*Proceedings of the Royal Society of Edinburgh*, **149**, 877–903, *2019*
DOI:10.1017/prm.2018.51

# Autonomous and non-autonomous unbounded attractors under perturbations

**Alexandre N. Carvalho**
Instituto de Ciências Matemáticas e de Computação,
Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668,
13560-970 São Carlos - SP, Brazil (andcarva@icmc.usp.br)

**Juliana F. S. Pimentel**
Centro de Matemática, Computação e Cognição,
Universidade Federal do ABC, 09210-580, Santo André - SP, Brazil
(juliana.pimentel@ufabc.edu.br)

(MS received 19 October 2016; accepted 7 April 2017)

Pullback attractors with forwards unbounded behaviour are to be found in the literature, but not much is known about pullback attractors with each and every section being unbounded. In this paper, we introduce the concept of unbounded pullback attractor, for which the sections are not required to be compact. These objects are addressed in this paper in the context of a class of non-autonomous semilinear parabolic equations. The nonlinearities are assumed to be non-dissipative and, in addition, defined in such a way that the equation possesses unbounded solutions as the initial time goes to $-\infty$, for each elapsed time. Distinct regimes for the non-autonomous term are taken into account. Namely, we address the small non-autonomous perturbation and the asymptotically autonomous cases.

*Keywords:* Global attractors; asymptotic behavior; semilinear parabolic equations; nonautonomous equations; perturbation of attractors

2010 *Mathematics subject classification:* 35B41; 35B20; 35B40; 35B44; 37B55; 35K58

## 1. Introduction

In this paper, we investigate the dynamics of solutions of the following non-autonomous semilinear parabolic equation

$$\begin{aligned} u_t &= u_{xx} + b(t)u + g(u), \quad x \in [0,\pi],\ t > s \qquad (1)\\ u(t,0) &= u(t,\pi) = 0,\\ u(s,x) &= u_0(x), \end{aligned}$$

where $b \in C^1(\mathbb{R})$ is a bounded function with bounds $b_1$ and $b_2$ satisfying

$$1 < b_1 < b(t) < b_2, \quad \text{for all}\ \ t \in \mathbb{R}. \qquad (2)$$

We also assume that $g : \mathbb{R} \to \mathbb{R}$ is a bounded, locally Lipschitz $C^2$ function. The global well-posedness in $L^2([0,\pi])$ and $W_0^{1,2}([0,\pi])$ for equation (1) is known from

© 2018 The Royal Society of Edinburgh


Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

[1–3, 9]. The dynamics of the non-autonomous equation (1) is described by the evolution process $\{S(t,s) : t \geqslant s\}$, defined by

$$S(t,s)u_0 = u(t,s;u_0)$$

where $u(t,s;u_0)$ is the unique global solution of (1) with initial condition $u_0$.

We write (1) in the semilinear abstract form

$$u_t + Au = F(t,u)$$

where $A$ is the unbounded linear operator $A : D(A) \subset L^2([0,\pi]) \to L^2([0,\pi])$, with domain

$$D(A) = \left\{u \in W^{2,2}([0,\pi]) : u = 0 \text{ in } x = 0, \pi\right\},$$

given by $Au = -u_{xx}$. Let $X$ denote the $L^2([0,\pi])$ space. The operator $A$ being sectorial has an associated scale of fractional power spaces $X^\alpha$ given by $X^\alpha = D(A^\alpha)$ with graph norm

$$\|u\|_\alpha = \|A^\alpha u\|, \quad x \in X^\alpha.$$

The eigenvalues associated with the operator $A$, under Dirichlet boundary conditions, will be denoted by $\lambda_j$,

$$\lambda_j = j^2, \quad j = 1,2,3,\ldots.$$

The nonlinear evolution process $\{S(t,s) : t \geqslant s\}$ is well defined on $X^\alpha$, for $\alpha > 1/4$.

We let $f(t,u) := b(t)u + g(u)$ and define $F : \mathbb{R} \times X^\alpha \to X$ by

$$F(t,u)(x) = b(t)u(x) + g(u(x)).$$

Similarly, the operator $G : X^\alpha \to X$ may be defined by

$$G(u)(x) = g(u(x)).$$

As we are assuming that $g$ is bounded in $\mathbb{R}$, the function $G(u)(\cdot)$ is also bounded. We denote the bound by $\Gamma$

$$|G(u)(x)| \leqslant \Gamma, \quad \text{for all } \ x \in [0,\pi] \quad \text{ and } \ u \in X^\alpha. \tag{3}$$

In what follows, we remark two main features of equation (1). We first notice that the requirement $b(t) > 1$, for every $t \in \mathbb{R}$, destroys the dissipative properties usually imposed on the nonlinearities. Therefore, the associated pullback attractor $\mathcal{A}(\cdot)$, if it exists, necessarily possesses unbounded fibres $\mathcal{A}(t)$, for each time $t$. This is proved in lemma 3.1. Therefore, $\mathcal{A}(t)$ cannot be compact.

Autonomous dynamical systems with bounded absorbing set have been extensively studied by many authors. However, if solutions unbounded with time exist, the equation has no bounded attractor. Despite the fact that unbounded attractors are quite common objects in dynamical systems, much less is known on their regard. As far as we are concerned, they were first introduced in [10, 11], for semilinear equations and systems of parabolic type. See also [12] for the context of evolution equations.

The particular case of equation (1) with the parameter $b$ is independent of time was recently investigated. Such equations with linear growing nonlinearities are known as *slowly non-dissipative equations*. The case $f = f(u)$ is treated in [4] and, in

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

[**17**] the general case $f = f(x, u, u_x)$ is addressed. In this setting, the generated semigroup was proved to possess an unbounded attractor, referred to as *non-compact global attractor*, and its heteroclinic connectivity was completely described. Despite the non-dissipativity, the semigroup is gradient-like and, consequently, solutions remaining bounded converge to bounded equilibria. The unbounded solutions, on their turn, grow-up to infinity norm in the direction of eigenfunctions associated with negative eigenvalues of the linear sectorial operator $-\partial_{xx} - bI$.

The limiting objects of unbounded solutions at infinity are interpreted as *equilibria at infinity*. In addition to that, the *grow-up solutions*, or unbounded solutions, play the role of heteroclinic orbits connecting bounded equilibria to equilibria at infinity. This interpretation allows for a characterization of non-compact global attractors that is similar to that we find for compact global attractors: it decomposes into the set of bounded equilibria, the set of equilibria at infinity, and the set of their heteroclinic connections, including the extended notion we just described.

The second main feature of (1) is that it produces a non-autonomous dynamical system and, therefore, the recently developed theory on non-dissipative equations does apply directly. We know that non-autonomous equations have attracted much attention over the years and, unlike what happens in an autonomous setting, the analysis of the asymptotic behaviour for such systems may be carried out in different ways. In this setting, pullback attractors have shown to be an appropriate choice of analogues in non-autonomous contexts for capturing the asymptotic behaviour of solutions, even if we only wish to study forwards dynamics (see [**5**, **6**]).

Although pullback attractors with forwards unbounded behaviour are considered in the literature, for instance, in the context of non-autonomous scalar logistic equations [**15**], not much is known about pullback attractors $\mathcal{A}(\cdot)$ with unbounded sections $\mathcal{A}(t)$, for each time $t$.

Our main goal is to investigate the forwards and backwards dynamics of $S(t, s)$ under further assumptions on the non-autonomous term $b(t)$. Namely, we contemplate two situations: small non-autonomous perturbations and asymptotically autonomous systems. We expect to describe the structure on the associated unbounded pullback attractor, by assuming such distinct regimes for the parameter $b = b(t)$. Despite given detailed information on the behaviour of solutions only for the two cases listed above, all the results and remarks in §§ 3 and 4 hold for the general equation (1), with no further assumption on $b(t)$.

We organize the paper as follows. In § 2, we review the recently developed theory of non-dissipative autonomous reaction-diffusion equations. We focus on the class of equations possessing linearly growing nonlinearities, that is, the autonomous equation (1) with $b(t) \equiv b$.

In § 3, we explore the dynamics of solutions to equation (1) as much as we can, without assuming further conditions on $b(t)$. The sections $\mathcal{A}(t)$ on the associated pullback attractor $\mathcal{A}(\cdot)$ are proved to be unbounded for each and every time $t \in \mathbb{R}$, which requires the introduction of the concept of *unbounded pullback attractor*. In addition, we ensure that solutions which are unbounded as the initial time $s$ goes to $-\infty$, for each elapsed time $t$, converge to 'autonomous equilibria at infinity'.

The dynamical structure at infinity is addressed in § 4. The study is firstly motivated by the analysis of the linear problem with $g \equiv 0$, which suggests that the structure of $\mathcal{A}(\cdot)$ for arbitrarily small initial times is, in some sense, independent of

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

time $t$. This statement is then confirmed by applying a common technique, used for investigating the dynamics of unbounded attractors. It basically consists of considering Poincaré projections of the phase space, aiming at the compactification of the dynamics. We conclude the section with some remarks on the number of equilibria at infinity contained at each section $\mathcal{A}(t)$ of the unbounded pullback attractor.

Further properties of solutions to equation (1), with the additional assumption that $b(t)$ is a small non-autonomous perturbation of a fixed parameter $b$, are investigated in § 5. We prove that, under small perturbations of this nature, the unbounded pullback attractor satisfies robustness properties.

Section 6 is devoted to the study of the dynamics of the unbounded pullback attractor when assuming $b(t)$ is backwards asymptotically autonomous. We additionally make some comments on the case where $b(t)$ is autonomous forwards in time.

The forwards dynamics of (1), under fairly general assumptions on $b(t)$, is addressed in § 7. We comment on the existence and description of the related unbounded uniform attractor.

## 2. The autonomous setting

In this section, we follow [**4**, **17**] to present the full characterization of the non-compact global attractor related to equation (1) with time independent $b$, that is, to the following equation:

$$\begin{aligned} u_t &= u_{xx} + bu + g(u), \quad x \in [0, \pi] \\ u(t,0) &= u(t,\pi) = 0 \\ u(0,x) &= u_0(x). \end{aligned} \tag{4}$$

Although the results in [**17**] are proved for the corresponding Neumann problem, they hold similarly for the Dirichlet equation (4).

