}$ be a simple collection of connectors of a graph $H \in \mathcal{H}$. If $\tilde{H}_{\mathcal{C}}$ is a forest, then H admits a 2-decomposition $\mathcal{D} = (F, M)$ such that the following holds.*

1. $|M \cap E(C)| = 1$ *for every basic cycle C of H; and*
2. *each $u \in \mathcal{C}$ is either a full vertex in $\mathcal{D}$ or is adjacent to a full vertex in $\mathcal{D}$.*

Proof. Let $\mathcal{C}$ and $H \in \mathcal{H}$ be as in the statement. The proof follows by induction on $n = |V(H)|$. Since $H \in \mathcal{H}$, it follows that $V_2(H)$ is a stable set and every vertex in $V_3(H)$ has exactly one neighbor in $V_2(H)$. Thus, we have $|V_3(H)| = 2|V_2(H)|$. If $|V_2(H)| = 1$, then $|V_3(H)| = 2$, and hence H has parallel edges, a contradiction. Therefore, we may assume $|V_2(H)| \geq 2$, which implies $|V_3(H)| \geq 4$. By Remark 2.1, $H - V_2(H)$ is a collection of (basic) cycles. First, suppose that H has exactly one basic cycle, say C. In this case, H has no connectors which implies that $\mathcal{C} = \emptyset$ and that every vertex in C is the end of a 2-chord. Let x_1 and x_2 be two adjacent vertices in C which are the ends of two distinct 2-chords in H. Let $x_1y_1z_1$ and $x_2y_2z_2$ be the 2-chords containing x_1 and x_2, respectively. For each $y \in V_2(H) - \{y_1, y_2\}$, let e_y be an arbitrary edge incident to y, and let

$$M = \{x_1x_2, y_1z_1, y_2z_2\} \cup \{e_y : y \in V_2(H) - \{y_1, y_2\}\} \text{ and } F = G - M.$$

Note that (F, M) is a 2-decomposition of H as desired. Therefore, we may assume that H has at least two basic cycles.

In what follows, we say that a basic cycle C is of *type 1* if no 2-chord has both ends in C; otherwise, we say C is of *type 2*. The following claim on basic cycles of type 1 arises naturally. Owing to space limitation, we leave its proof to the reader.

Claim 2.4. *If C is a basic cycle of type 1 in H, then C is a cut vertex of $\tilde{H}$.*

Since $\tilde{H}$ is connected and $\tilde{H}_{\mathcal{C}} \subseteq \tilde{H}$ is a forest, there is a spanning tree T of $\tilde{H}$ such that $\tilde{H}_{\mathcal{C}} \subseteq T$. By Claim 2.4, the leaves of T are basic cycles of type 2. Now, let $\mathcal{C}^*$ be the collection of connectors so that $\tilde{H}_{\mathcal{C}^*} = T$. Since $\tilde{H}_{\mathcal{C}} \subseteq T = \tilde{H}_{\mathcal{C}^*}$, we may assume $\mathcal{C} \subseteq \mathcal{C}^*$. In what follows we prove that H admits a 2-decomposition $\mathcal{D} = (F, M)$ such that (a) $|M \cap E(C)| = 1$ for every basic cycle C of H; and (b) each $u \in \mathcal{C}^*$ is either a full vertex in $\mathcal{D}$ or is adjacent to a full vertex in $\mathcal{D}$. Note that, since $\mathcal{C} \subseteq \mathcal{C}^*$, (b) implies (ii), and hence the result follows.

Let $V_2(H) = \{y_1, \ldots, y_\ell\}$ and, for each $y_i \in V_2(H)$, let x_i and z_i be the neighbors of y_i. Note that $V(H)$ is the disjoint union of the sets $\{x_i, y_i, z_i\}$ for $i \in \{1, \ldots, \ell\}$. Let C be a leaf of $T = \tilde{H}_{\mathcal{C}^*}$, and put

$$I = \{y_i \in V_2(H) : |\{x_i, z_i\} \cap V(C)| = 1\} \quad \text{and} \quad J = \{y_i \in V_2(H) : |\{x_i, z_i\} \cap V(C)| = 2\}.$$

We may assume, without loss of generality, that $x_i \in V(C)$ and $z_i \notin V(C)$ for every $y_i \in I$. Note that $I \neq \emptyset$, otherwise either T is disconnected or T has only one vertex, namely C, which implies that H has only one basic cycle, a contradiction. Thus, we may assume, without loss of generality, that $y_1 \in \mathcal{C}^*$ and $x_1 \in V(C)$. Let $C = u_1u_2 \ldots u_ku_1$, where $u_1 = x_1$. Now, we split the proof into two cases depending on whether the vertex x_1 is adjacent to a vertex with a neighbor that belongs to J.

