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KENDALL'S PROBLEM ON A SPHERE

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Abstract

In this paper we discuss Kendall's problem on a sphere. In rather physical than mathematical terms, a sphere Σ of radius 1, which rolls without slipping and without twisting over a sphere M of radius $R > 1$, is considered, and a fixed orthonormal referential $\mathcal{R}$ is placed on Σ. Given two points p and q on M, not necessarily different, and two states of $\mathcal{R}$, $\mathcal{R}_p$ and $\mathcal{R}_q$, also not necessarily different, it is supposed that Σ is on p with the referential $\mathcal{R}$ in the position $\mathcal{R}_p$, and one wishes to move Σ by a sequence of displacements without slipping and without spinning over geodesics of M to the point q, where the referential $\mathcal{R}$ must be in the state $\mathcal{R}_q$. We prove that it is possible to do this in a sequence of no more than 4 movements.

1 Introduction

Initially suggested on a special occasion as a challenge for good students (see Hammersley [1]), the so called Kendall's Problem admits several settings, and can be approached in various ways.

The original problem dealt with a sphere of radius $\rho > 0$ moving without slipping and without twisting over an horizontal plane Π. A positive orthonormal referential is attached to the centre of the sphere, and each "state" of the problem is a pair $(p, \mathcal{R})$ given by the contact point p of the sphere with the plane and by the position of the referential attached to it. The problem was to establish the smallest natural N such that it is possible to take the sphere from an initial state $(p, \mathcal{R}_p)$ to a final state $(q, \mathcal{R}_q)$ by a sequence of N movements, wherein the sphere is rolling without slipping and without twisting over a segment in Π.

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In [1] it is shown, in a quite elaborate and non-constructible way, that $N = 3$, and several generalizations are suggested. One of the most natural among these is to tackle a sufficiently regular surface M instead of the plane Π, and to study the controllability of taking a state $(p, \mathcal{R}_p)$ to a state $(q, \mathcal{R}_q)$ in a finite number of movements, in each of which the sphere rolls without slipping and without twisting on a geodesic of M.

In her Ph.D. thesis ([3]), Laura Biscolla proves this controlability when M is an analytical surface of revolution and the sphere which rolls without slipping and without twisting has a small enough radius ρ. This remains the best result in this direction. Biscolla also describes how to solve Kendall's original problem, on the plane, in 4 movements, by a simple geometric way, so that each movement is constructible with compass and ruler. More recently, very elegantly [4] showed another manner of solving this problem on the plane, in 3 movements.

However, it remains to determine whether given a surface M, there is a natural $N = N(M)$ such that it is possible by a succession of at most N movements to take the sphere from a point p with the positive orthonormal referencial $\mathcal{R}_p$ to a point q with the positive orthonormal referencial $\mathcal{R}_q$, the sphere rolling without slipping and without twisting in each of these movements over geodesics of M.

Here we consider this problem when M is a sphere with radius $R > 1$, and the rolling sphere Σ has radius $\rho = 1$ and rolls over the outside of M. The main result is that, in this case, $3 \leq N \leq 4$. The non-trivial part is to show that $N \leq 4$, and this is achieved describing explicitly a way to perform 4 geometrically constructible movements which solve the problem. In [4] it is shown that their result for the problem on a plane, in 3 movements, cannot be used when M is a sphere. This problem also is considered in [1] and [2].

Henceforth, M is a sphere with radius $R > 1$, and Σ is a sphere with radius $\rho = 1$ which rolls without slipping and without twisting over the outside of M, with a positive orthonormal referential attached to its centre.

For this problem, $M \times SO(3)$ is a natural phase space, where $SO(3)$ is the set of ordered orthonormal and $\Theta = (\lambda, \mu, \nu)$ are positive triads. Therefore, the clause "a (p, Θ) state" means that Σ touches the point $p \in M$, and the movable referential attached to its centre is in the state Θ. The canonical positive orthonormal referential of $\mathbb{R}^3$ will be referred to as $R_0 = (e_x, e_y, e_z)$.

