mathematics

Article

About the Structure of Attractors for a Nonlocal Chafee-Infante Problem

Rubén Caballero [1], Alexandre N. Carvalho [2], Pedro Marín-Rubio [3] and José Valero [1,*]

1 Centro de Investigación Operativa, Universidad Miguel Hernández de Elche, Avenida Universidad s/n, 03202 Elche, Spain; ruben.caballero@umh.es
2 Instituto de Ciências Matemáticas e de Computaçao, Universidade de São Paulo, Campus de São Carlos, Caixa Postal 668, São Carlos 13560-970, Brazil; andcarva@icmc.usp.br
3 Departamento Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, C/Tarfia, 41012 Sevilla, Spain; pmr@us.es
* Correspondence: jvalero@umh.es

Abstract: In this paper, we study the structure of the global attractor for the multivalued semiflow generated by a nonlocal reaction-diffusion equation in which we cannot guarantee the uniqueness of the Cauchy problem. First, we analyse the existence and properties of stationary points, showing that the problem undergoes the same cascade of bifurcations as in the Chafee-Infante equation. Second, we study the stability of the fixed points and establish that the semiflow is a dynamic gradient. We prove that the attractor consists of the stationary points and their heteroclinic connections and analyse some of the possible connections.

Keywords: reaction-diffusion equations; nonlocal equations; global attractors; multivalued dynamical systems; structure of the attractor; stability; Morse decomposition

MSC: 35B40; 35B41; 35B51; 35K55; 35K57

Citation: Caballero, R.; Carvalho, A.N.; Marín-Rubio, P; Valero, J. About the Structure of Attractors for a Nonlocal Chafee-Infante Problem. *Mathematics* **2021**, *9*, 353. https://doi.org/10.3390/math9040353

Academic Editor: Jose Cánovas
Received: 30 December 2020
Accepted: 5 February 2021
Published: 10 February 2021

1. Introduction

Ordinary and partial differential equations play a key role in modelling for all sciences: engineering, physics, chemistry, biology, medicine, economics and many others. The right understanding of the behaviour of solutions (in particular, well-posedness versus blow-up) means not only to predict the future of trajectories but also to establish strategies for control (i.e., optimisation). Concerning PDE and economics, it is interesting to cite the nice survey [1] and the references therein on many different problems dealing with effects such as aggregation and repulsion, optimal control, mean-field games and so on as applications.

Parabolic PDE models reflect the diffusion phenomena due to local touching of molecules and dissipation of energy, and when different internal and external factors come into play, they link naturally to some reaction-diffusion models, such as the growth versus capacity of the environment in biology or the endogenous growth versus the neo-classical theories in economics. In particular, capital accumulation distribution in space and time following spatial extensions of the continuous Ramsey model [2] by Brito [3–5] and others later use the semilinear parabolic PDE

$$\partial_t u - \alpha \Delta u = f(u) - c.$$

This spatiality introduces important issues about the steady states distribution and the dynamic evolution, convergence, local interaction among local agents and so on.

Not for the sake of generality but for real modelling purposes, in the last two decades the increment of nonlocal PDE models that attempt to capture in a more accurate way the real spreading of the problem (density of population, capital accumulation, consumption or prices and innovation indexes and so on) has been very important. Firstly we might comment about extensions by using some nonlocal operators acting in the right-hand side of the PDE and/or the boundary conditions as integral operators, leading to integro-differential equations. Among others, we can cite [6] for a system coupling capital and the pollution stock model; a population dynamic model in [7]

$$\partial_t u - \alpha \Delta u = u\left(f(u) - \alpha \int_{\mathbb{R}^N} g(x-y)u(y,t)dy \right);$$

the elliptic (stationary) counterpart in population/physics models of the Fischer-KPP [8]; and a logistic model [9]. Secondly, we wish to point out that the nonlocal extensions have also been performed on the diffusion operators as well. The literature about fractional Laplacian is vast nowadays. However, let us concentrate in an intermediate step. Coming originally from modelling of bacterial populations in biology, the introduction of a non-local viscosity in front of the Laplacian has become an interesting problem for different applications and for its mathematical study, as for example occurs in the equation

$$u_t - a(\int_\Omega g(y)u(t,y)dy)\Delta u = f(t).$$

In this way, the spreading (or aggregating/concentrating) effects are given by the increasing (resp. non-increasing) function a as a viscosity nonlocal coefficient. One should cite Prof. Chipot and his collaborators [10–16] among others for a detailed analysis, including existence, uniqueness, steady states and convergence of evolutionary solutions to equilibria.

When the reaction term f depends on the unknown u

$$u_t - a(\Phi_\Omega(u(t))\Delta u = f(t,u) \tag{1}$$

(here the functional Φ_Ω may represent a general nonlocal functional acting over the whole domain Ω, for instance, $\|u(t)\|^2_{H^1_0}$ or $\int_\Omega g(y)u(t,y)dy$), equilibria are difficult to analyse. Oppositely to ordinary differential equations, the analysis of the existence of stationary states for the above problem is much more involved. Additionally, comparing the reaction-diffusion equations with local diffusion, another difficulty is that in general a Lyapunov functional is not known to exist in most cases.

The dynamical analysis of problem (1) and in particular the existence of global attractors has been established till now in several papers (cf. [17–21]). Other differential operators such as the p-Laplacian coupled with nonlocal viscosity has also been considered (cf. [21–23]). However, in general little is known about the internal structure of the attractor, which is very important as it gives us a deep insight into the long-term dynamics of the problem. When we manage to obtain a Lyapunov functional some insights can be obtained.

If we consider the non-local equation

$$\frac{\partial u}{\partial t} - a(\|u\|^2_{H^1_0})\frac{\partial^2 u}{\partial x^2} = \lambda f(u) \tag{2}$$

with Dirichlet boundary conditions, then it is possible to define a suitable Lypaunov functional. In [18] it is shown that regular and strong solutions generate (possibly) multivalued semiflows having a global attractor which is described by the unstable set of the stationary points. Although this is already a good piece of information, our goal is to describe the structure of the attractor as accurately as possible. For this aim we need to study the particular situation where the domain is one-dimensional and the function f is of the type of the standard Chafee-Infante problem, for which the dynamics inside the attractor has been completely understood [24].

The first step when studying the structure of the attractor consists of analysing the stationary points. In the case where the function f is odd and Equation (2) generates a continuous semigroup, the existence of fixed points of the type given in the Chafee-Infante problem was established in [25]. Moreover, if a is non-decreasing, then they coincide with the ones in the Chafee-Infante problem, and moreover, in [26] the stability and hyperbolicity of the fixed points was studied. In this paper we extend these results for a more general function f (not necessarily odd and for which we do not known whether the Cauchy problem has a unique solution or not), showing that Equation (2) undergoes the same cascade of bifurcations as the Chafee-Infante equation. Moreover, when we allow the function a to decrease, though the problem possesses at least the same fixed point as in the Chafee-Infante problem, we show that more equilibria can appear. For a non-decreasing function a and an odd function f we prove also that even when uniqueness fails, the stability of the fixed points is the same as for the corresponding ones in the Chafee-Infante problem. Finally, we are able to prove that in this last case we have a dynamically gradient semiflow with respect to the disjoint family of isolated weakly invariant sets generated by the equilibria, which is ordered by the number of zeros of the fixed points. More precisely, the attractor consists of the set of equilibria and their heteroclinic connections and a connection from a fixed point to another is allowed only if the number of zeros of the first one is greater.

In Section 3 we study the existence of strong solutions of the Cauchy problem in the space H_0^1. In Section 4 we prove that strong solutions generate a multivalued semiflow in H_0^1 having a global attractor which is equal to the unstable set of the stationary points. In Section 5 we study the existence and properties of equilibria. In Section 6 we analyse the stability of the fixed points and establish that the semiflow is dynamically gradient.

2. Setting of the Problem

Let us consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|^2_{H_0^1})\dfrac{\partial^2 u}{\partial x^2} = \lambda f(u) + h(t), \quad t > 0, x \in \Omega, \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{3}$$

where $\Omega = (0,1)$ and $\lambda > 0$. Throughout the paper we will use the following conditions (but not all of them at the same time):

(A1) $f \in C(\mathbb{R})$.
(A2) $f(0) = 0$.
(A3) $f'(0)$ exists and $f'(0) = 1$.
(A4) f is strictly concave if $u > 0$ and strictly convex if $u < 0$.
(A5) Growth and dissipation conditions: for $p \geq 2$, $C_i > 0$, $i = 1, .., 4$, we have

$$|f(u)| \leq C_1 + C_2|u|^{p-1}, \tag{4}$$

$$f(u)u \leq C_3 - C_4|u|^p, \text{ if } p > 2, \tag{5}$$

$$\limsup_{u \to \pm\infty} \frac{f(u)}{u} \leq 0, \text{ if } p = 2. \tag{6}$$

(A6) The function $a \in C(\mathbb{R}^+)$ satisfies:

$$a(s) \geq m > 0.$$

(A7) The function $a \in C(\mathbb{R}^+)$ satisfies:

$$a(s) \leq M_1, \quad \forall s \geq 0,$$

where $M_1 > 0$.
(A8) The function $a \in C(\mathbb{R}^+)$ is non-decreasing.
(A9) $h \in L^2_{loc}(0, +\infty; L^2(\Omega))$.
(A10) h does not depend on time and $h \in L^2(\Omega)$.

We define the function $\mathcal{F}(u) = \int_0^u f(s)ds$. We observe that from (4) we have

$$|\mathcal{F}(s)| \leq \widetilde{C}(1 + |s|^p) \quad \forall s \in \mathbb{R}, \tag{7}$$

whereas (5) implies

$$\mathcal{F}(s) \leq \widetilde{\kappa} - \widetilde{\alpha}_1 |s|^p. \tag{8}$$

Additionally, from condition (6) it follows that for all $\varepsilon > 0$, there exists a constant $M > 0$ such that $\frac{f(u)}{u} \leq \varepsilon$, for all $|u| \geq M$. Hence, there exists $m_\varepsilon > 0$ such that

$$f(u)u \leq m_\varepsilon + \varepsilon u^2, \quad \forall u \in \mathbb{R}. \tag{9}$$

In addition, it follows that

$$\mathcal{F}(u) \leq \varepsilon u^2 + C_\varepsilon, \tag{10}$$

where $C_\varepsilon > 0$. These two inequaities are also true under condition (5).

The main aim of this paper consists of describing in as much detail as possible the internal structure of the global attractor in a similar way as for the classical Chafee-Infante equation.

Some of these conditions will be used all the time, whereas other ones will be used only in certain results. In particular, the function h will be considered as a time-dependent function satisfying (A9) only for establishing the existence of solution for problem (3). However, since we will study the asymptotic behaviour of solutions in the autonomous situation, for the second part concerning the existence and properties of global attractors, the function h will be time-independent, so assumption (A10) will be used instead. Finally, in order to study the structure of the global attractors in terms of the stationary points and their possible heteroclinic connections, we will assume that $h \equiv 0$.

Throughout the paper, $\|\cdot\|_X$ will denote the norm in the Banach space X.

3. Existence of Solutions

In this section we will establish the existence of strong solutions for problem (3) with an initial condition in the phase space $H_0^1(\Omega)$. Although we will follow along the same lines as a similar result given in [18], we would like to point out that in the present case, as we are working in a one-dimensional problem, the assumptions for the function f are much weaker. In particular, we do not need to impose a growth assumption of any kind.

Definition 1. *For $u_0 \in L^2(\Omega)$, a weak solution to (3) is an element $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, for any $T > 0$, such that*

$$\frac{d}{dt}(u,v) + a(\|u\|^2_{H_0^1})(\nabla u, \nabla v) = \lambda(f(u),v) + (h(t),v) \quad \forall v \in H_0^1(\Omega), \tag{11}$$

where the equation is understood in the sense of distributions.

As usual, let $A : D(A) \to H,\ D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, be the operator $A = -\dfrac{d^2}{dx^2}$ with Dirichlet boundary conditions. This operator is the generator of a C_0-semigroup $T(t) = e^{-At}$.

Definition 2. *For $u_0 \in H_0^1(\Omega)$, a strong solution to (3) is a weak solution with the extra regularity $u \in L^\infty(0,T;H_0^1(\Omega))$, $u \in L^2(0,T;D(A))$ and $\dfrac{du}{dt} \in L^2(0,T;L^2(\Omega))$ for any $T > 0$.*

Remark 1. *We observe that if u is a strong solution, then $u \in C([0,T];H_0^1(\Omega))$ (see [27] p.102). This way, the initial condition makes sense.*

Remark 2. *Since* $\frac{du}{dt} \in L^2(0,T;L^2(\Omega))$ *for any strong solution, in this case equality (11) is equivalent to the following one:*

$$\int_0^T \int_\Omega \frac{du(t,x)}{dt}\xi(t,x)dxdt - \int_0^T a(\|u(t)\|^2_{H_0^1}) \int_\Omega \frac{\partial^2 u}{\partial x^2}\xi dxdt \\ = \int_0^T \int_\Omega \lambda f(u(t,x))\xi(t,x)dxdt + \int_0^T \int_\Omega h(t,x)\xi(t,x)dxdt, \tag{12}$$

for all $\xi \in L^2(0,T;L^2(\Omega))$.

Theorem 1. *Assume conditions (A1), (A6) and (A9). Assume also the existence of constants* $\beta, \gamma > 0$ *such that*

$$f(u)u \leq \gamma + \beta u^2 \text{ for all } u \in \mathbb{R}. \tag{13}$$

Then, for any $u_0 \in H_0^1(\Omega)$, *problem (3) has at least one strong solution.*

Remark 3. *Assumption (13) is weaker than the dissipative property (9) as the constant* ε *is arbitrarily small. Due to the fact that we are working in a one-dimensional domain, no growth condition of the type given in (A5) is necessary in order to prove existence of solutions. Additionally, (13) implies that*

$$F(u) \leq \widetilde{\gamma} + \widetilde{\beta}u^2 \tag{14}$$

for some constants $\widetilde{\gamma}, \widetilde{\beta} > 0$.

Proof. Consider a fixed value $T > 0$. In order to use the Faedo–Galerkin method, let $\{w_j\}_{j\geq 1}$ be the sequence of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ with homogeneous Dirichlet boundary conditions, which forms a special basis of $L^2(\Omega)$. Since Ω is a bounded regular domain, it is known that $\{w_j\} \subset H_0^1(\Omega)$ and that $\cup_{n\in\mathbb{N}} V_n$ is dense in the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$, where $V_n = span[w_1, \ldots, w_n]$. As usual, P_n will be the orthogonal projection in $L^2(\Omega)$, that is,

$$z_n := P_n z = \sum_{j=1}^n (z, w_j)w_j,$$

and λ_j will be the eigenvalues associated with the eigenfunctions w_j. For each integer $n \geq 1$, we consider the Galerkin approximations

$$u_n(t) = \sum_{j=1}^n \gamma_{nj}(t)w_j,$$

which are given by the following nonlinear ODE system:

$$\begin{cases} \frac{d}{dt}(u_n, w_i) + a(\|u_n\|^2_{H_0^1})(\nabla u_n, \nabla w_i) = \lambda(f(u_n), w_i) + (h, w_i) \quad \forall i = 1, \ldots, n, \\ u_n(0) = P_n u_0. \end{cases} \tag{15}$$

We observe that $P_n u_0 \to u_0$ in $H_0^1(\Omega)$. This Cauchy problem possesses a solution on some interval $[0, t_n)$ and by the estimates in the space $L^2(\Omega)$ of the sequence $\{u_n\}$ given below for any $T > 0$, such a solution can be extended to the whole interval $[0, T]$.

Firstly, multiplying the equation in (15) by $\gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2} + a(\|u_n\|^2_{H_0^1})\|u_n(t)\|^2_{H_0^1} = \lambda(f(u_n(t), u_n(t)) + (h(t), u_n(t)) \quad \text{for a.e. } t \in (0, t_n). \tag{16}$$

By using the Young and Poincaré inequalities we deduce that

$$(h(t), u_n(t)) \leq \frac{m}{2}\|u_n(t)\|^2_{H^1_0} + \frac{1}{2\lambda_1 m}\|h(t)\|^2_{L^2},$$

where m is the constant from (A6). Hence, from (A6), (13) and (16) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2} + \frac{m}{2}\|u_n(t)\|^2_{H^1_0} \leq \lambda\gamma|\Omega| + \beta\lambda\|u_n(t)\|^2_{L^2} + \frac{1}{2\lambda_1 m}\|h(t)\|^2_{L^2}.$$

We infer that

$$\begin{aligned}\|u_n(t)\|^2_{L^2} &\leq \|u_n(0)\|^2_{L^2}e^{2\beta\lambda t} + \int_0^t e^{2\beta\lambda(t-s)}\left(2\lambda\gamma|\Omega| + \frac{1}{\lambda_1 m}\|h(s)\|^2_{L^2}\right)ds \\ &\leq \|u_n(0)\|^2_{L^2}e^{2\beta\lambda T} + K_1(T).\end{aligned} \tag{17}$$

Therefore, the solution exists on any given interval $[0, T]$ and

$$\{u_n\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)). \tag{18}$$

Now, we multiply Equation (3) by $\frac{du_n}{dt}$ to obtain

$$\|\frac{du_n}{dt}(t)\|^2_{L^2} + a(\|u_n\|^2_{H^1_0})\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H^1_0} = \frac{d}{dt}\int_\Omega \lambda\mathcal{F}(u_n)dx + (h(t), \frac{du_n}{dt}).$$

Introducing

$$A(s) = \int_0^s a(r)dr \tag{19}$$

we have

$$\frac{1}{2}\|\frac{du_n}{dt}(t)\|^2_{L^2} + \frac{d}{dt}\left(\frac{1}{2}A(\|u_n\|^2_{H^1_0}) - \int_\Omega \lambda\mathcal{F}(u_n)dx\right) \leq \frac{1}{2}\|h(t)\|^2_{L^2}.$$

Integrating the previous expression between 0 and t we get

$$\begin{aligned}&\frac{1}{2}A(\|u_n(t)\|^2_{H^1_0}) + \lambda\int_\Omega \mathcal{F}(u_n(0))dx + \frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|^2_{L^2}ds \\ &\leq \frac{1}{2}A(\|u_n(0)\|^2_{H^1_0}) + \lambda\int_\Omega \mathcal{F}(u_n(t))dx + \frac{1}{2}\int_0^t \|h(s)\|^2_{L^2}ds.\end{aligned} \tag{20}$$

By (A6), (14) and (17) it follows that

$$\begin{aligned}&\frac{m}{2}\|u_n(t)\|^2_{H^1_0} + \lambda\int_\Omega \mathcal{F}(u_n(0))dx + \frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|^2_{L^2}ds \\ &\leq \frac{1}{2}A(\|u_n(0)\|^2_{H^1_0}) + \lambda\widetilde{\beta}\|u_n(t)\|^2_{L^2} + \lambda\widetilde{\gamma}|\Omega| + K_2(T) \\ &\leq \frac{1}{2}A(\|u_n(0)\|^2_{H^1_0}) + \lambda\widetilde{\beta}e^{2\beta\lambda T}\|u_n(0)\|^2_{L^2} + K_3(T).\end{aligned} \tag{21}$$

Since $dim(\Omega) = 1$, $H^1_0(\Omega) \subset L^\infty(\Omega)$, so $u_n(0)$ is bounded in $L^\infty(\Omega)$. Thus, as f maps bounded sets of $\mathbb{R}$ into bounded ones, $\mathcal{F}(u_n(0))$ is bounded in $L^\infty(\Omega)$ as well. Therefore, we deduce that

$$\{u_n\} \text{ is bounded in } L^\infty(0, T; H^1_0(\Omega))$$

and

$$\frac{du_n}{dt} \text{ is bounded in } L^2(0, T; L^2(\Omega)). \tag{22}$$

By using again the embedding $H_0^1(\Omega) \subset L^\infty(\Omega)$ we obtain that u_n is bounded in the space $L^\infty(0,T;L^\infty(\Omega))$. Thus,

$$f(u_n) \text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)). \tag{23}$$

Additionally, we deduce that $\|u_n(t)\|^2_{H_0^1}$ is uniformly bounded in $[0,T]$, and then by the continuity of the function $a(\cdot)$ we get that the sequence $a\Big(\|u_n(t)\|^2_{H_0^1}\Big)$ is also uniformly bounded in $[0,T]$.