Case 1. *x_1 is adjacent to a vertex with a neighbor that belongs to J (see Fig. 1a)).*

Suppose, without loss of generality, that $u_2 = x_2$ and $y_2 \in J$. In what follows, we obtain a graph $H' \in \mathcal{H}$ such that $|V(H')| < |V(H)|$. Let H' be the graph obtained from $H - V(C) - J$ by subdividing, for every $y_i \in I$, the edge $z_iz_i^+$, obtaining the vertex z_i', and adding the edge y_iz_i' (see Fig. 1b)). Note that $V_2(H') = V_2(H) - J$.

Let $B \neq C$ be a basic cycle of H. Note that if $BC \notin E(\tilde{H})$, then B is a (basic) cycle in H'. On the other hand, if $BC \in E(\tilde{H})$, then B is not a cycle in H but a subdivision B' of B is a (basic) cycle in H'. Let $\varphi(B) = B$, if $BC \notin E(\tilde{H})$,

a) **b)**

Fig. 1: Reduction from a graph H 1a) to the graph H' 1b) in Case 1. In 1b), we use gray to indicate the elements from H that we removed to create H'.

and $\varphi(B) = B'$, otherwise. Now we show that φ is an isomorphism between $\tilde{H} - C$ and $\tilde{H}'$. It is not hard to see by the construction of H' that φ is a bijective function. Now, if $XY \in E(\tilde{H} - C)$, then there is a connector, say y, joining the basic cycles X and Y in H. The only connectors affected by the construction of H' are those that contain an end in C, and since $X \neq C$ and $Y \neq C$, it follows that y is a connector joining $\varphi(X)$ to $\varphi(Y)$ in H'. Thus $\varphi(X)\varphi(Y) \in E(\tilde{H}')$. Now, suppose that $\varphi(X)\varphi(X) \in E(\tilde{H}')$. By the construction of H', no connector is created. This implies that the connector vertices of H' is a subset of the connector vertices of H. Thus, if $\varphi(X)\varphi(Y) \in E(\tilde{H}')$, then there exist a connector vertex, say y, joining the basic cycle $\varphi(X)$ and $\varphi(Y)$. Since y is also a connector in H, $X \neq C$, and $Y \neq C$, it follows that $XY \in E(\tilde{H} - C)$. Therefore, $\tilde{H} - C$ and $\tilde{H}'$ are isomorphic.

Let $W = V(H) - (I \cup J \cup V(C))$, and note that $V(H) = W \cup I \cup J \cup V(C)$. Moreover, note that $V(H') = W \cup I \cup \{z_i' : y_i \in I\}$. Since $J \neq \emptyset$, it follows that $|V(H')| < |V(H)|$. Now we show that $H' \in \mathcal{H}$. First, since C is a leaf of the spanning tree T of $\tilde{H}$, the graph $T - C$ is connected, and since $\tilde{H}'$ is isomorphic to $\tilde{H} - C$, we have $T - C$ is a spanning tree of $\tilde{H}'$, and hence H' is connected. Also, by construction, H' is simple, each vertex in $V_3(H')$ has exactly one neighbor in $V_2(H')$, and $V_2(H')$ is a stable set. It remains to prove that every cycle in H' is a separating cycle. First, note that every basic cycle C' adjacent to C in $\tilde{H}$ yields a basic cycle of type 2 in $\tilde{H}'$. Moreover, note that a cycle containing a vertex with degree 2 or a basic cycle of type 2 is a separating cycle. Thus, we can focus on the basic cycles of type 1 in H'. Let C' be one of such cycles. By Claim 2.4 the cycle C' is a cut vertex of $\tilde{H}$. By the construction of H', the graph $\tilde{H} - C$ is isomorphic to $\tilde{H}'$. Thus, C' is a cut vertex of $\tilde{H}'$, which implies that C' is a separating cycle of H'. Therefore, we conclude that $H' \in \mathcal{H}$.