We assume $b > 1$ and $g \in C^2(\mathbb{R})$ is a bounded Lipschitz function. We let $X = L^2([0,\pi])$ and consider the sectorial operator $\tilde{A} := -\partial_{xx} - bI$. Then the fractional power spaces associated with the operator $A_1 = \tilde{A} + (b+1)I$ are well-defined and given by

$$X^\alpha := D\left(A_1^\alpha\right),$$

for each $\alpha \geqslant 0$, with the graph norm

$$\|x\|_\alpha := \|A_1^\alpha x\|, \quad x \in X^\alpha.$$

Since $b > 1$, it follows from an analysis of the Fourier decomposition of solutions that there exists at least one initial condition with corresponding solution blowing-up in infinite time, see [**4**, **17**]. Hence, equation (4) is non-dissipative and, therefore, it does not have a global attractor for the induced semiflow. The object to be considered is *non-compact global attractor*, defined as the nonempty minimal set which is positively invariant and attracts all bounded sets in the state space.

Moreover, the existence of a Lyapunov functional for equation (4) yields a gradient-like structure. As a consequence, any solution $u(t,x)$ of (4) either converges to a bounded equilibrium or is a grow-up solution. It also follows that the

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

non-compact global attractor $\mathcal{A}$ for equation (4) can be characterized by

$$\mathcal{A} = \cup_{v \in E^b} W^u(v),$$

where $E^b$ denotes the set of bounded equilibria and $W^u(v)$ denotes the associated unstable manifold of $v$.

Let $\{\varphi_j(\cdot) : j = 1, 2, \dots\}$ be the orthonormal basis in $L^2([0, \pi])$ composed by eigenfunctions of the operator $\tilde{A}$ under Dirichlet boundary conditions. The corresponding eigenvectors are denoted by $\tilde{\lambda}_j$, for $j = 1, 2, \dots$.

Regarding the asymptotic behaviour of grow-up solutions, the following result was obtained, [**4**, **17**].

LEMMA 2.1. *Let $u(t)$ be a grow-up solution of (4). Then $u(t)$ goes to infinity with time, being attracted to the following finite dimensional subspace of $L^2([0, \pi])$*

$$\text{span}\{\varphi_1, \varphi_2 \dots, \varphi_{[\sqrt{b}]}\},$$

*where $[\cdot]$ denotes the integer part.*

Even more can be said about the longtime behaviour of grow-up solutions $u(t)$ of (4). They go to infinity asymptotically approaching the eigenvector direction of a unique $\varphi_j(\cdot)$, for some $j \in \{1, \dots, [\sqrt{b}]\}$ depending on $u(t)$. See [**4**, **17**].

In addition, the inertial manifold theory along with nodal properties of such unbounded solutions provide the exact eigenspace $E_j$, corresponding to $\tilde{\lambda}_j$, they are being attracted to. Therefore, the projections of $\varphi_1, \varphi_2, \dots, \varphi_{[\sqrt{b}]}$ to infinity norm are objects at infinity regarded as *equilibria at infinity.* A Poincaré projection analysis of the phase space confirms these objects, in fact, correspond to equilibria of the projected equation. This justifies the term equilibria at infinity. See [**4**, **17**] for the details.

Regarding the backwards behaviour of unbounded solutions lying on the attractor, it follows from the gradient structure of (4) that they all converge to bounded equilibria. Therefore, the non-compact global attractor contains only trajectories that are bounded in the past.

The following result, obtained in [**4**, **17**], provides a detailed characterization for the non-compact global attractor related to equation (4).

THEOREM 2.1. *The non-compact global attractor $\mathcal{A}$ of (4) is given by*

$$\mathcal{A} = E^b \cup E^\infty \cup \{\textit{heteroclinic connections}\},$$

*where $E^\infty$ denotes the set of all equilibria at infinity.*

The characterization in theorem 2.1 is similar to that of compact global attractors if the grow-up solutions are understood as heteroclinic orbits connecting bounded equilibria to equilibria at infinity. The connecting orbit structure as displayed in theorem 2.1 is obtained in terms of blocking principles typically considered in the scalar reaction-diffusion equation setting. The argument regarding the dynamics at infinity includes the nondegeneracy condition $b \neq n^2$, to ensure hyperbolicity for the equilibria at infinity. In addition to theorem 2.1, a combinatorial characterization of $\mathcal{A}$, in terms of a permutation, is obtained in [**17**].

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

## 3. Unbounded pullback attractor

We present a preliminary result that leads to the unboundedness of the pullback attractor. More precisely, we guarantee that the evolution process $\{S(t,s) : t \geqslant s\}$ contains global solutions $u(t,s;u_0)$ which are unbounded as $s \to -\infty$, for each time $t$.

LEMMA 3.1. *If $b(t) > 1$, for every $t \in \mathbb{R}$, then there exists at least one initial condition $u_0$ such that the corresponding solution $u(t,s;u_0)$ of equation (1) satisfies*

$$\lim_{s\to-\infty} \|u(t,s;u_0)\|_{L^2([0,\pi])} = \infty,$$

*for each elapsed time $t \in \mathbb{R}$.*

*Proof.* Let $u(t,s;u_0)$ be a solution of (1). We can write

$$u(t,s;u_0) = \sum_{j=1}^{\infty} \hat{u}_j(t,s;u_0)\varphi_j(\cdot)$$

where $\hat{u}_j(t,s;u_0)$ satisfies the non-autonomous ODE

$$(\hat{u}_j)_t(t) = -\lambda_j \hat{u}_j(t) + b(t)\hat{u}_j(t) + \hat{g}_j(t) \tag{5}$$

with $\hat{u}_j(t,s;u_0) = \langle u(t,s;u_0), \varphi_j(\cdot)\rangle$ and $\hat{g}_j(t) = \langle G(u)(\cdot), \varphi_j(\cdot)\rangle$.

Each eigenmode of $u(t,s,u_0)$ is given by

$$\hat{u}_j(t,s,u_0) = \hat{u}_{0,j}\mathrm{e}^{-\int_s^t(\lambda_j - b(r))\mathrm{d}r} + \int_s^t \mathrm{e}^{-\int_r^t(\lambda_j-b(\theta))\mathrm{d}\theta}\hat{g}_j(r)\mathrm{d}r, \tag{6}$$

with $\hat{u}_{0,j} = \langle u_0(\cdot), \varphi_j(\cdot)\rangle$. Then we write (6) in the form

$$\hat{u}_j(t,s,u_0) = \left(\hat{u}_{0,j} - \int_\infty^s \mathrm{e}^{-\int_r^s(\lambda_j-b(\theta))\mathrm{d}\theta}\hat{g}_j(r)\mathrm{d}r\right)\mathrm{e}^{-\int_s^t(\lambda_j-b(r))\mathrm{d}r} + \int_\infty^t \mathrm{e}^{-\int_r^t(\lambda_j-b(\theta))\mathrm{d}\theta}\hat{g}_j(r)\mathrm{d}r. \tag{7}$$

We assume $\lambda_j - b(\theta) < 0$, for every $\theta$. In order to get this, we can take, for instance, $j = 0$. Indeed, $b(t) > b_1 > 1 = \lambda_0$, for every $t \in \mathbb{R}$, then

$$\lambda_0 - b(t) < 0, \quad t \in \mathbb{R}.$$

Notice that we have the following bound for the second term in (7)

$$\begin{aligned}\left|\int_\infty^t \mathrm{e}^{-\int_r^t(\lambda_j-b(\theta))\mathrm{d}\theta}\hat{g}_j(r)\mathrm{d}r\right| &\leqslant \Gamma\left|\frac{1}{\lambda_j - b(t)} - \lim_{r\to\infty}\frac{\mathrm{e}^{-\int_r^t(\lambda_j-b(\theta))\mathrm{d}\theta}}{\lambda_j - b(r)}\right| \\ &= \Gamma\left|\frac{1}{\lambda_j - b(t)} - \lim_{r\to\infty}\frac{\mathrm{e}^{\int_t^r(\lambda_j-b(\theta))\mathrm{d}\theta}}{\lambda_j - b(r)}\right| \\ &= \Gamma\left(\frac{1}{b(t)-\lambda_j}\right),\end{aligned}$$

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

where $\Gamma$ is defined in (3). Of course, the same holds for

$$\int_{\infty}^{s} \mathrm{e}^{-\int_r^s (\lambda_j - b(\theta))\mathrm{d}\theta} \hat{g}_j(r)\mathrm{d}r.$$

Hence, we consider an initial condition $u_0$ such that

$$\hat{u}_{0,j} \neq \lim_{s\to -\infty} \int_{\infty}^{s} \mathrm{e}^{-\int_r^s (\lambda_j - b(\theta))\mathrm{d}\theta} \hat{g}_j(r)\mathrm{d}r,$$

The corresponding solution $u(t, s; u_0)$ then satisfies, for each time $t \in \mathbb{R}$,

$$\lim_{s\to -\infty} |\hat{u}_j(t, s; u_0)| = \infty,$$

since

$$\lim_{s\to -\infty} \mathrm{e}^{-\int_s^t (\lambda_j - b(r))\mathrm{d}r} = \infty.$$

Therefore, we conclude that the $L^2$-norm of the corresponding solution $u(t, s; u_0)$

$$\sum_{j=0}^{\infty} |u(t, s; u_0)|^2$$

goes to infinity as $s \to -\infty$, for any time $t$. □

Remark 3.1. It is worth mentioning that in order to ensure the existence of unbounded fibres in the pullback attractor, the lower bound for $b(t)$ was only needed, in lemma 3.1, for arbitrarily large or arbitrarily small times.

This lemma suggests the need for a different notion of pullback attractor where the sections $A(t)$, $t \in \mathbb{R}$, are not required to be compact. We are then led to introduce the definition of unbounded pullback attractors.

We denote by $\mathrm{dist}(A, B)$ the Hausdorff semidistance between $A$ and $B$ subsets of the phase space $X^\alpha$, defined as

$$\mathrm{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_\alpha$$

For the non-dissipative non-autonomous equation (1), we define the following.

Definition 3.1. An unbounded pullback attractor $\mathcal{A}(\cdot)$ for the process $S(\cdot, \cdot)$ is a family of subsets $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ in the phase space $X^\alpha$ such that

(i) $\mathcal{A}(\cdot)$ is invariant with respect to $S(\cdot, \cdot)$, that is,

$$S(t, s)\mathcal{A}(s) = \mathcal{A}(t) \quad \text{for all } t \geqslant s;$$

(ii) for each $t \in \mathbb{R}$, $\mathcal{A}(t)$ pullback attracts bounded sets at time $t$

$$\lim_{s\to -\infty} \mathrm{dist}(S(t, s)B, \mathcal{A}(t)) = 0$$

for any bounded set $B \subset X^\alpha$; and

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

(iii) $\mathcal{A}(\cdot)$ is the invariant family of closed sets minimal with respect to property $(ii)$.