Finally, multiplying (15) by $\lambda_j\gamma_{ni}(t)$ and summing from $i=1$ to n we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H_0^1} + m\|\Delta u_n\|^2_{L^2} \le \lambda(f(u_n), -\Delta u_n) + (h(t), -\Delta u).$$

By (23) and applying the Young inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H_0^1} + m\|\Delta u_n\|^2_{L^2} \le \frac{\lambda^2}{m}\|f(u_n)\|^2_{L^2} + \frac{m}{4}\|\Delta u_n\|^2_{L^2} + \frac{1}{m}\|h(t)\|^2_{L^2} + \frac{m}{4}\|\Delta u\|^2_{L^2}.$$

Integrating the previous expression between 0 and t, it follows that

$$\|u_n(t)\|^2_{H_0^1} + m\int_0^t \|\Delta u_n(s)\|^2_{L^2}ds \le \|u_n(0)\|^2_{H_0^1} + \frac{2\lambda^2}{m}\int_0^t \|f(u_n(s))\|^2_{L^2}ds + \frac{2}{m}\int_0^t \|h(s)\|^2_{L^2}ds.$$

Taking into account (23), the last inequality implies that

$$u_n \text{ is bounded in } L^2(0,T;D(A)), \tag{24}$$

so $\{-\Delta u_n\}$ and $\{a(\|u_n\|^2_{H_0^1})\Delta u_n\}$ are bounded in $L^2(0,T;L^2(\Omega))$.

As a consequence, there exist $u \in L^\infty(0,T;H_0^1(\Omega))$ and a subsequence u_n (relabelled the same) such that

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)),\\ u_n &\rightharpoonup u \text{ in } L^2(0,T;D(A)),\\ f(u_n) &\overset{*}{\rightharpoonup} \chi \text{ in } L^\infty(0,T;L^\infty(\Omega)),\\ a(\|u_n\|^2_{H_0^1}) &\overset{*}{\rightharpoonup} b \text{ in } L^\infty(0,T), \end{aligned} \tag{25}$$

where $\rightharpoonup$ ($\overset{*}{\rightharpoonup}$) stands for the weak (weak star) convergence. By (22) and (24) the Aubin–Lions compactness lemma gives that $u_n \to u$ in $L^2(0,T;H_0^1(\Omega))$, so $u_n(t) \to u(t)$ in $H_0^1(\Omega)$ a.e. on $(0,T)$. Consequently, there exists a subsequence u_n, relabelled the same, such that $u_n(t,x) \to u(t,x)$ a.e. in $\Omega \times (0,T)$.

Moreover, thanks to the inequality

$$\|u_n(t_2) - u_n(t_1)\|^2_{L^2} = \left\|\int_{t_1}^{t_2} \frac{d}{dt}u_n(s)ds\right\|^2_{L^2} \le \|\frac{d}{dt}u_n\|^2_{L^2(0,T;L^2(\Omega))}\,|t_2-t_1| \quad \forall t_1,t_2 \in [0,T],$$

(21), (22) and $H_0^1(\Omega) \subset\subset L^2(\Omega)$, the Ascoli-Arzelà theorem implies that $\{u_n\}$ converges strongly in $C([0,T];L^2(\Omega))$ for all $T>0$. Therefore, we obtain from (21) that $u_n(t) \rightharpoonup u(t)$ in $H_0^1(\Omega)$, for any $t \ge 0$.

Additionally, by (25) we have that $P_n f(u_n)) \rightharpoonup \chi$ in $L^q(0,T;L^q(\Omega))$ for any $q \ge 1$ (see [28] p.224). Since f is continuous, it follows that $f(u_n(t,x)) \to f(u(t,x))$ a.e. in $\Omega \times (0,T)$. Therefore, in view of (25), by ([29] Lemma 1.3) we have that $\chi = f(u)$.

As a consequence, by the continuity of a we get that

$$a(\|u_n(t)\|^2_{H_0^1}) \to a(\|u(t)\|^2_{H_0^1}) \quad \text{a.e. on } (0,T).$$

Since the sequence is uniformly bounded, by Lebesgue's theorem this convergence takes place in $L^2(0,T)$, so $b = a(\|u\|^2_{H^1_0})$. Thus,

$$a(\|u_n\|^2_{H^1_0})\Delta u_n \rightharpoonup a(\|u\|^2_{H^1_0})\Delta u, \quad \text{in} \quad L^2(0,T;L^2(\Omega)).$$

Therefore, we can pass to the limit to conclude that u is a strong solution.

It remains to show that $u(0) = u_0$ which makes sense since $u \in C([0,T];H^1_0(\Omega))$ (see Remark 4). Indeed, let be $\phi \in C^1([0,T];H^1_0(\Omega))$ with $\phi(T) = 0$, $\phi(0) \neq 0$. We multiply the equation in (3) and (15) by ϕ and integrate by parts in the t variable to obtain that

$$\int_0^T \Big(-(u(t),\phi'(t)) - a(\|u(t)\|^2_{H^1_0})(\Delta u(t),\phi(t)) \Big) dt \\ = \int_0^T (\lambda f(u(t)) + h(t),\phi(t))dt + (u(0),\phi(0)), \tag{26}$$

$$\int_0^T \Big(-(u_n(t),\phi'(t)) - a(\|u_n(t)\|^2_{H^1_0})(\Delta u_n(t),\phi(t)) \Big) dt \\ = \int_0^T (\lambda f(u_n(t)) + h(t),\phi(t))dt + (u_n(0),\phi(0)). \tag{27}$$

In view of the previous convergences, we can pass to the limit in (27). Taking into account (26) and bearing in mind $u_n(0) = P_n u_0 \to u_0$, since $\phi(0) \in H^1_0(\Omega)$ is arbitrary, we infer that $u(0) = u_0$. □

4. The Existence and Structure of Attractors

In this section, we will prove the existence of a global attractor for the semiflow generated by strong solutions in the autonomous case. Thus, the function h will be an independent of time function satisfying (A10) instead of (A9). Additionally, we will establish that the attractor is equal to the unstable set of the stationary points (see the definition in (45)).

Throughout this section, for a metric space X with metric d we will denote by $dist_X(C,D)$ the Hausdorff semidistance from C to D, that is,

$$dist_X(C,D) = \sup_{c\in C} \inf_{d\in D} \rho(c,d).$$

Let us consider the phase space $X = H^1_0(\Omega)$ and the sets

$$K(u_0) = \{u(\cdot) : u \text{ is a strong solution of (3) such that } u(0) = u_0\},$$

$$\mathcal{R} = \cup_{u_0 \in X} K(u_0).$$

Denote by $P(X)$ the class of nonempty subsets of X. We define the (possibly multivalued) map $G : \mathbb{R}^+ \times X \to P(X)$ by

$$G(t,u_0) = \{u(t) : u \in \mathcal{R} \text{ and } u(0) = u_0\}. \tag{28}$$

In order to study the map G let us consider the following axiomatic properties of the set $\mathcal{R}$:

(K1) For every $x \in X$ there is $\phi \in \mathcal{R}$ satisfying $\phi(0) = x$.
(K2) $\phi_\tau(\cdot) := \phi(\cdot + \tau) \in \mathcal{R}$ for every $\tau \geq 0$ and $\phi \in \mathcal{R}$ (translation property).
(K3) Let $\phi_1, \phi_2 \in \mathcal{R}$ be such that $\phi_2(0) = \phi_1(s)$ for some $s > 0$. Then, the function ϕ defined by

$$\phi(t) = \begin{cases} \phi_1(t) & 0 \leq t \leq s, \\ \phi_2(t-s) & s \leq t, \end{cases}$$

belongs to $\mathcal{R}$ (concatenation property).

(K4) For every sequence $\{\phi^n\} \subset \mathcal{R}$ satisfying $\phi^n(0) \to x_0$ in X, there is a subsequence $\{\phi^{n_k}\}$ and $\phi \in \mathcal{R}$ such that $\phi^{n_k}(t) \to \phi(t)$ for every $t \geq 0$.

Assuming conditions (A1), (A6), (A10) and (13), property (K1) follows from Theorem 1, whereas (K2) and (K3) can be proved easily using equality (12). By ([30] Proposition 2) or ([31] Lemma 9) we know that $\mathcal{R}$ fulfilling (K1) and (K2) gives rise to a multivalued semiflow G through (28) (m-semiflow for short), which means that:

- $G(0, x) = x$ for all $x \in X$;
- $G(t + s, x) \subset G(t, G(s, x))$ for all $t, s \geq 0$ and $x \in X$.

Moreover, (K3) implies that the m-semiflow is strict, that is, $G(t+s, x) = G(t, G(s, x))$ for all $t, s \geq 0$ and $x \in X$.

We will show first that the m-semiflow G possesses a bounded absorbing set in the space $L^2(\Omega)$ and that property (K4) is satisfied.

Lemma 1. *Assume conditions (A1), (A6), (A10) and (13). Given $\{u^n\} \subset \mathcal{R}$, $u^n(0) \to u_0$ weakly in $H_0^1(\Omega)$, there exists a subsequence of $\{u^n\}$ (relabelled the same) and $u \in K(u_0)$ such that*

$$u^n(t) \to u(t) \text{ in } H_0^1(\Omega),\ \forall t > 0.$$

Additionally, if $u^n(0) \to u_0$ strongly in $H_0^1(\Omega)$, then for $t_n \to 0$ we get $u^n(t_n) \to u_0$ strongly in $H_0^1(\Omega)$.

Proof. Since $\dfrac{du^n}{dt} \in L^2(0, T; L^2(\Omega))$ and $u^n \in L^2(0, T; H_0^1(\Omega))$, we have by ([27] p. 102) that

$$\frac{d}{dt}\|u^n\|_{H_0^1}^2 = 2(-\Delta u^n, u_t^n) \text{ for a.a. } t \tag{29}$$

and $u^n \in C([0, T]; H_0^1(\Omega))$. Additionally, as $f(u^n) \in L^\infty(0, T; L^\infty(\Omega))$, by regularization one can show that $(F(u^n(t)), 1)$ is an absolutely continuous function on $[0, T]$ and

$$\frac{d}{dt}(F(u^n(t)), 1) = (f(u^n(t)), \frac{du^n}{dt}) \text{ for a.a. } t > 0. \tag{30}$$

By a similar argument as in Theorem 1, there is a subsequence of u^n such that

$$\begin{aligned} u^n &\text{ is bounded in } L^\infty(0, T; L^\infty(\Omega)), \\ u^n &\text{ is bounded in } L^\infty(0, T; H_0^1(\Omega)), \\ f(u^n) &\text{ is bounded in } L^\infty(0, T; L^\infty(\Omega)), \\ u^n &\text{ is bounded in } L^2(0, T; D(A)). \end{aligned} \tag{31}$$

Therefore, arguing as in the proof of Theorem 1, there exist $u \in K(u_0)$ and a subsequence u^n, relabelled the same, such that

$$\begin{aligned} u^n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^2(0,T;D(A)) \\ f(u^n) &\overset{*}{\rightharpoonup} f(u) \text{ in } L^\infty(0,T;L^\infty(\Omega)) \\ \frac{du^n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;L^2(\Omega)) \\ a(\|u^n\|^2_{H_0^1})\Delta u^n &\rightharpoonup a(\|u\|^2_{H_0^1})\Delta u \text{ in } L^2(0,T;L^2(\Omega)), \\ u^n &\to u \text{ in } L^2(0,T;H_0^1(\Omega)), \\ u^n &\to u \text{ in } C([0,T],L^2(\Omega)), \\ u^n(t) &\rightharpoonup u(t) \text{ in } H_0^1(\Omega) \quad \forall t \in (0,T]. \end{aligned} \tag{32}$$

We also need to prove that $u^n(t) \to u(t)$ in $H_0^1(\Omega)$ for all $t \in (0,T]$. To that end, we multiply (3) by u_t^n, and using (A10), (29) and (31) we have

$$\frac{1}{2}\left\|\frac{du^n}{dt}\right\|^2_{L^2} + \frac{d}{dt}\left(\frac{1}{2}A(\|u^n(t)\|^2_{H_0^1})\right) \le C.$$

Thus, we obtain

$$A(\|u^n(t)\|^2_{H_0^1}) \le A(\|u^n(s)\|^2_{H_0^1}) + 2C(t-s), \quad t \ge s \ge 0.$$

Since this inequality is also true for $u(\cdot)$, the functions $Q_n(t) = A(\|u^n(t)\|^2_{H_0^1}) - 2Ct$, $Q(t) = A(\|u(t)\|^2_{H_0^1}) - 2Ct$ are continuous and non-increasing in $[0,T]$. Moreover, from (32) we deduce that

$$Q_n(t) \to Q(t) \quad \text{for a.e. } t \in (0,T).$$

Take $0 < t \le T$ and $0 < t_j < t$ such that $t_j \to t$ and $Q_n(t_j) \to Q(t_j)$ for all j. Then

$$Q_n(t) - Q(t) \le Q_n(t_j) - Q(t) \le |Q_n(t_j) - Q(t_j)| + |Q(t_j) - Q(t)|.$$

For any $\delta > 0$ there exist $j(\delta)$ and $N(j(\delta))$ such that $Q_n(t) - Q(t) \le \delta$ if $n \ge N$. Then $\limsup Q_n(t) \le Q(t)$, so $\limsup \|u^n(t)\|^2_{H_0^1} \le \|u(t)\|^2_{H_0^1}$, which follows by contradiction using the continuity of the function $A(s)$. As $u^n(t) \to u(t)$ weakly in $H_0^1(\Omega)$ implies that $\liminf \|u^n(t)\|^2_{H_0^1} \ge \|u(t)\|^2_{H_0^1}$, we obtain

$$\|u^n(t)\|^2_{H_0^1} \to \|u(t)\|^2_{H_0^1},$$

so that $u^n(t) \to u(t)$ strongly in $H_0^1(\Omega)$.

Finally, if $u^n(0) \to u_0$ strongly in $H_0^1(\Omega)$ and we take $t_n \to 0$, then

$$Q_n(t_n) - Q(0) \le Q_n(0) - Q(0) = A(\|u^n(0)\|^2_{H_0^1}) - A(\|u_0\|^2_{H_0^1}) \to 0,$$

so $\limsup Q_n(t_n) \le Q(0)$. Repeating the above argument, we infer that $u^n(t_n) \to u_0$ strongly in $H_0^1(\Omega)$. □

Corollary 1. *Assume the conditions of Lemma 1. Then the set $\mathcal{R}$ satisfies condition $(K4)$.*

The map $t \mapsto G(t,x)$ is said to be upper semicontinuous if for every $x \in X$ and for an arbitrary neighbourhood $O(G(t,x))$ in X there is $\delta > 0$ such that as soon as $d(y,x) < \delta$, we have $G(t,y) \subset O$.

Proposition 1. *Assume the conditions of Lemma 1. The multivalued semiflow G is upper semicontinuous for all $t \geq 0$. Additionally, it has compact values.*

Proof. By contradiction let us assume that there exist $t \geq 0$, $u_0 \in H_0^1(\Omega)$, a neighbourhood $O(G(t,u_0))$ and sequences $\{y_n\}$, $\{u_0^n\}$ such that $y_n \in G(t,u_0^n)$, u_0^n converges strongly to u_0 in $H_0^1(\Omega)$ and $y_n \notin O(G(t,u_n))$ for all $n \in \mathbb{N}$. Thus, there exists $u^n \in K(u_0^n)$ such that $y_n = u^n(t)$. From Lemma 1 there exists a subsequence of y_n which converges to some $y \in G(t,u_0)$. This contradicts $y_n \notin O(G(t,u_0))$ for any $n \in \mathbb{N}$. □

In order to prove the existence of an absorbing set in the space $L^2(\Omega)$ we need to use the stronger condition (A5) instead of (13).

