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Organising the module category

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Abstract
We describe here the work developed with my students along the years in the IME-USP.

Keywords Module category · Auslander–Reiten theory · Tilting theory

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When the São Paulo Journal of Mathematical Sciences announced a special issue celebrating the golden jubilee of the Institute of Mathematics and Statistics of the University of São Paulo (IME-USP), I could not avoid thinking about my own history in the institute. A significant 41 years since I started my undergraduate studies there in 1979 followed by some graduate studies and as a teacher/researcher since March, 1985. I am proud to say that I am a full professor there since December, 2003, and by the fact that I was his Director in the quadrienium 2010–2014.

Among all the unforgettable memories of this period, I decided to concentrate here in describing the work I have done with my students since I became a doctorate supervisor in the early 90's of the last century. My main research area is Representation Theory of Algebras and so are the works I have supervised in this period. Here, I will survey shortly the work done with (listed here in chronological order and indicating the year of the corresponding thesis): Sonia Trepode (1995), Rosana R. Signorelli Vargas (1999), Angela Marta P. D. Savioli (2000), Marcelo Lanzilotta (2000), Edson R. Álvares (2002), Cecilia Tosar Escuder (2004), Clézio A. Braga (2005), Marcia Aguiar (2005), Heily Wagner (2012), Danilo Dias da Silva (2013) and Viktor Chust (2020).

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Besides these students with whom I have published works, I should also mention the graduate students Eliza H. Miyazaki (2005) and Antonio Marcos C. Neri (2006).

Despite some variety on the specific work done with each of them, one could think that they are concentrated on the idea of organising the module category of an algebra. I have arranged this survey not in chronological order but by subjects. Section 1 includes some preliminaries recalling notions which will be useful along the text.

I would also like to take the opportunity to dedicate this work to my dear friend Ibrahim Assem on the occasion of his retirement. We have been collaborating since 1992 and, apart from frequent mutual visits, five of the above mentioned students have spent some time at the *Université de Sherbrooke* working with him. In addition, one of his students spent a year in São Paulo under my supervision.

1 Preliminaries

The representation theory of algebras deals ultimately with the description of the category of modules over a (in general) finite dimensional k-algebra, where k is a field (in general, algebraically closed). Several techniques were developed in the last fifty years aiming an organisation of such categories and we shall mention some of them here: the description of classes of algebras and modules through quivers; the Auslander–Reiten theory; tilting theory or, more generally, techniques using homological invariants. Clearly, these mentioned techniques do not cover all those currently in use in the area. We decided to concentrate on those because they fairly cover the subjects of the works we would like to discuss here.

The rest of the section is devoted to discuss some basic results and notations and, whenever needed, we shall recall other concepts along the text. For further details on these basic materials we indicate the books [9, 19, 22].

Along this survey, by an algebra we will understand a basic finite dimensional (associative with unity) algebra over an algebraically closed field k. We choose this specific setting by simplicity because it englobes all the work discussed here but it is important to say that some of the results are valid in more general contexts. Whenever it is really important to mention such context we will do it.

Given an algebra A, the category of finitely generated (right) A-modules will be indicated by mod$A$ and by ind$A$ we indicate the category having one representative of each isoclass of indecomposable A-modules. We say that an algebra A is representation-finite provided ind$A$ has only finitely many objects.

Also, for a module M, we shall indicate by $\mathrm{pd}_A M$ and by $\mathrm{id}_A M$ its projective and its injective dimension, respectively. The global and the finitistic dimensions of an algebra A are, respectively,

$$\mathrm{gl.dim}\, A = sup\{\ \mathrm{pd}_A M \ :\ M \in\ \mathrm{mod}\, A\}$$
$$\mathrm{fin.dim}\, A = sup\{\ \mathrm{pd}_A M \ :\ M \in\ \mathrm{mod}\, A \text{ and } \ \mathrm{pd}_A M < \infty\}$$

1.1 Quivers

A quiver is given by a quadruple $Q = (Q_0, Q_1, s, e)$ where Q_0 is the set of vertices, Q_1 is the set of arrows, and $s, e : Q_1 \longrightarrow Q_0$ are functions which indicate, for each arrow α, its source $s(\alpha)$ and its target $e(\alpha)$. Visually, we indicate it locally as $s(\alpha)\overset{\alpha}{\longrightarrow}e(\alpha)$.

Let Q be a quiver. We say that it is **finite** if both Q_0 and Q_1 are finite. Given $x, y \in Q_0$, a path from x **to** y of length n is given by $\gamma = \alpha_1 \cdots \alpha_n$, where $\alpha_i \in Q_1$ is such that $s(\alpha_i) = e(\alpha_{i-1})$, for $i = 2, \cdots, n$, $s(\alpha_1) = x$ and $e(\alpha_n) = y$. By convenience, we assign to each vertex $x \in Q_0$ a path of length zero ϵ_x.

For each arrow $\alpha \in Q_1$, we associate a formal inverse α^{-1} with source $e(\alpha)$ and target $s(\alpha)$. A walk in Q is given by $\beta_1^{\nu_1} \cdots \beta_m^{\nu_m}$ $(m \geq 1)$, where, for each i, $\beta_i \in Q_1$ and $\nu_i \in \{1, -1\}$ and for each $i < l$, we have $\{s(\beta_i), e(\beta_i)\} \cap \{s(\beta_{i+1}), e(\beta_{i+1})\} \neq \emptyset$. We say that Q is connected if for all $x, y \in Q_0$, there exists a walk $\beta_1^{\nu_1} \cdots \beta_m^{\nu_m}$ such that $x = s(\beta_1^{\nu_1})$ and $y = s(\beta_m^{\nu_m})$

1.2 Path algebras

Given a field k and a finite quiver $Q = (Q_0, Q_1, s, e)$, we can easily assign an algebra kQ defined as follows. As a k-vector space, it is choosen the set of all paths in Q as a basis. Note that such a basis includes the paths of length zero. A natural concatenation of paths then induces an associative, but generally not commutative, multiplication in kQ.

The corresponding algebra kQ is associative and it has an unity (given by the sum of the paths of length zero) and, if Q has no oriented cycles (that is, no paths of positive length starting and ending at the same vertex), then it would be finite dimensional. Denote by J_Q the ideal of kQ generated by all paths of positive length.