As in the compact setting, the minimality assumption is crucial for uniqueness of the unbounded pullback attractor. In order to illustrate this assertion, we consider the evolution process

$$S(t,s)x = e^{(t-s)}x, \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad t \geqslant 0.$$

Any family $K(\cdot)$ of compact sets of the form

$$K(t) = [-e^{(t-s)}a, e^{(t-s)}a], \quad t \in \mathbb{R},$$

with a fixed positive real number $a$, is trivially invariant, in the non-autonomous sense $(i)$. Moreover, $K(t)$ pullback attracts bounded sets of $\mathbb{R}$ at time $t$.

Although being unbounded, the pullback attractor $\mathcal{A}(\cdot)$ related to (1) is locally compact. This follows from the fact that $\mathcal{A}(\cdot)$ lies in a finite-dimensional attracting invariant manifold. Indeed, it results from [**14**, Chap. 6] the existence of an invariant manifold

$$M = \{(t,p,q) \in \mathbb{R} \times P_N X \times Q_N X : p = \sigma(t,q)\}$$

where $\sigma : \mathbb{R} \times Q_N X \to X^\alpha$ satisfies

$$\|\sigma(t,q)\|_\alpha \leqslant C, \quad \|\sigma(t,q_1) - \sigma(t,q_2)\|_\alpha \leqslant L\|q_1 - q_2\|,$$

for constants $C$ and $L$, with $P_N$ being the orthogonal projection onto the $N$ first eigenvalues of $A$ and

$$Q_N = I - P_N.$$

In addition, $g$ being locally Lipschitz in $u$ implies the invariant manifold $M$ is exponentially attracting, see [**14**, Chap. 6]. The fact that $M(t)$ contains $\mathcal{A}(t)$, for every $t \in \mathbb{R}$, then follows. We also refer to [**16**] for sharp conditions on the existence of invariant manifolds related to more general non-autonomous equations.

It is worth mentioning that each section $M(t)$ of $M$ is non-compact. This is due to the invariance property of $M$ in the context of non-dissipative equations with unbounded attractors.

Although definition 3.1 allows unbounded pullback attractors with more complex dynamical structures, we understand that those that are locally compact or with finite fractal dimension play a more fundamental role in the theory.

Lemma 3.2. *Suppose $b(t)$ satisfies (2). Let $u(t,s;u_0)$ be any solution of (1) that is unbounded as $s$ goes to $-\infty$, for any $t$. Then, there exists $j \leqslant [\sqrt{b_2}]$ such that the normalized solution*

$$\frac{u(t,s;u_0)}{\|u(t,s;u_0)\|_{L^2([0,\pi])}}$$

*pullback converges to $\pm\varphi_j(\cdot)$ in the $L^2$-norm, where the integer $j$ and the sign of $\pm\varphi_j(\cdot)$ depend only on $u(t,s;u_0)$.*

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

*Proof.* We first prove that $\hat{u}_j(t,s;u_0)$ remains bounded as $s \to -\infty$ as long as $j > [\sqrt{b_2}]$. As in lemma 3.1, we write any solution $u(t,s;u_0)$ in the form

$$u(t,s;u_0) = \sum_{j=1}^{\infty} \hat{u}_j(t,s;u_0)\varphi_j(\cdot).$$

with eigenmodes $\hat{u}_j(t,s;u_0)$ given as in (6). By assuming that $j^2 > b_2$, the first term in (6) is obviously bounded as $s \to -\infty$, since

$$\lim_{s\to-\infty} \mathrm{e}^{-\int_s^t(\lambda_j - b(r))\mathrm{d}r} = 0, \qquad \text{for all} \quad t \in \mathbb{R}.$$

For the second term in (6), we get a bound as follows:

$$\begin{aligned}\left|\int_s^t \mathrm{e}^{-\int_r^t(\lambda_j - b(\theta))\mathrm{d}\theta}\hat{g}_j(r)\mathrm{d}r\right| &\leqslant \Gamma\left|\int_s^t \mathrm{e}^{-\int_r^t(\lambda_j - b(\theta))\mathrm{d}\theta}\mathrm{d}r\right| \\ &= \Gamma\left|\frac{1}{\lambda_j - b(t)} - \frac{\mathrm{e}^{-\int_s^t(\lambda_j - b(\theta))\mathrm{d}\theta}}{\lambda_j - b(s)}\right|,\end{aligned}$$

which is bounded as $s \to -\infty$, since

$$\lim_{s\to-\infty} -\int_s^t (\lambda_j - b(\theta))\mathrm{d}\theta = -\infty, \qquad \text{for all} \quad t \in \mathbb{R}.$$

In the remaining part of this proof, we show that, for any grow-up solution $u(t,s;u_0)$

$$\lim_{s\to-\infty} \frac{u(t,s;u_0)}{\|u(t,s;u_0)\|_{L^2([0,\pi])}} = \pm\varphi_j(\cdot),$$

for some $j$ with $j^2 \leqslant b_2$, depending on $u(t,s;u_0)$.

We take, for simplicity, $b(t)$ oscillating between two gaps of the spectrum of $A$. That is to assume there exists a fixed integer $N$ such that

$$\lambda_{N-1} < b_1 < b(t) < b_2 < \lambda_{N+1}. \tag{8}$$

The general case ensues from a similar argument.

It follows from lemma 3.1 that, for any unbounded solution $u(t,s;u_0)$ the eigenmodes $\hat{u}_j(t,s;u_0)$ with $\lambda_j < b(t)$, for every $t$, can grow to infinity norm as $s \to -\infty$. Therefore, the modes $\hat{u}_j(t,s;u_0)$ with $j \leqslant N-1$ are allowed to become unbounded as $s \to -\infty$. On the contrary, we know from the above calculations that $\hat{u}_j(t,s;u_0)$ remains bounded if $j^2 > b_2$. Hence, $\hat{u}_j(t,s;u_0)$ is uniformly bounded in $s$ if $j \geqslant N+1$.

By assuming (8), we cannot affirm whether or not $\hat{u}_N(t,s;u_0)$ remains bounded as $s \to -\infty$. We can, however, conclude the following on the possible growth behaviour of $\hat{u}_j(t,s;u_0)$, for any $j \leqslant N$. It follows from lemma 3.1 that, if $\hat{u}_j(t,s;u_0)$ is a

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

growing mode, then the first term in (7) gives the growth rate of $\hat{u}_j(t,s;u_0)$, that is, if $\hat{u}_j(t,s;u_0)$ grows to infinity norm as $s \to -\infty$ then the growth is given by

$$\mathrm{e}^{-\int_s^t (\lambda_j - b(r))\mathrm{d}r},$$

for any time $t$. As a result, the smaller $\lambda_j$ is, the faster $\hat{u}_j(t,s;u_0)$ grows to infinity norm as $s$ runs backwards in time. In particular, if $\hat{u}_N(t,s;u_s)$ becomes unbounded as $s \to -\infty$, it is necessarily the growing mode with the slowest growth.

We keep all the above information about backwards behaviour of each $\hat{u}_j(t,s;u_s)$, and write the following equality

$$\Big\| \frac{u(t,s;u_0)}{\|u(t,s;u_0)\|_{L^2([0,\pi])}} \mp \varphi_j(\cdot) \Big\|^2_{L^2([0,\pi])} = 2 \pm 2 \frac{\hat{u}_j(t,s;u_0)}{\|u(t,s;u_0)\|_{L^2([0,\pi])}}$$

As a consequence,

$$\lim_{s\to-\infty} \frac{u(t,s;u_0)}{\|u(t,s;u_0)\|_{L^2}} = \pm\varphi_j(\cdot)$$

if, and only if,

$$\lim_{s\to-\infty} \frac{\hat{u}_j(t,s;u_0)}{\|u(t,s;u_0)\|_{L^2([0,\pi])}} = \mp 1.$$

The previous discussion then leads to the following conclusion: if a solution $u(t,s;u_0)$ becomes unbounded as $s \to -\infty$, then the normalized solution converges to $\pm\varphi_j(\cdot)$ with $j$ corresponding to the growing eigenmode $\hat{u}_j(t,s;u_0)$ with the lowest subscript.

In addition, the sign of the limiting function $\pm\varphi_j(\cdot)$ should be the same as $u(t,s;u_0)(0)$ for all $s \in (\infty, s_0)$, for some $s_0 < 0$. □

The above lemma gives us a direction of pullback attraction for unbounded solutions, figuring as a non-autonomous analogue of lemma 2.1.

It was established in [**12**] the existence of non-compact global attractors, referred to as *maximal attractors*, for general evolution equations not necessarily satisfying dissipativity conditions. On the next result, similar non-autonomous arguments are used to obtain the existence of an unbounded pullback attractor for equation (1).

Theorem 3.1. *Let $b$ and $g$ satisfy the conditions we have asserted. Then the evolution process $S(\cdot,\cdot)$ of (1) possesses an unbounded pullback attractor $\mathcal{A}(\cdot)$.*

It is important to mention that a characterization for the attractor as in [**12**] cannot be directly obtained, as it requires a much more detailed investigation of the dynamics of solutions. This is expected to be obtained, for evolution processes in general settings, in a forthcoming paper.

We also stress the fact that the existence of an unbounded pullback attractor for equation (1) does not follow immediately. The invariance property requires solutions on the attractor to be globally defined, but this is not verified in regularizing processes for arbitrary initial data.

The next step is to obtain a better description of backwards asymptotic dynamics of unbounded solutions and, consequently, a more accurate characterization of the

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

dynamics at infinity on the unbounded pullback attractor. The next section is devoted to obtaining useful information on this regard.

## 4. Autonomous behaviour at infinity

In this section, we investigate the dynamics of (1) at the infinity of the phase space. In order to do that, we first restrict ourselves to the linear version of (1), that is, the case $g \equiv 0$. Since $g$ is taken to be bounded, we do not expect that the dynamics of (1), for arbitrarily large initial conditions will be affected by $g$. Then, at this point, we are interested in the following equation

$$\begin{aligned} u_t &= u_{xx} + b(t)u, \quad x \in [0, \pi] \\ u(t, 0) &= u(t, \pi) = 0 \end{aligned} \tag{9}$$

where $b(t)$ satisfies (2).