It is supposed that the reader is acquainted with the cinematic concepts of rigid body behind facts 1 and 2, stated below. A discussion of these results can be found in [3]. Therein can also be found a proof, following ideas of Carlos E. Harle, of the second fact.

Fact 1 *If a sphere S rolls without slipping and without twisting over a surface M and the point of contact draws a curve of length ℓ on M, then the point of contact draws on S a curve also of lenght ℓ.*

Fact 2 *If a sphere S rolls without slipping and without twisting over a surface M and the point of contact follows a geodesic in M, then the point of contact draws on S also a geodesic.*

In section 2, the purely cinematic results, which will be used in the proof that it is possible to solve Kendall's problem on a sphere in at most 4 movements, are discussed. The result itself is presented in section 3.

2 Cinematic lemmata

These results shall be quite useful to understand what happens with the referential that moves along Σ when it rolls without slipping and without twisting over a geodesic of M.

Fact 3 *Let γ be a great circle of M, and v a versor perpendicular to the plane determined by γ, and suppose that Σ rolls without slipping and without twisting over γ from the state $(p, (\lambda, \mu, \nu))$ to $(q, (\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}))$. Then $\widetilde{\lambda}$ is on the circle formed by the intersection of the unitary sphere with the cone of directrix v and generatrix λ.*

Proof Consider $R^* = (v, w, z) \in SO(3)$ and the state (p, R^*). Since v is perpendicular to the plane determined by γ, at the end of the movement under consideration, (p, R^*) is taken to (q, R^*_q), where $R^*_q = (v, w_q, z_q)$.

From the definition of a rigid body it follows that the angle between the first vectors of (λ, μ, ν) and R is equal to the angle between the first vectors of $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$ and R^*. Q.E.D.

Fact 4 *Let $(p, (\lambda, \mu, \nu)) \in M \times SO(3)$, where ν is normal to the sphere in p. If Σ rolls without slipping and without twisting over a geodesic of M between the states $(p, (\lambda, \mu, \nu))$ and $(q, (\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}))$, describing an arc measuring α on M, then the angle between ν and $\widetilde{\nu}$, measured in the direction of the movement performed by Σ, is $\alpha(R+1)$.*

Proof Without loss of generality, let p be the north pole of M and $\nu = e_z$.

Let π be the plane which contains the geodesic over which Σ rolls. While Σ rolls without slipping and without twisting over γ between $(p, (\lambda, \mu, \nu))$

and $(q, (\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}))$, the third vector of the referential moving along Σ clearly draws an arc between ν and $\widetilde{\nu}$, contained in π.

Since the arc drawn by Σ when moving from p to q measures α, the length of the geodesic arc described by the point of contact during this movement is αR, and since the rolling is without slipping and without twisting it follows from Fact 1 that the geodesic arc drawn by the point of contact in Σ also measures αR.

Consider now the final position, after the movement. Let O and O' be the centres of M and Σ, respectively, and mark on Σ the points p^* and q^* which are, respectively, the points of contact with M at the beginning and end of the aforesaid movement. Furthermore, let k^* and s^* be the northern and southern poles of Σ at the end of the motion.

It is clear that the points O, O', p, q, p^*, q^*, r^* and s^* lie on π, and besides this:

[i] $\widetilde{\nu}$ is the antipodal point of p^*;

[ii] the angle $s^*O'q^*$ measures α;

[iii] the angle $q^*O'p^*$ measures αR (since Σ has unitary radius);

[iv] The angles $s^*O'p^*$ and $k^*O'\nu$ are congruent.

Therefore, [ii] and [iii] yield that $s^*O'p^*$ measures $\alpha + \alpha R$ and the result follows from [iv]. Q.E.D.

For what follows, two particular situations of Fact 4 are extremely relevant .