Next result, due to Gabriel, justifies the utility of dealing with path algebras (in fact, with their quotients) since they somehow represent a very big class of algebras.

Theorem 1.1 *Let A be a basic finite dimensional algebra over an algebraically closed field k. Then there exists a unique quiver Q_A (called the* ordinary quiver of A *) such that $A \cong \frac{kQ_A}{I}$ where I is an ideal such that $J_Q^n \subset I \subset J_Q^2$ for some $n \geq 2$ (such an ideal is called* admissible).

We refer to [9] for details.

1.3 AR-theory

We shall briefly recall some notions on the Auslander–Reiten theory, referring the reader to [9] for details.

We start with the notion of radical morphisms. Let $X = \oplus_{i=1}^{n} X_i$ and$Y = \oplus_{j=1}^{m} Y_j$ be A-modules and their decompositions into indecomposable modules. A morphism $f : X \longrightarrow Y$ is a radical morphism provided for each inclusion $\iota_i : X_i \longrightarrow X$

and each projection $\pi_j : Y \longrightarrow Y_j$, the composition $\pi_j f \iota_i$ is not an isomorphism. In particular, if X and Y are indecomposable, radical morphisms from X to Y coincide with the nonisomorphisms between them. We denote by $\text{rad}_A(X, Y)$ the set of radical morphisms from X to Y. Clearly, rad_A is an ideal of the category mod$A$. More generally, for each $i > 1$, denote by $\text{rad}^i_A(X, Y)$ the *i-th* power of $\text{rad}_A(X, Y)$ and by $\text{rad}^\infty_A(X, Y)$ the intersection of all its powers.

A morphism $f : X \longrightarrow Y$ is called irreducible provided it is neither a section or a retraction and each decomposition $f = gh$ implies that either h is section or g is a retraction. Clearly, irreducible morphisms are radical morphisms which do not lie in rad^2_A.

A short exact sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is called almost split provided f and g are irreducible morphisms. They are also known in the literature as Auslander–Reiten sequences. If $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is an almost split sequence, then

(a) L is indecomposable (since g is irreducible).
(b) N is indecomposable (since f is irreducible).
(c) for each $h : X \longrightarrow N$ which is not a retraction, there exists $h' : X \longrightarrow M$ such that $h = gh'$.
(d) for each $h : L \longrightarrow X$ which is not a section, there exists $h' : M \longrightarrow X$ such that $h = h'f$.

Next results collect further important features on such sequences.

Theorem 1.2 *Let L, N be indecomposable A-modules.*

(a) *If L is not injective, then there exists a unique (up to isomorphism) almost split sequence* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$. *In particular, N is uniquely determined by L, up to isomorphism, and we shall denote it by* $N = \tau^{-1}L$.
(b) *If N is not projective, then there exists a unique (up to isomorphism) almost split sequence* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$. *Also, L is uniquely determined, up to isomorphism, by N and we denote* $L = \tau N$.

The unicity, in both case, also implies $\tau\tau^{-1}L \cong L$ *and* $\tau^{-1}\tau N \cong N$.

We normally refer to τ stated in the last theorem as the Auslander–Reiten translation.

Theorem 1.3 *Let L, N be indecomposable modules.*

(a) *If L is injective, then* $f' : L \longrightarrow M'$ *is an irreducible morphism if and only if there exists* $f'' : L \longrightarrow M''$ *such that* $(f', f'')^t : L \longrightarrow M' \oplus M''$ *is isomorphic to the natural projection* $\pi : L \longrightarrow L/\text{soc}L$.
(b) *If L is not injective, then* $f' : L \longrightarrow M'$ *is an irreducible morphism if and only if there exists* $f'' : L \longrightarrow M''$ *such that the sequence*

$$0 \longrightarrow L \xrightarrow{(f',f'')^t} M' \oplus M'' \longrightarrow \text{Coker}((f',f'')^t) \longrightarrow 0$$

is an almost split sequence starting at L.

(c) *If N is projective, then* $f' : M' \longrightarrow N$ *is an irreducible morphism if and only if there exists* $f'' : M'' \longrightarrow N$ *such that* $(f',f'') : M' \oplus M'' \longrightarrow N$ *is isomorphic to the natural inclusion* $\imath : \text{rad}N \longrightarrow N$.

(d) *If L is not projective, then* $f' : M' \longrightarrow N$ *is an irreducible morphism if and only if there exists* $f'' : M'' \longrightarrow N$ *such that the sequence*

$$0 \longrightarrow \text{Ker}(f',f'') \longrightarrow M' \oplus M'' \xrightarrow{(f',f'')} N \longrightarrow 0$$

is an almost split sequence ending at N.

The theorems above show that there exists a way to organise the module category over an algebra A using quivers through indecomposable modules and irreducible morphisms. Such a quiver is called the Auslander–Reiten quiver of A and will be indicated here as Γ_A. Its vertices are in one-to-one correspondence to the isoclasses of indecomposable A-modules. Because of that, we shall indicate both indecomposable and its corresponding vertex by the same letter. Given vertices X, Y in Γ_A, the number of arrows of type $X \longrightarrow Y$ equals the dimension of the k-vector space of irreducible morphisms from X to Y.

Because of the existence of the Auslander–Reiten translation, such a quiver is called a **translation quiver**. The study of the structure of the Auslander–Reiten quiver plays an important rôle in the Representation Theory of Algebras.

1.4 Composition of irreducible morphisms

As seen above, a nonzero morphism $f : X \longrightarrow Y$, with X, Y indecomposable, is irreducible if and only if it lies in $\text{rad}_A(X,Y) \setminus \text{rad}^2_A(X,Y)$. One could ask if there would be some similar pattern for the paths of irreducible morphisms. Specifically, if

$$X = X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots X_{m-1} \xrightarrow{f_m} X_m = Y$$

is a path of irreducible morphisms through indecomposable modules of length m, can we assure that the composition $f_m \cdots f_1$ is either zero or belongs to $\text{rad}^m_A(X,Y) \setminus \text{rad}^{m+1}_A(X,Y)$? The answer is, in general, no, and this indeed fails for $m = 2$ (see [29]). There has been an intense investigation to understand what is behind it. It is not our aim here to be extensive in this line of investigation, but we will mention for future use the work of Liu [63] on the degrees of irreducible morphisms and the more recent one started in a joint work with Chaio and Trepode [29–32] and later on by Chaio and others (see for instance [33]).