We start by recalling the concept of non-autonomous equilibria for evolution processes. These objects are defined as follows. See [**8**, **9**].

Definition 4.1. Let $\xi : \mathbb{R} \to X^\alpha$ be a global solution for equation (1). We say that $\xi$ is a non-autonomous equilibrium if the zeros of $\xi(t, \cdot)$ are independent of $t$.

For any non-autonomous equilibrium $\xi$, we denote by $W^u(\xi(\cdot))(\cdot)$ the unstable manifold of $\xi$ defined by

$$W^u(\xi(\cdot))(t) = \{x \in X^\alpha : \text{there exists a global solution } \phi : \mathbb{R} \to X^\alpha \text{ such that } \phi(t) = x \quad \text{and} \quad \lim_{s\to-\infty} \operatorname{dist}(\phi(s), \xi(s)) = 0\},$$

for each time $t$.

We stress the fact that (9) satisfies the following convenient condition. The projected equation onto the eigenspace $E_j$, associated with $\lambda_j$, has the form

$$(\hat{u}_j)_t = -\lambda \hat{u}_j + b(t)\hat{u}_j,$$

for each $j = 1, 2, \ldots$. Therefore, as it occurs in the autonomous setting $b(t) \equiv b$, each eigenspace $E_j$ is invariant under the flow (9).

We fix the real constant $b > 1$ and take the integer $N$ satisfying

$$N = [\sqrt{b}]. \tag{10}$$

As we may recall from § 2, the autonomous linear equation

$$u_t = u_{xx} + bu, \quad x \in [0, \pi] \tag{11}$$

possesses a non-compact global attractor $\mathcal{A}$. In addition, if $N$ satisfies (10), $\mathcal{A}$ is the unbounded finite-dimensional invariant subset $W^u(0)$, whose basis is composed by the $N$ first eigenfunctions $\varphi_j(\cdot)$.

When dealing with non-autonomous problems, the notion of hyperbolicity of equilibria is expressed in terms of the exponential dichotomy of bounded global

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

solutions. In the sequel, we recall the definition of linear evolution processes with exponential dichotomy.

DEFINITION 4.2. Let $T(t,s)$ be a linear evolution process. We say that $T(t,s)$ has an exponential dichotomy with projection $P(\cdot) = \{P(t) : t \in \mathbb{R}\}$, exponent $\omega$ and constant $M$, if the following conditions are satisfied:

(i) $P(t)T(t,s) = T(t,s)P(s)$, for all $t \geqslant s$;

(ii) the restriction of $T(t,s)$ to the image of $P(s)$ is an isomorphism onto the image of $P(t)$;

(iii)

$$\|T(t,s)(I - P(s))\| \leqslant Me^{-\omega(t-s)}, \quad t \geqslant s$$

and

$$\|T(t,s)(P(t))\| \leqslant Me^{\omega(t-s)}, \quad t \leqslant s.$$

Let $T(\cdot)$ be the semigroup induced by the autonomous equation (11). Then $T(\cdot)$ is given by $e^{-\tilde{A}t}$, where

$$\tilde{A} = -\partial_{xx} - bI.$$

The linear process $S_T(\cdot,\cdot)$ associated with $T(\cdot)$ has exponential dichotomy with projection $P(t) = P$, $t \in \mathbb{R}$, where $P$ is the orthogonal projection onto the $N$ first eigenfunctions $\varphi_j$ of $A$ with Dirichlet boundary conditions. The related exponent $\omega$ is given by $\tilde{\lambda}_N = N^2 - b$ (see, for instance, [**9**]).

A global bounded solution of a nonlinear process is said to be *hyperbolic* if the associated linearized process has an exponential dichotomy. Linear processes with exponential dichotomy have, of course, all the global bounded solutions being hyperbolic. Similarly to the autonomous counterpart, the evolution process related to (9) has exponential dichotomy with exponent $\omega$ depending on the bounds $b_1$ and $b_2$ for $b(t)$. Therefore, any bounded non-autonomous equilibrium for (9) is hyperbolic. In particular, $\xi \equiv 0$ is a hyperbolic non-autonomous equilibrium of (9) and, moreover, the related unstable manifold $W^u(\xi \equiv 0)(\cdot)$ satisfies

$$W^u(\xi \equiv 0)(t) \subset \mathcal{A}(t), \qquad \text{for all} \;\; t \in \mathbb{R}.$$

where $\mathcal{A}(\cdot)$ is the unbounded pullback attractor of (9).

In what regards the equilibria at infinity, for the autonomous equation (11), the nondegeneracy condition $b \neq n^2$ is assumed. With this condition imposed, (11) has exactly $2N$ equilibria at infinity, for $N$ given as in (10).

For the non-autonomous equation (9), the following can be asserted regarding the dynamics at infinity. It follows from lemma 3.2 that equation (9), with $b(t)$ satisfying (2), has at least $2N_1$ equilibria at infinity, where $N_1 = [\sqrt{b_1}]$, contained in each section $\mathcal{A}(t)$ of the pullback attractor. In fact, any unbounded solution $u(t,s;u_0)$ has at least $2N_1$ possible limiting object at infinity, as $s \to -\infty$. Besides, it follows from the proof of Lemma 3.1 that we can always construct an unbounded solution converging to any of these equilibria at infinity.

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

If we assume that

$$\lambda_N < b_1 < b(t) < b_2 < \lambda_{N+1}, \qquad \text{for all} \quad t \in \mathbb{R}. \tag{12}$$

Then, each section $\mathcal{A}(t)$ will contain exactly $2N$ equilibria at infinity. This shows, in particular, that the number of equilibria at infinity of equation (9) does not depend on $b(t)$, if it varies as in (12). Also, we may prove that the dynamics at infinity of (9) is not affected by 'nonsmall' perturbations, even including the intra-infinite heteroclinics, as long as $b(t)$ lies on a unique gap as in (12). In addition, each section on the unstable manifold $W^u(\xi \equiv 0)(\cdot)$ of $\xi \equiv 0$ has dimension $N$, if $b(t)$ satisfies (12).

As we can notice, the image of $b(t)$ in $\mathbb{R}$ is paramount for determining the number of equilibria at infinity and, more generally, the dynamics at infinity. Some particular cases where $b(t)$ may vary inside two gaps are considered in § 6.

The above analysis of the linear non-autonomous equation (9) suggests that the longtime dynamics, as $s$ goes to $-\infty$, is autonomous. The remaining part of this section is devoted to confirm and clarify this statement. More specifically, if $E^\infty(t)$ denotes the set of all equilibria at infinity contained in each section $\mathcal{A}(t)$, then we claim that regarding the structure at infinity, only the cardinality of the set $E^\infty(t)$ may vary with time $t$.

As in [**13**], we rely on Poincaré projections of the phase space to get a better understanding of the asymptotic profile of unbounded solutions. In what follows, reproduce the discussion in [**13**] for the context of our nonlinear non-autonomous equation (1).

Firstly, we identify $X^\alpha$ with the hyperplane $X^\alpha \times \{1\} \subset X^\alpha \times \mathbb{R}$. The inner product in $X^\alpha$ is defined as

$$\langle u, v\rangle_\alpha = \langle A^\alpha u, A^\alpha v\rangle_{L^2}, \quad u, v \in X^\alpha.$$

Then, the space $X^\alpha \times \{1\}$ is projected onto the infinite dimensional upper hemisphere

$$\mathcal{H} = \big\{(\chi, z) \in X^\alpha \times \mathbb{R} : \langle \chi, \chi\rangle_\alpha + z^2 = 1, z \geqslant 0\big\}$$

which is tangent to the hyperplane $X^\alpha \times \{1\}$ at its north pole.

We denote by $\mathcal{P}(M)$ the Poincaré projection of any point $M$ on the hyperplane. As we allow $M$ to go to infinity, its image $\mathcal{P}(M)$ goes to the equator of $\mathcal{H}$, that is, to the subset

$$\mathcal{H}_e = \{(\chi, z) \in \mathcal{H} : z = 0\}$$

The coordinates of $\mathcal{P}(M) = (\chi, z)$, for $M = (u, 1)$, may be explicitly computed:

$$\chi = \frac{u}{(1 + \langle u, u\rangle_\alpha)^{1/2}}, \quad z = \frac{1}{(1 + \langle u, u\rangle_\alpha)^{1/2}}.$$

To ease the computation, we choose, as it is done in [**13**], to work on planes rather than spheres. For each fixed eigenvector $e \in \{\varphi_j^\alpha : j = 1, 2, \ldots\}$, $\varphi_j^\alpha := A^{-\alpha}\varphi_j$, we project $\mathcal{M}$ again onto the vertical hyperplane $C$ which is tangent to $\mathcal{H}$ at the point $(e, 0) \in \mathcal{H}_e$. This projection is defined only for those points in $\mathcal{H}$ such that the line

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

through $M$, $\mathcal{P}(M)$ and the origin $(0,0) \in X^\alpha \times \mathbb{R}$ intersects the hyperplane $C$. If we denote by $(\xi, \zeta)$ the coordinates of $\mathcal{P}(M)$ projected on $C$, we have

$$\xi = \frac{u}{\langle u, e\rangle_\alpha}, \quad \zeta = \frac{1}{\langle u, e\rangle_\alpha}.$$

We define the non-autonomous operator

$$\mathcal{L}(u,t) = u_{xx} + b(t)u + g(u)$$

for $u \in X^\alpha$ and $t \in \mathbb{R}$. In addition, we let $\mathcal{L}_\zeta(\xi, t)$ be the homothety of $\mathcal{L}$ with factor $\zeta$, that is,

$$\mathcal{L}_\zeta(\xi, t) = \zeta\mathcal{L}(\zeta^{-1}\xi, t).$$

A direct calculation then yields

$$\begin{aligned}\xi_t &= \frac{u_t\langle u, e\rangle_\alpha - u\langle u_t, e\rangle_\alpha}{\langle u, e\rangle_\alpha^2}\\ &= \mathcal{L}(u,t)\zeta - u\zeta\langle\mathcal{L}(u,t)\zeta, e\rangle_\alpha\\ &= \mathcal{L}_\zeta(\xi,t) - \langle\mathcal{L}_\zeta(\xi,t), e\rangle_\alpha\xi\end{aligned}$$

and

$$\begin{aligned}\zeta_t &= -\frac{\langle u_t, e\rangle_\alpha}{\langle u, e\rangle_\alpha^2}\\ &= -\zeta\langle\mathcal{L}_\zeta(\xi,t), e\rangle_\alpha.\end{aligned}$$