Theorem 1 *Let $(p, \Theta) \in M \times SO(3)$ and consider q a point of M such that there is a geodesic arc γ connecting p and q, measuring $n\frac{2\pi}{R+1}$ for some $n \in \mathbb{N}$ (i.e. the length of the arc in the great circle connecting p and q is $n\frac{2\pi}{R+1}R$). Then, by rolling Σ without slipping and without twisting from (p, Θ) over γ, the same referential Θ is obtained in q.*

Moreover, the reciprocal of this assertion also holds, that is, if $\widetilde{q} \in M$ is a point such that by rolling Σ on M from (p, Θ) over a geodesic arc $\widetilde{\gamma}$ connecting p to $\widetilde{q}$ the state $(\widetilde{q}, \Theta)$ is reached, then the length of $\widetilde{\gamma}$ is $n\frac{2\pi}{R+1}R$, for some natural n.

Proof Without loss of generality, assume that p is the north pole of M, and that $\Theta = (\lambda, \mu, \nu)$ is such that $\nu = e_z$, and μ is orthogonal to the plane containing the great circle to be followed between p and q. Let $(q, \widetilde{\Theta})$ be the state after the movement, with $\widetilde{\Theta} = (\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$.

Because μ and the plane where the movement happens are orthogonal, it is evident that $\widetilde{\mu} = \mu$, and so $\widetilde{\Theta} = \Theta$ if, and only if, $\widetilde{\nu} = \nu$. It follows straightforwardly from Fact 4 that this happens if, and only if, the geodesic arc described during the movement between p e q measures $n\frac{2\pi}{R+1}$, for some $n \in \mathbb{N}$. Q.E.D.

Theorem 2 *Consider $(p, (\lambda, \mu, \nu))$, with ν normal to the sphere in p. Then, when rolling Σ without slipping and without twisting from p via a geodesic γ so that the length of the great circle arc being drawn is $\frac{\pi}{R+1}R + n\frac{2\pi}{R+1}R$, for some $n \in \mathbb{N}$, Σ reaches a final state (q, Θ_q), wherein $\Theta_q = (\widetilde{\lambda}, \widetilde{\mu}, -\nu)$.*

Proof It suffices to apply Fact 4. Q.E.D.

3 Solving Kendall's problem on a sphere

Here it shall be shown that it is possible to solve Kendall's problem on a sphere in at most 4 movements. The case of same referentials at the starting and final points is dealt with first. The general conclusion is then achieved by combining the results of subsection 3.1 and of section 2.

3.1 Same referentials

Let p and q be points of M, and $\Theta \in SO(3)$ a referential. The aim is to take the state (p, Θ) to (q, Θ). Clearly, without any loss of generality, it is allowed to call p the northern pole of M and so shall be done occasionally, in order to profit from a "geographical" language.

If x and y are points on M, let Λ_{xy} be the set of geodesic arcs in M connecting x and y, regardless of the amount of turns happening around M. For $\gamma \in \Lambda_{xy}$, the length of γ shall be represented by $d_\gamma(x, y)$.

In view of the Theorem 1, it is natural to consider, for a point $x \in M$, the following set:

$$C_x := \left\{ y \in M \colon \exists \gamma \in \Lambda_{xy},\ d_\gamma(x, y) = n\frac{2\pi}{R+1}R,\ n \in \mathbb{N} \right\}. \tag{1}$$

Of course, if x is a pole of M, then C_x is a union of "parallels" (whereby x itself and its antipodal are also considered parallels). Whether R is rational or irrational, C_x is a finite or ennumerable set of parallels, and, in the latter situation, C_x is dense on M. The following description of C_x, even if completely blatant, will be quite important.

Fact 5 *Let $x \in M$ and γ be a great circle passing through x. The set C_x is an ennumerable reunion of parallels in M and, on every arc of length $\frac{2\pi}{R+1}R$, γ crosses at least one of these parallels.*

Proof It follows immediately from the definition of C_x. Q.E.D.

Fact 6 *Let p and q be points of M such that $C_p \cap C_q \neq \emptyset$, and pick $\Theta \in SO(3)$. Then it is possible to take the state (p, Θ) to the state (q, Θ) in 2 movements.*

Proof Just take y on this intersection and use Theorem 1, so as to see that two movements are enough. Q.E.D.