Let $f : X \longrightarrow Y$ be an irreducible morphism where X is indecomposable. Following Liu [63] we say that the **left degree of** f is the least positive integer $n \geq 1$, when exists, such that there exists a morphism $h : Z \longrightarrow X$ in $\text{rad}^n_A(Z,X) \setminus \text{rad}^{n+1}_A(Z,X)$ and $fh \in \text{rad}^{n+2}_A(X,Y)$. This degree will be said to be ∞ if there is no such n. Dually, one can define right degree of f. These notions have been successfully used in the description of components of the Auslander–Reiten quivers of an algebra (see, for instance, [63]).

1.5 Tilting theory

Tilting theory was introduced in the early 80's of last century by Happel and Ringel [58] upon a seminal work by Brenner-Butler [28] and, since then, it has played a very important rôle in the Representation Theory of Algebras. Also because of that, many generalisation of this concept has been tried in different directions. The *classical* tilting theory deals with finitely generated modules of projective dimension at most one over finite dimensional algebras (see below for definitions). Clearly, one can relax it in many directions and stages either allowing, for instance, modules of greater projective dimensions, or working with infinitely generated modules or with more general algebras and rings (see for instance [6, 7]). The possibilities are endless and we are not going to enter in details here but concentrate on the work of Sonia Trepode and Clézio Braga done in their thesis. Trepode worked with problems of more classical nature, while Braga did with problems involving infinitely generated tilting modules. We first recall the classical notion.

Definition 1.4 An A-module T is tilting if

(a) $\mathrm{pd}_A T \leq 1$.
(b) $\mathrm{Ext}^1_A(T, T) = 0$.
(c) There exists a short exact sequence $0 \to A \to T_1 \to T_2 \to 0$ with T_i in $\mathrm{add}T$.

A trivial example of a tilting module is given by the regular A-module A_A.

Let now T be a tilting A-module and consider the endomorphism algebra$B = \mathrm{End}_A T$. One can then consider torsion pairs $(\mathcal{T}(T), \mathcal{F}(T))$ in mod$A$ and$(\mathcal{X}(T), \mathcal{Y}(T))$ in mod$B$ as follows

- $\mathcal{T}(T) = \{M \in \text{ mod } A \ : \ \mathrm{Ext}^1_A(T, M) = 0\}$
- $\mathcal{F}(T) = \{M \in \text{ mod } A \ : \ \mathrm{Hom}_A(T, M) = 0\}$
- $\mathcal{X}(T) = \{N \in \text{ mod } B \ : \ T \otimes_B N = 0\}$
- $\mathcal{Y}(T) = \{N \in \text{ mod } B \ : \ \mathrm{Tor}^B_1(T, N) = 0\}$

The so-called Brenner Butler Theorem proven in [28] establishes that the functor $\mathrm{Hom}_A(T, -)$ is an equivalence between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$ while the functor $\mathrm{Ext}^1_A(T, -)$ is an equivalence between $\mathcal{F}(T)$ and $(\mathcal{X}(T)$.

A particular important case occurs when the tilting module T is splitting, that is, when the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ splits. Consequently, each indecomposable B-module lies either in $\mathcal{X}(T)$ or in $\mathcal{Y}(T)$. This is the case, for instance, when the algebra A is hereditary, that is, an algebra such that any submodule of a projective module is also projective. We observe that the ideal I in Theorem 1.1 is zero if and only if A is hereditary, that is, hereditary algebras are the path algebras kQ with Q a finite acyclic quiver.

Definition 1.5 *(Happel-Ringel* [58]*)* Let $A = kQ$, where Q is a finite acyclic quiver, be an (hereditary) algebra and T be a tilting A-module. Then $\mathrm{End}_A(T)$ is called a tilted algebra of type Q .

Finally, we say that a tilting A-module T is separating provided the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ splits.

2 Iterated tilted algebras and the Roldán's theorem

The Ph.D.'s work of Sonia Trepode [18, 67] was on the class of iterated tilted algebras, that is, algebras defined upon an iteration of the tilting process. To be more precise, an algebra A is called iterated tilted of type Q , where Q is a finite connected quiver without oriented cycles, provided there exists a sequence of algebras $A = A_0, A_1, \dots, A_m = kQ$ and a sequence of tilting modules $T^{(i)}_{A_i}$, with $0 \le i < m$, such that $\mathrm{End}\, T^{(i)}_{A_i} = A_{i+1}$ and each $T^{(i)}$ is separating. Observe that, in this case, if A_i is representation-finite, then so is A_j for $0 \le j \le i$.

In a private communication, O. Roldán conjectured that if Q is an euclidean quiver and A is a representation-finite iterated tilted algebra of type Q, then there exists a sequence as above but with A_{m-1} being representation-finite. If true, this result would reduce the study of representation-finite iterated tilted algebras of type Q to that of representation-finite tilted algebras of the same type. However, this conjecture does not stand as shown by an example in [18].

However, replacing the condition of the tilting modules to be separating by modules of type APR-tiltings and APR-cotiltings, the result will stand as stated below. This is the main result of Trepode's thesis.

Recall that an APR-tilting module is defined as follows. Suppose the algebra A has a simple projective module S which is not injective and denote by P the sum of the other indecomposable projective modules. The module $\tau^{-1}S \oplus P$, where τ is the Auslander–Reiten translate, is a tilting module and it is called APR-tilting. APR stands for Auslander, Platzeck and Reiten who in the paper [21] first considered this kind of construction. There, with other terminology, they showed in particular that APR-tilting modules are separating. Modules of type APR-cotilting modules are defined dually.

Recall also that a quiver is of euclidean type if it is of one of the following types: $\widetilde{\mathbb{A}}_n$ $(n \ge 1)$, $\widetilde{\mathbb{D}}_m$ $(m \ge 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$ (see [9]).