By induction hypothesis, the graph H' has a 2-decomposition (F', M') satisfying (1) and (2) with respect to the simple collection of connectors $C^* - \{y_1\}$. We now describe in four steps how to obtain the desired 2-decomposition (F, M) of H from (F', M') (see Fig. 2):

(a) We put x_1x_2 in M and all the edges of $C - x_1x_2$ in F. We put y_2x_2 in F, y_2z_2 in M and, for each $y_i \in J - \{y_2\}$, we put edges x_iy_i, y_iz_i in distinct sets of $\{M, F\}$.

(b) We put edges x_1y_1 and y_1z_1 in F. In addition, for each $y_i \in I - \{y_1\}$, we put the edge x_iy_i in M and the edge y_iz_i in F.

(c) We put each edge

$$e \in E(G) - \Big(E(C) \cup \{y_ix_i, y_iz_i : y_i \in I \cup J\} \cup \{z_iz_i^+ : y_i \in I\}\Big)$$

in F if $e \in F'$. Otherwise, we put e in M.

(d) Finally, for each $y_i \in I$, we put the edge $z_iz_i^+ \in E(G)$ in M if $z_i'z_i^+ \in M'$. Otherwise, we put $z_iz_i^+$ in F.

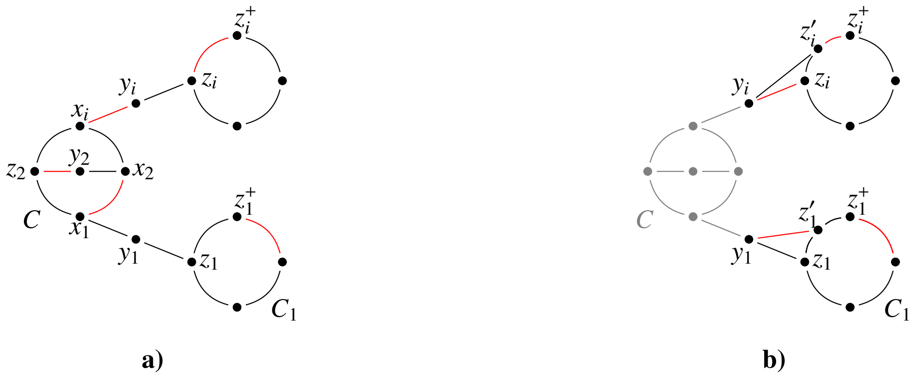

Fig. 2: Reduction from a graph H 2a) to the graph H' 2b) in Case 1. In both figures, for a 2-decomposition (F, M), the edges in F (resp. M) are colored black (resp. red). In 2b), we use gray to indicate the elements from H that we removed to create H'.

We first note that in the steps (a)-(d), each edge of $E(H)$ has been put either in M or F: in steps (a) and (b) the edges of the set $E(C) \cup \{y_iz_i, y_ix_i \colon y_i \in I \cup J\}$ are covered, in step (d) the edges of the set $\{z_iz_i^+ \colon y_i \in I\}$ are covered and all the remaining edges are covered in step (c). We make the following useful claim.

Claim 2.5. *Edge $z_iz_i' \in F'$ for all $y_i \in I$.*

Proof. Let B be the basic cycle of H' that contains z_iz_i'. If z_iz_i' belongs to M', then due to (1) all the edges in $E(B) - \{z_iz_i'\}$ belong to F'. Hence, the cycle $\{z_iy_i, y_iz_i'\} \cup (E(B) - \{z_iz_i'\})$ belongs to F', a contradiction. □

It is straightforward from the assignments in the steps (a)-(d) that F is a forest and M is a matching (for the edges that were subdivided, item (1) with respect to M' and Claim 2.5 ensure that M is a matching and that F has no cycles). We now check that items (1) and (2) hold for (F, M). Due to step (a), we have that $|M \cap E(C)| = 1$ and due to steps (c)-(d), we have that $|M \cap E(C')| = 1$ for every other basic cycle C' of H with $C' \neq C$. Hence, (1) holds. Due to step (b), we have that y_1 is a full vertex in (F, M). Let $y \in C^* - \{y_1\}$. Since C is a leaf of $T = \tilde{H}_{C^*}$, it follows that $y \notin I$. Thus, if y is full in (F', M'), then due to the step (c), it is also full in (F, M). So, suppose that y is not full in (F', M') and hence, it has a neighbor x which is a full vertex in (F', M'). Let C' be the basic cycle of H that contains x. If $x \neq z_i^+$ for each $y_i \in I - \{y_1\}$, then due to step (c), we have that x is a full vertex in (F, M) as well. Suppose that $x = z_i^+$ for some $i \in I - \{y_1\}$. Since x is a full vertex in (F', M'), we have that xy, xz_i' and xx' belong to F', where x' is the neighbor of x distinct of z_i' in the basic cycle C'. Due to step (c), xy and xx' belong to F, and due to step (d) the edge $z_iz_i^+$ belongs to F (since $z_i'x \in F'$). Therefore item (2) holds. This finishes the proof of Case 1.