By taking the coordinates of $\xi$ in the orthonormal basis $\{\varphi_j^\alpha : j = 1, 2, \ldots\}$,

$$\xi_n = \langle\xi, \varphi_n^\alpha\rangle_\alpha,$$

we get

$$\begin{aligned}(\xi_n)_t &= \langle\mathcal{L}_\zeta(\xi,t), \varphi_n^\alpha\rangle_\alpha - \langle\mathcal{L}_\zeta(\xi,t), e\rangle_\alpha\xi_n\\ &= \langle\xi_{xx} + b(t)\xi + g_\zeta(\xi), \varphi_n^\alpha\rangle_\alpha - \langle\xi_{xx} + b(t)\xi + g_\zeta(\xi), e\rangle_\alpha\xi_n,\end{aligned}$$

where $g_\zeta$ is defined by

$$g_\zeta(\xi) := \zeta g(\zeta^{-1}\xi).$$

If we take, for instance, $e = \pm\varphi_i^\alpha$, the hyperplane $C = C_i^\pm$ is given by

$$C_i^\pm = \{(\xi, \zeta) \in X^\alpha \times \mathbb{R} : \xi_i = \pm 1\}.$$

Then, $\xi_n$ satisfies the following equation on $C_i^\pm$

$$\begin{aligned}(\xi_n)_t &= (-\lambda_n \pm \lambda_i\xi_i)\xi_n + (b(t) \mp b(t)\xi_i)\xi_n + \langle g_\zeta(\xi), \varphi_n^\alpha\rangle_\alpha \mp \langle g_\zeta(\xi), \varphi_n^\alpha\rangle_\alpha\xi_n\\ &= (\lambda_i - \lambda_n)\xi_n + \langle g_\zeta(\xi), \varphi_n^\alpha\rangle_\alpha \mp \langle g_\zeta(\xi), \varphi_n^\alpha\rangle_\alpha\xi_n, \qquad (13)\end{aligned}$$

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

for all $n \neq i$. The coordinate $\zeta$ satisfies the equation below on the hyperplane $C_i^{\pm}$

$$\begin{aligned}(\zeta)_t &= \zeta\langle \xi_{xx} + b(t)\xi + g_\zeta(\xi),\ \pm\varphi_i^\alpha\rangle_\alpha \\ &= -\lambda_i\zeta \mp \langle g_\zeta(\xi), \varphi_i\rangle_\alpha \zeta. \end{aligned} \tag{14}$$

In addition, because $g$ is bounded, the nonlinear terms in (13) and (14) go to zero as $\zeta \to 0$. Hence, the equations on the equator $\mathcal{H}_e$ become

$$(\xi_n)_t = (i^2 - n^2)\xi_n, \quad \zeta_t = 0$$

for all $i \neq n$.

By projecting the points $M \in X^\alpha \times \{1\}$ onto the hyperplanes $C_j^{\pm}$, corresponding to each of eigenvectors $\pm\varphi_j(\cdot)$, we obtain the following equilibria lying on the equator

$$\left\{\Phi_j^{\pm,\infty} : j = 1, 2, \ldots\right\}, \tag{15}$$

where each $\Phi_j^{\pm,\infty}$ is given by

$$\Phi_j^{\pm,\infty} = \{(\chi, z) \in \mathcal{H} : \chi_j = \pm 1,\ z = 0 \quad \text{and} \quad \chi_n = 0, \quad \forall n \neq j\}.$$

Moreover, the heteroclinic connectivity at infinity is comparable with the Chafee-Infante structure, as in the autonomous setting. See [**4**, **17**].

We recall additionally that grow-up solutions of (1) have only a finite number of possible limiting objects at infinity. As a consequence, each section $\mathcal{A}(t)$ of the pullback attractor of (1) contains only a finite number of those. As we know, from lemma 3.2, such number depends only on $b(t)$. The general case where the non-autonomous term $b(t)$ is only required to be bounded, as in (2), is such that the set of equilibria at infinity $E^\infty(t)$ contained in the section $\mathcal{A}(t)$ is composed of $2N$ objects where $[\sqrt{b_1}] \leqslant N \leqslant [\sqrt{b_2}]$ and $N$ varies with $t$.

The cases where $b(t)$ is a small non-autonomous perturbation of a fixed $b$ or $b(t)$ is asymptotically autonomous are discussed in §§ 5 and 6, respectively.

## 5. Structure of the unbounded pullback attractor for small non-autonomous perturbation

This section is devoted to the analysis of the dynamics related to equation (1) when $b(t)$ is a small non-autonomous perturbation of a fixed $b$. More precisely, we are interested in the equation

$$\begin{aligned} u_t &= u_{xx} + b_\epsilon(t)u + g(u), \quad x \in [0, \pi], \quad t > s \\ u(0, t) &= u(\pi, t) = 0, \\ u(s, x) &= u_0(x), \end{aligned} \tag{16}$$

where the non-autonomous term $b_\epsilon(t)$ satisfies, additionally to the hypotheses described in § 1, the following

$$\lim_{\epsilon \to 0} \sup_{t \in \mathbb{R}} |b_\epsilon(t) - b| = 0, \tag{17}$$

for some fixed constant $b > 1$.

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

Before we treat the general form of (16), suppose $b_\epsilon(t) \equiv b_\epsilon$ is independent of $t$, that is, suppose we are considering the equation

$$u_t = u_{xx} + b_\epsilon u + g(u). \tag{18}$$

In this particular case, we know from the results in [**4**, **17**] that, for each $\epsilon > 0$, equation (18) is a slowly non-dissipative equation with non-compact global attractor decomposing as

$$\mathcal{A}_\epsilon = E^c_\epsilon \cup E^\infty_\epsilon \cup \{\text{heteroclinic connections}\}_\epsilon$$

where $E^c_\epsilon$ is the set of equilibria

$$E^c_\epsilon = \left\{v^\epsilon_1, v^\epsilon_2, \ldots, v^\epsilon_{n_\epsilon}\right\}$$

of (18), $E^\infty_\epsilon$ is the set of equilibria at infinity

$$E^\infty_\epsilon = \left\{\Phi^{\pm,\infty}_1, \ldots, \Phi^{\pm,\infty}_{N_\epsilon}\right\}$$

and the set of heteroclinic connections between equilibria, which also includes the grow-up solutions.

If we consider $\epsilon_0$ sufficiently small such that

$$[\sqrt{b_{\epsilon_0}}] = [\sqrt{b}]$$

then $N_\epsilon = N = [\sqrt{b}]$, for any $\epsilon < \epsilon_0$. If this is the case, then the dynamics at infinity, for each $\epsilon < \epsilon_0$, does not depend on $\epsilon$. We observe that the nondegeneracy condition $b \neq n^2$, pointed out in § 2, is fundamental at this point.

Then we proceed as in [**17**] to decompose the non-compact global attractor $\mathcal{A}_\epsilon$ as

$$\mathcal{A}_\epsilon = \mathcal{A}^c_\epsilon \cup \mathcal{A}^\infty_\epsilon$$

where $\mathcal{A}^c_\epsilon$ is the maximal compact invariant subset in $X^\alpha$ and $\mathcal{A}^\infty_\epsilon$ is composed by the set of equilibria at infinity

$$E^\infty_\epsilon = E^\infty = \left\{\Phi^{\pm,\infty}_1, \ldots, \Phi^{\pm,\infty}_N\right\}$$

and the set of grow-up solutions of (16). It follows that $\mathcal{A}^c_\epsilon$ contains the set of bounded equilibria $E^c_\epsilon$ and their heteroclinic orbit connections.

We want to prove that $N_\epsilon$ does not depend on $\epsilon$ and, moreover, the heteroclinic connectivity on the limiting attractor $\mathcal{A}$ is preserved by the perturbation (16).

Let us define the following subsets of $X^\alpha$

$$\mathcal{U}^\epsilon = \{u_0 \in X^\alpha : \|u(t;\epsilon,u_0)\| \to \infty, \quad \text{as} \quad t \to \infty\}$$

for each $\epsilon$, where $u(t;\epsilon,u_0)$ denotes the unique solution of the (18) with initial condition $u_0$.

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

We claim that $\mathcal{U}^\epsilon$ does not depend on the parameter $\epsilon$, if $\epsilon$ is sufficiently small. Indeed, if $u_0 \in \mathcal{U}^\epsilon$ then

$$\lim_{t\to\infty} \|u(t;\epsilon,u_0)\|_{L^2} = \infty.$$

Consequently, there exists some $j < [\sqrt{b_\epsilon}] = N$ such that the norm $|\hat{u}_j(t;\epsilon,u_0)|$ goes to infinity as $t \to \infty$. It follows from lemma 3.1 that

$$\hat{u}_j(0;\epsilon,u_0) \neq \int_\infty^0 \mathrm{e}^{(\lambda_j-b_\epsilon)s}\hat{g}_j(s)\mathrm{d}s$$

and, therefore,

$$\hat{u}_j(0;0,u_0) = \langle u(0;0,u_0),\varphi_j(\cdot)\rangle = \langle u_0(\cdot),\varphi_j(\cdot)\rangle = \hat{u}_j(0;\epsilon,u_0) \neq \int_\infty^0 \mathrm{e}^{(\lambda_j-b_\epsilon)s}\hat{g}_j(s)\mathrm{d}s.$$

If $\epsilon$ is sufficiently small, then we also have

$$\hat{u}_j(0;0,u_0) \neq \int_\infty^0 \mathrm{e}^{(\lambda_j-b)s}\hat{g}_j(s)\mathrm{d}s. \tag{19}$$

Again lemma 3.1 is applied to obtain, from (19), that $|\hat{u}_j(t;0,u_0)|$ goes to infinity as $t \to \infty$ and, therefore,

$$\lim_{t\to\infty} \|u(t;0,u_0)\|_{L^2} = \infty.$$

We have then concluded that $u_0 \in \mathcal{U}^0 := \mathcal{U}$, that is to say that $\mathcal{U}^\epsilon \subset \mathcal{U}$. Similarly, we obtain that $\mathcal{U}^\epsilon \subset \mathcal{U}$, for $\epsilon$ small. Hence,

$$\mathcal{U}^\epsilon = \mathcal{U},$$

for all $\epsilon$ sufficiently small.

