A VERSION OF CURVE SELECTION LEMMA FOR REAL ANALYTIC SETS

Manuel Valentim de Pera Garcia
Departamento de Matemática Aplicada
Instituto de Matemática e Estatística - Universidade de São Paulo

§0-Introduction: We consider an analytic set S in $\mathbf{R}^n$ with $O \in S$. Lojaciewicz has proved (see [L-1]) that in a neighborhood V of O there are a finite number of analytic manifolds M_i, $1 \le i \le r$ in V each one of them connected, semi-analytic and adherent to the origin such that

$$S \setminus \{O\} = \bigcup_{i=1}^{r} M_i;$$

we present here a demonstration of that there are, for $1 \le i \le r$, *analytic* curves $\gamma_i: [0;\varepsilon] \longrightarrow \mathbf{R}^n$ such that $\gamma_i(0) = O$ and $\gamma_i(t) \in M_i$, for $0 < t \le \varepsilon$. Except by the analyticity of γ_i at zero this is a result of Bruhat and Whitney in a joint work of 1959 (see [B-W-1]). In that work the curves are analytic in $]0;\varepsilon]$ and continuous in $[0;\varepsilon]$. Our result in the context of real algebraic sets is a classic theorem of John Milnor (see [M-1]) who gives it the name of *curve selection lemma*.

§1- Some preliminaries: We give in this section the basic definitions of analytic sets and some results above them that we will use later. We don't prove the facts expressed in this section. The demonstrations of these statments can be found, for instance, in [N-1].

In this text Ω will be an open subset of $\mathbf{R}^n$.

Definition 1.0: *A subset S of Ω is* **analytic in** Ω *if for each point $p \in \Omega$ there are a neighborhood $U_p \subseteq \Omega$ of p and real analytic functions $f_1, \ldots, f_r$ defined in U_p such that*

$$S \cap U_p = \{x \in U_p : f_i(x) = 0,\ 1 \le i \le r\}.$$

The functions $f_1, \ldots, f_r$ depend on the point p.

If S is an analytic set in Ω and $p \in \Omega$ then we denote by $(S;p)$ the germ of S at p. We call a germ of analytic set S by *analytic germ*.

Definition 1.1: *We say that an analytic set S of Ω is* **irreducible at** $p \in \Omega$ *if when we have $(S;p) = (S_1;p) \cup (S_2;p)$, with S_1 and S_2 analytic sets in Ω then there is $i \in \{1;2\}$ such that $(S;p) = (S_i;p)$.*

We say that the germ $(S;p)$ is irreducible if some representant of this germ is irreducible at p.

We have the

Fact 0: *If S is analytic in Ω and $p \in \Omega$ then there are a finite number k of irreducible analytic germs $(S_i;p)$, $1 \le i \le k$ such that*

(i) $(S;p) = \bigcup_{i=1}^{k} (S_i;p)$

(ii) $(S_j;p) \not\subset \bigcup_{i=1, i\neq j}^{k} (S_i;p)$, $\forall j \in \{1;\ldots;k\}$.

Moreover, except by the ordenation of the germs $(S_i;p)$, this decomposition is unique.

The germs $(S_i;p)$ determined above are called **irreducible components of S in p**.

Definition 1.2: *Let S be an anaytic set in Ω and let $p \in S$. We say that p* **is m-regular** *if there is an open neighborhood $U \subseteq \Omega$ of p such that $S \cap U$ is an analytic manifold of dimension m. We say that p is a* **singular point** *of S if, for all $m \in \mathbb{N}$, p isn't m-regular.*

It follows immediately from this definition that

Fact 1: *Let S be an analytic set in $\Omega \subseteq \mathbb{R}^n$ and let m be a natural number. with $m < n$. A point $p \in S$ is m-regular if, and only if, there are an open neighborhood of $p, U \subseteq \Omega$, and real analytic functions $f_{m+1}, \ldots, f_n$ defined in U such that $S \cap U = \{x \in U: f_i(x) = 0\}$ and $Df_{m+1}(x), \ldots, Df_n(x)$ are linearly independent for all $x \in U$.*

By using this concept one can define the notion of dimension of an irreducible anlyitic germ as follows