Theorem 2.1 [18, 67] *Let Q be a euclidean quiver without oriented cycles, and A be a representation-finite iterated tilted algebra of type Q. Then there exist a sequence of algebras $A = A_0, A_1, \dots, A_m$ and a sequence of modules $T^{(i)}_{A_i}$, where $0 \le i < m$, such that each $T^{(i)}$ is an APR-tilting A_i-module, or an APR-cotilting A_i -module, $\mathrm{End}\, T^{(i)}_{A_i} = A_{i+1}$ and A_m is a tilted representation-finite algebra of type Q.*

In her thesis, Trepode showed the above result for the case $\widetilde{\mathbb{A}}_n$ $(n \ge 1)$ using the classification of the tilted and iterated tilted algebras of such type, and a weaker version for the other cases as well. The proof provided in [18] for the general case does not use any classification result. It is also worth to mention that the above theorem is no longer true if Q is a wild quiver.

Sonia Trepode is a very successful argentinian mathematician in the *Universidad Nacional de Mar del Plata (UNMDP)* (Argentina) having published more than 50 papers. Our collaboration along the last twenty years includes joint works with I. Assem [13–16], C. Chaio [29–32] (on the composition of irreducible morphisms), with J. A. de la Peña [47–48] (on minimal and tilt-critical algebras) and others [2, 12].

3 Infinitely generated tilting modules

As mentioned in Sect. 1, the classical tilting theory inspired many investigations in more general settings. Our work with Clézio Braga [26, 27] deals with one of these situations: general ring and infinitely generated modules.

Let R be a ring with unity and denote by Mod$R$ the category of all R-modules (not necessarily finitely generated). We say that a module T in Mod$R$ is tilting provided:

(1) $\mathrm{pd}_R T < \infty$;
(2) $\mathrm{Ext}^i_R(T, T^{(I)}) = 0$ for each $i \geq 1$ and all set I; and
(3) there exists an exact sequence

$$0 \longrightarrow R \xrightarrow{f_0} T_0 \xrightarrow{f_1} \cdots \xrightarrow{f_r} T_r \longrightarrow 0$$

with $T_i \in \mathrm{Add}T$ for each $0 \leq i \leq r$.

For a module M, denote by $(M)^{\perp}$ the intersection $\cap_{i \geq 1} \mathrm{Ker}\, \mathrm{Ext}^i(M, \_)$. The main result of [27] is as follows.

Theorem 3.1 [27] *Let R be a ring and $\{T^i\}_{i\in\mathbb{N}}$ be a sequence of tilting modules such that $\mathrm{Add}T^i \neq \mathrm{Add}T^j$ if $i \neq j$, $T^{i+1} \in (T^i)^{\perp}$ and $\mathrm{pd}_R T^i \leq n$. Then there exists another sequence of tilting modules $\{\overline{T}^i\}_{i\in\mathbb{N}}$ with $\mathrm{Add}\overline{T}^i = \mathrm{Add}T^i$, $\overline{T}^{i+1} \in (\overline{T}^i)^{\perp}$ and $\mathrm{pd}_R \overline{T}^i \leq n$ for some $n \geq 1$. This latter sequence consists of a direct system of monomorphisms such that $T = \varinjlim_{i\in\mathbb{N}} \overline{T}^i$ is a tilting module in* Mod$R$ *and* $\mathrm{pd}_R T \leq n+1$.

In [26], we apply this result to the class of tilted algebras to construct infinitely generated tilting modules.

Clézio Braga works nowadays in the *Universidade Estadual do Oeste do Paraná (UNIOESTE)* and is presently involved in some problems on the representation dimension of an algebra (see below for a discussion on this homological invariant).

4 Shod and weakly shod algebras

The work with Marcelo Lanzilotta led us ultimately to the introduction and further study of several classes of algebras such as shod, laura, ada and so on (see below). Its origin goes back to the work by Happel-Reiten-Smalø [57] on quasitilted algebras. There, the authors considered a generalization of the concept of tilting objects in abelian categories. The endomorphism algebra of such an object is called quasitilted. Tilted algebras are, clearly, quasitilted but the converse is not true. We refer to [57] for such a construction. We are here more interested in a characterisation proved there and which is the starting point of our work.

In such a fundamental paper, Happel, Reiten and Smalø show that a quasitilted algebra A can be characterised by the properties: (QT1) $\text{gl.dim}A \leq 2$; and (QT2) for each indecomposable module M, either $\text{pd}_A M \leq 1$ or $\text{id}_A M \leq 1$. They have also shown that (QT1) and (QT2) are independent and that (QT2) implies that $\text{gl.dim}A \leq 3$.

Lanzilotta and I started, then, to consider the class of algebras satisfying property (QT2) and we call them shod algebras (for **s**mall homological dimension algebras). Following [40], where this notion was introduced, we published [41] exploring the existence of paths of irreducible morphisms from an injective to a projective for shod algebras, and [42] where we study a larger class called weakly shod algebras.

In order to recall the main results of such works, let us recall a fundamental result proven by Happel-Reiten-Smalø in [57]: the existence of a trisection in the category of a quasitilted algebra. For an algebra A, denote by $\mathcal{L}_A$ (respectively, $\mathcal{R}_A$) the subcategory of $\text{ind}A$ whose predecessors (respectively, successors) have projective dimension (respectively, injective dimension) at most one. If A is quasitilted, then $(\mathcal{L}_A \setminus \mathcal{R}_A) \cup (\mathcal{L}_A \cap \mathcal{R}_A) \cup (\mathcal{R}_A \setminus \mathcal{L}_A)$ gives a trisection in $\text{ind}A$. Observe also that if $\text{ind}A = \mathcal{L}_A \cup \mathcal{R}_A$, then clearly property (QT2) is true while the inverse is not so clear at first.

Before we state the main result of [40] we shall need another concept. Let $(*) : X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t$ be a path of irreducible morphisms in $\text{ind}A$. If $\tau_A X_{j+1} = X_{j-1}$ for some $1 \leq j \leq t-1$, then we say that $(*)$ has a hook at X_j. Sectional paths are paths of irreducible morphisms without hooks.