Remark 1 *When M is a plane, C_x may be analogously set up as the ennumerable reunion of circles with centre on x and radius $n2\pi R$, and then clearly $C_x \cap C_y$ always is not the empty set. Thus, when M is a plane, it is always possible to take (p, Θ) to (q, Θ) in two movements, as [3] does. The following example shows that this is not always the case on a sphere. It is presented here because it highlights the difference of setting the problem on a plane or on a sphere, and because it follows immediately from it that the minimum amount of movements needed to solve Kendall's problem on a sphere is $N \geq 3$.*

Example 1 Example of close standing conjugated points

Let the radius of M be $R = 2m$, for some $m \in \mathbb{N}$, and assume either

[i] p, q poles, or

[ii] keeping p on the north pole, let $\tilde{q}$ be quite close to the south pole.

In [i], it is easy to see by Theorem 2, that Σ, leaving p with the standard referential and moving over only one great circle, always reaches q bearing a mirrored referential when compared to the standard referential of $\mathbb{R}^3$ (and vice-versa). Because of the proportion between the radii of Σ and M, C_p and C_q are a finite collection of parallel circles. Therefore

$$\nexists n, m : C_{p,n} \cap C_{q,m} \neq \emptyset \tag{2}$$

In [ii], $C_{\tilde{q}}$ is, as before, a finite collection of circles which, by continuity, lie close to those forming C_q (above), and, as in case [i], where C_p and C_q are finite collections of parallels with empty intersections, now, for $\tilde{q}$ close to the south pole, also $C_{\tilde{q}} \cap C_p = \emptyset$.

Fact 7 *If the radius of M is $R > 1$, there are states $(p, \mathcal{R}_p)$ and $(q, \mathcal{R}_q)$ which cannot be taken one to another in one or two movements.*

Proof A preliminary remark: if p and q are, respectively, the north and south poles of M, then for Σ to go in two movements from p to q, rolling without slipping and without twisting over geodesics, and to "change" geodesics it must stop in p or q.

If R is irrational, let p be the north pole, q the south pole, and $\mathcal{R}$ the standard referential. Obviously it is then impossible to take $(p, \mathcal{R})$ to $(q, \mathcal{R})$ in two or fewer movements.

If R is rational, let p and q be as before, let $\mathcal{R}_p$ be the standard referential, and $\alpha \in \mathbb{R}$ be such that $\frac{\alpha}{2\pi} \notin \mathbb{Q}$. Pick $R_q = (\lambda, \mu\nu)$ where the angle between ν and e_z measures α. Taking the previous remark into account, it is easy to see that it is not possible to take $(p, \mathcal{R}_p)$ to $(q, \mathcal{R}_q)$ in one or two movements. Q.E.D.

Despite the previous example, (p, Θ) can always be moved to (q, Θ) in at most 3 movements, as shall be shown next.

Let $x \in M$, and $\beta \in (0, 2\pi)$, then the spherical cap of centre x and angle β wil be called $V_\beta(x) := \{y \in M : \exists \gamma \in \Lambda_{xy},\ d_\gamma(x, y) < \beta R\}$. It is noteworthy that if $\widetilde{x}$ is the antipodal point of x, then $V_\beta(x) = M \setminus \overline{V}_{2\pi-\beta}(\widetilde{x})$.

Fact 8 *For $p \in M$ let $\widetilde{p}$ be the antipodal point of p. If $q \notin V_{\frac{\pi}{R+1}}(\widetilde{p})$, then $C_p \cap C_q \neq \emptyset$.*

Proof Just notice that if γ is a great circle going through p, it contains $\widetilde{p}$, and $\gamma \cap V_{\frac{\pi}{R+1}}(\widetilde{p})$ is an arc of lenght $2\frac{\pi}{R+1}R$ centred on $\widetilde{p}$. The conclusion follows immediately from Fact 5. Q.E.D.