Definition 1.3: *Let p be a point of Ω. If $(S;p)$ is an irreducible analytic germ. not empty, then the maximum of the set*

$$\{q \in \mathbb{N}: \text{ all representant of } (S;p) \text{ has a sequence of } q\text{-regular points that converges to } p\}$$

is called **dimension of the germ** *$(S;p)$. Define the dimention of $(\emptyset;p)$ as $-\infty$.*

One can put us the question of the concept of dimension of an analytic germ is well defined. This is a classical result of analytic sets and can be state as

Fact 2: *If $p \in \Omega$ and S is a representant of the irreducible anlytic germ $(S;p)$ then the set of points of S which are m-regular, for some $m \geq 0$, is dense in S.*

Now we can state

Definition 1.4: *Let $p \in \Omega$ and let $(S;p)$ be an analytic germ in p. Take $(S_i;p)$, $1 \leq i \leq r$, the irreducible components of $(S;p)$. We define the* **dimension of** *$(S;p)$ as the maximum of the dimensions of $(S_i;p)$, $1 \leq i \leq r$. We will denote this number by $dim(S;p)$*

One can prove (but we don't known any elementary proof) that

Fact 3: *Let S be an analytic set in $\Omega \subseteq \mathbb{R}^n$ and let $p \in S$. such that $(S;p)$ is irreducible. If $dim(S;p) = q < n$ and $M_q = \{x \in \Omega: x \text{ is } q\text{-regular }\}$ then there is a neigborhood $U \subseteq \Omega$ of p and there are real analytic functions $f_0; f_{q+1}; \ldots; f_n$ defined in U such that*

(i) $U \cap M_q = \{x \in U: f_0(x) \neq 0$ and $f_i(x) = 0,\ q+1 \leq i \leq n\}$ and $Df_{q+1}(x); \ldots; Df_n(x)$ are linerly independent for $x \in M_q$.

(ii) $U \cap M_q = \{x \in U: x \in S$ and $f_0(x) \neq 0\}$.

As a direct consequence of this we have

Fact 4: *If S is an analytic set of Ω and $p \in S$ with $(S;p)$ irreducible and $dim(S;p) = q$, then there is an open neighborhood $U \subseteq \Omega$ of p such that the set*

$$S_1 := \{x \in S \cap U: x \text{ isn't } q\text{-regular }\}$$

is an analytic set in U and $dim(S_1;p) < q$.

In section 1 we will need two another classical result of analytic sets.

Fact 5: *Suppose that $p \in \Omega$ and that $(S;p)$ is an irreducible analytic germ of dimension $q \geq 0$. If $(S_1;p)$ is an analytic germ with $(S_1;p) \subsetneq (S;p)$ then $dim(S_1;p) < q$.*

To finish this section of preliminaries we state a result of local decomposition of real analytic sets which is more simple that the Lojaciewicz one about this. and that it is enough to ours purposes in sections 2 and 3. We need the Lojaciewicz's result olny in section 4 and we will state it there.

Fact 6: *Let S be an analytic subset of Ω and let $p \in S$, with $dim(S;p) \geq 1$. Then there is a neighborhood U of p and there are a finite number of connected analytic manifolds $M_1,\ldots,M_s$, with $dimM_i = m_i \in \{1;\ldots;dim(S;p)\}$, such that p is adherent to M_i, $1 \leq m_i \leq dim(S;p)$ and*

$$(S \setminus \{p\}) \cap U = \bigcup_{i=1}^{s} M_i.$$

§2 - **Reduction to the case of dimension 1:** In this section we will state our main result and we show that it is enough to prove it for analytic germes of dimension 1.

The central result of this work is

Theorem 0: *Let Ω be an open subset of $\mathbf{R}^n$ with $O \in \Omega$ and let S be an analytic set in Ω. Consider a finite number of real analytic functions $g_1,\ldots,g_\ell$ defined in Ω and put $U = \{x \in \Omega: g_i(x) > 0,\ 1 \leq i \leq \ell\}$. If $O \in \overline{U \cap S}$ then there are $\varepsilon > 0$ and an analytic curve $\gamma: [0;\varepsilon] \longrightarrow \Omega$ such that $\gamma(0) = O$ and $\gamma(t) \in U \cap S$ for $t > 0$.*

Note that, since S is a closed set in Ω and $O \in Omega$, it follows from these hypotheses that $O \in S$. Furthermore, it's clear that $dim(S;O) \geq 1$.

The proof of this theorem is done in two steps.