Theorem 4.1 [40] *The following are equivalent for an algebra A:*

(a) *A is shod, that is, for each indecomposable module M, either* $\text{pd}M \leq 1$ *or* $\text{id}M \leq 1$.
(b) $\text{ind}A = \mathcal{L}_A \cup \mathcal{R}_A$.
(c) $(\text{add}\,\mathcal{R}_A, \text{add}(\mathcal{L}_A \setminus \mathcal{R}_A))$ *is a torsion pair in* $\text{mod}A$.
(d) $(\text{add}\,(\mathcal{R}_A \setminus \mathcal{L}_A), \text{add}\,\mathcal{L}_A)$ *is a torsion pair in* $\text{mod}A$.
(e) *there exists a torsion pair $(\mathcal{X}, \mathcal{Y})$ in* $\text{mod}A$ *such that* $\text{pd}_A Y \leq 1$ *for each $Y \in \mathcal{Y}$ and* $\text{id}_A X \leq 1$ *for each $X \in \mathcal{X}$.*
(f) *any path from an indecomposable injective module to an indecomposable projective module can be refined to a path of irreducible morphisms and any such refinement has at most two hooks, and, in case there are two, they are consecutive.*

Observe that it follows from the last condition of this theorem that, for a shod algebra, the following condition is true: (WSA) there exists a positive integer n_0

such that any path of irreducible morphisms in ind$A$ from an injective to a projective has length bounded by n_0. This gives interesting informations on the Auslander–Reiten quiver of a shod algebra and was further explored in [41].

Next step in our considerations was to discuss the algebras which satisfy the condition (WSA) above. We shall refer to such an algebra as weakly shod algebra and describe them in [42]. Clearly, shod algebras are weakly shod. Without entering in more details, we should mention that a weakly shod algebra can be seen as an iteration of one point extensions (see below) from a tilted algebra and this allows us to completely describe their Auslander–Reiten quivers. We shall mention that, in parallel, Reiten-Skowronski also studied weakly shod algebras under the name double tilted algebras (see [65]).

As mentioned above, our work with Lanzilotta leads to the study of several classes of algebras defined using homological properties based on the ideas studied in [41–42]. I should mention here the laura algebras [11], the left and right glued algebras [10], the left and the right supported algebras [14] or ada algebras [8]. We refer to two surveys for further details, even though they are not completely updated: [12, 37].

Marcelo Lanzilotta presented his thesis at the *Universidad de La República*, Montevideo (Uruguay), where he is nowadays a well-established researcher. The thesis was co-supervised by Alfredo Jones, who also worked at IME-USP for many years. Some years ago, Lanzilotta noticed the potential of further exploring what he called the Igusa-Todorov functions (see [59]) and its connection with the (still open) *finitistic conjecture* (see below). He has been successfully working in this direction and related questions.

5 One-point extension of shod algebras

With my student Angela Marta P. D. Savioli, we studied two problems concerning shod algebras [43, 50]. In a first work, this also jointly with M. Lanzilotta, we looked at its Hochschild cohomology, while in the second we study the one-point extensions of shod algebras. Let us describe them.

The Hochschild cohomology groups $\mathrm{H}^i(A)$ of an algebra A gives important homological informations being, therefore, much studied in the area of Representation Theory of Algebras. In [43], we proved that if A is a strict weakly shod algebra (that is, a weakly shod algebra which is not quasitilted), then $\mathrm{H}^i(A) = 0$ for $i \geq 2$. The result cannot be extended to arbitrary weakly shod algebras since there are quasitilted algebras with the second Hochschild cohomology nonzero.

In order to describe the second problem, we shall need some notions. Recall now that, given an algebra B and a B-module M, one can construct the one-point extension of B by M as the matrix algebra

$$A = B[M] = \begin{pmatrix} B & 0 \\ M & k \end{pmatrix}$$

with its usual operations. A typical problem here is to look at informations which can be exchanged between A and B upon special properties on M. It is not difficult to see that if $A = B[M]$ is shod, then so is B. The problem we pose in [42] is the reverse. To give the flavour of the results we got, we shall mention two of them.

We say that an algebra is strict shod provided it is shod but not quasitilted. In this case, there exists a path of irreducible morphisms in ind$A$ from an injective to a projective having one hook. We then say that such a projective is end of a hook. Also, an indecomposable module M is called directing provided it does not lie in a path $M = M_0 \longrightarrow \cdots \longrightarrow M_t = M$ (with $t \geq 2$) of radical morphisms.

Theorem 5.1 [50] *Let B be a strict shod algebra and $M = \mathrm{rad}P$ where P is an indecomposable projective module end of a hook. Then*

(a) $\mathrm{Ext}^i(M, M) = 0$ *for* $i = 1, 2$.
(b) *If M contains an indecomposable no projective direct summand in $\mathcal{R}_A$, then $B[M]$ is strict shod if and only if M is a directing module lying in* $\mathrm{add}\mathcal{R}_A$ *and* $\tau M \in \mathrm{add}\mathcal{L}_A$.

Let $A = B[M]$ be a one-point extension of B by M. It is well-known that an A-module can be described as a triple (k^t, X, f), where X lie in mod$B$, and $f : k^t \otimes_k M \longrightarrow X$ is a B-homomorphism.

Theorem 5.2 [50] *Let B be a shod algebra and M be a projective B-module. Then $A = B[M]$ is shod if and only if for each indecomposable A-module (k^t, X, f) we have $X \in \mathrm{add}\mathcal{L}_B$ or $X \in \mathrm{add}\mathcal{R}_B$.*

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6 Derived categories and quasitilted algebras

The use of derived categories in representation theory goes back essentially to the work of Happel [56] and has been a powerful tool to understand the categories of algebras. We refer to such a work for basic results on derived and triangular categories.

With Cecilia Tosar Escuder, we studied the derived categories over quasitilted algebras [54]. We shall recall briefly the main results discussed there.

Let A be an algebra and let X^* be a complex in the bounded derived category $\mathrm{D}^b(A)$ of A. Define

$$J_{X^*} = \{i \in \mathbb{Z} : H^i(X^*) \neq 0\} \quad \text{and set} \quad l(X^*) = \max(J_{X^*}) - \min(J_{X^*}).$$

Rephrasing Happel's characterisation of hereditary algebra, one can say that A is hereditary if and only if $l(X^*) \leq 1$ for each complex X^*. Recall also that the finitistic conjecture (still open) states that the invariant

$$\text{fin.dim. } A \;=\; \sup \{ \, \text{pd}_A X \; : \; \text{pd}_A X < \infty \}$$

is always finite.