Proposition 2. *Assume that conditions (A1), (A5), (A6) and (A10) hold. Then the m-semiflow G has a bounded absorbing set in $L^2(\Omega)$; that is, there exists a constant $K > 0$ such that for any $R > 0$ there is a time $t_0 = t_0(R)$ such that*

$$\|y\|_{L^2} \leq K \quad \textit{for all} \quad t \geq t_0,\ y \in G(t,u_0), \tag{33}$$

where $\|u_0\|_{L^2} \leq R$. Moreover, there is $L > 0$ such that

$$\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq L \quad \textit{for all} \quad t \geq t_0,\ u \in K(u_0). \tag{34}$$

Proof. By multiplying Equation (3) by u and using (A6) and (9), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + m\|u(t)\|_{H_0^1}^2 &\leq (f(u),u) + (h,u) \\ &\leq m_\varepsilon|\Omega| + \varepsilon\|u(t)\|_{L^2}^2 + \frac{1}{2\lambda_1 m}\|h\|_{L^2}^2 + \frac{\lambda_1 m}{2}\|u\|_{L^2}^2. \end{aligned} \tag{35}$$

By using the Poincaré inequality it follows that

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq 2m_\varepsilon|\Omega| + 2(\varepsilon - \frac{m}{2}\lambda_1)\|u(t)\|_{L^2}^2 + \frac{1}{\lambda_1 m}\|h\|_{L^2}^2 = -\delta\|u(t)\|_{L^2}^2 + \kappa,$$

where $\delta = m\lambda_1 - 2\varepsilon$, $\kappa = 2m_\varepsilon|\Omega| + \frac{1}{\lambda_1 m}\|h\|_{L^2}^2$. We take a small enough $\varepsilon > 0$ so that $\delta > 0$. Then Gronwall's lemma gives

$$\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 e^{-\delta t} + \frac{\kappa}{\delta}. \tag{36}$$

Hence, taking

$$t \geq t_0 = \frac{1}{\delta}\ln\left(\frac{\delta R^2}{\kappa}\right)$$

we get (33) for $K = \sqrt{\frac{2\kappa}{\delta}}$.

On the other hand, using again the Poincaré inequality from (35) we get

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + \left(\frac{m\lambda_1 - 2\varepsilon}{\lambda_1}\right)\|u(t)\|_{H_0^1}^2 \leq \kappa$$

and integrating from t to $t+1$ we obtain

$$\left(\frac{m\lambda_1 - 2\varepsilon}{\lambda_1}\right)\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq \|u(t)\|_{L^2}^2 + \kappa.$$

Therefore, applying (33) and (34) follows. □

Further, in order to obtain an absorbing set in $H_0^1(\Omega)$ we need to assume additionally that either the function $a(\cdot)$ is bounded above or that it is non-decreasing.

Proposition 3. *Assume the conditions in Proposition 2 and that either (A7) or (A8) holds true. Then there exists an absorbing set B_1 for G, which is compact in $H_0^1(\Omega)$.*

Proof. In view of Proposition 2 we have an absorbing set B_0 in $L^2(\Omega)$. Let $K > 0$ be such that $\|y\| \le K$ for all $y \in B_0$.

Through multiplying (3) by u and using (9) and (36) we get

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + a\left(\|u(t)\|_{H_0^1}^2\right)\|u(t)\|_{H_0^1}^2 \le 2m_\varepsilon|\Omega| + 2\varepsilon\|u(t)\|_{L^2}^2 + \frac{1}{\lambda_1 m}\|h\|_{L^2}^2$$
$$\le K_1 + K_2\|u(0)\|_{L^2}^2.$$

Thus, integrating between t and $t+r$, $0 < r \le 1$, we deduce by using (36) again that

$$\begin{aligned} &\|u(t+r)\|_{L^2}^2 + \int_t^{t+r} a\left(\|u(s)\|_{H_0^1}^2\right)\|u(s)\|_{H_0^1}^2 ds \\ &\le K_1 + K_2\|u(0)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 \le K_3\|u(0)\|_{L^2}^2 + K_4. \end{aligned} \tag{37}$$

Additionally, if $p > 2$ in (A5), we multiply again by (3) by u and use (5) and (A6) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \frac{m}{2}\|u(t)\|_{H_0^1}^2 + C_4\|u(t)\|_{L^p}^p \le C_3 + \frac{1}{2\lambda_1 m}\|h\|_{L^2}^2.$$

Integrating over $(t, t+r)$ we have

$$\|u(t+r)\|_{L^2}^2 + 2C_4\int_t^{t+r}\|u(s)\|_{L^p}^p ds \le K_5 + \|u(t)\|_{L^2}^2 \le K_6 + \|u(0)\|_{L^2}^2. \tag{38}$$

If we assume (A7), by (37) and (A6) we have that

$$\int_t^{t+r} A(\|u(s)\|_{H_0^1}^2)ds \le \int_t^{t+r} M_1\|u(s)\|_{H_0^1}^2 ds \le K_7(1 + \|u(0)\|_{L^2}^2). \tag{39}$$

If we assume (A8), by (37) we obtain

$$\begin{aligned} \int_t^{t+r} A(\|u(s)\|_{H_0^1}^2)ds &= \int_t^{t+r}\int_0^{\|u(s)\|_{H_0^1}^2} a(r)drds \\ &\le \int_t^{t+r} a\left(\|u(s)\|_{H_0^1}^2\right)\|u(s)\|_{H_0^1}^2 ds \le K_3\|u(0)\|_{L^2}^2 + K_4. \end{aligned} \tag{40}$$

On the other hand, by (7) we get

$$-\int_\Omega F(u(t))dx \ge -\widetilde{C}\int_\Omega (1 + |u(t)|^p)dx. \tag{41}$$

By using (29) and (30) we can argue as in Theorem 1 to obtain

$$\frac{1}{2}\|u_t\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}A(\|u(t)\|_{H_0^1}^2 - \int_\Omega \lambda\mathcal{F}(u_n)dx\right) \le \frac{1}{2}\|h\|_{L^2}^2.$$

Since (38)–(41) imply that

$$\int_t^{t+r}\left(\frac{1}{2}A(\|u(s)\|_{H_0^1}^2 - \int_\Omega \lambda\mathcal{F}(u(s))dx\right)ds \le K_8 + K_9\|u(0)\|_{L^2}^2,$$

we can apply the uniform Gronwall lemma to get

$$\frac{1}{2}A(\|u(t+r)\|^2_{H^1_0}) - \int_\Omega \lambda\mathcal{F}(u(t+r))dx \leq \frac{K_8 + K_9\|u(0)\|^2_{L^2}}{r} + K_{10}, \quad \text{for all } t \geq 0,$$

so by condition $(A6)$, (10) and (36) it follows that

$$\|u(t+1)\|^2_{H^1_0} \leq K_{11} + K_{12}\|u(0)\|^2_{L^2},$$

for all $t \geq 0$. In particular,

$$\|u(1)\|^2_{H^1_0} \leq K_{11} + K_{12}\|u(0)\|^2_{L^2},$$

for any strong solution $u(\cdot)$ with initial condition $u(0)$.

For any $u_0 \in H^1_0(\Omega)$ with $\|u_0\|_{H^1_0} \leq R$ and any $u \in \mathcal{R}$ such that $u(0) = u_0$, the semi-flow property $G(t+1, u_0) \subset G(1, G(t, u_0))$ and $G(t, u_0) \subset B_0$, if $t \geq t_0(R)$, imply that

$$\|u(t+1)\|^2_{H^1_0} \leq C(1+K^2) \; \forall t \geq t_0(R).$$

Then there exists $M > 0$ such that the closed ball B_M in $H^1_0(\Omega)$ centred at 0 with radius M is absorbing for G.

By Lemma 1 the set $B_1 = \overline{G(1, B_M)}$ is an absorbing set which is compact in $H^1_0(\Omega)$. □

Given an m-semiflow G, a set $B \subset X$ is said to be negatively (positively) invariant if $B \subset G(t, B)$ $(G(t, B) \subset B)$ for all $t \geq 0$, and strictly invariant (or, simply, invariant) if it is both negatively and positively invariant.

We recall that a set $\mathcal{A} \subset X$ is called a global attractor for the m-semiflow G if it is negatively invariant and attracts all bounded subsets; i.e., $dist_X(G(t, B), \mathcal{A}) \to 0$ as $t \to +\infty$. When $\mathcal{A}$ is compact, it is the minimal closed attracting set ([32] Remark 5).

Theorem 2. *Assume the conditions of Proposition 3. Then the multivalued semiflow G possesses a global compact invariant attractor $\mathcal{A}$.*

Proof. From Propositions 1 and 3 we deduce that the multivalued semiflow G is upper semicontinuous with closed values and the existence of an absorbing which is compact in $H^1_0(\Omega)$. Therefore, by ([32] Theorem 4 and Remark 8) the existence of the global invariant attractor and its compactness in $H^1_0(\Omega)$ follow. □

We recall some concepts which are necessary to study the structure of the global attractor.

Definition 3. *A map $\phi : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if $\phi(\cdot + s) \mid_{[0,\infty)} \in \mathcal{R}$ for all $s \in \mathbb{R}$. It is a complete trajectory of G if $\phi(t+s) \in G(t, \phi(s))$ for every $s \in \mathbb{R}$, $t \geq 0$.*

An element $z \in X$ is a fixed point of $\mathcal{R}$ if $\varphi(\cdot) \equiv z \in \mathcal{R}$. We denote the set of all fixed points by $\mathfrak{R}_{\mathcal{R}}$.

An element $z \in X$ is a fixed point of G if $z \in G(t, z)$ for every $t \geq 0$.

Several properties concerning fixed points, complete trajectories and global attractors are summarised in the following results [33].

Lemma 2. *Let (K1)-(K2) hold. Then each fixed point (complete trajectory) of $\mathcal{R}$ is also a fixed point (complete trajectory) of G.*

Let (K1)-(K4) hold. Then the fixed points of $\mathcal{R}$ and G are the same. In addition, a map $\phi : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if and only if it is continuous and a complete trajectory of G.

The standard well-known result in the single-valued case for describing the attractor as the union of bounded complete trajectories reads in the multivalued case as follows.

Theorem 3. *Suppose that (K1) and (K2) are satisfied and that either (K3) or (K4) holds true. The semiflow G is assumed to have a compact global attractor $\mathcal{A}$. Then*

$$\mathcal{A} = \{\gamma(0) : \gamma \in \mathbb{K}\} = \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma \in \mathbb{K}\}, \tag{42}$$

where $\mathbb{K}$ stands for the set of all bounded complete trajectories in $\mathcal{R}$.

In view of Theorem 3, as $\mathcal{R}$ satisfies (K3) and (K4) (by Corollary 1), the global attractor is characterised in terms of bounded complete trajectories, so (42) follows.

The set B is said to be weakly invariant if for any $x \in B$ there exists a complete trajectory γ of $\mathcal{R}$ contained in B such that $\gamma(0) = x$. Characterisation (42) implies that the attractor $\mathcal{A}$ is weakly invariant.

The set of fixed points $\mathfrak{R}_{\mathcal{R}}$ is characterised as follows.

Lemma 3. *Assume the conditions of Lemma 1. Let $\mathfrak{R}$ be the set of $z \in H^2(\Omega) \cap H_0^1(\Omega)$ such that*

$$-a(\|z\|^2_{H_0^1})\frac{d^2z}{dx^2} = \lambda f(z) + h \quad \text{in } L^2(\Omega). \tag{43}$$

Then $\mathfrak{R}_{\mathcal{R}} = \mathfrak{R}$.

Proof. If $z \in \mathfrak{R}_{\mathcal{R}}$, then $u(t) \equiv z \in \mathcal{R}$. Thus, $u(\cdot)$ satisfies (12) and $\frac{du}{dt} = 0$ in $L^2(0, T; L^2(\Omega))$, so (43) is satisfied. Let $z \in \mathfrak{R}$. Then the map $u(t) \equiv z$ satisfies (43) for any $t \geq 0$ and $\frac{du}{dt} = 0$ in $L^2(0, T; L^2(\Omega))$, so (12) holds true. □

Finally, we shall obtain the characterisation of the global attractor in terms of the unstable and stable sets of the stationary points.

Theorem 4. *Assume the conditions of Proposition 3. Then it holds that*

$$\mathcal{A} = M^+(\mathfrak{R}) = M^-(\mathfrak{R}),$$

where

$$M^+(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{K},\ \gamma(0) = z,\ dist_{\,H_0^1}(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty\}, \tag{44}$$

$$M^-(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{F},\ \gamma(0) = z,\ dist_{\,H_0^1}(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty\}, \tag{45}$$

and $\mathbb{F}$ denotes the set of all complete trajectories of $\mathcal{R}$ (see Definition 3).

Remark 4. *In (45) it is equivalent to use $\mathbb{K}$ instead of $\mathbb{F}$ because all the solutions are bounded forward in time.*

Proof. We consider the function $E : \mathcal{A} \to \mathbb{R}$

$$E(y) = \frac{1}{2}A(\|y\|^2_{H_0^1}) - \lambda\int_\Omega F(y(x))dx - \int_\Omega h(x)y(x)dx. \tag{46}$$

Note that $E(y)$ is continuous in $H_0^1(\Omega)$. Indeed, the maps $y \mapsto \frac{1}{2}A(\|y\|^2_{H_0^1})$ and $y \mapsto \int_\Omega h(x)y(x)dx$ are obviously continuous in $H_0^1(\Omega)$. On the other hand, by the embedding $H_0^1(\Omega) \subset L^\infty(\Omega)$ and using Lebesgue's theorem, the continuity of $y \to \int_\Omega F(y(x))dx$ follows.

By using (29)–(30) and multiplying Equation (3) by $\frac{du}{dt}$ for any $u \in \mathcal{R}$, we can obtain the following energy equality:

$$\int_s^t \| \frac{d}{dr} u(r) \|_{L^2}^2 dr + E(u(t)) = E(u(s)) \quad \text{for all } t \geq s \geq 0.$$

Hence, $E(u(t))$ is non-increasing, and by $(A6)$, (10) and the boundedness of $\mathcal{A}$, it is bounded from below. Thus $E(u(t)) \to l$, and $t \to +\infty$, for some $l \in \mathbb{R}$.

Let $z \in \mathcal{A}$ and $u \in \mathbb{K}$ be such that $u(0) = z$. By contradiction, suppose the existence of $\varepsilon > 0$ and $u(t_n)$, where $t_n \to +\infty$, for which $dist_{H_0^1}(u(t_n), \mathfrak{R}) > \varepsilon$. Since $\mathcal{A}$ is compact in $H_0^1(\Omega)$, we can take a converging subsequence (relabelled the same) such that $u(t_n) \to y$ in $H_0^1(\Omega)$, where $t_n \to \infty$. By the continuity of the function E, it follows that $E(y) = l$. We will obtain a contradiction by proving that $y \in \mathfrak{R}$. Define $v_n(\cdot) = u(\cdot + t_n)$. By Lemma 1, there exist $v \in \mathcal{R}$ and a subsequence satisfying $v(0) = y$ and $v_n(t) \to v(t)$ in $H_0^1(\Omega)$ for $t \geq 0$. Thus, from $E(v_n(t)) \to E(v(t))$ we infer that $E(v(t)) = l$. Additionally, $v(\cdot)$ satisfies the energy equality, so that

$$l + \int_0^t \|v_r\|_{L^2}^2 dr = E(v(t)) + \int_0^t \|v_r\|_{L^2}^2 dr = E(v(0)) = E(y) = l.$$

Therefore, $\frac{dv}{dt}(s) = 0$ for a.a. s, and then by Lemma 3 we have $y \in \mathfrak{R}_{\mathcal{R}} = \mathfrak{R}$. As a consequence, $\mathcal{A} \subset M^+(\mathfrak{R})$. The converse inclusion follows from (42).

As before, take arbitrary $z \in \mathcal{A}$ and $u \in \mathbb{K}$ satisfying $u(0) = z$. Since by the embedding $H_0^1(\Omega) \subset C([0,1])$ the energy function is bounded from above in $\mathcal{A}$, $E(u(t)) \to l$, as $t \to -\infty$, for some $l \in \mathbb{R}$. Suppose that there are $\varepsilon > 0$ and $u(t_n)$, where $t_n \to +\infty$, such that $dist_{H_0^1}(u(-t_n), \mathfrak{R}) > \varepsilon$. Up to a subsequence we have that $u(-t_n) \to y$ in $H_0^1(\Omega)$, $E(y) = l$. Moreover, for $v_n(\cdot) = u(\cdot - t_n)$ there are $v \in \mathcal{R}$ and a subsequence such that $v(0) = y$ and $v_n(t) \to v(t)$ in $H_0^1(\Omega)$ for $t \geq 0$. Therefore, $E(v_n(t)) \to E(v(t))$ gives $E(v(t)) = l$ and then by the above arguments we get a contradiction because $y \in \mathfrak{R}$. Hence, $\mathcal{A} \subset M^-(\mathfrak{R})$ and we deduce the converse inclusion from (42). □

Finally, we are able to obtain that the global attractor is compact in the space $C^1([0,1])$. This property will be important in order to study a more precise structure of the global attractor in terms of the stationary points and their heteroclinic connections.

We define the function $w(t) = u(\alpha^{-1}(t))$, where $\alpha(t) = \int_0^t a(\|u(s)\|_{H_0^1}^2) ds$, which is under the conditions of Proposition 3 (see [18] for more details) a strong solution to the problem

$$\begin{cases} \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = \frac{f(w) + h}{a(\|w\|_{H_0^1}^2)}, \text{ in } (0,\infty) \times \Omega, \\ w = 0 \quad \text{on } (0,\infty) \times \partial\Omega, \\ w(0,x) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{47}$$

Let $V^{2r} = D(A^r)$, $r \geq 0$. We will prove first that the attractor is compact in any space V^{2r} with $0 \leq r < 1$. For this aim we will need the concept of mild solution. We consider the auxiliary problem

$$\begin{cases} \frac{dv}{dt} + Av(t) = g(t), \ t > 0, \\ v(0) = u_0, \end{cases} \tag{48}$$

where $g \in L^2_{loc}(0, +\infty; L^2(\Omega))$. The function $u \in C([0,+\infty), L^2(\Omega))$ is called a mild solution to problem (48) if

$$v(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} g(s) ds, \forall t \geq 0. \tag{49}$$

In the same way as in Lemma 2 in [34] we obtain that a strong solution to problem (47) is a mild solution to problem (48) with $g(t) = (f(w(t)) + h)/a(\|w(t)\|^2_{H^1_0})$.