In the first one of them we prove that if $dim(S;O) > 1$ then there is an analytic subset S_1 of an open neighborhood Ω_1 of the origin, with $S_1 \subsetneq S$, $O \in \overline{S_1 \cap U}$ and $dim(S_1;O) < dim(S;O)$. In the second step we prove Theorem 0 when $dim(S;O) = 1$. Of course, this ends the proof.

In the rest of this section we will prove the first of these steps. We start with

Fact 7: *Let S be an analytic set in $\Omega \subseteq \mathbf{R}^n$ and let $M_q = \{x \in S: x$ is q-regular$\}$. If $g: \Omega \longrightarrow \mathbf{R}$ is analytic and $V = \left\{x \in M_q: x \text{ is a critical point of } g|_{M_q}\right\}$ then there is an open set $W \subseteq \Omega$ such that V is analytic in W.*

Dem.: The statment is obvious when $q = n$. Then we suppose that $q < n$.

Let be $x_0 \in V \subseteq M_q$. It folows from Fact 1 that there are analytic functions $f_{q+1},\ldots,f_n$ which are defined in an open neighborhood $W_{x_0} \subseteq \Omega$ of x_0 such that

$$S \cap W_{x_0} = \{x \in W_{x_0}: f_i(x) = 0,\ q+1 \leq i \leq n\}.$$

and the rank of $A(x) = [Df_{q+1}(x),\ldots,Df_n(x)]$ is equal to $n-q$, for all $x \in W_{x_0}$.

Now, if we remember that M_q is an analytic manifold of dimension q, it follows that for $x \in W_{x_0} \cap M_q$ we have $T_xM_q = ker A(x)$.

From this we can conclude that if $B(x) = [\,Dg(x) \quad A(x)\,]$ then

$$V \cap W_{x_0} = \{x \in W_{x_0}: \text{rank } B(x) \leq n-q\}.$$

In fact, the rank of $B(x)$ is equal to $n-q$ (since the rank of $A(x)$ is $n-q$) but this is superfulous here.

This shows that $V \cap W_{x_0}$ is an analytical set in W_{x_0}.

Now take $W = \cup_{x \in M_q} W_x$. It's immediate that W is an open subset of Ω and that V is analytic in W. ∎

So we have

Fact 8: *In the hypotheses of the Fact 7 if $\Omega_1 = \mathring{\Omega}_1$ is a bounded subset of Ω with $\overline{\Omega_1} \subset \Omega$ then set of critical values of $g|_{M_q \cap \overline{\Omega_1}}$ is finite.*

Dem.: Let x_0 be a critical point of $g|_{M_q \cap \overline{\Omega_1}}$ and take $W \subseteq \Omega$. a neighborhood of x_0 such taht $V \cap W$ is analytic in W.

By the local decomposition of analytic sets that we have seen in Fact 6 we can assume. w.l.g., that there are connected analytic manifolds $M_1, \dots, M_s$ such that each M_i is adherent to the origin and

$$(V \setminus \{x_0\}) \cap W = \cup_{i=1}^{s} M_i.$$

Then. since V is the set of the critical points of g restricted to the analytic manifold M_q, one can conclude that $g(x) = g(x_0)$. for all $x \in V \cap W$.

Now. use Fact that $\overline{\Omega_1}$ is compact to conclude the thesis. ■

By appling the previous result to the function $r(x) = \|x - x_0\|^2$. we can conclude imediately that

Fact 9: *If x_0 is a q-regular point of the analytic set S of Ω. then there is an $\varepsilon_0 > 0$ such that $S_\varepsilon(x_0) = \{x \in \mathbf{R}^n : \|x - x_0\| = \varepsilon\}$ is transversal to S, for $0 < \varepsilon < \varepsilon_0$.*

Now we can prove the result that finishs this section.

Fact 10: *Let S be is an analytic set of $\Omega \subseteq \mathbf{R}^n$ with $O \in \Omega$ and consider the open $U = \{x \in \mathbf{R}^n : g_i(x) > 0,\ 1 \leq i \leq \ell\}$. where $g_i : \Omega \longrightarrow \mathbf{R}$ is an analytic function for $1 \leq i \leq \ell$. Suppose that $O \in \overline{S \cap U}$ and $dim(S{:}O) = q > 1$. Then there is an analytic set S_1 in an open neighborhood of the origin. V. such that $S_1 \subsetneq S$. $O \in \overline{S_1 \cap U}$ and $1 \leq dim(S_1;O) < q$.*

Dem.: If $q = n$ the result is evident. then we'll suppose $q < n$.