The following ought to be highlighted. For $x \in M$, choose $\Gamma : [0; +\infty[$ a geodesic in M starting on x. Let γ be the image of Γ, i.e. a great circle, and $y \in \gamma \cap C_x$. Then there is $\Gamma^{-1}\{y\} \neq \emptyset$, but it can happen that $d_\Gamma(x\Gamma(s)) \neq n\frac{2\pi}{R+1}R$, for all $s \in \Gamma^{-1}\{y\}$ and $n \in \mathbb{N}$. That $y \in C_x$ means that *there is a geodesic arc $\sigma \in \Lambda_{xy}$ of lenght $n\frac{2\pi}{R+1}R$, but σ must not be an arc of* Γ. There are indeed circumstances where σ is not a subarc of Γ. Though easily seen, it is stated for clarity:

Fact 9 *For $q \in M$ and $\Gamma : [0; +\infty[\longrightarrow M$ a geodesic of M such that $\Gamma(0) = q$ and with image γ. Consider the sets $X = \gamma \cap C_q$ and $X^+ = \{y \in \gamma : \exists (s, n) \in \Gamma^{-1}\{y\} \times \mathbb{N},\ d_\Gamma(q\Gamma(s)) = n\frac{2\pi}{R+1}R\}$.*

Then $X = X^+$ if, an only if, R is rational, and, for R irrational, X^+ is dense in X.

Proof Assume on γ the orientation given by the movement of Γ, and place the origin of the arcs on q.

Easily it is seen that the points of X are the end-points y of the oriented arcs qy, measuring $n\frac{2\pi}{R+1}$, $n \in \mathbb{Z}$, while the points of X^+ are the end-points y of the oriented arcs qy measuring $n\frac{2\pi}{R+1}$, $n \in \mathbb{N}$.

It is clear that $X^+ \subset X$, as well as that, for R irrational, X^+ is dense on X.

Let $y_0 \in \gamma$ be such that the lenght of the oriented arc qy_0 is $n\frac{2\pi}{R+1}$, where $-n \in \mathbb{N}$. The sets X e X^+ will be the same if, and only if, $y_0 \in X^+$.

Note that $y_0 \in X^+$ if, and only if, there is $k \in \mathbb{N}$ such that $n\frac{2\pi}{R+1} \equiv_{2\pi} k\frac{2\pi}{R+1}$.

Therefore $X = X^+$ if, and only if, $(k-n)\frac{2\pi}{R+1} = \ell 2\pi$, for some $\ell \in \mathbb{Z}$. Since k e $-n$ are naturals, it follows that $y_0 \in X^+$ if, and only if, $R \in \mathbb{Q}$. Q.E.D.

The next important result means, freely speaking, that for any two points p and q on M, and a great circle γ crossing q, there is always a point x on $\gamma \cap C_q$ which is "sufficiently close" to p, in the sense given by fact 8, and which can, furthermore, be reached leaving q in any direction on γ.

Fact 10 *For $p, q \in M$ and $\Gamma : [0, +\infty[$ a geodesic with $\Gamma(0) = q$, there is $x = \Gamma(T)$ such that $d_\Gamma(qx) = n\frac{2\pi}{R+1}R$ and $C_x \cap C_p \neq \emptyset$.*

Proof Let γ be the image of Γ, and, as in Fact 9, take $X^+ = \{y \in \gamma : \exists(s,n) \in \Gamma^{-1}\{y\} \times \mathbb{N},\ d_\Gamma(q\Gamma(s)) = n\frac{2\pi}{R+1}R\}$. The desired point x must be in X^+.

Notice that if $q \in M$, and H is any closed hemisphere of M, then $C_q \cap (H \cap \gamma) \neq \emptyset$. It will be shown that, in fact, $X^+ \cap H \neq \emptyset$. This follows directly from Fact 9. For R rational, $C_q \cap \gamma = X^+$ and so $X^+ \cap H \neq \emptyset$. For R irrational, since X^+ is dense on γ, and the interior of the arc $H \cap \gamma$ is not empty, it follows, also in this case, that $X^+ \cap H \neq \emptyset$.