Lemma 4. *Assume the conditions of Proposition 3. Then the global attractor $\mathcal{A}$ is compact in V^{2r} for every $0 \le r < 1$.*

Proof. Let $z \in \mathcal{A}$ be arbitrary. Since $\mathcal{A}$ is invariant, there exist $u_0 \in \mathcal{A}$ and $u \in \mathcal{R}$ such that $z = u(1)$ and $u(t) \in \mathcal{A}$ for all $t \ge 0$. Since $w(t) = u(\alpha^{-1}(t))$ is a mild solution of (48) with $g(t) = (f(w(t)) + h)/a(\|w(t)\|^2_{H^1_0})$, the variation of constants formula (49) gives

$$z = w(\alpha(1)) = e^{-A\alpha(1)}u_0 + \int_0^{\alpha(1)} e^{-A(\alpha(1)-s)}g(s)ds.$$

As $\mathcal{A}$ is bounded in $H^1_0(\Omega)$ (and then in $L^\infty(\Omega)$), condition (A6) and the continuity of f imply that

$$\|u_0\|_{L^2} \le C, \ \|g\|_{L^\infty(0,\alpha(1);L^2(\Omega))} \le C,$$

where $C > 0$ does not depend on z. The standard estimate $\|e^{-At}\|_{\mathcal{L}(L^2(\Omega),D(A^r))} \le M_r t^{-r}e^{-at}$, $M_r, a > 0$ ([27] Theorem 37.5), implies that

$$\begin{aligned}\|A^r z\|_{L^2} &\le \left\|A^r e^{-A\alpha(1)}u_0\right\|_{L^2} + \int_0^{\alpha(1)} \left\|A^r e^{-A(\alpha(1)-s)}g(s)\right\|_{L^2} ds \\ &\le M_r e^{-a\alpha(1)}\alpha(1)^{-r}C + M_r C\int_0^{\alpha(1)} (\alpha(1)-s)^{-r}ds,\end{aligned}$$

so $\mathcal{A}$ is bounded in V^{2r} for every $0 \le r < 1$.

From the compact embedding $V^\alpha \subset V^\beta$, for $\alpha > \beta$, and the fact that $\mathcal{A}$ is closed in any V^{2r} we obtain the result. □

Corollary 2. *Assume the conditions of Proposition 3. Then the global attractor $\mathcal{A}$ is compact in $C^1([0,1])$.*

Proof. We obtain by Lemma 37.8 in [27] the continuous embedding

$$V^{2r} \subset C^1([0,1]) \text{ if } r > \frac{3}{4}.$$

Hence, the statement follows from Lemma 4. □

5. Fixed Points

In this section we are interested in studying the fixed points of problem (3) when $h \equiv 0$, that is, the solutions of the boundary-value problem

$$\begin{cases} -a(\|u\|^2_{H^1_0})\dfrac{d^2u}{dx^2} = \lambda f(u), \ 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{50}$$

For this aim we will use the properties of the fixed points of the standard Chafee-Infante equation. In order to do that, for any $d \ge 0$ we will study the)–(Aollowing boundary-value problem.

$$\begin{cases} -a(d)\dfrac{d^2u}{dx^2} = \lambda f(u), \ 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{51}$$

as it is obvious that $u(\cdot)$ is solution to problem (50) if and only if $u(\cdot)$ is a solution to problem (51) with $d = \|u\|^2_{H^1_0}$.

5.1. Dependence on the Parameters of the Fixed Points for the Chafee-Infante Equation

Denoting $\widetilde{\lambda} = \dfrac{\lambda}{a(d)}$ problem (51) becomes

$$\begin{cases} -\dfrac{d^2u}{dx^2} = \widetilde{\lambda} f(u),\ 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{52}$$

Assuming conditions (A1)–(A5), it is known [35] that if $n^2\pi^2 < \widetilde{\lambda} \leq (n+1)^2\pi^2$, then this problem has exactly $2n+1$ solutions, denoted by $v_0 \equiv 0,\ v_1^{\pm}, ..., v_n^{\pm}$. The function $v_k^{\pm}$ has $k+1$ simple zeros in $[0,1]$.

We need to study the dependence of the norm of these fixed points on the parameter $\widetilde{\lambda}$. First, we will show that the H^1-norm of the fixed points of problem (52) is strictly increasing with respect to the parameter $\widetilde{\lambda}$.

Lemma 5. *Assume conditions (A1)–(A5). Let $v_1 = v^+_{k,\lambda_1}$, $v_2 = v^+_{k,\lambda_2}$ with $k^2\pi^2 < \lambda_1 < \lambda_2$. Then $\|v_1\|_{H_0^1} < \|v_2\|_{H_0^1}$.*

Proof. We consider the equivalent norm in $H_0^1(\Omega)$ given by $\|v'\|_{L^2}$. The fixed points are the solutions of the initial value problem

$$\begin{cases} \dfrac{d^2u}{dx^2} + \widetilde{\lambda} f(u) = 0, \\ u(0) = 0,\ u'(0) = v_0 \end{cases} \tag{53}$$

such that $u(1) = 0$. The solutions of (53) satisfy the relation

$$\frac{(u'(x))^2}{2} + \widetilde{\lambda} F(u(x)) = \widetilde{\lambda} E,\ 0 \leq x \leq 1, \tag{54}$$

for some constant $E \geq 0$. Denote $u_{\widetilde{\lambda}} = v^+_{k,\widetilde{\lambda}}$. By Theorem 7 in [35] we have that $u_{\widetilde{\lambda}}$ is associated with a unique value $E = E_k^+(\widetilde{\lambda}) > 0$. Moreover, $E_k^+(\widetilde{\lambda})$ is a solution of one of the following equations:

$$\begin{aligned} m\tau_+^{\widetilde{\lambda}}(E) + (m-1)\tau_-^{\widetilde{\lambda}}(E) &= \frac{1}{\sqrt{2}}, \\ m\tau_-^{\widetilde{\lambda}}(E) + (m-1)\tau_+^{\widetilde{\lambda}}(E) &= \frac{1}{\sqrt{2}}, \\ m\tau_+^{\widetilde{\lambda}}(E) + m\tau_-^{\widetilde{\lambda}}(E) &= \frac{1}{\sqrt{2}}, \end{aligned} \tag{55}$$

where either $k = 2m-1$ or $k = 2m$ and

$$\tau_+^{\widetilde{\lambda}}(E) = \widetilde{\lambda}^{-1/2} \int_0^{U_+(E)} (E - F(u))^{-1/2}\, du, \tag{56}$$

$$\tau_-^{\widetilde{\lambda}}(E) = \widetilde{\lambda}^{-1/2} \int_{U_-(E)}^0 (E - F(u))^{-1/2}\, du, \tag{57}$$

being $U_+(E)$ $(U_-(E))$ the positive (negative) inverse of F at E. It is obvious that for E fixed the functions $\tau_+^{\widetilde{\lambda}}(E)$, $\tau_-^{\widetilde{\lambda}}(E)$ are strictly decreasing with respect to $\widetilde{\lambda}$. Then from (55) we deduce that the root $E_k^+(\widetilde{\lambda})$ is strictly increasing with respect to $\widetilde{\lambda}$. Thus, If $\lambda_1 < \lambda_2$, we have

$$\sqrt{2\lambda_1(E_k^+(\lambda_1) - F(u))} < \sqrt{2\lambda_2(E_k^+(\lambda_2) - F(u))},\quad U^-(E_k^+(\lambda_1)) \leq u \leq U^+(E_k^+(\lambda_1)). \tag{58}$$

We will prove now that $\|u'_{\widetilde{\lambda}}\|_{L^2}$ is strictly increasing in $\widetilde{\lambda}$.

The function $u_{\widetilde{\lambda}}$ has $k+1$ simple zeros in $[0,1]$ and $u_{\widetilde{\lambda}}$ is positive in the first subinterval. Let $T_+(E_k^+(\lambda))$ be the x-time necessary to go from the initial condition $u_\lambda(0) = 0$ to the point where $u'_\lambda(T_+(E_k^+(\lambda))) = 0$. Then the length of the first subinterval is $2T_+(E_k^+(\lambda))$ [35]. By (54),

$$(u'_{\widetilde{\lambda}}(x))^2 = \sqrt{2\widetilde{\lambda}}\sqrt{E_k^+(\widetilde{\lambda}) - F(u_{\widetilde{\lambda}}(x))}u'_{\widetilde{\lambda}}(x),$$

so we have

$$\int_0^{T_+(E_k^+(\widetilde{\lambda}))}(u'_{\widetilde{\lambda}}(x))^2dx = \int_0^{T_+(E_k^+(\widetilde{\lambda}))}\sqrt{2\widetilde{\lambda}}\sqrt{E_k^+(\widetilde{\lambda}) - F(u_{\widetilde{\lambda}}(x))}u'_{\widetilde{\lambda}}(x)dx.$$

By the change of variable $v = u_{\widetilde{\lambda}}(x)$ we obtain

$$\int_0^{T_+(E_k^+(\widetilde{\lambda}))}(u'_{\widetilde{\lambda}}(x))^2dx = \int_0^{U^+(E_k^+(\widetilde{\lambda}))}\sqrt{2\widetilde{\lambda}}\sqrt{E_k^+(\widetilde{\lambda}) - F(v)}dv = g(\widetilde{\lambda}).$$

Since $\widetilde{\lambda} \mapsto U^+(E_k^+(\widetilde{\lambda}))$ is strictly increasing and by using (58), we conclude that the function $g(\widetilde{\lambda})$ is strictly increasing. Hence, by putting $x_1(\widetilde{\lambda}) = 2T_+(E_k^+(\widetilde{\lambda}))$ we obtain that the norm of $u_{\widetilde{\lambda}}$ in the first subinterval, $\left\|u'_{\widetilde{\lambda}}\right\|_{L^2(0,x_1(\widetilde{\lambda}))}$, is strictly increasing. By arguing in the same way as for the other subintervals, we obtain that $\widetilde{\lambda} \mapsto \|u'_{\widetilde{\lambda}}\|_{L^2}$ is strictly increasing. □

Let us prove the same result but with respect to the norm $\|u_{\widetilde{\lambda}}\|_{L^p}$ with $p \geq 1$.

Lemma 6. *Assume conditions (A1)–(A5) and let f be odd. Let $v_1 = v^+_{k,\lambda_1}$, $v_2 = v^+_{k,\lambda_2}$ with $k^2\pi^2 < \lambda_1 < \lambda_2$. Then $\|v_1\|_{L^p} < \|v_2\|_{L^p}$ for any $p \geq 1$.*

Proof. As in the previous lemma, denote $u_{\widetilde{\lambda}} = v^+_{k,\widetilde{\lambda}}$. The function $u_{\widetilde{\lambda}}$ has $k+1$ zeros in $[0,1]$ at the points $0 < x_1 < x_2 < ... < x_{k-1} < 1$. When f is odd, by symmetry, the length of all subintervals has to be the same, so $x_j = \frac{j}{k}$ regardless the value of $\widetilde{\lambda}$.

We shall prove that in the first subinterval we have that $u_{\lambda_1}(x) < u_{\lambda_2}(x)$, for all $x \in \left(0, \frac{1}{k}\right)$. By (54) for $x \in [0, \frac{1}{2k}]$ we have

$$x = \int_0^x ds = \int_0^{u_{\widetilde{\lambda}}(x)} \frac{du}{\sqrt{2\widetilde{\lambda}\left(E_k^+\left(\widetilde{\lambda}\right) - F(u)\right)}},$$

so (58) yields

$$\begin{aligned} x = \int_0^{u_{\lambda_2}(x)} \frac{du}{\sqrt{2\lambda_2(E_k^+(\lambda_2) - F(u))}} &= \int_0^{u_{\lambda_1}(x)} \frac{du}{\sqrt{2\lambda_1(E_k^+(\lambda_1) - F(u))}} \\ > \int_0^{u_{\lambda_1}(x)} \frac{du}{\sqrt{2\lambda_2(E_k^+(\lambda_2) - F(u))}}&, \text{ if } x \in (0, \frac{1}{2k}]. \end{aligned}$$

Thus, $u_{\lambda_1}(x) < u_{\lambda_2}(x)$, for all $x \in (0, \frac{1}{2k}]$. By symmetry we obtain that the inequality is true in $\left(0, \frac{1}{k}\right)$.

Repeating the same argument in the other subintervals we get that

$$|u_{\lambda_1}(x)| < |u_{\lambda_2}(x)| \text{ for all } x \in (0,1),\ x \neq \frac{j}{k},\ j = 1, ...k-1.$$

This implies that $\|u_{\lambda_1}\|_{L^p} < \|u_{\lambda_2}\|_{L^p}$ for any $p \geq 1$. $\square$

Remark 5. *The statements in Lemmas 5 and 6 are also true for $v^-_{k,\tilde{\lambda}}$, because $v^-_{k,\tilde{\lambda}}(x) = v^+_{k,\tilde{\lambda}}(1-x)$, so the H^1_0 and L^p norms of $v^-_{k,\tilde{\lambda}}$ and $v^+_{k,\tilde{\lambda}}$ are the same.*

5.2. Nonlocal Fixed Points

Although in this paper we are mainly interested in problem (3), we will study the existence of stationary points for an elliptic problem with a more general nonlocal term than in (50). Namely, let us consider the following problem:

$$\begin{cases} -a(l(u))u_{xx} = \lambda f(u),\ 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{59}$$

where

$$l(u) = \|u\|^r_{H^1_0} \text{ or } \|u\|^r_{L^p},\ p \geq 1, r > 0.$$

Let

$$d_k = \sup\{d : \lambda > a\left(\overline{d}\right)\pi^2 k^2\ \forall \overline{d} \leq d\}.$$

Then for any $d < d_k$ there exists the fixed point u^d_k of (51), where u^d_k is either equal to u^+_k or u^-_k.

It is obvious that any solution of (59) is a solution of (51) with $d = l(u)$. Therefore, all the solutions to problem (59) have to be solutions u^d_k to problem (51) for a suitable d.

Theorem 5. *Assume conditions (A1)–(A6) and, additionally, that*

$$a(0)\pi^2 k^2 < \lambda. \tag{60}$$

Then:

- *For any $1 \leq j \leq k$ there exists $d^*_j < d_k$ such that $u_j^{d^*_j}$ is a fixed point of problem (59).*
- *If $\lambda \leq a(0)\pi^2(k+1)^2$ and $a(0) = \min_{s\geq 0}\{a(s)\}$, there are no fixed points for $j > k$.*
- *If $N \geq k$ is the first integer such that $\lambda \leq \inf_{s\geq 0}\{a(s)\pi^2(N+1)^2\}$, there are no fixed points for $j > N$.*
- *If $l(u) = \|u\|^r_{H^1_0}$, $\lambda \leq a(0)\pi^2(k+1)^2$ and a is non-decreasing, there are exactly $2k+1$ solutions to problem (59): $0,\ u^{\pm}_{1,d^*_1}, ..., u^{\pm}_{k,d^*_k}$.*
- *If $l(u) = \|u\|^r_{L^p}$, $\lambda \leq a(0)\pi^2(k+1)^2$, f is odd and a is non-decreasing, there are exactly $2k+1$ solutions to problem (59): $0,\ u^{\pm}_{1,d^*_1}, ..., u^{\pm}_{k,d^*_k}$.*

Proof. For the first statement, it is enough to prove the result for $j = k$. By condition (60) we have that $d_k \in (0, +\infty]$.

Consider first the case where d_k is finite. We need to obtain the existence of $d^*_k < d_k$ such that $l\left(u^{d^*_k}_k\right) = d^*_k$. When $d = 0$ it is clear that $l(u^0_k) > 0$. Additionally, we know that $l\left(u^{d_k}_k\right) = 0$. Through multiplying (51) by u^d_k and using (9), (A6) and the Poincaré inequality we obtain

$$\left\|\left(u^d_k\right)'\right\|^2_{L^2} \leq \frac{\lambda}{a(d)}\left(f\left(u^d_k\right), u^d_k\right) \leq \frac{\lambda}{m}\left(m_\varepsilon + \varepsilon\left\|u^d_k\right\|^2_{L^2}\right) \leq K_1 + \frac{1}{2}\left\|\left(u^d_k\right)'\right\|^2_{L^2},$$

so, by using the embedding $H^1_0(\Omega) \subset L^\infty(\Omega)$, $l\left(u^d_k\right)$ is bounded in d. This implies that the function $g(d) = l\left(u^d_k\right)$ has to intersect the line $y(d) = d$ at some point d^*_k. It remains to check that $d^*_k < d_k$. For this aim we prove first that $u^d_k \underset{d\to d_k}{\to} 0$ strongly in $H^1_0(\Omega)$. Indeed, as u_k^d is bounded in $H_0^1(\Omega)$, there exist v and a sequence $\{u_k^{d_j}\}$ such that $u_k^{d_j} \to v$ in $L^2(\Omega)$. The embedding $H_0^1(\Omega) \subset C([0,1])$ and the continuity of the function $f(u)$ imply that $\{f(u_k^{d_j})\}$ is bounded in $C([0,1])$, so from

$$\left\| \left(u_k^{d_j}\right)'' \right\|_{L^2} \le \frac{\lambda}{a(d_j)} \left\| f\left(u_k^{d_j}\right) \right\|_{L^2} \le \frac{\lambda}{m} \left\| f\left(u_k^{d_j}\right) \right\|_{L^2} \le C$$

we deduce that $\{u_k^{d_j}\}$ is bounded in $H^2(\Omega)$. Hence, $u_k^{d_j} \to v$ in $H_0^1(\Omega)$ and $C^1([0,1])$. Additionally, $f(u_k^{d_j}) \to f(v)$ in $C([0,1])$. Therefore, for any $\psi \in H_0^1(\Omega)$ we have that

$$\begin{array}{ccc} \left(\left(u_k^{d_j}\right)', \psi' \right) & = & \frac{\lambda}{a(d_j)} \left(f\left(u_k^{d_j}\right), \psi \right) \\ \downarrow & & \downarrow \\ (v', \psi') & = & \frac{\lambda}{a(d_k)} (f(v), \psi), \end{array}$$

which implies that v is a solution to problem (51) with $d = d_k$. However, from $u_k^{d_j} \to v$ in $C^1([0,1])$ it follows that v cannot be a point with less than $k+1$ simple zeros in $[0,1]$ and then $\lambda / a(d_k) = k^2\pi^2$ implies that $v \equiv 0$. As the limit is the same for every converging subsequence, $u_k^d \underset{d \to d_k}{\to} 0$ strongly in $H_0^1(\Omega)$. Thus, $d_k > 0$ and $\lim_{d \to d_k} \left\| \left(u_k^d\right)' \right\|_{L^2} = 0$ imply that $d_k^* < d_k$.