Case 2. *x_1 is not adjacent to a vertex with a neighbor in J.*

In this case, both neighbors of x_1 in C, namely u_2 and u_k, are adjacent to a vertex in I. Let C_1 be the basic cycle that contains z_1 and let ℓ be the smallest $i \in \{1, \ldots, k\}$ for which u_{i+1} is an end of a 2-chord. We may assume, without loss of generality, that $u_\ell = x_2$ and $u_{\ell+1} = x_3$. Let C_2 be the basic cycle that contains z_2 (possibly $C_1 = C_2$).

In what follows, analogously to Case 1, we obtain a graph $H' \in \mathcal{H}$ such that $|V(H')| < |V(H)|$. Let H' be the graph obtained from $H - V(C) - J$ by (i) identifying the vertices y_1 and y_2 into a new vertex y, and (ii) for every $y_i \in I - \{y_1, y_2\}$, subdividing the edge $z_iz_i^+$, obtaining the vertex z_i', and adding the edge y_iz_i' (see Fig. 3). Now, note that $V_2(H') = \{y\} \cup (V_2(H) - (J \cup \{y_1, y_2\}))$.

Let $B \neq C$ be a basic cycle of H. Note that if $BC \notin E(\tilde{H})$, then B is a (basic) cycle in H'. On the other hand, if $BC \in E(\tilde{H})$, then B is not a cycle in H', but one subdivision B' of B is. Let $\varphi(B) = B$, if $BC \notin E(\tilde{H})$, and $\varphi(B) = B'$, otherwise. Now we show that, if $C_1 = C_2$, then φ is an isomorphism between $\tilde{H}'$ and $\tilde{H} - C$; otherwise, we show that φ is an isomorphism between $\tilde{H}'$ and $(\tilde{H} - C) \cup C_1C_2$ (here, we only add the edge C_1C_2 to $\tilde{H} - C$ if this action results in a simple graph). By the construction of H', it is not hard to check that φ is a bijective function. If $XY \in E(\tilde{H} - C)$, then there is a connector, say y', joining the basic cycles X and Y in H. The only connectors affected by the construction of H' are those that contain an end in C and, since $X \neq C$ and $Y \neq C$, it follows that y' is a connector joining $\varphi(X)$ to $\varphi(Y)$

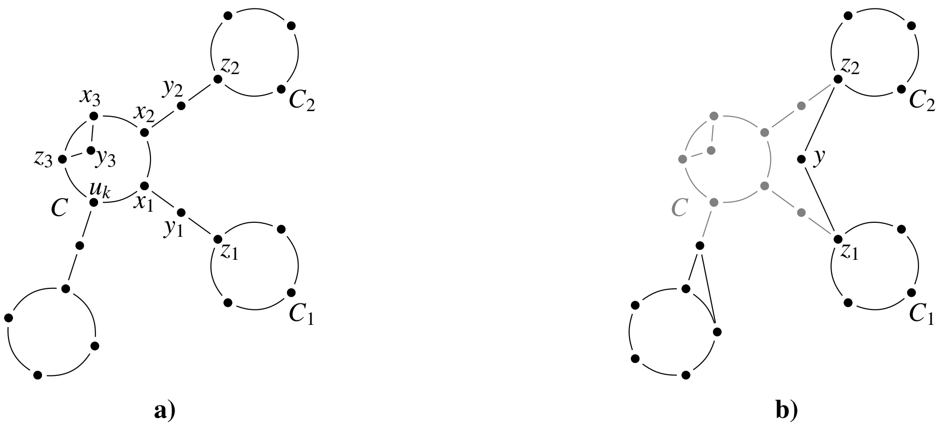

Fig. 3: Reduction from a graph H 3a) to the graph H' 3b) in Case 2. In 3b), we use gray to indicate the elements from H that we removed to create H'.