Since $R > 1$, if the antipodal point of p is $\widetilde{p}$, then $V_{\frac{\pi}{R+1}}(\widetilde{p})$ is inside the open hemisphere H_0 formed by the points of M lying strictly closer to $\widetilde{p}$ than to p. So, if y is a point on the closed half-plane $H = M \setminus H_0$, then $C_y \cap C_p \neq \emptyset$. Thus, it is enough to take $x \in X^+ \cap H$ to finish. Q.E.D.

Now it is easy to prove the assertion made at the beginning of this subsection.

Theorem 3 *Let p and q be points of M, and Θ a positive orthonormal referential. It is possible to move the state (p, Θ) to (q, Θ) in, at most, 3 movements.*

Proof Let γ be any great circle crossing q, and choose $x \in \gamma \cap C_q$ given by Fact 10.

Then, in view of the definition of C_q and from Theorem 1, it is possible to roll Σ without slipping and without twisting over γ, taking the state (q, Θ) to (x, Θ).

Using again that x is given by Fact 10 it follows, by Fact 8, that it is possible to take (q, Θ) to (x, Θ) in two movements. Q.E.D.

3.2 The general case

Another cinematic fact to start with:

Fact 11 *Let (λ, μ, ν) and $(\bar{\lambda}, \bar{\mu}, \nu)$ be two referentials such that $\nu = e_z$, the angle between λ and $\bar{\lambda}$ is $\phi, 0 < \phi < \pi/2$, and let p be the north pole of M.*

If Σ rolls without slipping and without twisting from the state $(p, (\lambda, \mu, \nu))$ drawing an arc of length $\frac{\pi}{R+1}R$ over the meridian determined by the perpendicular to the bisectrix of ϕ (i.e. over the meridian perpendicular to the bisectrix of the acute angle between λ and $\bar{\lambda}$), then the referential obtained at the final point of this movement is $(\bar{\lambda}, -\bar{\mu}, -\nu)$.

Proof Since $\nu = e_z$ in both referentials, (λ, μ) as well as $(\bar{\lambda}, \bar{\mu})$ are referentials of the plane (e_x, e_y), and so reference to the aforementioned angle ϕ means the acute angle between λ and $\bar{\lambda}$ in this plane.

Let v be the versor of the bisectrix of the acute angle between λ and $\bar{\lambda}$.

Note that if ℓ is the perpendicular to the bisectrix of this angle, and if Σ rolls without slipping and without twisting from the north pole over the meridian determined by ℓ (i.e. the meridian going through the centre of M and through ℓ), it follows from Fact 3 that the vector λ of the referential attached to Σ moves in a cone of directrix v and geractrix λ. Since v is the versor of the bisectrix of the acute angle between λ and $\bar{\lambda}$, this cone intercepts the plane $z = 0$ in the half-lines $t\lambda$ and $t\bar{\lambda}$, $t \geq 0$.

On the other hand, if Σ lies on the north pole with referential (λ, μ, ν), and $\nu = e_z$, and it rolls without slipping and without twisting for $\frac{\pi}{R+1}$ over the meridian determined by ℓ, then by Theorem 2, at the end of these movements ν has been taken to $-\nu = -e_z$ and (λ, μ) to $(\hat{\lambda}, \hat{\mu})$ which being perpendicular to $-\nu$, lies on the $z = 0$ plane .

Therefore, $\hat{\lambda} = \overline{\lambda}$ (since in view of the previous remark of what happens to λ during this movement, $\hat{\lambda}$ must be equal to λ or $\overline{\lambda}$, and clearly the first alternative cannot happen), and so $\hat{\mu} = -\overline{\mu}$. Q.E.D.