Second, let $d_k = +\infty$. Then the existence of $d_k^* < +\infty$ follows by the same argument as before.

The second and third statements are a consequence of

$$\lambda \le a(0)\pi^2(k+1)^2 \le a(d)\pi^2(k+1)^2 \text{ for any } d \ge 0$$

and

$$\lambda \le \inf_{s \ge 0}\{a(s)\}\pi^2(N+1)^2 \le a(d)\pi^2(N+1)^2 \text{ for any } d \ge 0,$$

respectively, because in such a case for problem (51) the fixed points $v_j^{\pm}$, $j > k$ (respectively $j > N$), do not exist.

The last two statements are a consequence of the first two statements and of the fact that the points of intersection of the functions $g(d) = l\left(u_k^d\right)$ and $y(d) = d$ has to be unique, because if a is non-decreasing, then $g(d)$ is non-increasing by Lemmas 5 and 6. $\square$

In view of this theorem, we have exactly the same equilibria and bifurcations as in the classical Chafee-Infante equation (see [24,35]) when the function $a(d)$ is non-decreasing, because in this case in view of the monotone dependence between the functions $a(d)$ and $g(d)$, there is only one intersection point of the function $g(d)$ with the bisector, as it is shown in Figure 1. This follows from the fact that $g(d) - d$ is strictly decreasing, but there may be weaker conditions on $a(\cdot)$ that would lead $g(d) - d$ to be strictly decreasing.

When the function $a(\cdot)$ is not assumed to be monotone, an interesting situation appears. More precisely, it is possible to have more than two equilibria with the same number of zeros. If $l(u) = \|u\|_{H_0^1}^2$, for the equilibria with $k+1$ zeros in $[0,1]$ this happens when the equation

$$d = \int_0^1 \left| \frac{du_k^d(x)}{dx} \right|^2 dx = g(d) \quad (61)$$

has more than one solution. For instance, if $a(0) = a(\bar{d})$ for some $0 < \bar{d} < g(0)$, then $g(0) = g(\bar{d})$. Assuming that there are $0 < d_k^1 < d_k^2 < \bar{d}$ such that $a(d_k^2) = a(d_k^1) = \frac{\lambda}{\pi^2 k^2}$, there must exist $0 < d_1^* < d_k^1 < d_k^2 < d_2^* < \bar{d}$ such that $g(d_i^*) = d_i^*$. Now, by the argument in Theorem 5, there must exist a $d_3^* > \bar{d}$ such that $g(d_3^*) = d_3^*$, obtaining six fixed points with

$k+1$ zeros in $[0,1]$. This situation is shown in Figure 2, where d_1^*, d_2^* and d_3^* are solutions of (61), that is, there are three intersection points with the bisector. We notice that when $a(d) > \lambda/(\pi^2 k^2)$, the function $g(d)$ is not defined since the condition for such equilibria to exist is not satisfied, but we can make this function continuous by putting $g(d) = 0$ whenever $a(d) \geq \lambda/(\pi^2 k^2)$. This procedure establishes that, having fixed a natural number k, for any $j \in \mathbb{N}$ we may construct $a(\cdot)$ in such a way that we have $2(2j+1)$ equilibria with $k+1$ zeros in $[0,1]$.

At least there is always one intersection point with the bisector, but the function $g(d)$ could be even tangent to the bisector at some point or not cut it again.

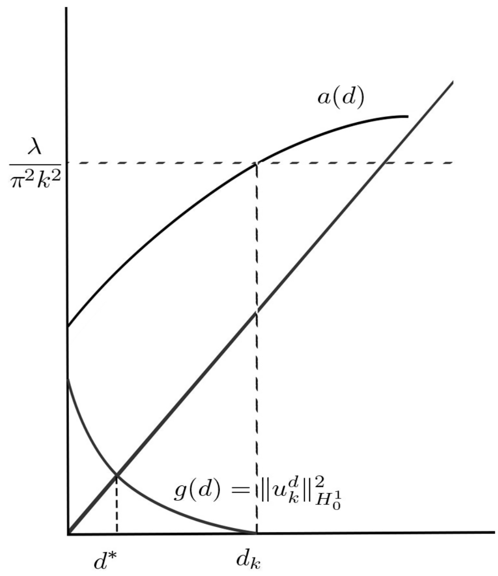

Figure 1. $a(d)$ non-decreasing.

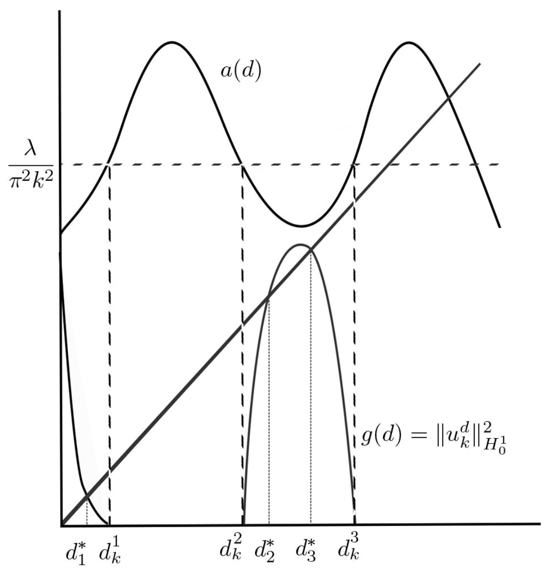

Figure 2. $a(d)$ whatever.

5.3. Lap Number and Some Forbidden Connections

With Theorem 5 at hand we can improve the description of the global attractor given in Theorem 4.

Under conditions (A1)–(A6), (A8) and $h \equiv 0$, if

$$a(0)\pi^2 n^2 < \lambda \le a(0)\pi^2 (n+1)^2 \quad (62)$$

then problem (3) possesses exactly $2n+1$ fixed points: $v_0 = 0,\ u^{\pm}_{1,d_1^*}, ..., u^{\pm}_{n,d_n^*}$.

Let ϕ be a bounded complete trajectory. We know by Theorem 4 that

$$dist_{H_0^1}(\phi(t), \mathfrak{R}) \to 0, \text{ as } t \to \pm\infty.$$

As the number of fixed points is finite, we will prove that in fact the solution has to converge to one fixed point forwards and backwards. We recall the omega and alpha limit sets of ϕ, given by

$$\omega(\phi) = \{y : \exists t_n \to +\infty \text{ such that } \phi(t_n) \to y\},$$
$$\alpha(\phi) = \{y : \exists t_n \to -\infty \text{ such that } \phi(t_n) \to y\},$$

are non-empty, compact and connected ([36] Lemma 3.4 and Proposition 4.1). Additionally, $dist_{H_0^1}(\phi(t), \omega(\phi)) \underset{t\to+\infty}{\to} 0$, $dist_{H_0^1}(\phi(t), \alpha(\phi)) \underset{t\to-\infty}{\to} 0$. Since $\omega(\phi), \alpha(\phi) \subset \mathfrak{R}$ and $\mathfrak{R}$ is finite, the only possibility is that $\omega(\phi) = z_1 \in \mathfrak{R}$, $\alpha(\phi) = z_2 \in \mathfrak{R}$.

Thus, we have established the following result.

Theorem 6. *Let us assume conditions (A1)–(A6), (A8), (62) and $h \equiv 0$. Then*

$$\mathcal{A} = \cup_{k=0}^{2n+1} M^+(v_k) = \cup_{k=0}^{2n+1} M^-(v_k),$$

where n is given in (62) and $v_0 = 0$, $v_1 = u^+_{1,d_1^*}$, $v_2 = u^-_{1,d_1^*}$, ...

In other words, the global attractor $\mathcal{A}$ consists of the set of stationary points $\mathfrak{R}$ (which has $2n+1$ elements) and the bounded complete trajectories that connect them (the heteroclinic connections).

Remark 6. *As the Lyapunov function (46) is strictly decreasing along a trajectory ϕ which is not a fixed point, then there cannot exist homoclinic connections for any fixed point. This implies in particular that if $n = 0$, then $\mathcal{A} = \{0\}$.*

Remark 7. *If we use condition (A7) instead of (A8), then we cannot guarantee that the number of fixed points is finite. However, if we suppose that this is the case, then the result remains valid. In this situation, there could be more than two fixed points with the same number of zeros.*

Lemma 7. *Let us assume conditions (A1)–(A6), $h \equiv 0$ and either (A7) or (A8). Let $u^+_{k,d_k^*}, u^-_{k,d_k^*}$ be a pair of fixed points corresponding to the same value d_k^*. Then there cannot be an heteroclinic connection between them.*

Proof. The function $v(x) = u^+_{k,d_k^*}(1-x)$ is a fixed point corresponding to d_k^* as

$$-\frac{\partial^2 v}{\partial x^2}(x) = -\frac{\partial^2 u^+_{k,d_k^*}}{\partial x^2}(1-x) = \frac{\lambda}{a(d_k^*)} f\left(u^+_{k,d_k^*}(1-x)\right) = \frac{\lambda}{a(d_k^*)} f(v(x)),$$

so $u^-_{k,d_k^*}(x) = v(x) = u^+_{k,d_k^*}(1-x)$. The equalities

$$\int_0^1 \left(\frac{\partial u^-_{k,d_k^*}}{\partial x}(x)\right)^2 dx = \int_0^1 \left(\frac{\partial u^+_{k,d_k^*}}{\partial x}(1-x)\right)^2 dx = \int_0^1 \left(\frac{\partial u^+_{k,d_k^*}}{\partial x}(y)\right)^2 dy,$$

$$\int_0^1 \int_0^{u^-_{d_k^*}(x)} f(s)dsdx = \int_0^1 \int_0^{u^+_{d_k^*}(1-x)} f(s)dsdx = \int_0^1 \int_0^{u^+_{d_k^*}(y)} f(s)dsdy$$

imply that $E(u^{-}_{k,d_k^*}) = E\left(u^{+}_{k,d_k^*}\right)$, where E is the Lyapunov function (46). Since this function is strictly decreasing along a trajectory ϕ which is not a fixed point, there cannot exist a heteroclinic connection between these two points. □

Remark 8. *In the case where condition (A7) is assumed, there could be more than two equilibria with $k+1$ zeros in $[0,1]$. In this case there could exist connections between fixed points with different values of the constant d.*

By using the concept of lap number of the solutions we can discard some more heteroclinic connections.

We consider the function $w(t) = u(\alpha^{-1}(t))$, which is a strong solution to problem (47). For any strong solution $u(\cdot)$ conditions (A1), (A3), (A6) and $u \in C([0,+\infty), H_0^1(\Omega))$ imply that the function

$$r(t,x) = \frac{\lambda}{a(\|w(t)\|^2_{H_0^1})}\frac{f(w(t,x))}{w(t,x)}$$

is continuous and $w(\cdot)$ is a solution of the linear equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = r(t,x)w. \tag{63}$$

Thus, by Theorem A3 in the Appendix A (see also Theorem C in [37]) the number of zeros of $w(t)$ in $[0,1]$ is a nonincreasing function of t. Since $\alpha^{-1}(t)$ is an increasing function of time, the result is also true for the solution $u(\cdot)$. Making use of this property we will prove the following result.

Lemma 8. *Let us assume conditions (A1)–(A6), $h \equiv 0$ and either (A7) or (A8). Then if $n > k$, there cannot exist a connection from the fixed point $u^{\pm}_{k,d_k^*}$ to the fixed point $u^{\pm}_{n,d_n^*}$, that is, there cannot exist a bounded complete trajectory ϕ such that*

$$\phi(t) \to u^{\pm}_{n,d_n^*} \text{ as } t \to +\infty,\ \phi(t) \to u^{\pm}_{k,d_k^*} \text{ as } t \to -\infty.$$

Proof. By contradiction assume that such complete trajectory exists. Denote by $l(z)$ the number of zeros of z in $[0,1]$. By using the compactness of the attractor in $C^1([0,1])$ (see Corollary 2) we obtain that

$$\phi(t) \to u^{\pm}_{n,d_n^*} \text{ in } C^1([0,1]) \text{ as } t \to +\infty,$$
$$\phi(t) \to u^{\pm}_{k,d_k^*} \text{ in } C^1([0,1]) \text{ as } t \to -\infty.$$

Then, as the zeros are simple, we can choose $t_1 > 0$ large enough such that $l(\phi(-t_1)) = l\left(u^{\pm}_{k,d_k^*}\right) = k+1$. Put $u(t) = \phi(t-t_1)$, which is a strong solution of (3). Now we choose $t_2 > 0$ such that $l(u(t_2)) = l\left(u^{\pm}_{n,d_n^*}\right) = n+1$. Then $l(u(0)) = k+1$ and $l(u(t_2)) = n+1 > k+1$. This contradicts the fact that the number of zeros of $u(t)$ is non-increasing. □

6. Morse Decomposition

In this section we study in more detail the structure of the global attractor in the case where the function f is odd. More precisely, we obtain a dynamically gradient m-semiflow G, which is equivalent to saying that there is a Morse decomposition of the attractor [38], and we study the stability of the fixed points.

6.1. Approximations

We consider now the situation when conditions (A1)–(A6), $h = 0$ and either (A7) or (A8) are satisfied, and moreover, the function f is odd.

In this section we consider the following problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|^2_{H^1_0})\dfrac{\partial^2 u}{\partial x^2} = \lambda f_{\varepsilon_n}(u), \quad t > 0,\ x \in (0,1), \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{64}$$

where the function f_{ε_n} is defined below and $\varepsilon_n \to 0$, as $n \to \infty$.

Let $\rho_{\varepsilon_n}(\cdot)$ be a mollifier in $\mathbb{R}$. We define the function $f^{\varepsilon_n}(u) = \int_{\mathbb{R}} \rho_{\varepsilon_n}(s) f(u-s) ds$. It is well known that $f^{\varepsilon_n}(\cdot) \in C^\infty(\mathbb{R})$ and that for any compact subset $A \subset \mathbb{R}$ we have $f^{\varepsilon_n} \to f$ uniformly on A. It is clear that for $u > \varepsilon_n$ the function $f^{\varepsilon_n}(u)$ is strictly concave.

We need the approximation to fulfil (A2) and (A3). To that end, we consider the approximation except on the interval $[-\varepsilon_n, \varepsilon_n]$, for any $\varepsilon_n > 0$. There exists a polynomial of sixth degree $p(x)$ such that

$$\begin{array}{ll} p(0) = 0, & p(\varepsilon_n) = h(\varepsilon_n), \\ p'(0) = 1, & p'(\varepsilon_n) = h'(\varepsilon_n), \\ p''(0) = 0, & p''(\varepsilon_n) = h''(\varepsilon_n), \\ p'''(0) = -1. & \end{array}$$

We choose $\gamma > 0$ such that $p''(s) < 0$ for all $s \in (0, \gamma]$. We can assume that $\varepsilon_n < \gamma$ for all n.

Thus, by construction the function

$$f_{\varepsilon_n}(x) = \begin{cases} -f^{\varepsilon_n}(-x) & if \quad x < -\varepsilon_n, \\ -p(-x) & if \quad -\varepsilon_n \le x \le 0, \\ p(x) & if \quad 0 \le x \le \varepsilon_n, \\ f^{\varepsilon_n}(x) & if \quad x > \varepsilon_n \end{cases} \tag{65}$$

approximates the function f uniformly in compact sets, that is, for any $[-M, M]$ and $\delta > 0$ there exists $n_0(M, \delta) \in \mathbb{N}$ such that

$$|f(x) - f_{\varepsilon_n}(x)| < \delta, \quad \text{for all } n \ge n_0,\ x \in [-M, M]. \tag{66}$$

Additionally, it satisfies the following properties:

(B1) $f_{\varepsilon_n} \in C^2(\mathbb{R})$;
(B2) $f_{\varepsilon_n}(0) = 0$;
(B3) $f'_{\varepsilon_n}(0) = 1$;
(B4) f_{ε_n} is strictly concave if $u > 0$ and strictly convex if $u < 0$;
(B5) f_{ε_n} is odd.