In particular, we have shown that the set of initial conditions corresponding to unbounded solutions of the limiting equation

$$u_t = u_{xx} + bu + g(u) \tag{20}$$

besides being stable under small perturbations, it is also invariant as we vary $\epsilon$, for small values of $\epsilon$.

Next, we prove that $\mathcal{U}$ is an open subset of $X^\alpha$. More precisely, if $u_0'$ is sufficiently close to $u_0 \in \mathcal{U}$, in the $X^\alpha$-norm, we are claiming that the corresponding solution $u'(t;0,u_0')$ satisfies

$$\lim_{t\to\infty} \|u'(t;0,u_0')\|_{L^2} = \infty.$$

Indeed, the unique solution $u(t;u_0)$ through $u_0 \in \mathcal{U}$ satisfies

$$\hat{u}_j(0;u_0) = \int_0^\pi u_0(x)\varphi_j(x)\mathrm{d}x \neq \int_\infty^0 \mathrm{e}^{(\lambda_j-b)s}\hat{g}_j(s)\mathrm{d}s.$$

Hence,

$$\hat{u}_j'(0;u_0') = \int_0^\pi u_0'(x)\varphi_j(x)\mathrm{d}x \neq \int_\infty^0 \mathrm{e}^{(\lambda_j-b)s}\hat{g}_j(s)\mathrm{d}s. \tag{21}$$

as long as $u_0'$ is sufficiently close to $u_0$.

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

We define the following subset of $X^\alpha$, which will be very useful for our investigation of the solution dynamics for equation (18)

$$\mathcal{B} = X^\alpha \setminus \mathcal{U}.$$

As defined, $\mathcal{B} \subset X^\alpha$ is a closed subset. In particular, $\mathcal{B}$ is a complete metric space.

We let $T_\epsilon(\cdot)$ be the semigroup induced by equation (18). If we consider $T_\epsilon(\cdot)$ restricted to $\mathcal{B}$, we may apply the standard theory for compact global attractors and dissipative semigroups (see, for instance, [**14**]).

It follows that $T_\epsilon(\cdot)$ restricted to $\mathcal{B}$ is a gradient semigroup with respect to the set of hyperbolic equilibria

$$E_\epsilon^c = \{v_1^\epsilon, v_2^\epsilon, \dots, v_n\},$$

for every $\epsilon$ sufficiently small. Notice, in particular, that $n_\epsilon = n$ is independent on $\epsilon$. Moreover, the associated global attractor is given by

$$\mathcal{A}_\epsilon^c = \left( \bigcup_{i=1}^n W^u\left(v_i^\epsilon\right) \right) \cap \mathcal{B} \tag{22}$$

and

$$\lim_{\epsilon \to 0} \|v_i^\epsilon(\cdot) - v_i(\cdot)\|_\alpha = 0, \tag{23}$$

where $\{v_1, v_2, \dots, v_n\}$ is the set of equilibria for the limiting equation (20).

Hence, the non-compact global attractor $\mathcal{A}_\epsilon$ of equation (18) decomposes as

$$\mathcal{A}_\epsilon = \mathcal{A}_\epsilon^c \cup \mathcal{A}_\epsilon^\infty$$

where $\mathcal{A}_\epsilon^\infty$ is an unbounded subset containing the set of equilibria at infinity in $E^\infty$ and grow-up solutions. The compact set $\mathcal{A}_\epsilon^c$ is given as in (22). Moreover, $\mathcal{A}_\epsilon$ decomposes into the union of $E_\epsilon^c$, $E^\infty$ and the orbits connecting each of these equilibria.

As we will see below, a few more can be said about the structure of solutions within the non-compact global attractor $\mathcal{A}_\epsilon$.

The heteroclinic connectivity on $\mathcal{A}_\epsilon^\infty$ is determined, for each $\epsilon$, from the information on the zero number of each of the grow-up solutions. This is proved in [**17**] using inertial manifold theory. Hence, if $\epsilon$ is small enough, the connections to infinity are necessarily preserved. That is to say that bounded equilibria connected to an equilibria at infinity $\Phi_j^{\pm,\infty}$ via a solution of the limiting equation (20), remains connected to $\Phi_j^{\pm,\infty}$ via a solution of the perturbed equation (18).

In what regards the heteroclinic connectivity on $\mathcal{A}_\epsilon^c$, it is also preserved by small perturbations. This follows from the Morse-Smale property of (18). Indeed, the semiflow induced by (18) satisfies the so-called Sturm property (or zero number or lap number property): for any solution $v(t,x)$, the zero number $z(v(t,\cdot))$, which is the number of strict sign changes of $v(t,\cdot)$, is nonincreasing in $t$. One of the conditions that Morse-Smale semiflows satisfy is that unstable and unstable manifolds $W^u$ and $W^s$ of any two (bounded) equilibria intersect transversely. This fundamental feature follows from the above nodal property. Moreover, Morse-Smale systems

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

are structurally stable. Therefore, we conclude that bounded connections of (18) are also preserved.

We summarize the above findings in the following result.

THEOREM 5.1. *Suppose $\epsilon > 0$ is sufficiently small. Let $b_\epsilon(t) \equiv b_\epsilon$ satisfies (17). Then the global attractor $\mathcal{A}_\epsilon$ of equation (18) decomposes into bounded equilibria*

$$E^c_\epsilon = \{v^\epsilon_1, \ldots, v^\epsilon_n\},$$

*equilibria at infinity*

$$E^\infty = \{\Phi^{\pm,\infty}_1, \ldots, \Phi^{\pm,\infty}_N\}$$

*and heteroclinic connecting orbits, where the compact subset $\mathcal{A}^c_\epsilon$ satisfies (22) and $E^c_\epsilon$ satisfies (23). Moreover, the heteroclinic connections in the attractor $\mathcal{A}$ of (20) are preserved in $\mathcal{A}_\epsilon$.*

The investigation of the non-autonomous perturbation (16) is carried in a similar way. The non-compact global attractor for the limiting autonomous equation (20) is given by

$$\mathcal{A} = \mathcal{A}^c \cup \mathcal{A}^\infty$$

where $\mathcal{A}^c = (\cup^n_{i=1} W^u(v_i)) \cap \mathcal{B}$ and $\mathcal{B} = X^\alpha \setminus \mathcal{U}$, with $\mathcal{U}$ defined as

$$\mathcal{U} = \{u_0 \in X^\alpha : \|u(t; s, 0, u_0)\| \to \infty, \quad s \to -\infty\}.$$

Here, $u(t; s, 0, u_0)$ denotes the unique solution of (20) with initial condition $u_0$ at time $s$.

We assume $\epsilon_0$ is sufficiently small such that

$$[\sqrt{b_\epsilon}(t)] = [\sqrt{b}], \quad \text{for all } t \in \mathbb{R},$$

for all $\epsilon < \epsilon_0$. Lemma 3.2 then implies that backwards unbounded solutions $u(t; s, 0, u_0)$ converge to one of the following equilibria at infinity

$$E^\infty = \{\Phi^{\pm,\infty}_1, \ldots, \Phi^{\pm,\infty}_N\}$$

for any time $t \in \mathbb{R}$.

Similarly to the autonomous case, we define

$$\mathcal{U}^\epsilon = \{u_0 \in X^\alpha : \|u(t; s, \epsilon, u_0)\| \to \infty, \quad s \to -\infty\},$$

where $u(t; s, \epsilon, u_0)$ is the unique solution of (16) with initial condition $u_0$. Our claim is that, as in the autonomous setting, $\mathcal{U}_\epsilon$ does not depend on $\epsilon$, if $\epsilon$ is sufficiently small. Indeed, if $u_0 \in \mathcal{U}^\epsilon$ then there exists some integer $j < N$ such that the

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

corresponding solution $u(t; s, \epsilon, u_0)$ satisfies

$$\lim_{s\to-\infty} |\hat{u}_j(t; s, \epsilon, u_0)| = \infty.$$

Lemma 3.1 then yields

$$\lim_{s\to-\infty} \int_{\infty}^{s} \mathrm{e}^{-\int_r^s(\lambda_j - b_\epsilon(\theta))\mathrm{d}\theta} \hat{g}_j(r)\mathrm{d}r \neq \langle u(0; s, \epsilon, u_0), \varphi_j(\cdot)\rangle$$

If we choose $\epsilon$ sufficiently small, the following also holds

$$\langle \hat{u}_j(0; s, 0, u_0), \varphi_j(\cdot)\rangle \neq \lim_{s\to-\infty} \int_{\infty}^{s} \mathrm{e}^{-\int_r^s(\lambda_j - b)\mathrm{d}\theta} \hat{g}_j(r)\mathrm{d}r.$$

Then, we apply lemma 3.1 to conclude that

$$\lim_{s\to-\infty} \|u(t; s, 0, u_0)\| = \infty$$

and, therefore, $u_0 \in \mathcal{U}$. An analogous argument shows that $\mathcal{U} \subset \mathcal{U}^\epsilon$ for small values of $\epsilon$. In addition to $\mathcal{U}^\epsilon = \mathcal{U}$ be independent of $\epsilon$, it does not depend on time $t$ either.

Also, $\mathcal{U}$ is an open subset of $X^\alpha$. The proof follows as in the autonomous case. Suppose $u_0 \in \mathcal{U}^\epsilon$, then

$$(\hat{u}_0)_j \neq \lim_{s\to-\infty} \int_{\infty}^{s} \mathrm{e}^{(\lambda_j - b_\epsilon(\theta))\mathrm{d}\theta} \hat{g}_j(r)\mathrm{d}r.$$

If $u_0'$ is close to $u_0$ in $X^\alpha$, then

$$(\hat{u}_0')_j = \int_0^\pi u_0'(x)\varphi_j(x)\mathrm{d}x \neq \lim_{s\to-\infty} \int_{\infty}^{s} \mathrm{e}^{(\lambda_j - b_\epsilon(\theta))\mathrm{d}\theta} \hat{g}_j(r)\mathrm{d}r.$$

We conclude that $\mathcal{U}$ is open in $X^\alpha$ and, therefore,

$$\mathcal{B} = X^\alpha \setminus \mathcal{U}$$

is a closed subset of $X^\alpha$. In order to obtain information on the dynamics of bounded solutions, we restrict our attention to the subset $\mathcal{B}$, as we did in the autonomous case.