Another observation: if $H_k = \{x = (x_1; \dots; x_n) \in \mathbf{R}^n : x_k = 0\}$. we can assume that $S \not\subset H_k$, for $k = 1; \dots; n$. In fact. if this doesn't happen one can take r as the number of k_s'. such that $S \subseteq H_k$ and consider S an analyitic subset of an open subset of $\mathbf{R}^{n-r}$.

We can suppose also that $(S;O)$ is irreducible. since if this fails we can take an irreducible component $(\check{S};O)$ of $(S;O)$ given by Fact 0 and then we work with $\check{S}$ (a representant of $(\check{S};O)$) in place of S.

If we put. for $\varepsilon > 0$. $\Sigma_\varepsilon(S) = \{x \in S : \|x\| < \varepsilon$ and x is not q-regular $\}$ we can suppose that there is $\varepsilon_0 > 0$ such that $\Sigma_{\varepsilon_0}(S) \cap U = \emptyset$. In fact. if this doesn't occur. we use Fact 4 to choose $\varepsilon_1 > 0$ such that $\Sigma_{\varepsilon_1}(S)$ is analytic and since. also by Fact 4. $dim(S_1{:}O) < dim(S;O)$ we are done.

Then we are working under the hypothesis of that $(S;O)$ is an irreducible analytic germ of dimension $q \leq n - 1$ and all the points of $U \cap S$ are q-regular.

By using Fact 3 and Fact 9 we can choose an open neighborhood V of O such that. if $r(x) = \|x\|^2$,

(i) There are analytic functions f_0 and $f_j : V \longrightarrow \mathbf{R}$, $q + 1 \leq j \leq n$ such that $S \cap V \cap U = \{x \in U \cap V : f_0(x) \neq 0$ and $f_j(x) = 0,\ q + 1 \leq j \leq n\}$ and. moreover. the rank of $[Df_{q+1}(x) \ \ \dots \ \ Df_n(x)]$ is $n - q$. $\forall x \in U \cap V \cap S$ (to see this. use Fact 3).

(ii) The rank of $[Df_{q+1}(x) \ \ \dots \ \ Df_n(x) \ \ Dr(x)]$ is $n - q + 1$. $\forall x \in U \cap V \cap S$ (this is a consequence of Fact 9).

Consider $U_1 = \{x \in V : g_i(x) > 0.\ 1 \leq i \leq \ell$ and $(f_0(x))^2 > 0\}$. We have $U_1 \subseteq U$ and, by (i). one can see that $U \cap V \cap S = U_1 \cap S$ therefore $O \in \overline{U_1 \cap S}$.

We will prove that there is an anaytic set S_1 in V such that $S_1 \subsetneq S$. $1 \leq dim(S_1;O) \leq q-1$ and $O \in \overline{U_1 \cap S}$. Of curse, this will complete the proof.

For this we'll construct $n+1$ subsets of V – $S';S'_1;\ldots;S'_n$ –. which will be analytitc sets in V and we will prove that, at least one of them satisfies our thesis.

Put $g_0(x) = (f_0(x))^2$ and consider

$$g(x) = \prod_{i=0}^{\ell} g_i(x) \text{ and } r(x) = \|x\|^2.$$

We will define

$S' = \{x \in S \cap V\colon \text{rank of } [Df_{q+1}(x) \ \ldots \ Df_n(x) \ \ Dr(x) \ \ Dg(x)] \leq n-q+1\}$,

Of course S' is an analytic subset of V and $S' \subset S$.

We will prove now that $O \in \overline{S' \cap U_1}$.

For this define, for $\varepsilon > 0$. $K_\varepsilon = \{x \in S_\varepsilon \cap S\colon g_i(x) \geq 0,\ 0 \leq i \leq \ell\}$.

Since $O \in \overline{S \cap U_1}$ there is a sequence $\varepsilon_m \downarrow 0$ such that $K_{\varepsilon_m} \neq \emptyset$. and there is $x^m \in K_{\varepsilon_m}$ with $g_i(x^m) > 0$ for $0 \leq i \leq \ell$.