Lemma 9. *Let f satisfy (A5). Then the functions f_{ε_n} satisfy condition (A5) and (9) with independent constants of ε_n.*

Proof. We assume without loss of generality that $\varepsilon_n < 1$. In order to check (4) and (5) we only need to consider u outside the interval $[-1, 1]$, because the sequence $\{f_{\varepsilon_n}\}$ is uniformly bounded in any compact set of $\mathbb{R}$. Then for $u \notin [-1,1]$ the Hölder inequality and $\int_{\mathbb{R}} \rho_{\varepsilon_n}(s)ds = 1$ give

$$\begin{aligned}|f_{\varepsilon_n}(u)| &= \left|\int_{\mathbb{R}} f(u-s)\rho_{\varepsilon_n}(s)ds\right| \le \int_{\mathbb{R}} |f(u-s)|\rho_{\varepsilon_n}(s)ds \\ &\le \int_{\mathbb{R}} \left(C_1 + C_2|u-s|^{p-1}\right)\rho_{\varepsilon_n}(s)ds \\ &\le C_1 + C_2 2^{p-2}\left(\int_{-\varepsilon_n}^{\varepsilon_n}\left(|u|^{p-1} + |s|^{p-1}\right)\rho_{\varepsilon_n}(s)ds\right) \\ &\le \widetilde{C}_1 + \widetilde{C}_2|u|^{p-1}.\end{aligned}$$

If f satisfies (5), then

$$\begin{aligned}f_{\varepsilon_n}(u)u &= \int_{\mathbb{R}} f(u-s)(u-s)\rho_{\varepsilon_n}(s)ds + \int_{\mathbb{R}} f(u-s)s\rho_{\varepsilon_n}(s)ds \\ &\le \int_{\mathbb{R}} (C_3 - C_4|u-s|^p)\rho_{\varepsilon_n}(s)ds + \int_{\mathbb{R}}\left(C_1 + C_2|u-s|^{p-1}\right)s\rho_{\varepsilon_n}(s)ds \\ &\le K_1 - C_4\int_{\mathbb{R}}\left(2^{1-p}|u|^p - |s|^p\right)\rho_{\varepsilon_n}(s)ds \\ &+ C_2 2^{p-2}\int_{\mathbb{R}}\left(|u|^{p-1} + |s|^{p-1}\right)s\rho_{\varepsilon_n}(s)ds \\ &\le \widetilde{C}_3 - \widetilde{C}_4|u|^p,\end{aligned}$$

where we have used $|u|^p \le 2^{p-1}\left(|s^p| + |u-s|^p\right)$ and the Young inequality.

For (9) we put in the above inequality $p = 2$, $C_3 = m_\varepsilon$, $C_4 = -\varepsilon$ and obtain

$$f_{\varepsilon_n}(u)u \le \widetilde{m}_\varepsilon + \varepsilon u^2,$$

which obviously implies (6). □

Our next aim is to focus on the convergence of solutions of the approximations.

Theorem 7. *Let conditions (A1)–(A6), $h = 0$ and either (A7) or (A8) be satisfied and let, moreover, the function f be odd. If $u_{\varepsilon_n,0} \to u_0$ in $H_0^1(\Omega)$ as $\varepsilon_n \to 0$, then for any sequence of solutions of (64) $u_{\varepsilon_n}(\cdot)$ with $u_{\varepsilon_n}(0) = u_{\varepsilon_n,0}$ there exists a subsequence of ε_n such that u_{ε_n} converges to some strong solution $u(\cdot)$ of (3) in the space $C([0,T], H_0^1(\Omega))$, for any $T > 0$.*

Proof. By using (29) and (30) we can repeat the same lines of the proof of Theorem 1 and obtain the existence of a function $u(\cdot)$ and a subsequence of u_{ε_n} such that

$$\begin{aligned}u_{\varepsilon_n} &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)), \\ u_{\varepsilon_n} &\rightharpoonup u \text{ in } L^2(0,T;D(A)), \\ \frac{du_{\varepsilon_n}}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;L^2(\Omega)), \\ u_{\varepsilon_n} &\to u \text{ in } C([0,T];L^2(\Omega)), \\ u_{\varepsilon_n} &\to u \text{ in } L^2(0,T;H_0^1(\Omega)), \\ f_{\varepsilon_n}(u_{n_\varepsilon}) &\overset{*}{\rightharpoonup} f(u) \text{ in } L^\infty(0,T;L^\infty(\Omega)), \\ a(\|u_{\varepsilon_n}\|^2_{H_0^1})\Delta u_{\varepsilon_n} &\rightharpoonup a(\|u\|^2_{H_0^1})\Delta u \text{ in } L^2(0,T;L^2(\Omega)).\end{aligned}$$

Additionally, in the same way we prove that $u(\cdot)$ is a strong solution to problem (3) such that $u(0) = u_0$.

The uniform estimate in the space $H_0^1(\Omega)$ implies also that if $t_n \to t_0$, then $u_{\varepsilon_n}(t_n) \rightharpoonup u(t_0)$ in $H_0^1(\Omega)$. We need to prove that this convergence is in fact strong, proving then the convergence in $C([0,T], H_0^1(\Omega))$ for any $T > 0$.

In the same way as in the proof of Lemma 1 we deduce that for some $C > 0$ the functions $Q_n(t) = A(\|u_{\varepsilon_n}(t)\|^2_{H_0^1}) - 2Ct$, $Q(t) = A(\|u(t)\|^2_{H_0^1}) - 2Ct$ are continuous and non-increasing in $[0,T]$. Moreover, $Q_n(t) \to Q(t)$ for a.e. $t \in (0,T)$. Let first $t_0 > \dot{0}$. Then we take $0 < t_j < t_0$ such that $t_j \to t_0$ and $Q_n(t_j) \to Q(t_j)$ for all j. Then

$$Q_n(t_n) - Q(t_0) \leq Q_n(t_j) - Q(t_0) \leq |Q_n(t_j) - Q(t_j)| + |Q(t_j) - Q(t_0)| \text{ for } t_n > t_j.$$

For any $\delta > 0$ there exist $j(\delta)$ and $N(j(\delta))$ such that $Q_n(t_n) - Q(t_0) \leq \delta$ if $n \geq N$, so $\limsup Q_n(t_n) \leq Q(t_0)$. Hence, a contradiction argument using the continuity of $A(s)$ shows that $\limsup \|u_{\varepsilon_n}(t_n)\|^2_{H_0^1} \leq \|u(t_0)\|^2_{H_0^1}$. This, together with $\liminf \|u_{\varepsilon_n}(t_n)\|^2_{H_0^1} \geq \|u(t_0)\|^2_{H_0^1}$, implies that $\|u_{\varepsilon_n}(t_n)\|^2_{H_0^1} \to \|u(t_0)\|^2_{H_0^1}$, so that $u_{\varepsilon_n}(t_n) \to u(t_0)$ strongly in $H_0^1(\Omega)$. For the case when $t_0 = 0$ we use the same argument as in Lemma 1. □

We denote by $\mathcal{A}_{\varepsilon_n}$ the global attractor for the semiflow G_{ε_n} corresponding to problem (64).

Lemma 10. *Assume the condition of Theorem 7. Then $\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$ is bounded in $H_0^1(\Omega)$. Hence, the set $\overline{\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}}$ is compact in $L^2(\Omega)$.*

Proof. By Lemma 9 inequality (9) is satisfied for any n with constants which are independent of ε_n, so inequality (36) holds true with constants independent of ε_n. Thus, there a exists a common absorbing ball B_0 in $L^2(\Omega)$ (with radius $K > 0$) for problems (64). Further, by repeating the same steps as in Proposition 3 we obtain a common absorbing ball in $H_0^1(\Omega)$ (with radius $\widetilde{K} > 0$), as by Lemma 9 the constants which are involved are independent of ε_n. Thus, $\|y\|_{H_0^1} \leq \widetilde{K}$ for any $y \in \cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$. □

Lemma 11. *Assume the condition of Theorem 7. Then $\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$ is bounded in V^{2r} for any $0 \leq r < 1$. Hence, $\overline{\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}}$ is compact in V^{2r} and $C^1([0,1])$.*

Proof. By using Lemma 10 we obtain the boundedness of $\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$ in V^{2r} by repeating the same lines in Lemma 4. The rest of the proof follows from the compact embedding $V^\alpha \subset V^\beta$, $\alpha > \beta$, and the continuous embedding $V^{2r} \subset C^1([0,1])$ if $r > \frac{3}{4}$. □

Corollary 3. *Assume the condition of Theorem 7. Then any sequence $\xi_n \in \mathcal{A}_{\varepsilon_n}$ with $\varepsilon_n \to 0$ is relatively compact in $C^1([0,1])$.*

Lemma 12. *Assume the condition of Theorem 7. Then up to a subsequence any bounded complete trajectory u_{ε_n} of (64) converges to a bounded complete trajectory u of (3) in $C([-T,T], H_0^1(\Omega))$ for any $T > 0$. On top of that, if $y_n \in \mathcal{A}_{\varepsilon_n}$, then passing to a subsequence $y_n \to y \in \mathcal{A}$ in $H_0^1(\Omega)$. Hence,*

$$dist_{H_0^1}(\mathcal{A}_{\varepsilon_n}, \mathcal{A}) \to 0 \text{ as } n \to \infty. \tag{67}$$

Proof. Let us fix $T > 0$. By Corollary 3 $u_{\varepsilon_n}(-T) \to y$ in $H_0^1(\Omega)$ up to a subsequence. Theorem 7 implies that u_{ε_n} converges in $C([-T,T], H_0^1(\Omega))$ to some solution u of (3). If we choose successive subsequences for $-2T, -3T\ldots$ and apply the standard diagonal procedure, we obtain that a subsequence u_{ε_n} converges to a complete trajectory u of (3) in $C([-T,T], H_0^1(\Omega))$ for any $T > 0$. Finally, from Lemma 10 this trajectory is bounded.

If $y_n \in \mathcal{A}_{\varepsilon_n}$, by Corollary 3 we can extract a subsequence converging to some y. If we take a sequence of bounded complete trajectories $\phi_n(\cdot)$ of (64) such that $\phi_n(0) = y_n$, then by the previous result it converges in $C([-T,T], H_0^1(\Omega))$ to some bounded complete trajectory $\phi(\cdot)$ of (3), so $y \in \mathcal{A}$.

Finally, if (67) was not true, there would exist $\delta > 0$ and a sequence $y_n \in \mathcal{A}_{\varepsilon_n}$ such that $dist_{H_0^1}(y, \mathcal{A}) > \delta$. However, passing to a subsequence $y_n \to y \in \mathcal{A}$, which is a contradiction. □

Lemma 13. *Assume the conditions of Theorem 7. Let $\tau_{\pm}^{d_n,\varepsilon_n}$ be the functions (56)–(57) for problem (51) but replacing f by f_{ε_n} and d by d_n. Let $d_n, E_n \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \tau_{\pm}^{d_n,\varepsilon_n}(E_n) = \frac{\sqrt{a(0)}\pi}{\sqrt{2\lambda}}.$$

Proof. Let us consider $f_{d_n,\varepsilon_n}(u) = \frac{\lambda f_{\varepsilon_n}(u)}{a(d_n)}$. In view of property $(B4)$ and (66), since $f'_{\varepsilon_n}(0) = f'(0) = 1$ and $f_{\varepsilon_n}(0) = f(0) = 0$, given $\gamma \in (0,1)$ there exists $\delta > 0$ (independent of ε_n) such that

$$\begin{array}{cc} (1-\gamma)u \le f_{\varepsilon_n}(u) \le (1+\gamma)u, & \text{for any } u \in (0,\delta). \\ \frac{1}{1+\gamma} \le \frac{u}{f_{\varepsilon_n}(u)} \le \frac{1}{1-\gamma}, & \text{for any } u \in (0,\delta). \end{array} \quad (68)$$

The sequence $\mathcal{F}_{\varepsilon_n}(\cdot)$ converges uniformly to $\mathcal{F}(\cdot)$ in compact sets. Moreover, as $U_+(E)$ is continuous and using ([39] p. 60), given $\delta > 0$, there exists $\eta > 0$ such that $U_+^{\varepsilon_n}(E) \le \delta$ for any $0 < E \le \eta$. Now, if we integrate the first inequality in (68) between 0 and u we obtain

$$\frac{1}{2}(1-\gamma)u^2 \le \mathcal{F}_{\varepsilon_n}(u) \le \frac{1}{2}(1+\gamma)u^2, \quad \text{for any } 0 \le u \le \delta.$$

By using the change of variable $E_n y^2 = \mathcal{F}_{\varepsilon_n}(u)$, we have

$$\left(\frac{1-\gamma}{2E_n}\right)^{1/2} u \le y \le \left(\frac{1+\gamma}{2E_n}\right)^{1/2} u, \quad \text{if } 0 < E_n \le \eta,\ 0 \le y \le 1.$$

Dividing the previous expression by $\sqrt{\frac{\lambda}{a(d_n)}} f_{d_n,\varepsilon_n}(u)$ and using (68) we obtain

$$\left(\frac{a(d_n)(1-\gamma)}{2\lambda E_n(1+\gamma)^2}\right)^{1/2} \le \frac{\sqrt{a(d_n)}y}{\sqrt{\lambda} f_{d_n,\varepsilon_n}(u)} \le \left(\frac{a(d_n)(1+\gamma)}{2\lambda E_n(1-\gamma)^2}\right)^{1/2} \text{if } 0 < E_n \le \eta,\ 0 \le y \le 1.$$

Now if we multiply by $2\sqrt{E_n}(1-y^2)^{-\frac{1}{2}}$ and integrate from 0 to 1, we get

$$\pi\left(\frac{a(d_n)(1-\gamma)}{2\lambda(1+\gamma)^2}\right)^{1/2} \le \tau_+^{\varepsilon_n}(E_n) \le \pi\left(\frac{a(d_n)(1+\gamma)}{2\lambda(1-\gamma)^2}\right)^{1/2}, \quad \text{if } 0 < E_n \le \eta.$$

Then the theorem follows as $a(d_n) \to a(0)$ when $n \to \infty$. The proof for $\tau_-^{\varepsilon_n}$ is analogous. □

Under the conditions of Theorem 7, if (A8) is satisfied and

$$a(0)\pi^2 k^2 < \lambda \le a(0)\pi^2(k+1)^2,\ k \in \mathbb{Z},\ k \ge 0, \quad (69)$$

holds, then by Theorem 5 problem (64) has exactly $2k+1$ fixed points (denoted by $v_0 = 0,\ v^{\pm}_{1,d_1^{\varepsilon_n}}, \ldots, v^{\pm}_{k,d_k^{\varepsilon_n}}$) and $v^{\pm}_{m,d_m^{\varepsilon_n}}$ has $m+1$ zeros in $[0,1]$ for each $1 \le m \le k$. The same is valid for problem (3) and we denote the $2k+1$ fixed points by $v_0 = 0,\ u^{\pm}_{1,d_1^*}, \ldots, u^{\pm}_{k,d_k^*}$.

Lemma 14. *Assume the conditions of Theorem 7, (A8) and (69). Let $m \in \mathbb{N}$, $1 \le m \le k$, be fixed. Then $v^+_{m,d_m^{\varepsilon_n}}$ (resp. $v^-_{m,d_m^{\varepsilon_n}}$) do not converge to 0 in $H_0^1(\Omega)$ as $\varepsilon_n \to 0$.*

Proof. Assume that $v^+_{m,d_m^{\varepsilon_n}} \to 0$ in $H_0^1(0,1)$. Then it converges to 0 in $C([0,1])$ and the equality

$$-\frac{d^2 v^+_{m,d_m^{\varepsilon_n}}}{dx^2}(x) = \frac{\lambda f_{\varepsilon_n}\left(v^+_{m,d_m^{\varepsilon_n}}(x)\right)}{a(d_m^{\varepsilon_n})}$$

implies that $v^{+}_{m,d^{\varepsilon_n}_m} \to 0$ in $C^2([0,1])$. In particular, $\dfrac{dv^{+}_{m,d^{\varepsilon_n}_m}}{dx}(0) \to 0$ and $d^{\varepsilon_n}_m = \left\| v^{+}_{m,d^{\varepsilon_n}_m} \right\|^2_{H^1_0} \to 0$. The value E_n corresponding to the fixed point $v^{+}_{m,d^{\varepsilon_n}_m}$ is equal to $\dfrac{a(d^{\varepsilon_n}_m)}{2\lambda}\dfrac{dv^{+}_{m,d^{\varepsilon_n}_m}}{dx}(0)$, so $E_n \to 0$. We will show that this is not possible. We know by Lemma 13 that

$$\lim_{n\to\infty} \tau_{\pm}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) = \frac{\pi\sqrt{a(0)}}{\sqrt{2\lambda}}.$$

Additionally, since $v^{+}_{m,d^{\varepsilon_n}_m}$ is a fixed point with $d = d^{\varepsilon_n}_m$ one of the following conditions has to be satisfied (see (55)):

$$j\tau_{+}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) + (j-1)\tau_{-}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) = \left(\frac{1}{2}\right)^{\frac{1}{2}}, \tag{70}$$

$$j\tau_{-}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) + (j-1)\tau_{+}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) = \left(\frac{1}{2}\right)^{\frac{1}{2}}, \text{ if } m = 2j-1 \tag{71}$$

$$j\tau_{+}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) + j\tau_{-}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) = \left(\frac{1}{2}\right)^{\frac{1}{2}}, \text{ if } m = 2j. \tag{72}$$

Since $E_n \to 0$ and $\lambda > k^2\pi^2 a(0) \geq m^2\pi^2 a(0)$, there exists ε_{n_0} such that

$$\tau_{\pm}^{d^{\varepsilon_{n_0}}_m,\varepsilon_{n_0}}(E_{n_0}) < \frac{1}{\sqrt{2}m}.$$

Hence, neither of (70)–(72) is possible. □

Lemma 15. *Assume the conditions of Theorem 7, (A8) and (69). Let $m \in \mathbb{N}$, $1 \leq m \leq k$, be fixed. Then $v^{+}_{m,d^{\varepsilon_n}_m}$ (resp. $v^{-}_{m,d^{\varepsilon_n}_m}$) converges to u^{+}_{m,d^*_m} in $H^1_0(\Omega)$ (resp. u^{-}_{m,d^*_m}) as $\varepsilon_n \to 0$.*

Proof. We consider $v^{+}_{m,d^{\varepsilon_n}_m}$. In view of Corollary 3, $v^{+}_{m,d^{\varepsilon_n}_m}$ is relatively compact in $C^1([0,1])$, so up to a subsequence $v^{+}_{m,d^{\varepsilon_n}_m} \to v$ strongly in $C^1([0,1])$ and $d^{\varepsilon_n}_m \to d^* = \|v\|^2_{H^1_0}$. The proof will be finished if we prove that $v = u^{+}_{m,d^*_m}$. We observe that since in such a case every subsequence would have the same limit, the whole sequence would converge to u^{+}_{m,d^*_m}.