The semigroup $T(\cdot)$ induced by the limiting autonomous equation and the evolution process $S_\epsilon(\cdot,\cdot)$ of (16), both restricted to $\mathcal{B}$, satisfy dissipativeness and boundedness properties. Consequently, we may apply the standard theory of pullback attractors. In particular we may apply theorems 8.7 and 5.36 in [**9**] to obtain the following result.

THEOREM 5.2. *If $b_\epsilon(t)$ satisfies (17) and all the equilibria of the limiting autonomous equation are hyperbolic, then there are hyperbolic global solutions $\xi_{i,\epsilon}$*

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

*of* $S_\epsilon(\cdot,\cdot)$ *satisfying*

$$\lim_{\epsilon\to 0}\sup_{t\in\mathbb{R}}\|\xi_{i,\epsilon}(t)-v_i(\cdot)\|_\alpha = 0$$

*for* $i=1,2,\dots,n$. *Moreover, the pullback attractor of* $S_\epsilon(\cdot,\cdot)$ *with phase space* $\mathcal{B}$ *is given by*

$$\mathcal{A}^c_\epsilon(t) = (\cup_{i=1}^n W^u(\xi_{i,\epsilon}(\cdot))(t))\cap\mathcal{B},\quad t\in\mathbb{R},$$

*if* $\epsilon$ *is sufficiently small.*

As we mentioned before, lemma 3.2 implies that, for each time $t$, any unbounded solution $u(t;s,\epsilon,u_0)$ may only converge to one of the following equilibria

$$E^\infty_\epsilon = E^\infty = \{\Phi_1^{\pm,\infty},\dots,\Phi_N^{\pm,\infty}\}.$$

Also, lemma 3.1 implies that all of the equilibria at infinity lying in the set $E^\infty_\epsilon = E^\infty$ are in fact contained in each of the fibres $\mathcal{A}_\epsilon(t)$ of the pullback attractor. Then, $E^\infty$ is the set of equilibria at infinity for each $\mathcal{A}_\epsilon$.

Because $S_\epsilon(\cdot,\cdot)$ is dynamically gradient with respect to the family

$$\{\xi_{1,\epsilon}(\cdot),\dots,\xi_{n,\epsilon}(\cdot)\},$$

any global bounded solution $\xi$ on the pullback attractor $\mathcal{A}^c_\epsilon$ satisfies

$$\lim_{t\to-\infty}\|\xi(t)-\xi_{l,\epsilon}(t)\|_\alpha = 0,\quad\text{and}\quad \lim_{t\to+\infty}\|\xi(t)-\xi_{k,\epsilon}(t)\|_\alpha = 0, \tag{24}$$

for some $k,l$ satisfying $1\leqslant k<l\leqslant n$.

We claim that all bounded heteroclinic connections for the limiting semiflow $T(\cdot)$ are preserved by small non-autonomous perturbations as in (16). If this is the case, any two distinct equilibria $v_l$ and $v_k$ are connected by a solution contained in the limiting attractor $\mathcal{A}^c$ if, and only if, there is a global bounded solution on $\mathcal{A}^c_\epsilon$ connecting $\xi_{l,\epsilon}(\cdot)$ and $\xi_{k,\epsilon}(\cdot)$, in the sense of (24).

The argument follows as in the small autonomous perturbation case. However, we need a non-autonomous equivalent result for equation (16) stating that all intersections of stable and unstable manifolds of distinct equilibria are transverse. In fact, this assertion was recently proved in [**7**]. Therefore, we affirm that the heteroclinic connections within $\mathcal{A}^c$ are indeed preserved by the non-autonomous perturbation (16), as long as we take $\epsilon$ small enough.

## 6. Structure of the unbounded pullback attractor for asymptotically autonomous equations

In this section, we consider the non-autonomous equation (1) with $b(t)$ being backwards asymptotically autonomous. Since theorem 5.2 holds for small non-autonomous perturbations, we can apply it for obtaining the existence and description of the pullback attractor $\mathcal{A}(\cdot)$ in the setting of asymptotically autonomous evolution processes.

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

We investigate the dynamics of

$$u_t = u_{xx} + b(t)u + g(u) \tag{25}$$

with $b(t)$ satisfying

$$b_1 < b(t) < b_2 \quad \text{and} \quad \lim_{t\to-\infty} b(t) = b_1 \tag{26}$$

It is known that the backwards limiting equation

$$u_t = u_{xx} + b_1 u + g(u) \tag{27}$$

has a finite set of equilibria, which we also denote by $E^c = \{v_1, \ldots, v_n\}$, if all the equilibria are assumed to be hyperbolic. In this case, the non-compact global attractor is gradient and given by

$$\mathcal{A} = \cup_{i=1}^n W^u(v_i).$$

As we mentioned before, the limiting attractor $\mathcal{A}$ decomposes into a compact subset $\mathcal{A}^c$ and an unbounded subset $\mathcal{A}^\infty$. The grow-up solutions and the $2N_1$ equilibria at infinity are all contained on $\mathcal{A}^\infty$.

For the asymptotically autonomous problem (25) with $b(t)$ satisfying (26), we prove the following.

THEOREM 6.1. *Assume* (26) *holds and all the equilibria of* (27) *are hyperbolic. Then there are global hyperbolic solutions* $\xi(\cdot)$ *of* (25), $1 \leqslant i \leqslant n$, *satisfying*

$$\lim_{t\to-\infty} \|\xi_i(t) - v_i(\cdot)\|_\alpha = 0, \quad i = 1, 2, \ldots, n.$$

*Moreover, the evolution process* $S(\cdot,\cdot)$ *of* (25), *when restricted to the subset of bounded solutions* $\mathcal{B} \subset X^\alpha$, *has a pullback attractor given by*

$$\mathcal{A}^c(t) = (\cup_{i=1}^n W^u(\xi_i(\cdot))(t)) \cap \mathcal{B}, \quad t \in \mathbb{R}.$$

*Proof.* In order to obtain information on the evolution process related to (25), we apply theorem 5.2 to the forwards truncated evolution process $S_\tau(\cdot,\cdot)$ generated by

$$u_t = u_{xx} + b_\tau(t)u + g(u) \tag{28}$$

where

$$b_\tau(t) = \begin{cases} b(t), & \text{if } t \leqslant -\tau \\ b(\tau), & \text{if } t > -\tau. \end{cases}$$

and $\tau > 0$.

Because (28) is a small non-autonomous perturbation of (27), the subset $\mathcal{B}$ of bounded solutions of (28) may be defined as in § 5. In particular, $\mathcal{B}$ is independent of $\tau$.

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

Theorem 5.2 guarantees the existence of hyperbolic global solutions $\xi_{i,\tau}$ such that

$$\lim_{\tau\to\infty}\sup_{t\in\mathbb{R}}\|\xi_{i,\tau}(t)-v_i(\cdot)\|_\alpha=0$$

and the pullback attractor of $S_\tau(\cdot,\cdot)$ in the restricted phase space $\mathcal{B}\subset X^\alpha$ is the following

$$\mathcal{A}^c_\tau(t)=\big(\cup_{j=1}^n W^u(\xi_{i,\tau}(\cdot))(t)\big)\cap\mathcal{B},\quad t\in\mathbb{R}$$

for $\tau$ sufficiently small.

Let $S(t,s)$ be the evolution process of (25). Then, we observe that the processes $S_\tau(t,s)$ and $S(t,s)$ coincide if $s\leqslant t\leqslant -\tau$. A prompt conclusion is that their corresponding pullback attractors on $\mathcal{B}$ satisfy

$$\mathcal{A}^c(t)=\mathcal{A}^c_\tau(t),\qquad \text{for}\ \ t\leqslant-\tau.$$

At this point, we remark that $\mathcal{B}$ is also the subset of global bounded solutions for the process $S(t,s)$, as long as $s\leqslant t\leqslant-\tau$. This is due to the fact that

$$S_\tau(t,s)=S(t,s),\quad \text{if } s\leqslant t\leqslant-\tau.$$

In addition, $\mathcal{B}$ is independent of time, then the set of bounded solutions for each section $\mathcal{A}(t)$ is given by $\mathcal{B}$, for every time $t\in\mathbb{R}$. As a consequence, we may recover $\mathcal{A}^c(t)$, for $t\geqslant-\tau$, by appealing to the invariance property of $\mathcal{A}^c(t)$. We get

$$\mathcal{A}^c(t)=S(t,s)\mathcal{A}^c(s),\qquad \text{for all}\ \ s\leqslant-\tau\leqslant t.$$

□

Regarding the equilibria at infinity, the following can be stated. We know that the evolution process $S_\tau(t,s)$ of equation (28) coincides with $S(t,s)$, for any $s\leqslant t\leqslant-\tau$. Hence, if $t\leqslant-\tau$, any unbounded solution $u(t;s,\tau,u_0)$ converges to some $\Phi_j^{\pm,\infty}$, as $s\to-\infty$, where $j$ necessarily satisfies

$$j\leqslant N_1.$$

This is a consequence of $S_\tau(t,s)$ being a small non-autonomous perturbation of (27). We conclude that

$$E^\infty=\{\Phi_1^{\pm,\infty},\dots,\Phi_{N_1}^{\pm,\infty}\}$$

is the set of equilibria at infinity on $\mathcal{A}(t)$ for arbitrarily small times $t$.

However, notice that the number of objects on the set $E^\infty(t)\subset\mathcal{A}(t)$, may increase or decrease as $t$ varies. In the particular case where (25) is also forwards asymptotically autonomous, we may also obtain the set $E^\infty(t)$ for arbitrarily large times $t$.

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

Suppose, for instance, that $b(t)$ satisfies (26) and

$$\lim_{t\to\infty} b(t) = b_2.$$

In this particular setting, we can additionally conclude the following for equation (25). Let $k$ be a positive real number and $b_k(t)$ be defined as

$$b_k(t) = \begin{cases} b(t), & \text{if } t \geqslant k \\ b(k), & \text{if } t \leqslant k. \end{cases}$$

Then, if we choose $k$ sufficiently large, the process $S_k(t, s)$ induced by

$$u_t = u_{xx} + b_k(t)u + g(u)$$

is a small non-autonomous perturbation of

$$u_t = u_{xx} + b_2 u + g(u).$$

It follows from § 5 that, for any time $t$, the set $E^\infty \subset \mathcal{A}_k(t)$ contains $2N_2$ elements, where $N_2 = [\sqrt{b_2}]$. Because

$$S_k(t, s) = S(t, s), \quad \text{for } k \leqslant s \leqslant t,$$

the set of limiting objects an unbounded solution $u(t; s, u_0)$ may limit (as $s \to -\infty$) to is given by

$$E^\infty(t) = \left\{\Phi_1^{\pm,\infty}, \ldots, \Phi_{N_2}^{\pm,\infty}\right\},$$

for large times $t$.