in H'. If $C_1 \neq C_2$, then, by the construction of H', the vertex y is a connector in H' joining the basic cycles $\varphi(C_1)$ and $\varphi(C_2)$ and, as result, $\varphi(C_1)\varphi(C_2) \in E(\tilde{H}')$. Now, suppose that $\varphi(X)\varphi(Y) \in E(\tilde{H}')$, and hence there exists a connector y' in H' joining $\varphi(X)$ and $\varphi(Y)$. By the construction of H', the vertex y is the only connector that we can create, so the set connector vertices of H' distinct from y is a subset of the connector vertices of H. If $|\{X, Y\} \cap \{C_1, C_2\}| < 2$, then $y' \neq y$, and hence y' is a connector in H joining X and Y, and hence $XY \in E(\tilde{H} - C)$. Now, $|\{X, Y\} \cap \{C_1, C_2\}| = 2$, then, by construction, $XY \in (\tilde{H} - C) \cup C_1C_2$. Therefore, φ is an isomorphism between $\tilde{H}'$ and $\tilde{H} - C$, if $C_1 = C_2$, or between $\tilde{H}'$ and $(\tilde{H} - C) \cup C_1C_2$, otherwise.

Let $W = V(H) - (I \cup J \cup V(C))$, and hence $V(H) = W \cup I \cup J \cup V(C)$. Moreover, note that $V(H') = W \cup (I - \{y_1, y_2\}) \cup \{z'_i : y_i \in I - \{y_1, y_2\}\} \cup \{y\}$. It follows that $|V(H')| < |V(H)|$. Now we claim that $H' \in \mathcal{H}$. First, since C is a leaf of a spanning tree T of $\tilde{H}$, and either $\tilde{H} - C$ or $(\tilde{H} - C) \cup C_1C_2$ is isomorphic to $\tilde{H}'$, it follows that $T - C$ is a spanning tree of $\tilde{H}'$, and hence H' is connected. Also, by construction, H' is simple, each vertex in $V_3(H')$ has exactly one neighbor in $V_2(H')$, and $V_2(H')$ is a stable set. It remains to prove that every cycle in H' is a separating cycle. Again, if a cycle $C' \subset H'$ contains a vertex in $V_2(H')$ or has a 2-chord, then C' is a separating cycle. Thus, we can assume that $V(C') \subsetneq V_3(H')$ and that C' has no 2-chords. By the construction of H', C' must be a basic cycle of type 1 in H, and so by Claim 2.4, C' is a cut vertex of $\tilde{H}$. Now we show that C' is a cut vertex in $\tilde{H}'$. Let H_1 be the component in $\tilde{H} - C'$ containing the vertex C. Note that, if $C_i \in \tilde{H} - C'$ for $i \in \{1, 2\}$, then C_i belongs to H_1. Since $\tilde{H}'$ is isomorphic either to $\tilde{H} - C$ or to $(\tilde{H} - C) \cup C_1C_2$, to show that C' is a cut vertex in $\tilde{H}'$ it is sufficient to show that C' has neighbor in $V(H_1 - C)$ in the graph $\tilde{H}'$. If $C'C \notin E(\tilde{H})$, then clearly C' has neighbor in $V(H_1 - C)$. Thus, we may assume that $C'C \in E(\tilde{H})$. Now, note that $C' \in \{C_1, C_2\}$ and $C_1 \neq C_2$, otherwise, by the construction of H', the cycle C' would be a basic cycle of type 2. Suppose, without loss of generality, that $C' = C_1$. Therefore, by the construction of H', the edge $C'C_2 \in E(\tilde{H}')$. Hence C' has a neighbor in $V(H_1 - C)$ in the graph $\tilde{H}'$, which implies that C' is a cut vertex in $\tilde{H}'$ and, consequently, that C' is a separation cycle in H'.

By induction hypothesis, the graph H' admits a 2-decomposition (F', M') satisfying (1) and (2) with respect to the simple collection of connectors $C^* - \{y_1\}$. In what follows, we obtain from (F', M') a 2-decomposition (F, M) of G as desired (see Fig. 4).