Theorem 6.1 [54] *Let A be an algebra. If there exists a positive integer n such that every indecomposable complex* X^* *in* $\mathrm{D}^b(A)$ *satisfies* $l(X^*) \leq n$, *then fin.dim.*$A \leq n$.

Recall that an algebra A is gentle provided $A \cong \frac{kQ}{I}$ where: (i) each vertex of Q is the source of at most two arrows and the target of at most two arrows; (ii) for each arrow α there exists at most one arrow β and at most one arrow γ such that $\alpha\beta \notin I$ and $\gamma\alpha \notin I$; (iii) for each arrow α there exists at most one arrow δ and at most one arrow η such that $\alpha\eta \in I$ and $\delta\alpha \in I$; and (iv) the ideal I is generated by a set of paths of length two.

Theorem 6.2 [54] *Let A be a gentle algebra. Then A is quasitilted if and only if each indecomposable complex* X^* *in* $\mathrm{D}^b(A)$ *satisfies* $l(X^*) \leq 2$.

For the next results we shall need the following definition. Let $\mathcal{C}$ be a full subcategory of ind$A$. We say that a complex is a $\mathcal{C}$ -complex provided it is isomorphic to a complex with modules in add$\mathcal{C}$. We are particularly interested in the subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$ defined above (see Sect. 4).

Theorem 6.3 [54] *Let A be an algebra and B its left support (that is, the endomorphism algebra of the sum of indecomposable projective modules in* $\mathcal{L}_A$*). Then the full triangulated subcategory* $\mathrm{D}^b(A)$ *determined by the* $\mathcal{L}_A$*-complexes is the triangulated category* $\mathrm{D}^b(B)$.

As a consequence, we get new characterizations of quasitilted algebras and quasitilted gentle algebras.

Theorem 6.4 [54] *The following statements are equivalent for an algebra A*:

(a) *A is quasitilted.*
(b) *each complex in* $\mathrm{D}^b(A)$ *is an* $\mathcal{L}_A$*-complex.*
(c) *each complex in* $\mathrm{D}^b(A)$ *is an* $\mathcal{R}_A$*-complex.*

Theorem 6.5 [54] *Let A be a gentle algebra. If each complex in* $\mathrm{D}^b(A)$ *is either an* $\mathcal{L}_A$*-complex or an* $\mathcal{R}_A$*-complex, then A is quasitilted.*

Cecilia Tosar has also published at the time an interesting work on string shod algebras jointly with J. Bélanger [25]. She is currently working in the

Departamento de Matemáticas del Instituto de Educación Secundaria Alonso de Madrigal (Ávila, Spain).

7 Pullbacks of algebras

The aim of our work with Heily Wagner was to study pullbacks of algebras taking into account some homological properties. During her Ph.D., she spent a year in the University of Sherbrooke under the supervision of Ibrahim Assem and this work was published in the papers [17, 24, 55].

A possible overall strategy in the representation theory is to transfer informations on some class of algebras which is reasonable well understood to others via some specific constructions. A good example already discussed above is the tilting process, another been the one-point of extension technique. Here, the construction is given via the pullback of algebras.

Pullbacks of algebras and rings have been studied from various points of view and we shall not extend ourself here into details. The interested reader will have no difficult to find works concerning them. However, there were few works using recent techniques of representation theory of algebras and this was our starting point.

Given two surjective homomorphisms of algebras $f : A \longrightarrow B$ and $g : C \longrightarrow B$, one can usually assigned a new algebra $R = \{(a, c) \in A \times C : f(a) = g(c)\}$ which is called the pullback of A and C (or, if it is needed to stress the morphisms, of f and g). The overall problem is then: which properties can be transfered from the initial algebras A, B and C to the new one R? Stating in this general way, the problem can be very hard to be solved and, so, it is important to find suitable restrictions on the original algebras in order to get nice results. Because of the description of our algebras in terms of quivers, a good strategy would be to impose conditions directly on them, which is, somehow, visually convenient.

Some results in this direction was proven by Igusa-Platzeck-Todorov-Zacharia [60] and Levesque [61] and the ideia was to further explore them. To mention one, Levesque proved, for instance, that if the pullback is of *Nakayama oriented type* (that is, the ordinary quiver of B is $\mathbb{A}_n$ linearly oriented) and if A, C are hereditary, then R is tilted. We generalised such a result relaxing the condition on B by allowing it to be of tree type and not only of type $\mathbb{A}_n$ [24].

In this direction, we also mention the following result. Let R is the Dynkin oriented pullback of $A \longrightarrow B$ and $C \longrightarrow B$ (that is, assume that B is of Dynkin type with a unique source). Then $\mathrm{ind}R = \mathrm{ind}A \cup \mathrm{ind}C$.

With such a description, it is possible to have a good control on the indecomposable R-modules in terms of those which are A-modules or C-modules. This, clearly, gives interesting informations on the relations between the Auslander–Reiten quivers of pullbacks and those of the initial algebras. Moreover, this reasoning induces a trisection in the category of $\mathrm{ind}R$ of the form $(\mathrm{ind}A \setminus \mathrm{ind}B,\ \mathrm{ind}B, \mathrm{ind}C \setminus \mathrm{ind}B)$. Without entering in much details, one can explore such a trisection to get homological informations on the algebras involved. For instance, using the already mentioned subcategories $\mathcal{R}_A, \mathcal{L}_A$ (Sect. 4 above), one can get sufficient conditions for R to be left or right glued algebras or even algebras of type laura.

Another problem dealt was on the representation dimension of an algebra. We refer to [38] for a brief account on such homological invariant but, philosophically, it should, in the saying of Auslander, *measure the complexity of a module category, in particular how far an algebra is from being representation finite* (see [20]). Representation-finite algebras are characterised by the fact that their representation dimension is two. On the other hand, many of the *good* classes of algebras have representation dimension at most three. By *good* we just mean well-understood. It is fair to mention its connection with the finitistic conjecture [59].

In fact, the results proven concerning representation dimension could be stated in a more general setting using again trisections on the category of the indecomposable modules and we shall refer to [17] for details. They, when stated in terms of pullbacks, implies that, for instance, ada algebras [1] have representation dimension at most three. Also, with some not so restrictive extra conditions, if R is the pullback of $A \longrightarrow B$ and $C \longrightarrow B$, then the representation dimension of R cannot excede those of A and C.