## 7. The uniform attractor for the non-autonomous dynamical system

It is well known that in opposition to the behaviour of autonomous systems, the forwards and pullback dynamics are distinct, in general, for non-autonomous systems. The analysis of both asymptotics provides, however, complementary information on the dynamics of a non-autonomous problem. Since all the previous sections have mostly been concerned to the pullback behaviour of the system (1), we dedicate this section to the forwards analysis.

As regards the forward dynamics within infinity, the following may be affirmed. Analogues of lemmas 3.1 and 3.2 also hold true when taking into account the behaviour of solutions as $t \to \infty$, for each initial time $s$. That is to say that, the existence of forwards unbounded solutions of (1) is guaranteed, for every initial time $s$. Then, by performing the Poincaré projection analysis, one can find the number of equilibria at infinity a forwards unbounded solution may converge to. As in the pullback realm, the number of equilibria will depend on the specific behaviour of $b(t)$.

If the main features of the pullback attractor are the invariance and the pullback attraction, the uniform attractor, on its turn, is a non-invariant set but it satisfies the crucial forwards attraction property. Therefore, a description of the uniform

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

attractor related to (1) would greatly complement the information provided in the previous sections.

We now follow the recent literature on the asymptotic behaviour of non-autonomous systems. See [**5**]. A family of nonlinearities is defined as a base semiflow to be driven by a time shift operator $\theta_t$. More precisely, the nonlinearity $f(t,\cdot) \in X$ is a bounded continuous function of $t \in \mathbb{R}^+$,

$$f \in C_b(\mathbb{R}^+, X),$$

then we let $\Sigma_0$ denote the set of all translates of $f$

$$\Sigma_0(f) = \{\theta_t(f) := f(\cdot + t); t \in \mathbb{R}^+\}.$$

We take the metric $\rho$ to be given by the uniform convergence on compact subintervals. The hull of the function $f$ is given by

$$\Sigma := \Sigma_\rho(f) = \text{ closure of } \Sigma_0(f) \text{ in } C_b(\mathbb{R}, X) \text{ with respect to } \rho.$$

The continuity of $\theta_t$ on $\Sigma$ follows from the continuity of $\Sigma_0$. The semiflow $\{\theta_t\}_{t\geqslant 0}$ on $\Sigma$ will be the base semiflow, referred to as the *driving semigroup*.

In addition, for each $\sigma \in \Sigma$, consider the semiflow

$$\begin{aligned} \varphi : \mathbb{R}^+ \times X \to X \\ (t, u_0) \mapsto \varphi(t,\sigma)u_0 = S(t, 0; \sigma)u_0 \end{aligned}$$

where $S(t, s; \sigma)$ is the process related to (1) with $f$ replaced by the translate $\sigma$. Then, $\varphi$ defines a semiflow

$$\varphi : \mathbb{R}^+ \times \Sigma \times X \to X$$

called the *cocyle semiflow*. The semiflows $\theta$ and $\varphi$ define the *non-autonomous dynamical system* $(\varphi, \theta)$ on $(X, \Sigma)$. The associated *skew product semiflow* $\{\Pi(t) : t \geqslant 0\}$ given by

$$\Pi(t)(u,\sigma) = (\varphi(t,\sigma)\,,\, \theta_t\sigma)$$

is a semigroup on $X \times \Sigma$.

It follows from the forwards analogues of lemmas 3.1 and 3.2 that the compactness assumption on the uniform attractors has to be dropped in the setting of equation (1).

Definition 7.1. The minimal subset of $X$ that is forwards uniformly attracting for all bounded subsets of $X$ and $\Sigma$, uniformly with respect to the initial time, is called the unbounded uniform attractor for the non-autonomous dynamical system $(\varphi, \theta)_{(X,\Sigma)}$

If we assume that $b'(t)$ is bounded on $\mathbb{R}$ then, as we will see in theorem 7.1, equation (1) possesses a uniform attractor. The boundedness on $b'(t)$ implies $b(t)$

Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

is globally Lipschitz. As a consequence, any sequence of translations

$$\theta_{t_n}(b) = b(t + t_n)$$

with $t_n \to \infty$ as $n \to \infty$, has a subsequence converging uniformly on compact intervals. Then, we conclude that the driving semigroup $\theta_t$ on $\Sigma$ with metric $\rho$ possesses a global attractor $\mathcal{S}$.

In addition to the existence of an attractor for $\theta_t$, we have the following. For each $\sigma \in \Sigma$, the evolution process $S(t, s; \sigma)$ possesses a finite dimensional attracting invariant manifold $M_\sigma(\cdot)$, with unbounded sections $M_\sigma(t)$ contained in $X$.

By combining both statements we can obtain the existence of an attracting unbounded subset in $X \times \Sigma$ for $\{\Pi(t) : t \geqslant 0\}$. As a consequence, the skew product semiflow $\Pi(t) = (\varphi(t, \sigma), \theta_t\sigma)$ has an unbounded global attractor.

We will prove in a future work that, as in [**5**, theorem 2.7] the non-autonomous dynamical system $(\varphi, \theta)_{(X,\Sigma)}$ related to equation (1) has an unbounded uniform attractor $\mathbb{A}$ given by

$$\mathbb{A} = \bigcup_{\eta \in \Gamma} \bigcup_{t \in \mathbb{R}} \mathcal{A}_\eta(t) \tag{29}$$

where $\Gamma$ is the set of all bounded solutions $\eta : \mathbb{R} \to \mathcal{S}$ for $\theta_t$ and $\{\mathcal{A}_\eta(t)\}_{t \in \mathbb{R}}$ is the unbounded pullback attractor for the evolution process

$$S_\eta(t, s)u := \varphi(t - s, \eta(s))u, \quad u \in X, \quad t \geqslant s.$$

The following result will be proved.

THEOREM 7.1. *Suppose $b'(t)$ is bounded in $\mathbb{R}$. Then the non-autonomous dynamical system $(\varphi, \theta)$ related to equation (1) has an unbounded uniform attractor $\mathbb{A}$ given by (29).*

It is worth noticing that the non-autonomous attracting invariant manifolds, existing for each $\sigma$ in the attractor of $\theta_t$, have all the same attraction exponent. Since $b$ satisfies (2), the rate of attraction for any limiting translate function will only depend on the upper bound $b_2$ and on $g'$. However, the precise behaviour at infinity on each manifold, and also on $\mathbb{A}$, is still to be described. The details will be too long to insert here.

## References


1 J. Arrieta and A. Carvalho. Abstract parabolic problems with critical nonlinearities and applications to Navier Stokes and Heat Equations. *Trans. Amer. Math. Soc.* **352** (2000), 285–310.

2 J. Arrieta, A. Carvalho and A. Rodríguez-Bernal. Parabolic problems with nonlinear boundary conditions and critical nonlinearities. *J. Differ. Equ.* **156** (1999), 376–406.

3 J. Arrieta, A. Carvalho and A. Rodríguez-Bernal. Attractors for parabolic problems with nonlinear boundary condition. Uniform bounds. *Comm. Partial Differ. Equ.* **25** (2000), 1–37.

4 N. Ben-Gal. Grow-Up Solutions and Heteroclinics to Infinity for Scalar Parabolic PDEs. *PhD Thesis, Division of Applied Mathematics, Brown University*, 2010.

5 M. Bortolan, T. Caraballo, A. Carvalho and J. Langa. Skew product semiflows and Morse decomposition. *J. Differ. Equ.* **255** (2013), 2436–2462.


Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51

6 M. Bortolan, A. Carvalho and J. Langa. Structure of attractors for skew product semiflows. *J. Differ. Equ.* **257** (2014), 490–522.

7 M. Bortolan, A. Carvalho, J. Langa and G. Raugel. Non-autonomous perturbations of Morse-Smale semigroups: stability of the phase diagram. Preprint.

8 A. Carvalho, J. Langa and J. Robinson. Structure and bifurcation of pullback attractors in a non-autonomous Chafee-Infante equation. *Proc. Amer. Math. Soc.* **140** (2011), 2357–2373.

9 A. Carvalho, J. Langa and J. Robinson. Attractors for infinite-dimensional non-autonomous dynamical systems. In *Applied mathematical sciences* (ed. S.S. Antman, P. Holmes and K. Sreenivasan), vol.182 (New York: Springer-Verlag, 2013).

10 V. V. Chepyzhov. On unbounded invariant sets and attractors of some semilinear equations and systems of parabolic type. *Russian Math. Surveys* **42** (1987), 167–168.

11 V. V. Chepyzhov. Unbounded attractors of some parabolic systems of differential equations and estimates on their dimension. *Soviet Math. Dokl.* **38** (1989), 4–49.

12 V. V. Chepyzhov and A. Yu. Goritskii. Unbounded attractors of evolution equations, Properties of Global Attractors of Partial Differential Equations. *Adv. Soviet Math.* **10**, Amer. Math. Soc., Providence, RI (1992), 85–128.

13 J. Hell. Conley index at infinity. *Topol. Methods Nonlinear Anal.* **42** (2013), 137–167.

14 D. Henry. *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, 840 (Berlin-New York: Springer-Verlag, 1981).

15 J. Langa, J. Robinson, A. Rodríguez-Bernal, A. Suárez and A. Vidal-López. Existence and nonexistence of unbounded forwards attractor for a class of non-autonomous reaction diffusion equations. *Discrete Contin. Dyn. Syst.* **18** (2007), 483–497.

16 M. Miklavčič. A sharp condition for existence of an inertial manifold. *J. Dynam. Differ. Equ.* **3** (1991), 437–456.

17 J. Pimentel and C. Rocha. A permutation related to non-compact global attractors for slowly non-dissipative systems. *J. Dynam. Differ. Equ.* **28** (2016), 1–28.


Downloaded from https://www.cambridge.org/core. USP Sao Carlos-ICMC-Biblioteca do Instituto de Ciencias Matematicas e da Computacao, on 30 Jan 2020 at 17:56:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/prm.2018.51