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# A Note on Conformal Geometry

Elisha Falbel

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Elisha Falbel
Instituto de Matemática e Estatística
Universidade de São Paulo
CP20570-São Paulo-Brasil

## 1 Introduction

There are several approaches to conformal geometry. We will restrict our atention to the equivalence problem, that is, the problem of finding appropriate connections in appropriate spaces. A modern introduction to it by means of second order frames is in [K].

The treatment of conformal geometry presented in this work was inspired by E. Cartan's treatment of Cauchy-Riemann structures of dimention 3 [C] by the introduction of a line bundle. This approach, in the case of CR-structures was generalized by Chern in [CM]. In this work we consider the line bundle, which we call $E$, of all conformal metrics to a fixed one over a manifold, and over this line bundle an appropriate coframe bundle Y which solves the equivalence problem.

The main advantage of this formulation is the first order treatment of the equivalence problem over the line bundle, and moreover, we belive that this paper could be usefull as an introduction to the CR equivalence problem in a simpler but analogous situation.

As an application, we compute the Chern-Simons class of a conformal structure on a 3-manifold pulling-back by sections of this line-bundle a Chern-Simons class on Y.

The line bundle E was used in [FG] to obtain conformal invariants using a Ricci flat riemannian metric defined on the product $E \times I$, where $I = (-1, 1)$. The process to obtain the Chern-Simons class is the same used in [BE] to obtain a secondary class for CR-structures. The author would like to thank Profs. A. M. Rodrigues and J. A. Verderesi for enlightenning discussions and CNPq for partial support.

## 2 The Conformal Group and Conformal Structure

In this section we review the conformal group for the sake of completeness. Let S be the following matrix of order n+2

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The conformal group is defined as $L = O(n+1,1) = \{X \in GL(n+2, \mathbb{R})\,;\ X^TSX = S\}$.

This group acts naturaly in the so-called Möbius space $S^n$, thought as the subspace of the projective space $\{[x] \in RP^{n+1}$ *with* $x^TSx = 0\ \}$, x being a colunm vector in $\mathbb{R}^{n+2}$. The isotropy of this action is the group

$$L_0 = \left\{ \begin{pmatrix} a^{-1} & 0 & 0 \\ v & A & 0 \\ b & \xi & a \end{pmatrix} \in O(n+1,1) \right\}$$

Observe that in the matrix above we have $A \in O(n)$, $a \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ with $v = a^{-1}A\xi^T, b = \frac{\xi\xi^T}{2a}$.

The isotropy of first order is the isotropy of the adjoint action, and it is the set

$$L_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \xi^T & I_n & 0 \\ \frac{\xi\xi^T}{2} & \xi & 1 \end{pmatrix} \in O(n+1,1) \right\}$$

We call $G_1$ the group

$$G_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ A\xi^T & A & 0 \\ \frac{\xi\xi^T}{2} & \xi & 1 \end{pmatrix} \in O(n+1,1) \right\}$$

Observe that $L \supset L_0 \supset G_1 \supset L_1$.

We start stating the definition of a conformal structure. Let (M, g) be a riemannian manifold.

**Definition 2.1** *The conformal structure associated to (M, g) is the G-structure given by the set of 1-forms*

$$(1) \qquad \theta'^i = a^i_j\theta^j \qquad where\ (a^i_j) \in CO(m)$$

Geometrically $\theta^i$ is a basis of coframes satisfying $\theta^j(X_i) = \delta^j_i$ with $1 \le i, j \le m$ where $X_i$ is an orthogonal basis of TM with respect to g. It is well known that this is not a first order structure in the sense that we cannot find a canonical parallelism of this bundle in order to solve the equivalence problem. We introduce next a $G_1 -$ *bundle* over the line bundle of all conformal metrics to a fixed one and, on this bundle, we find a canonical parallelism.

## 3 The Line Bundle

Let E be the trivial line bundle, with fiber $R^+$ of all metrics conformal to g. We will define a $G_1$-structure over this line bundle. Let $\pi : E \to M$ be the natural projection.

Given a point $e \in E$, we define $\omega^i_e = \pi^*(\theta^i_e)$, where $\theta^i_e$ is a coframe for M at the point $\pi(e)$ with metric e.

Choose a local section $\theta^i$, this is equivalent to a local choice of a riemannian metric. This gives local coordinates to the line bundle. In those coordinates $\omega^i = \lambda\theta^i$. Diferentiating we get

$$d\omega^i = \omega^j \wedge \omega^i_j + \frac{d\lambda}{\lambda} \wedge \omega^i$$

where $\omega^i_j$ is the Levi-Civita connection defined by $\theta^i$. If $u\theta^i$, where u is a positive function on M, is another section, then

$$d\omega^i = \omega^j \wedge \omega^i_j + (\frac{d\lambda}{\lambda} + \frac{du}{u}) \wedge \omega^i$$

Therefore the form $d\lambda/\lambda$ is defined modulo horizontal forms.

We define the $G_1 - structure$ by the forms

$$(1) \quad \begin{cases} \phi' &= \phi + a_i\omega^i \quad \text{with } \phi = \frac{d\lambda}{\lambda} \text{ in a trivialization of E} \\ \omega^{i'} &= a^i_j\omega^j \quad \text{where } (a^i_j) \in O(m) \end{cases}$$

We will call this $G_1$-bundle over E to be Y. We have $Y \to E \to M$. Our next goal is to construct a canonical parallelism $Y \to E$.