In view of (66) $f_{\varepsilon_n}(v^{+}_{m,d^{\varepsilon_n}_m})$ converges to $f(v)$ in $C([0,1])$. It follows that

$$-\frac{\partial^2 v}{\partial x^2} = \frac{\lambda f(v)}{a(\|v\|^2_{H^1_0})}$$

and v is a solution of (50), so v is a fixed point of (3). We need to prove that $v = u^{+}_{m,d^*_m}$. By Lemma 14 $v \neq 0$, and then $v = u^{\pm}_{j,d^*_j}$ for some $1 \leq j \leq k$. Since $u^{\pm}_{j,d^*_j}$ has $j+1$ simple zeros, the convergence $v^{+}_{m,d^{\varepsilon_n}_m} \to u^{\pm}_{j,d^*_j}$ in $C^1([0,1])$ implies that $v^{+}_{m,d^{\varepsilon_n}_m}$ has $j+1$ zeros for $n \geq N$. However, $v^{+}_{m,d^{\varepsilon_n}_m}$ possesses $m+1$ zeros in $[0,1]$. Thus, $m = j$.

For the sequence $v^{-}_{m,d^{\varepsilon_n}_m}$ the proof is analogous. □

6.2. Instability

We will prove that the fixed points 0 and $u^{\pm}_{k,d^*_k}$, $k \geq 2$, are unstable under some additional assumptions on the functions f and a. For that aim we need to use the approximative problems (64).

Theorem 8. *Assume that the conditions (A1)–(A8), $h = 0$ and (69) with $k \geq 1$ are satisfied; and let the function $f(\cdot)$ be odd and $a(\cdot)$ be globally Lipschitz continuous. Then the equilibria $v_0 = 0$ and $u^{\pm}_{j,d^*_j}$, $2 \leq j \leq k$ (if $k \geq 2$), are unstable.*

Remark 9. *The condition that $a(\cdot)$ is globally Lipschitz continuous could be dropped, as we can replace $a(\cdot)$ in (64) by a sequence $a_{\varepsilon_n}(\cdot)$ of globally Lipschitz continuous functions.*

Proof. Problem (64) generates a single-valued semigroup $\{T_{\varepsilon_n}(t); t \geq 0\}$ with a finite number of fixed points: $v_0 = 0$, $v^{\pm}_{1,d_1^{\varepsilon_n}}, ..., v^{\pm}_{k,d_k^{\varepsilon_n}}$ [26]. We know by Theorems 3.5 and 3.6 in [26] that for any $v^{+}_{j,d_j^{\varepsilon_n}}$ with $j \geq 2$ and v_0 there exists a bounded complete trajectory u^{ε_n} such that

$$u^{\varepsilon_n}(t) \to v^{+}_{j,d_j^{\varepsilon_n}} \quad \text{as } t \to -\infty, \quad \text{for } k \geq 2,$$

so v_0, $v^{+}_{j,d_j^{\varepsilon_n}}$ are unstable. The same is valid for $v^{-}_{j,d_j^{\varepsilon_n}}$. On the other hand, by Lemma 15 we have

$$v^{\pm}_{j,d_j^{\varepsilon_n}} \to u^{\pm}_{j,d^*_j}, \tag{73}$$

where $u^{\pm}_{j,d^*_j}$ is a fixed point of problem (3) with $j+1$ zeros in $[0,1]$. We prove the result for u^{+}_{j,d^*_j}. For u^{-}_{j,d^*_j} and v_0 the proof is the same.

By Lemma 12 we obtain that up to a subsequence u^{ε_n} converges to a bounded complete trajectory u of problem (3) in the space $C([-T,T], H^1_0(\Omega))$ for every $T > 0$. Thus, either $u(\cdot)$ is a fixed point v_{-1} or by Theorem 4 there exists a fixed point v_{-1} of problem (3) such that

$$u(t) \to v_{-1} \quad \text{as } t \to -\infty \text{ in } H^1_0(\Omega).$$

In the second case, if $v_{-1} = u^{+}_{j,d^*_j}$, the proof would be finished, so let assume the opposite.

Assume first that either $u(\cdot)$ is not a fixed point or it is a fixed point but $v_{-1} \neq u^{+}_{j,d^*_j}$. We consider $r_0 > 0$ such that the neighbourhood $\mathcal{O}_{2r_0}(v_{-1})$ does not contain any other fixed point of problem (3). For any $r \leq r_0$ we can choose $t_r \to -\infty$ and n_r such that $u^{\varepsilon_n}(t_r) \in \mathcal{O}_r(v_{-1})$ for all $n \geq n_r$. On the other hand, since $u^{\varepsilon_n}(t) \to v^{+}_{j,d_j^{\varepsilon_n}}$, as $t \to -\infty$, and $v^{+}_{j,d_j^{\varepsilon_n}} \to u^{+}_{j,d^*_j} \notin B_{2r_0}(v_{-1})$, there exists $t'_r < t_r$ such that

$$u^{\varepsilon_{n_r}}(t) \in \mathcal{O}_{r_0}(v_{-1}) \text{ for } t \in (t'_r, t_r],$$

$$\left\| u^{\varepsilon_{n_r}}(t'_r) - v_{-1} \right\|_{H^1_0} = r_0.$$

Let first $t_t - t'_r \to +\infty$. We define the sequence $u_1^{\varepsilon_{n_r}}(t) = u^{\varepsilon_{n_r}}(t + t'_r)$, which passing to a subsequence converges to a bounded complete trajectory $\phi(t)$ such that $\phi(t) \in \mathcal{O}_{r_0}(v_{-1})$ for all $t \geq 0$. As there is no other fixed point in $\mathcal{O}_{2r_0}(v_{-1})$, $\phi(t) \to v_{-1}$ as $t \to +\infty$. However, $\|\phi(0) - v_{-1}\| = r_0$, so $\phi(\cdot)$ is not a fixed point. Then $\phi(t) \to v_{-2}$ as $t \to -\infty$, where v_{-2} is a fixed point different from v_{-1}. Second, let $|t_t - t'_r| \leq C$. Then put $u_1^{\varepsilon_{n_r}}(t) = u^{\varepsilon_{n_r}}(t + t_r)$. Passing to a subsequence we have that

$$u_1^{\varepsilon_{n_r}}(0) \to v_{-1},$$
$$t_r - t'_r \to t_0, \text{ as } r \to 0.$$

Additionally, $u_1^{\varepsilon_{n_r}}(\cdot)$ converges to a bounded complete trajectory $u^1(\cdot)$ of problem (3) such that $u^1(0) = v_{-1}$. Let

$$\psi_1(t) = \begin{cases} u^1(t) \text{ if } t \leq 0, \\ v_{-1} \text{ if } t \geq 0. \end{cases}$$

We note that $\|u^1(-t_0) - v_{-1}\|_{H_0^1} = r_0$ implies that $u^1(\cdot)$ is not a fixed point. Then ψ_1 is a bounded complete trajectory of problem (3) such that $\psi_1(t) \to v_{-2} \neq v_{-1}$ as $t \to -\infty$. If $v_{-2} = u^+_{j,d_j^*}$, the proof is finished.

If $v_{-2} \neq u^+_{j,d_j^*}$, we continue constructing by the same procedure a chain of connections in which the new fixed point is always different from the previous ones, because the existence of the Lyapunov function (46) avoids the existence of a cyclic chain of connections. Since the number of fixed points is finite, at some moment we obtain a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \to u^+_{j,d_j^*}$ as $t \to -\infty$, proving that u^+_{j,d_j^*} is unstable.

Now let $u(\cdot) = v_{-1} = u^+_{j,d_j^*}$. Defining the neighbourhood $\mathcal{O}_{2r_0}(v_{-1})$ as before, for any $r \leq r_0$ we can choose n_r such that $u^{\varepsilon_n}(0) \in \mathcal{O}_r(v_{-1})$ for all $n \geq n_r$. Additionally, since $u^{\varepsilon_n}(t) \to z_0^n$, as $t \to +\infty$, where $z_0^n \neq v^+_{j,d_j^{\varepsilon n}}$ is a fixed point of (64), there exists $t_r > 0$ such that

$$u^{\varepsilon_{n_r}}(t) \in \mathcal{O}_{r_0}(v_{-1}) \text{ for } t \in [0, t_r),$$
$$\|u^{\varepsilon_{n_r}}(t_r) - v_{-1}\|_{H_0^1} = r_0.$$

The sequence $\{t_r\}$ cannot be bounded. Indeed, if $t_r \to t_0$, then $u^{\varepsilon_{n_r}}(t_r) \to u(t_0) = v_{-1}$, which is a contradiction with $\|u^{\varepsilon_{n_r}}(t_0) - v_{-1}\|_{H_0^1} = r_0$. Then $t_r \to +\infty$. We define the functions $u_1^{\varepsilon_{n_r}}(t) = u^{\varepsilon_{n_r}}(t + t_r)$, which satisfy that $u_1^{\varepsilon_{n_r}}(t) \in \mathcal{O}_{r_0}(v_{-1})$ for all $t \in [-t_r, 0)$. Passing to a subsequence it converges to a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \in \mathcal{O}_{r_0}(v_{-1})$ for all $t \leq 0$. This trajectory is not a fixed point as $\|\phi(0) - v_{-1}\|_{H_0^1} = r_0$ and $\phi(t) \to u^+_{j,d_j^*}$ as $t \to -\infty$, so u^+_{j,d_j^*} is unstable. □

Further, we will prove that there is also a connection from 0 to the point $u^{\pm}_{k,d_k^*}$.

Theorem 9. *Assume the conditions of Theorem 8. Then there exists a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \underset{t\to-\infty}{\to} 0$, $\phi(t) \underset{t\to+\infty}{\to} u^+_{k,d_k^*}$ (and the same is valid for u^-_{k,d_k^*}). Thus, $E(0) = 0 > E(u^{\pm}_{k,d_k^*})$.*

Proof. We start with the case where $k = 1$. We have three fixed points: 0, $u^+_{1,d_1^*}, u^-_{1,d_1^*}$. By Theorem 8 there exists a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \underset{t\to-\infty}{\to} 0$, whereas Theorem 4 and Remark 6 imply that it has to converge forward to a fixed point different from 0, that is, to either u^+_{1,d_1^*} or u^-_{1,d_1^*}. If, for example, $\phi(t) \underset{t\to+\infty}{\to} u^+_{1,d_1^*}$, then as the function f is odd, $\psi(t) = -\phi(t)$ is another bounded complete trajectory and $\psi(t) \underset{t\to+\infty}{\to} -u^+_{1,d_1^*} = u^-_{1,d_1^*}$.

Further we consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|^2_{H_0^1})\dfrac{\partial^2 u}{\partial x^2} = \lambda f_k(u), \quad t > 0,\ 0 < x < \frac{1}{k}, \\ u(t,0) = u(t, \frac{1}{k}) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{74}$$

where $f_k(u) = \sqrt{k} f\left(u/\sqrt{k}\right)$ satisfies (A1)–(A5). In this problem, condition (69) implies that there are again three fixed points: 0, $u^+_{1,d_1^*,\frac{1}{k}}, u^-_{1,d_1^*,\frac{1}{k}}$. By the above argument there is a connection $\phi_{\frac{1}{k}}(\cdot)$ from 0 to $u^+_{1,d_1^*,\frac{1}{k}}$ (also to $u^-_{1,d_1^*,\frac{1}{k}}$). Since the function f is odd, $u^+_{k,d_k^*}(x)$ is equal to $\frac{1}{\sqrt{k}} u^+_{1,d_1^*,\frac{1}{k}}(x)$ on $[0, \frac{1}{k}]$, to $-\frac{1}{\sqrt{k}} u^+_{1,d_1^*,\frac{1}{k}}\left(x - \frac{1}{k}\right)$ on $[\frac{1}{k}, \frac{2}{k}]$, etc. Then the function $\phi(\cdot)$ such that $\phi(t,x) = \frac{(-1)^j}{\sqrt{k}} \phi_{\frac{1}{k}}\left(t, x - \frac{j}{k}\right)$ on $[\frac{j}{k}, \frac{j+1}{k}]$, $j = 0, 1, ..., k-1$, is a bounded complete trajectory of problem (3) which goes from 0 to u^+_{k,d_k^*}. □

Remark 10. *When $k = 1$ the structure of the global attractor is the same as in the Chafee-Infante equation.*

6.3. Gradient Structure

We will obtain that the m-semiflow G is dynamically gradient. Let us recall this concept.

A weakly invariant set M of X is isolated if there is a neighbourhood O of M such that M is the maximal weakly invariant subset on $\mathcal{O}$. If M belongs to the global attractor $\mathcal{A}$, then it is compact ([38] Lemma 19). In this case, it is equivalent to use a δ-neighbourhood $\mathcal{O}_\delta(M) = \{y \in X : dist(y, M) < \delta\}$.

Suppose that there is a finite disjoint family of isolated weakly invariant sets $\mathcal{M} = \{M_1, \ldots, M_m\}$ in $\mathcal{A}$, that is, for every $j \in \{1, \ldots, n\}$ there is $\epsilon_j > 0$ such that $M_j \subset \mathcal{A}$ is the maximal weakly invariant set on $\mathcal{O}_{\epsilon_j}(M_j)$, and suppose that there exists $\delta > 0$ such that $\mathcal{O}_\delta(M_i) \cap \mathcal{O}_\delta(M_j) = \emptyset$, if $i \neq j$.

Definition 4. *We say the m-semiflow $G\colon \mathbb{R}^+ \times X \to P(X)$ is dynamically gradient with respect to the disjoint family of isolated weakly invariant sets $\mathcal{M} = \{M_1, \ldots, M_m\}$ if for every complete and bounded trajectory ψ of $\mathcal{R}$ we have that either $\psi(\mathbb{R}) \subset M_j$, for some $j \in \{1, \ldots, m\}$, or $\alpha(\psi) \subset M_i$ and $\omega(\psi) \subset M_j$ with $1 \le j < i \le m$.*

Let us consider the case when the conditions of Theorem 8 hold. Then (3) possesses exactly $2k+1$ fixed points: $v_0 = 0$, $u^{\pm}_{1,d_1^*}, \ldots, u^{\pm}_{k,d_k^*}$. Additionally, as f is odd, $u^{+}_{j,d_j^*} = -u^{-}_{j,d_j^*}$ for any j. We define the following sets:

$$M_1 = \{u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}\}, \ldots,\ M_k = \{u^{+}_{k,d_k^*}, u^{-}_{k,d_k^*}\},\ M_{k+1} = \{0\}. \tag{75}$$

They are weakly invariant and using Lemma 7 we deduce easily that they are isolated. Then the family $\mathcal{M} = \{M_1, \ldots, M_{k+1}\}$ is a finite disjoint family of isolated weakly invariant sets.

Proposition 4. *Assume the conditions of Theorem 8. Then G is a dynamically gradient semiflow with respect to the family (75) after (possibly) reordering them.*

Proof. We reorder the family (75) in such a way that if the value of the Lyapunov function E given in (46) is equal to L_i for the set $\widetilde{M}_i$, then $L_j \le L_n$ for $j < n$. Then Theorem 25 in [38] implies that G is dynamically gradient with respect to this family. □

We will obtain then that the fixed points $u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}$ are asymptotically stable. The compact set $M \subset \mathcal{A}$ is a local attractor for G in X if there is $\varepsilon > 0$ such that $\omega(O_\varepsilon(M)) = M$, where

$$\omega(B) = \{y : \exists t_n \to +\infty,\ y_n \in G(t_n, B) \text{ such that } y_n \to y\}$$

is the ω-limit set of B. By Lemma 14 in [38] if M is a local attractor in X, then it is stable. Thus, a local attractor is asymptotically stable.

Theorem 10. *Assume the conditions of Theorem 8. Then the stationary points $u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}$ are asymptotically stable.*

Proof. By ([38] Theorem 23 and Lemma 15) $\widetilde{M}_1$ is a local attractor in X, so it is asymptotically stable. By Theorem 8 the sets M_j, $j \ge 2$, are unstable. Thus, $\widetilde{M}_1 = M_1$. As M_1 consists of the two elements $u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}$, which are obviously disjoint, they are asymptotically stable as well. □

We will prove that there is a connection from 0 to any other fixed point $u^{\pm}_{j,d_j^*}$.

Theorem 11. *Assume the conditions of Theorem 8. Then there exists a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \underset{t\to-\infty}{\to} 0$, $\phi(t) \underset{t\to+\infty}{\to} u^+_{j,d^*_j}$ for all $1 \leq j \leq k$ (and the same is valid for u^-_{j,d^*_j}).*

Proof. Let us consider problem (74) with $k = j$. The function $u^+_{1,d^*_j,\frac{1}{j}}(x) = \sqrt{j}u^+_{j,d^*_j}(x)$, $x \in [0, \frac{1}{j}]$, is the unique positive fixed point of problem (74). Let $X^+_j = \{u \in H^1_0\left(0, \frac{1}{j}\right) : u(x) \geq 0\ \forall x \in [0, \frac{1}{j}]\}$ be the positive cone of $H^1_0\left(0, \frac{1}{j}\right)$. If we consider the restriction of the semigroup $T^{\varepsilon_n}_j(\cdot)$ of problem (64) in the interval $\left(0, \frac{1}{j}\right)$ to X^+_j, denoted by $T^{\varepsilon_n,+}_j(\cdot)$, then there exists a global attractor $\mathcal{A}^+_{n,j}$ [25]. Since 0 and $v^+_{1,d^{\varepsilon_n}_j,\frac{1}{j}} = \sqrt{j}v^+_{j,d^{\varepsilon_n}_j}\mid_{[0,\frac{1}{j}]}$ are the unique fixed points of $T^{\varepsilon_n,+}_j$, $\mathcal{A}^+_{n,j}$ is connected, $v^+_{1,d^{\varepsilon_n}_1,\frac{1}{j}}$ is stable [26] and $\mathcal{A}^+_{n,j}$ consists of the fixed points and their heteroclinic connections, there must exist a bounded complete trajectory $\phi^{\varepsilon_n}_j(\cdot)$ of $T^{\varepsilon_n,+}_j$ which goes from 0 to $v^+_{1,d^{\varepsilon_n}_j,\frac{1}{j}}$. By Lemma 12 up to a subsequence it converges to a bounded complete trajectory $\phi_j(\cdot)$ of problem (74) with $k = j$ such that $\phi_j(t) \geq 0$ for all $t \in \mathbb{R}$. Since by Theorem 10 the fixed point $u^+_{1,d^*_j,\frac{1}{j}}$ is stable, the only possibility is that $\phi_j(t) \to 0$, as $t \to -\infty$, $\phi_j(t) \to u^+_{1,d^*_j,\frac{1}{j}}$, as $t \to +\infty$. Then the function $\phi(\cdot)$ such that $\phi(t,x) = \frac{(-1)^i}{\sqrt{j}}\phi_j\left(t, x - \frac{i}{j}\right)$ on $[\frac{i}{j}, \frac{i+1}{j}]$, $i = 0, 1, ..., j-1$, is a bounded complete trajectory of problem (3) which goes from 0 to u^+_{j,d^*_j}.