But K_{ε_m} is compact them there is a $\overline{x}^m \in K_{\varepsilon_m}$ which maximizes $g|_{K_{\varepsilon_m}}$. Therefore $g(\overline{x}^m) > 0$.

Then $g_i(\overline{x}^m) > 0$, for $0 \leq i \leq \ell$ and we have proved that $\overline{x}^m \in U_1$.

Now we say that $\overline{x}^m \in S'$. This is a direct consequence of Fact that this point is a maximum point of $g|_{S \cap U \cap S_{\varepsilon_m}}$ and $U \cap S_{\varepsilon_m}$ is an analytic manifold of dimension $q-1$, by (ii).

Then, since $\lim_{m \to \infty} \overline{x}^m = O$. we have $O \in \overline{S' \cap U}$. As $\overline{x}^m \neq O$ this shows that $dim(S';O) \geq 1$.

For $1 \leq i \leq n$, let the functions $\lambda_i\colon \mathbf{R}^n \longrightarrow \mathbf{R}$, $\lambda_i(x) = x_i g(x)$.

Consider. for $1 \leq i \leq n$. the analytic subset of V defined by

$S'_i = \{x \in S \cap V\colon \text{rank of } [Df_{q+1}(x) \ \ldots \ Df_n(x) \ \ Dr(x) \ \ D\lambda_i(x)] \leq n-q+1\}$,

We claim that $O \in \overline{S'_i \cap U_1}$. for $1 \leq i \leq \ell$.

Fix an i. Remember that $S \not\subset H_i$. then we can suppose. without lost of generality, that there is a sequence $\varepsilon_m \downarrow 0$ such that

$$V^+_{\varepsilon_m} = S \cap U_1 \cap S_{\varepsilon_m} \cap \{x = (x_1;\ldots;x_n) \in \mathbf{R}^n\colon x_i > 0\} \neq \emptyset.$$

One can see without difficulty (remember the definition of U_1 and Fact that S and S_ε are closed) that $\lambda_i(x) > 0$. for $x \in V^+_{\varepsilon_m}$, and $\lambda_i(x) = 0$ for $x \in \partial\overline{V^+_{\varepsilon_m}}$.

Now, take the compact $\overline{V^+_{\varepsilon_m}}$ and consider $\overline{y}^m_i$ a maximum point of $\lambda_i|_{\overline{V^+_{\varepsilon_m}}}$.

It's obvious that $y^m_i \in \overline{S'_i \cap U_1}$. Then we have proved the claim. We have shown also, that $dim(S'_i;O) \geq 1$.

There are two possible situations:

(I) $S \neq S'$ or $S \neq S'_i$ for some $i \in \{1;\ldots;n\}$.

(II) $S = S' = S'_1 = \ldots = S'_n$.

We'll prove that the case (II) is impossible since this implies that $dim(S;O) = 1$ against our assumption.

Suppose that (II) happens and take $x \in S \cap V \cap U_1$ with $0 < \|x\| < \epsilon_0$.

Remember that (*ii*) is true. Then if $S = S'$ we conclude that $Dg(x)$ is a linear combination of $Df_{q+1}(x);\dots;Df_n(x);Dr(x)$ for all $x \in V \cap U_1 \cap S$.

Put now $\pi_i(x_1;\dots;x_n) = x_i$, for $1 \le i \le n$.

If $S = S'_i$, $D\lambda_i(x)$ is a linear combination of $Df_{q+1}(x);\dots;Df_n(x);Dr(x)$ for all $x \in S \cap U_1 \cap V$.

From the definition of λ_i it follows that $D\lambda_i(x) = \pi_i(g(x)) + x_i Dg(x)$.

Since $g(x) \neq 0$ in U we see that π_i is a linear combination of the linear transformations $Df_{q+1}(x);\dots;Df_n(x);Dr(x)$ for $x \in S \cap U_1 \cap V$, and for $1 \le i \le n$.

But the rank of $[\pi_1 \;\dots\; \pi_n]$ is equal to n, then, by (ii), $n - p + 1 = n$ and so $p = 1$.

Then (I) happens. Suppose, w.l.g., that $S \neq S'$. Since $S' \subset S$ and $(S;O)$ is irreducible we have, by Fact 5, that $dim(S';O) < dim(S;O)$. This ends the proof. ■

§3 - **The Proof in the Case of Dimension 1:** In this section we complete the proof of Theorem 0. We take an analytic set S with $dim(S;O) = 1$ and we prove it in this case. We use in this demonstration the following result which describes analytic sets of dimension 1.