Remark 2 *In [3], this result is not used when solving the problem geometrically on a plane, although it is also true there, with obvious adaptations. Indeed, the algorithm set out here provides another geometrical approach to show that Kendall's problem can be solved in 4 movements when M is a plane. However, in view of the solutions in 3 movements presented in [1] and [4], this lacks greater importance.*

Theorem 4 *Let $(p, (\lambda, \mu, \nu))$ and $(q, (\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$ be any two states in $M \times SO(3)$. It is possible to leave $(p, (\lambda, \mu, \nu))$ and reach $(q, (\overline{\lambda}, \overline{\mu}, \overline{\nu})$ in, at most, 4 movements.*

Proof Without loss of generality, assume that p is the north pole of M, and that the referential $(\lambda, \mu, \nu))$ is the canonical referential (e_x, e_y, e_z).

The main idea behind this proof is to show that there are sufficiently close points p_1 and q_1, and a positive orthonormal referential $(\hat{\lambda}, \hat{\mu}, -\nu)$ such that *it is possible to take $(p, (\lambda, \mu, \nu))$ to $(p_1, (\hat{\lambda}, \hat{\mu}, -\nu))$ in one movement, and also to take $(q, (\overline{\lambda}, \overline{\mu}, \overline{\nu}))$ to the state $(q_1, (\hat{\lambda}, \hat{\mu}, -\nu))$ in one movement.*

To achieve this, three steps shall be done:

1. Let r be the bisectrix between $\overline{\nu}$ and $-e_z$ (if $\overline{\nu} = -e_z$, just take $(\hat{\lambda}, \hat{\mu}, -\nu) = (\overline{\lambda}, \overline{\mu}, \overline{\nu})$, $q_1 = q$, and go to the next step).

   Let ℓ be the meridian of M which crosses q, and whose direction is perpendicular to r. From Fact 3, when the rolling sphere moves from the state $(q, (\overline{\lambda}, \overline{\mu}, \overline{\nu}))$ over the meridian ℓ, the third vector of the referential attached to this sphere moves in the circle C determined by the intersection of the cone of directrix r and generatrix $\overline{\nu}$ with the sphere of centre 0 and radius 1. Since $-e_z \in C$, there is a point $q_1 \in \ell$ such that the referential moving along the sphere has as third vector $-e_z$. Let $(\hat{\lambda}, \hat{\mu}, -e_z)$ be this referential.

2. Consider now the referential $(\hat{\lambda}, -\hat{\mu}, e_z)$. Clearly, since $(\hat{\lambda}, \hat{\mu}, -e_z)$ is positive and orthonormal, $(\hat{\lambda}, -\hat{\mu}, e_z)$ is also. From Fact 11, it is possible to choose a meridian $\widetilde{\ell}$ going through the north pole p, and putting Σ on p with the canonical referential attached to it, rolling it without slipping and without twisting over an arc $\widetilde{\ell}$ of lenght $\frac{\pi}{R+1}$, to arrive at a point $\widetilde{p}_1$ of $\widetilde{\ell}$ where Σ has the referential $(\hat{\lambda}, \hat{\mu}, -e_z)$.

3. Unfortunately, $\widetilde{p}_1$ can be not close enough (as stated in Fact 8) to q_1, but it suffices to employ Fact 10 with $p = q_1$, $q = \widetilde{p}_1$ and $\gamma = \widetilde{\ell}$ to continue the movement of Σ over $\widetilde{\ell}$ (keeping the same direction of the rolling from p to $\widetilde{p}_1$), and to obtain, with only one movement starting in p, a point $p_1 \in C_{\widetilde{p}_1} \cap \widetilde{\ell}$ such that $C_{q_1} \cap C_{p_1} \neq \emptyset$. Of course, when arriving in p_1, the referential attached to Σ is $(\hat{\lambda}, \hat{\mu}, -e_z)$.

This proves the emphasised statement above. To round this off, perceive that, since $d < 2\frac{\pi}{R+1}R$, the state $(p_1, (\hat{\lambda}, \hat{\mu}, -\nu))$ can be taken to the state $(q_1, (\hat{\lambda}, \hat{\mu}, -\nu))$ in two movements. Q.E.D.

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