For u^-_{j,d^*_j}, noting that $u^-_{j,d^*_j} = -u^+_{j,d^*_j}$, the result follows by choosing the bounded complete trajectory $\widetilde{\phi}(t) = -\phi(t)$. □

As a consequence we obtain that the order of the family $\mathcal{M}$ has to be the one given in (75).

Theorem 12. *The semiflow G is dynamically gradient with respect to the family $\mathcal{M}$ in the order given in (75), that is, $\widetilde{M}_i = M_i$ for any i.*

Proof. As by Theorem 11 there is a connection from 0 to $u^\pm_{j,d^*_j}$, $1 \leq j \leq k$, we have proved that $\widetilde{M}_{k+1} = \{0\} = M_{k+1}$. The fact that the order of the other sets is the one given in (75) follows from Lemma 8. □

Author Contributions: Conceptualization, R. Caballero, A.N. Carvalho, P. Marín-Rubio and J. Valero; methodology, R. Caballero, A.N. Carvalho, P. Marín-Rubio and J. Valero; validation, R. Caballero, A.N. Carvalho, P. Marín-Rubio and J. Valero.; formal analysis, R. Caballero, A.N. Carvalho, P. Marín-Rubio and J. Valero; investigation, R. Caballero, A.N. Carvalho, P. Marín-Rubio and J. Valero; writing—original draft preparation, R. Caballero, A.N. Carvalho, P. Marín-Rubio and J. Valero; writing—review and editing, R. Caballero, A.N. Carvalho, P. Marín-Rubio and J. Valero; supervision, R. Caballero, A.N. Carvalho, P. Marín-Rubio and J. Valero; funding acquisition, R. Caballero, A.N. Carvalho, P. Marín-Rubio, J. Valero. All authors have read and agreed to the published version of the manuscript.

Funding: The first author is a fellow of the FPU program of the Spanish Ministry of Education, Culture and Sport, reference FPU15/03080. This work has been partially supported by the Spanish Ministry of Science, Innovation and Universities, project PGC2018-096540-B-I00; by the Spanish Ministry of Science and Innovation, project PID2019-108654GB-I00; and by the Junta de Andalucía and FEDER, project P18-FR-4509.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to thank also the referees for their useful comments.
During the preparation of this manuscript our colleague and friend María José Garrido-Atienza passed away. She was a very kind and warm person and we will miss her a lot. We dedicate this paper to her memory.
We would like also to devote this work to the memory of Encarna López, Rubén's grandmother, with sorrow and love. She was always an example of goodness and willpower. Even in her last moments, she always had a smile in bad times. The whole family will miss her very much.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

In this appendix we generalise the lap number property of solutions of linear equations proved in [24] to the case when we do not have classical solutions. For this we will use a maximum principle for non-smooth functions from [40].

Let $\mathcal{O}$ be a region in $\mathbb{R}^2$ and let $(t_0, x_0) \in \mathcal{O}$ and $\rho, \sigma > 0$. We denote

$$Q_{\rho,\sigma} = \{(t, x) : t \in (t_0 - \sigma, t_0), |x - x_0| < \rho\},$$

where we assume that t_0, x_0, ρ, σ are such that $\overline{Q}_{\rho,\sigma} \subset \mathcal{O}$.

We denote by W the space of all functions from $L^2(\mathcal{O})$ such that

$$\int_{\mathcal{O}} \left(|u(t,x)|^2 + \left| \frac{\partial u}{\partial x}(t,x) \right|^2 \right) d\mu < +\infty.$$

As a particular case of Theorem 6.4 in [40] we obtain the following maximum and minimum principles.

Theorem A1. *(Maximum principle) Let $u \in W$ be such that*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \leq 0 \tag{A1}$$

in the sense of distributions. If

$$\sup ess_{(t,x) \in Q_{\rho\nu,\sigma_1}} u(t,x) = M,$$

for some ν, $0 < \nu < 1$, and any σ_1, where $0 < \sigma_1 < \sigma$, then $u(t,x) = M$ for a.a. $(t,x) \in Q_{\rho,\sigma}$.

Theorem A2. *(Minimum principle) Let $u \in W$ be such that*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \geq 0 \tag{A2}$$

in the sense of distributions. If

$$\inf ess_{(t,x) \in Q_{\rho\nu,\sigma_1}} u(t,x) = M,$$

for some ν, $0 < \nu < 1$, and any σ_1, where $0 < \sigma_1 < \sigma$, then $u(t,x) = M$ for a.a. $(t,x) \in Q_{\rho,\sigma}$.

We are ready to prove the lap-number property, saying that the number of zeros is a non-increasing function of time.

Theorem A3. *Let $r(t,x)$ be a continuous function and $u \in C([t_0, t_1], H_0^1(\Omega)) \cap L^2(t_0, t_1; H^2(\Omega))$ be such that $\dfrac{du}{dt} \in L^2(t_0, t_1; L^2(\Omega))$ and satisfies the equation*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = r(t,x)u, \; 0 < x < 1, \; t_0 < t \leq t_1. \tag{A3}$$

Then the number of components of

$$\{x : 0 < x < 1,\ u(t,x) \neq 0\}$$

is a non-increasing function of t.

Proof. We follow similar lines as in ([24] Theorem 6).

Denote $Q(t) = \{x \in (0,1) : u(t,x) \neq 0\}$. We need to show that there is an injective map from the components of $Q(t_1)$ to the components of $Q(t_0)$ if $t_1 > t_0$. If we denote by C a component of $Q(t_1)$ and by S_C the component of $[t_0, t_1] \times (0,1) \cap \{u(t,x) \neq 0)\}$ which contains C, then in order to obtain the injective map it is necessary to prove two facts:

1. $S_C \cap Q(t_0) \neq \varnothing$;
2. If C_1, C_2 are two components of $Q(t_1)$, then $S_{C_1} \cap S_{C_2} = \varnothing$.

Let us prove the first statement by contradiction, so assume that $S_C \cap Q(t_0) = \varnothing$. We can assume without loss of generality that $r(t,x) < 0$, because this property is satisfied for the function $W(t,x) = u(t,x)e^{-\lambda t}$ with $\lambda > 0$ large enough and the components of these two functions coincide. Consider, for example, that $u(t,x) > 0$ in S_C. Let $M = \max_{(t,x)\in S_C} u(t,x)$. By hypothesis and the Dirichlet boundary conditions this maximum has to be attained at a point (t', x') such that $t_0 < t' \leq t_1$, $0 < x' < 1$. Additionally, there has to be an $\varepsilon > 0$ such that if $(t,x) \in S_C$ and $t_0 < t \leq t_0 + \varepsilon$, then $u(t,x) < M$, as otherwise there would be a sequence $(t_n, x_n) \in S_C$, $t_n > t_0$, such that $t_n \to t_0$ and $u(t_n, x_n) = M$. By the continuity of u this would imply that $u(t_0, x_0) = M$ for some $(t_0, x_0) \in S_C$, which is a contradiction. Then we can choose t' as the first time when the maximum is attained, so $u(t,x) < M$ for all $(t,x) \in S_C$, $t_0 < t < t'$. By the continuity of u there exists a rectangle $R = [t' - \delta, t'] \times [x' - \gamma, x' + \gamma]$ such that R belongs to S_C. In order to apply Theorem A1 we put $\mathcal{O} = R$ and

$$Q_{\gamma,\delta} = \{(t,x) : t \in (t' - \delta, t'), |x - x'| < \gamma\}.$$

We have that

$$\sup_{(t,x)\in Q_{\nu\gamma,\sigma_1}} u(t,x) = M,$$

for some $0 < \nu < 1$ and any $0 < \sigma_1 < \delta$. Since u satisfies (A1), we conclude from Theorem A1 that $u(t,x) = M$ for all $(t,x) \in Q_{\rho,\sigma}$, which is a contradiction.

For the second statement suppose the existence of two disjoints components C_1, C_2 of $Q(t_1)$ such that $S_{C_1} \cap S_{C_2} \neq \varnothing$, which implies in fact that $S_{C_1} = S_{C_2}$. In this case we can assume that $r(t,x) > 0$, being this justified by the function $W(t,x) = u(t,x)e^{\lambda t}$ with $\lambda > 0$ large enough. Let, for example, $u(t,x) > 0$ in S_{C_1} and assume that the interval C_1 has lesser values than the interval C_2. Additionally, it is clear that between C_1 and C_2 there must exist a point (t_1, x_0) such that $u(t_1, x_0) = 0$. On the other hand, the set $S_{C_1} \cap (t_0, t_1) \times [0,1]$ is path connected. Thus, there exists a simple path ξ such that one end point is in $\{t_1\} \times C_1$ and the other one is in $\{t_1\} \times C_2$. Let us consider the set L of all points which are above the curve ξ and such that the function u vanishes at them. This set is non-empty because $(t_1, x_0) \in L$. Since L is compact, the function $g : L \to [t_0, t_1]$ given by $g(t,x) = t$ attains it minimum at a certain point $(t', x') \in L$ such that $t_0 < t'$. Then there exists a set $R = [t' - \delta, t') \times [x' - \gamma, x' + \gamma]$ which belongs to S_{C_1}. Let $\mathcal{O} = R$ and

$$Q_{\gamma,\delta} = \{(t,x) : t \in (t' - \delta, t'), |x - x'| < \gamma\}.$$

We have that

$$\inf_{(t,x)\in Q_{\nu\gamma,\sigma_1}} u(t,x) = 0,$$

for some $0 < \nu < 1$ and any $0 < \sigma_1 < \delta$. Since u satisfies (A2), we conclude from Theorem A2 that $u(t,x) = 0$ for all $(t,x) \in Q_{\rho,\sigma}$, which is a contradiction. □

References

1. Burger, M.; Caffarelli, L.; Markowich, P.A. Partial differential equation models in the socio-economic sciences. *Phil. Trans. R. Soc. A* **2014**, *372*, 2028.
2. Ramsey, F. A mathematical theory of saving. *Econ. J.* **1928**, *38*, 543–559.
3. Brito, P. A Bentham-Ramsey model for spatially heterogeneous growth. In *Working Papers of the Department of Economics*; ISEG, University of Lisboa: Lisbon, Portugal, 2001.
4. Brito, P. The dynamics of growth and distribution in a spatially heterogeneous world. In *Working Papers of the Department of Economics*; ISEG, University of Lisboa: Lisbon, Portugal, 2004.
5. Brito, P. Global endogeneous growth and distributional dynamics. In Proceedings of the Dynamics, Economic Growth, and International Trade, Milan, Italy, 13–14 September 2012.
6. Anita, S.; Capasso, V.; Kunze, H.; La Torre, D. Dynamics and optimal control in a spatially structured economy growth model with pollution diffusion and environmental taxation. *Appl. Math. Lett.* **2015**, *42*, 36–40.
7. Deng, K.; Wu, Y. Global stability for a nonlocal reaction-diffusion population model. *Nonlinear Anal. Real World Appl.* **2015**, *25*, 127–136.
8. Achleitner, F. ; Kuehn, C. On bounded positive stationary solutions for a nonlocal Fisher-KPP equation. *Nonlinear Anal.* **2015**, *112*, 15-29.
9. Delgado, M.; Molina-Becerra, M.; Santos Júnior, J.R.; Suárez, A. A non-local perturbation of the logistic equation in $\mathbb{R}^N$. *Nonlinear Anal.* **2019**, *187*, 147–158.
10. Chipot M.; Lovat, B. Some remarks on nonlocal elliptic and parabolic problems. *Nonlinear Anal.* **1997**, *30*, 461–627.
11. Chipot M.; Lovat, B. On the asymptotic behaviour of some nonlocal problems. *Positivity* **1999**, *3*, 65–81.
12. Chipot, M.; Molinet, L. Asymptotic behaviour of some nonlocal diffusion problems. *Appl. Anal.* **2001**, *80*, 273–315.
13. Chipot, M.; Rodrigues, J.F. On a class of nonlocal nonlinear elliptic problems. *Math. Model. Numer. Anal.* **1992**, *26*, 447–467.
14. Chipot, M.; Siegwart, M. On the Asymptotic behaviour of some nonlocal mixed boundary value problems. In *Nonlinear Analysis and Applications: To V. Lakshmikantham on his 80th Birthday*; Agarwal, R.P., O'Regan, D., Eds; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2003; Volume 1, pp.431–449.
15. Chipot, M.; Valente, V.; Vergara Caffarelli, G. Remarks on a nonlocal problem involving the Dirichlet energy. *Rend.Sem. Mat. Univ. Padova* **2003**, *110*, 199–220.
16. Zheng, S.; Chipot, M. Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms. *Asymptot. Anal.* **2005**, *45*, 301–312.
17. Anh, C.T.; Tinh, L.T.; Toi, V.M. Global attractors for nonlocal parabolic equations with a new class of nonlinearities. *J. Korean Math. Soc.* **2018**, *55*, 531–551.
18. Caballero, R.; Marín-Rubio, P.; Valero, J. Existence and characterization of attractors for a nonlocal reaction-diffusion equation with an energy functional. *J. Dyn. Differ. Equ.* **2021**, in press.
19. Caraballo, T.; Herrera-Cobos, M.; Marín-Rubio, P. Long-time behavior of a non-autonomous parabolic equation with nonlocal diffusion and sublinear terms. *Nonlinear Anal.* **2015**, *121*, 3–18.
20. Caraballo, T.; Herrera-Cobos, M.; Marín-Rubio, P. Time-dependent attractors for non-autonomous non-local reaction-diffusion equations. *Proc. Roy. Soc. Edinburgh Sect. A* **2018**, *148A*, 957–981.
21. Caraballo, T.; Herrera-Cobos, M.; Marín-Rubio, P. Robustness of time-dependent attractors in H1-norm for nonlocal problems. *Discret. Contin. Dyn. Syst. Ser. B* **2018**, *23*, 1011–1036.
22. Caraballo, T.; Herrera-Cobos, M.; Marín-Rubio, P. Global attractor for a nonlocal p-laplacian equation without uniqueness of solution, *Discret. Contin. Dyn. Syst. Ser. B* **2017**, *17*, 1801–1816.
23. Caraballo, T.; Herrera-Cobos, M.; Marín-Rubio, P. Asymptotic behaviour of nonlocal p-Laplacian reaction-diffusion problems. *J. Math. Anal. Appl.* **2018**, *459*, 997–1015.
24. Henry, D. Some infinite-dimensional Morse-Smale systems defined by parabolic partial differential equations. *J. Differ. Equations* **1985**, *59*, 165–205.
25. Carvalho, A.N.; Li, Y.; Luna, T.L.M.; Moreira, E. A non-autonomous bifurcation problem for a non-local scalar one-dimensional parabolic equation. *Commun. Pure Appl. Anal.* **2020**, *19*, 5181–5196.
26. Carvalho, A.N.; Moreira, E. Stability and hyperbolicity of equilibria for a scalar nonlocal one-dimensional quasilinear parabolic problem. *arXiv* **2020**, arXiv:2005.07857.
27. Sell, G.R.; You, Y. *Dynamics of Evolutionary Equations;* Springer: New York, NY, USA, 2002.
28. Robinson, J.C. *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*; Cambridge University Press: Cambridge, UK, 2001.
29. Lions, J.L. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*; Gauthier-Villar: Paris, France, 1969.
30. Caraballo, T.; Marín-Rubio, P.; Robinson, J. A comparison between two theories for multi-valued semiflows and their asymptotic behaviour. *Set-Valued Anal.* **2003**, *11*, 297–322.
31. Kapustyan, O.V.; Pankov, V.; Valero, J. On global attractors of multivalued semiflows generated by the 3D Bénard system. *Set-Valued Var. Anal.* **2012**, *20*, 445–465.
32. Melnik, V.S.; Valero, J. On attractors of multi-valued semi-flows and differential inclusions. *Set-Valued Anal.* **1998**, *6*, 83–111.

33. Kapustyan, O.V.; Kasyanov, P.O.; Valero, J. Structure and regularity of the global attractor of a reacction-diffusion equation with non-smooth nonlinear term. *Discret. Contin. Dyn. Syst.* **2014**, *32*, 4155–4182.
34. Valero, J. On L^r-regularity of global attractors generated by strong solutions of reaction-diffusion equations. *Appl. Math. Nonlinear Sci.* **2016**, *1*, 375–390.
35. Caballero, R.; Carvalho, A.N.; Marín-Rubio, P.; Valero, J. Robustness of dynamically gradient multivalued dynamical systems. *Discret. Contin. Dyn. Syst. Ser. B* **2019**, *24*, 1049–1077.
36. Ball, J.M. Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations. *J. Nonlinear Sci.* **1997**, *7*, 475–502.
37. Angenent, S. The zero set of a solution of a parabolic equation. *J. Reine Angew. Math.* **1988**, *390*, 79–96.
38. Da Costa, H.B.; Valero, J. Morse decompositions and Lyapunov functions for dynamically gradient multivalued semiflows. *Nonlinear Dyn.* **2016**, *84*, 19–34.
39. De Souza, P.N.; Nuno, J. *Berkeley Problems in Mathematics*; Springer: New York, NY, USA, 2002.
40. Kadlec, J. Strong maximum principle for weak solutions of nonlinear parabolic differential inequalities. Č*asopis Pěst. Mat*. **1967**, *92*, 373–391.