Fact 11 (Local description of real analytic sets of dimension 1): *Let S be an analytic set of the open $\Omega \subseteq \mathbf{R}^n$ with $O \in S$ and $dim(S;O) = 1$. Then there exists an open neighborhood $W \subseteq \Omega$ of O and there are a finite number of analytic curves $\gamma_k:]\varepsilon;\varepsilon[\longrightarrow W$, $1 \le k \le s$ such that $S \cap W = \bigcup_{k=1}^{s} [\text{ Image } (\gamma_k)]$ and $\gamma_k(0) = O$ for all k.*

Dem.: See, for instance, [G-1].

So we have

Theorem 0: *Let Ω be an open subset of $\mathbf{R}^n$ with $O \in \Omega$ and let S be an analytic set in Ω. Consider a finite number of real analytic functions $g_1,\dots,g_\ell$ defined in Ω and put $U = \{x \in \Omega : g_i(x) > 0,\ 1 \le i \le \ell\}$. If $O \in \overline{U \cap S}$ then there are $\varepsilon > 0$ and an analytic curve $\gamma: [0;\varepsilon] \longrightarrow \Omega$ such that $\gamma(0) = O$ and $\gamma(t) \in U \cap S$ for $t > 0$.*

Dem.: By Fact 10 we need only consider the case $dim(S;O) = 1$.

Since $O \in \overline{U}$ we see that $g_j(O) \ge 0, 1 \le j \le \ell$. Put $I_0 = \{i \in \{1;\dots;\ell\} : g_i(O) = 0\}$.

Let W be the open neighborhood of O in $\mathbf{R}^n$ which is given by the Fact 11 and consider the analytic curves $\gamma_k:]-\varepsilon_1;\varepsilon_1[\longrightarrow W$, $1 \le k \le s$, given in that result, which describe $S \cap W$.

To complete the proof of the Theorem we need to show a $\overline{k} \in \{1;\dots;s\}$ and an $\varepsilon > 0$ such that $g_i(\gamma_{\overline{k}}(t)) > 0$ for $i = 1;\dots;\ell$ and $0 < t < \varepsilon$.

Define, for $(i;k) \in \{1;\dots;\ell\} \times \{1;\dots s\}$ the function $h_{ik} = g_i \circ \gamma_k$.

If $i \notin I_0$ then $h_{ik}(0) > 0$, for $1 \le k \le r$, and so we have a positive $\hat{\varepsilon}$ such that $g_i(\gamma_k(t)) > 0$ for $0 < t < \hat{\varepsilon}$ and for all k (and $i \in I \setminus I_0$). Note that if $I_0 = \emptyset$ this finishs the proof, since, in this case, any γ_k has the propriets that we want.

For $i \in I_0$ we have $h_{ik}(0) = O$, for $1 \le k \le s$. Since $O \in \overline{U \cap S}$ one can see that there exists a $\overline{k} \in \{1;\dots;s\}$ and there is a sequence $t_m \to 0$ such that $h_{i\overline{k}}(t_m) > 0$ for all $i \in I_0$.

We can suppose, w.l.g., that $t_m > 0$, and, since $\# I_0 < +\infty$ and $h_{i\overline{k}}$ is analytic, we conclude that is possible to choose a small enough $\varepsilon' > 0$ in order to hold that $h_{i\overline{k}}(t) > 0$ for $i \in I_0$ and $0 < t < \varepsilon'$.

If we put $\gamma = \gamma_{\overline{k}}$ and $\varepsilon = \min\{\hat{\varepsilon};\varepsilon'\}$ we have the thesis. ■

§4-An Application of The Curve Selection Lemma: Here we do a simple application of Theorem 0 to the Lojaciewicz decomposition of an analytic set.