Heily Wagner is currently working in the *Universidade Federal do Paraná (UFPR)*. She has returned recently from a post-doctorate in the *Universidad Nacional de Mar del Plata Argentina*.

8 Translation quivers

We have seen above that the Auslander–Reiten quiver of an algebra A is a very nice way of organising the category mod$A$. From that quiver, it is possible to infer several patterns, and, in many case, they follow solely from its combinatorial properties. It is natural then to consider a more general type of quiver which, in the essence, preserves such combinatorics. This is the idea behind the notion of translation quivers.

A pair (Γ, τ) is said to be a translation quiver provided:

(a) Γ is a quiver without loops and locally finite, that is, each vertex of Γ_0 is the source or the target of at most finitely many arrows in Γ; and
(b) τ is a bijection $\tau : \Gamma_0' \longrightarrow \Gamma_0''$, where Γ_0' and Γ_0'' are subsets of Γ_0 such that for each $x \in \Gamma_0'$ and each $y \in x^- = \{y \in \Gamma_0 : \exists \text{ an arrow } y \longrightarrow x\}$, the number of arrows from y to x equals the number of arrows from τx to y.

Such a bijection τ is called a translation in Γ. A vertex x in $\Gamma_0 \setminus \Gamma_0'$ (respectively, in $\Gamma_0 \setminus \Gamma_0''$) will be called projective (respectively, injective) and we write $\tau x = 0$ (respectively, $\tau^{-1}x = 0$). The symbol 0 does not belong to Γ_0. A translation quiver (Γ, τ) is called proper provided for each $x \in \Gamma_0$ such that $\tau x \neq 0$, then x^- is non-empty.

A translation quiver (Γ, τ) is called regular provided $\tau^n x$ is defined for each $x \in \Gamma$ and each $n \in \mathbb{Z}$. Also, if Γ does not have both projectives and injectives, then we say that Γ is semiregular. So, a non-semiregular quiver contains both projectives and injectives. Given a locally finite quiver Δ without loops, one can assign a regular translation quiver $\mathbb{Z}\Delta$ as follows:

- The set of vertices of $\mathbb{Z}\Delta$ is $(\mathbb{Z}\Delta)_0 = \{(n,x) : n \in \mathbb{Z}, x \in \Delta\}$.
- For each arrow $x \longrightarrow y$ and each $n \in \mathbb{Z}$ one defines the following two arrows in $(\mathbb{Z}\Delta)_1$: $(n,x) \longrightarrow (n,y)$ and $(n-1,y) \longrightarrow (n,x)$.
- The translation in $\mathbb{Z}\Delta$ is $\rho : \mathbb{Z}\Delta_0 \longrightarrow \mathbb{Z}\Delta_0$ given by $\rho(n,x) = (n-1,x)$ for each $(n,x) \in \mathbb{Z}\Delta_0$. It is easy to check that ρ is indeed a translation in $\mathbb{Z}\Delta$.

In [62], Li considered the problem of embedding a component Γ of Γ_A into translations quivers of type $\mathbb{Z}\Delta$. His main result states that such an embedding exists if and only if any path in Γ from an injective to a projective is sectional. The starting point of our work with Edson R. Álvares [3, 4] was to seek possible generalisations of such nice results. A first and natural result was to consider translation quivers in general instead of components of Auslander–Reiten quivers. Also, it would be interesting to look at quivers with possible cycles.

Let (Γ, τ) be a translation quiver. A weak section of Γ is a full convex subquiver Δ such that for each $x \in \Gamma$ there exists exactly one integer i such that $\tau^i x \in \Delta_0$. We mention the main result of [3].

Theorem 8.1 [3] *Let (Γ, τ) be a proper and connected non-semiregular translation quiver. The following statements are equivalent:*

(a) Γ *has a weak section.*
(b) *Each path from an injective to a projective is sectional.*
(c) *There exist a quiver Δ, an $n \in \mathbb{Z}$ and a $\Delta[n]$-complete embedding $\varphi : \Gamma \longrightarrow \mathbb{Z}\Delta$*
.

In [4], other embeddings of components of Γ_A into translation quivers of type $\mathbb{Z}\Delta$ were considered for special kind of algebras such as shod algebras. Recall from Theorem 4.1 that for a shod algebra any path of irreducible morphisms in ind$A$ from an injective module to a projective has at most two hooks. Moreover, if the algebra is strict shod (that is, it is shod but not quasitilted), then it is also true that there does exist such a path with at least one hook. This lead us to the notion of double section as specified by Reiten-Skowroński in [64].

In [4], we investigate similar notions in translation quivers, that is, we define a double section and what we call the *hip-property* as follows.

Definition 8.2 Let (Γ, τ) be a proper and connected translation quiver.

1. A full subquiver Δ of Γ is called a double section of Γ provided:

   (a) Δ has no oriented cycles.
   (b) Δ is convex in Γ.
   (c) For each τ-orbit $\mathcal{O}$ in Γ, $1 \le |\Delta_0 \cap \mathcal{O}| \le 2$.
   (d) If $\mathcal{O}$ is a τ-orbit in Γ such that $|\Delta_0 \cap \mathcal{O}| = 2$, the $\Delta_0 \cap \mathcal{O} = \{x, \tau x\}$ for some $x \in \Gamma_0$ and there are sectional paths $i \rightsquigarrow \tau x$ and $x \rightsquigarrow p$ where i is an injective vertex and p is a projective vertex.

2. Suppose Γ is non-semiregular. We say that it satisfies the **hip-property** provided each path in Γ from an injective to a projective passes through at most two hooks, and, in case it passes through two, they are consecutives.

We mention the following result from [4].

Theorem 8.3 [4] *Let* (Γ, τ) *be a non-semiregular proper, connected translation quiver without oriented cycles and suppose it satisfies the hip-property. Then*

(a) Γ *contains a double section.*
(b) *There exists an embedding of translation quivers from* Γ *into* $\mathbb{Z}\Delta$ *for some quiver without oriented cycles* Δ.