## 4 The Cartan Connection

If we choose a section of coframes $\omega^i, \phi$, we can write without loss of generality

$$(2) \quad \begin{cases} d\omega^i &= \omega^j \wedge \omega^i_j + \phi \wedge \omega^i \\ d\phi &= \omega^j \wedge \omega_j \end{cases}$$

where $\omega^i_j$ and $\omega_j$ are one-forms to be determined. We will impose conditions on them so that they will be uniquely determined.

Define

$$\Omega^i_k = d\omega^i_k + \omega^i_j \wedge \omega^j_k + \omega^i \wedge \omega_k + \omega_i \wedge \omega^k$$

and write

$$\Omega^i_k = \frac{1}{2}K^i_{kls}\omega^l \wedge \omega^s$$

Then we can state the following theorem.

**Theorem 4.1** *There are uniquely defined forms $\omega^i_j$ and $\omega_j$, such that*

$$(3) \qquad \begin{array}{rcl} \omega^i_j & = & -\omega^j_i \\ \sum K^i_{kis} & = & 0 \end{array}$$

**Proof :** The existence and uniqueness of the forms $\omega^i_j$ follows by the same reasoning as in riemannian geometry. It remains to find the forms $\omega_j$. Starting with arbitrary forms $\overline{\omega}_j$ defined by equation (2), we use Cartan's lemma to obtain that

$$\omega_j = \overline{\omega}_j + b_{jk}\omega^k$$

with $b_{jk} = b_{kj}$. Computing $\Omega^i_j - \overline{\Omega}^i_j$ we obtain

$$\sum K^i_{jik} - \sum \overline{K}^i_{jik} = (n-2)b_{jk} + \delta^j_k \sum b_{ii}$$

Imposing that $\sum K^i_{jik} = 0$, we can solve for $b_{jk}$:

$$b_{jk} = \frac{1}{n-2}\left(\frac{1}{2(n-1)}\delta^j_k \overline{K}^i_{jij} - \overline{K}^i_{jik}\right)$$

□

To define the Cartan connection we write the equations (2) as equations in Y

$$(4) \qquad \left\{ \begin{array}{rcl} d\theta & = & -\omega\wedge\theta + \phi\wedge\theta \\ d\phi & = & -w\wedge\theta \end{array} \right.$$

where we consider the tautological forms defined by $\omega^i$ in a column vector $\theta$.

**Theorem 4.2** *There are unique forms $w$ and $\omega$ defined on the fiber bundle $Y \to E$ by*

$$(5) \qquad \begin{array}{rcl} \omega^T & = & -\omega \\ \sum K^i_{kis} & = & 0 \end{array}$$

*and we have that*

$$(6) \qquad \pi = \begin{pmatrix} -\phi & \theta^{T'} & 0 \\ w^T & \omega & \theta \\ 0 & w & \phi \end{pmatrix}$$

*is a Cartan connection.*

**Proof :** We have to examine the change on the forms $\omega$ and $w$ when a change

$$\left\{ \begin{array}{rcl} \phi' & = & \phi + a\theta \\ \theta' & = & A\theta \quad \text{where } A \in O(m) \end{array} \right.$$

is made. By straightforward computations we obtain the following transformations

$$\omega = A^{-1}\omega' A + \theta a - a^T \theta^T$$

$$w = w' A - a\omega + a\phi + aa\theta - \frac{1}{2}\theta^T a a^T$$

We observe then, that those transformations laws are precisely

$$R_g^*(\omega) = ad(g^{-1})\omega$$

□

The curvature of the Cartan connection $\pi$ is given by

$$\Pi = d\pi - \pi \wedge \pi = \begin{pmatrix} 0 & 0 & 0 \\ W^T & \Omega & 0 \\ 0 & W & 0 \end{pmatrix}$$

Let $\sigma : M \to E$ be a section of the line-bundle. This is equivalent to fix a riemannian metric in the conformal class. We will show that there is a natural imbedding of the Levi-Civita connection of the riemannian structure in the Cartan connection over the line bundle.

Let $\theta^i$ be a orthonormal coframe for this metric. Observe that $\phi = d\lambda/\lambda$ is determined by $\sigma$. We determine $w$ using the curvature of the metric. From the structure equation $d\phi = -w \wedge \theta = 0$ we compute that

$$w_j = \frac{1}{n-2}\left(\frac{1}{2(n-1)}\delta_k^j R^i_{jij} - R^i_{jik}\right)\theta^k$$

where $R^i_{jkl}$ is the riemannian curvature of the metric.

Observe also that using this imbedding $\sigma^*(\Omega^i_j) = W^i_j$, the Weyl tensor.

To construct the Chern-Simons class consider the simplest case of 3-manifolds, and the Chern-class

$$c_2 = Tr(\Pi \wedge \Pi)$$

which lives in Y. Using the standard argument we construct the transgression on Y

$$T_2 = Tr(\pi \wedge \Pi) - \frac{1}{3}Tr(\pi \wedge \pi \wedge \pi)$$

Using a section $\sigma$, and the imbedding above, we compute the pull-back of the transgression on Y, to get

$$\sigma^*(T_2) = Tr(\omega \wedge \Omega_R) - \frac{1}{3}Tr(\omega \wedge \omega \wedge \omega)$$

where $\Omega_R$ is the riemannian curvature of the metric. This is precisely the Chern-Simons class for the conformal structure.

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Cidade Universitária "Armando de Salles Oliveira"
Rua do Matão, 1010 - Butantã
Caixa Postal - 20.570 (Ag. Iguatemi)
CEP: 01498 - São Paulo - Brasil