We start with the definition of *semi-analytic set.*

Definition 4.0: *A set $A \subseteq \Omega$ is* **semi-analytic** *in Ω if, for all $p \in \Omega$, there is an open neighborhood $U_p \subseteq \Omega$ of p and there exist real analytic functions $f_1;\dots;f_r$ and $g_1;\dots;g_\ell$ defined in U_p such that*

$$U_p \cap A = \{x \in U_p : f_i(x) = 0 \text{ and } g_j(x) > 0, \text{ for } 1 \le i \le r \text{ and } 1 \le j \le \ell\}.$$

Fact 12: *Suppose that $O \in \Omega$ and consider S and A, respectively, an analytic and a semi-analytic set in Ω with $O \in \overline{A \cap S}$. Then there exists an analytic curve $\gamma: [0;\varepsilon] \longrightarrow \Omega$ such that $\gamma(0) = O$ and $\gamma(t) \in A \cap S$, for $0 < t \le \varepsilon$.*

Dem.: Since $O \in \Omega$ we can consider W an open neighborhood of O and the real analytic functions $h_1;\dots;h_s$ such that

$$W \cap S = \{x \in W : h_k(x) = 0,\ 1 \le k \le s\}.$$

Analogously, let V be an open neighborhood of the origin and take $f_1;\dots;f_r$ and $g_1;\dots;g_\ell$ real analytic functions such that

$$V \cap A = \{x \in V : f_i(x) = 0,\ 1 \le i \le r \text{ and } g_j(x) > 0,\ 1 \le j \le \ell\}.$$

Now consider in the open neighborhood of O, $\Omega_1 = W \cap V$ the analytic set

$$S_1 = \{x \in \Omega_1 : h_k(x) = f_i(x) = 0,\ 1 \le k \le s \text{ and } 1 \le i \le r\},$$

and the open set

$$U = \{x \in \Omega_1 : g_j(x) > 0,\ 1 \le j \le \ell\}.$$

Of course, $(S \cap A) \cap \Omega_1 = S_1 \cap U$ and $O \in \overline{U \cap S_1}$.

The result follows from an application of Theorem 0 to S_1 and U. ■

The next example shows that we can't avoid the hypothesis of $O \in \Omega$ in Fact 12 (or in Theorem 0)

Exemple 4.0: Take $\Omega = A = U = \mathbf{R}^2 \setminus \{O\}$ and $S = \{(x;y) \in \Omega : y = \sin(\frac{1}{x})\}$. A is a semi-analytic subset of Ω, S is an anlaytic set in Ω, $O \in \overline{U \cap S} = \overline{A \cap S}$ and it's evident that there isn't a *continuous curve* $\gamma: [0;\varepsilon] \longrightarrow \mathbf{R}^2$ which satisfies Theorem 0 or Fact 12.

To finish this note we present the Lojaciewicz decomposition of an analytic set and an application of Fact 12.

Lojaciewicz's Decomposition of Analytic Sets: *Let S be an analytic set of Ω and $p \in S$. There is an open neighborhod U of p, $U \subseteq \Omega$ and analytic manifolds $W_1;\dots;W_s$ wich are semi-analytic sets in U such that $(S \setminus \{p\}) \cap U = \bigcup_{i=1}^{s} W_i$.*

Dem.: See [L-1]. ■

So, as an imediate consequence of this and Fact 12 we can prove

Fact 13: *Let be S a real analytic set of the open $\Omega \subseteq \mathbf{R}^n$ with $p \in S$. If $W_1;\dots;W_s$ are the components of the Lojaciewicz's decomposition of S at p then there are analytic curves $\gamma_i: [0;\varepsilon] \longrightarrow \Omega$, $1 \le i \le s$, such that $\gamma_i(0) = p$ and $\gamma_i(t) \in W_i$ for $0 < t \le \varepsilon$.*

References

[B-1] - Bruhat, F. & Whitney, H. - Quelques Propriétès Fondamentales des esembles analytiques-réels - *Comm. math. Helv.* 33 - 1959, pp 132-160

[L-1] - Lojaciewcz, S. - *Ensembles semi-Aanlytiques.* - I.H.E.S., 1965.

[G-1] - Garcia, Manuel V.P. - *Uma Versão do "Curve Sellection Lemma" Para Conjuntos Analíticos*, - Dissertação de Mestrado - IME-USP, 1982

[M-1] - Milnor, John - *Singular Points of Complex Hypersurfaces*. - Annals of math. Studies, nº 61, Princeton Univ. Press

[N-1] - Narasimhan, R. - *Introduction to the Theory of Analytic Spaces*. - Lecture Notes in Math. nº 25, Sprigler Verlag, 1966