(a) We put x_2x_3 in M and all the edges of $C - x_2x_3$ in F. We put y_3x_3 in F, y_3z_3 in M and, for each $y_i \in J - \{y_3\}$, we put edges x_iy_i, y_iz_i in distinct elements of $\{M, F\}$.

(b) We put edges x_1y_1 and x_2y_2 in F. In addition, for each $y_i \in I - \{y_1, y_2\}$, we put the edge x_iy_i in M and the edge y_iz_i in F.

(c) We put each edge

$$e \in E(G) - (E(C) \cup \{z_iz_i^+ : y_i \in I - \{y_1, y_2\}\} \cup \{y_iz_i, y_ix_i : y_i \in I \cup J\})$$

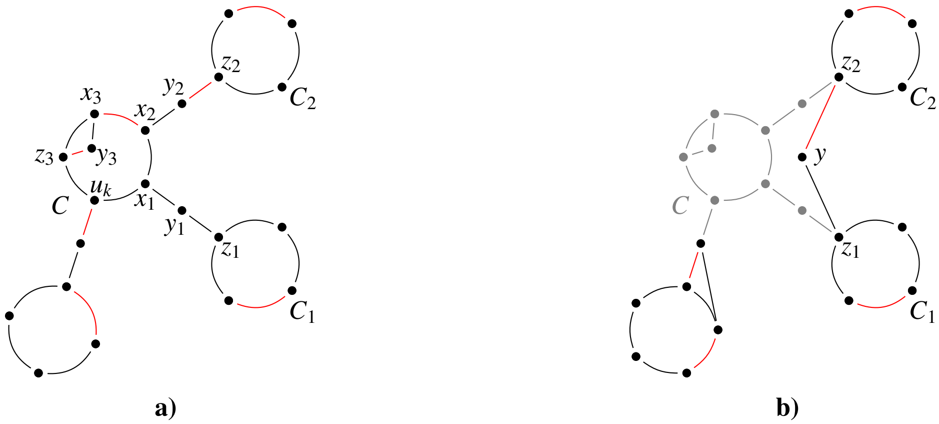

Fig. 4: Reduction from a graph H 4a) to the graph H' 4b) in Case 2. In both figures, for a 2-decomposition (F, M), the edges in F (resp. M) are colored black (resp. red). In 4b), we use gray to indicate the elements from H that we removed to create H'.

in F if $e \in F'$. Otherwise, we put e in M.

(d) For each $y_i \in I - \{y_1, y_2\}$, we put edge $z_i z_i^+$ in M if $z_i' z_i^+ \in M'$. Otherwise, we put $z_i z_i^+$ in F.

(e) Finally, for each $y_i \in \{y_1, y_2\}$, we put $y_i z_i$ in F if $y z_i \in F'$. Otherwise, we put $y_i z_i$ in M.

We now show that (F, M) is the 2-decomposition of H desired. It is straightforward that each edge of $E(H)$ is either in F or M. Analogously to **Case 1**, the following claim arises (the same proof applies).

Claim 2.6. *Edge $z_i z_i' \in F'$ for all $y_i \in I - \{y_1, y_2\}$.*

From the assignments in the steps (a)-(e), since item (1) holds for H', and by Claim 2.6, it is clear that M is a matching. We now check that F is a forest. Due to the assignments in the steps (a)-(d), it is clear that F is a forest in the graph $G - \{y_1 z_1, y_2 z_2\}$. The assignments in the step (e) ensure that edges $y_1 z_1, y_2 z_2$ are assigned to F or M without creating cycles in F: both $y_1 z_1, y_2 z_2$ are put in F if $y z_1, y z_2$ are in F', and one of them $y_1 z_1$, or $y_2 z_2$ is in M whenever its copy in H', namely $y z_1$ or $y z_2$, is in M'. Note that item (1) holds due to the assignments in the steps (a) and (c), and that item (2) follows from the assignments in the steps (a)-(c) and Claim 2.6. This finishes the proof of Case 2, and concludes the proof of Proposition 2.3. □

3. Concluding Remarks

In this paper we verified Conjecture 1.2 for subcubic graphs in $\mathcal{G}$ whose subgraph induced by vertices of degree 3, $G[V_3]$, is a collection of cacti in which every vertex belongs to a cycle. This is a preliminary result that might lead to more general ones, as now we can investigate further cases related to cacti, as they seem to be powerful structures to tackle this conjecture.

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