As a consequence, for a strict shod algebra, one can embed its non-semiregular component in a a translation quiver of type $\mathbb{Z}\Delta$ for some directed quiver without oriented cycle Δ

We have two other works with Edson Ribeiro Álvares [2, 5], who is nowadays a researcher in the *Universidade Federal do Paraná (UFPR)*, where he helped to organise the internal algebra group. Also, he has been involved in collaborations with many other groups and working in related topics.

9 Mesh algebras

As we saw above, given a quiver Q we can assign an algebra kQ. Moreover, we saw that any algebra A is a quotient of an algebra of the type kQ by an admissible ideal (Theorem 1.1).

Suppose now we consider a finite directed translation quiver (Γ, τ). We want to look at the quotient of the path algebra $k\Gamma$ by an ideal generated by the relations induced by the translations. Let us be more specific. Observe that given a vertex $x \in \Gamma_0$ which is not projective, then $\tau x \neq 0$ and $x^- = \tau x^+$. If $[y, x] = \{\alpha_1, \dots, \alpha_r\}$ is a complete set of the arrows $y \longrightarrow x$, then the condition above allow us to define a bijection σ between the arrows in $[y, x]$ and the arrows $\tau x \longrightarrow y$. The element $\Sigma_{i=1}^{r} \alpha_i \sigma(\alpha_i)$ is called a mesh relation in Γ. If one considers the ideal I of $k\Gamma$ generated by all possible mesh relations (once fixed such bijections defined above), then the corresponding algebra $k\Gamma/I$ will be called a mesh algebra.

If one starts with a representation-finite algebra B, then its Auslander–Reiten quiver Γ_B is a finite translation quiver. This particular situation has been first studied by Auslander and, so they are called Auslander algebra (see [20] for the introduction of such algebras, but with other focus).

Our work with Rosana Signorelli Vargas [53] aimed to study the class of mesh algebras under some homological aspects. Particularly, we were interested in characterising when a mesh algebra is simply connected and strongly simply connected. We are not going to enter in much detail here because of the technicalities on these concepts, but we would like to comment some of our findings.

Firstly, for such a class of algebras, the concept of simply connected and strongly simply connected coincide. As a consequence, we also characterise the completely separated mesh algebras. Recall that an algebra is completely separated provided it is Schurian and strongly simply connected. A subclass of mesh algebras defined using the so-called postprojective partition (see below for a definition) is also studied.

Finally, we give necessary conditions for the first Hochschild cohomology of a mesh algebra to vanish, extending a result by Happel for Auslander algebras. Due to the technicalities involved, we shall refrain from entering in much more details referring to [53] for them.

Rosana S. Vargas is a researcher at the *Escola de Artes, Ciências e Humanidades da USP (EACH-USP)*. We would like to mention also her works [1, 8] on the class of ADA algebras, that is, algebras such that every indecomposable projective module and every indecomposable injective module lies in $\mathcal{L}_A$ or in $\mathcal{R}_A$ (see Sect. 4 above).

10 Partitions

The work done with Danilo Dias da Silva (and published in [51, 52, 66]) involves in some sense two kinds of organisation of the module category. The first one is related to the structure of the Auslander–Reiten quiver, but viewed via the study of irreducible morphisms and paths of irreducible morphisms, and the second is its connections with the so-called postprojective partitions which we shall discuss below. We briefly recall here the main results.

In [33], Chaio-LeMeur-Trepode introduced a kind of *quiver covering* and proved that if $h : Z \longrightarrow X$ has depth m (that is, it lies in $\mathrm{rad}_A^m \setminus \mathrm{rad}_A^{m+1}$) and if $f : X \longrightarrow Y$ (with X, Y indecomposable) is an irreducible morphism such that $fh \in \mathrm{rad}_A^{m+2}$, then there exists $h' : Z \longrightarrow X$ also with depth m and $fh' = 0$. This leads to useful consequences on the study of the components of the Auslander–Reiten quiver (see [33] for details). At the end, they pose the following question: is it possible to change h' by a path of irreducible morphisms between indecomposable modules from Z to X? Clearly, such a, let us say, *more concrete* description of h' gives a further understanding of the AR-quiver.

The first result of our work with Silva is a positive answer to this question (see [51]). It is worthwhile mentioning that it is even proven that the above path has length equal the depth of h.

Moreover, this is done in a more general context. We generalise the notion of left degree of an irreducible morphism relative to a subcategory $\mathcal{C}$ of ind$A$, which expand the possibilities for the understanding of the whole category. We shall not enter in the technicalities here and refer to [51] for details, but we shall stress some interesting consequences.

One of the ideas is to look at the degree relative to some special subcategory related to the postprojective partition. Recall that a partition $\mathbf{P_0}, \mathbf{P_1}, \dots, \mathbf{P}_\infty$ of ind$A$ (A an algebra) is called postprojective provided:

(1) $\mathrm{ind}A = \bigcup_{i \leq \infty} \mathbf{P_i}$.

(2) For each $i < \infty$, $\mathbf{P_i}$ is finite and a cover for $\bigcup_{j\geq i} \mathbf{P_j}$.

To say that $\mathbf{P_i}$ is a cover for $\bigcup_{j\geq i} \mathbf{P_j}$ means that for each $X \in \bigcup_{j\geq i} \mathbf{P_j}$, there exists $Y \in \mathrm{add}\mathbf{P_i}$ and an epimorphism $f : Y \longrightarrow X$. It should be clear that $\mathbf{P_0}$ is the set of indecomposable projective modules. The notion of postprojective partition and its dual preinjective was introduced by Auslander–Smalø in [23] and its connection with the Auslander–Reiten quivers was considered, for instance, in [36].

When we consider the category $\mathcal{C}$ equals to $\mathbf{P_0}$, the notion of relative left degree can be characterised in such a way that it does not rely on the concept of radical (see Theorem 4.5 of [51]).

We continue the discussion of postprojective modules (that is, the modules lying in $\mathrm{add}(\bigcup_{i<\infty} \mathbf{P_i})$) related to irreducible morphisms and their degrees in [52]. One problem which arises concerning such partitions is when there exists irreducible morphisms $X \longrightarrow Y$ with $X \in \mathbf{P_i}$ and $Y \in \mathbf{P_j}$ with $i + 1 < j < \infty$. We give some sufficient conditions for that to happen in a given connected component.

In [66], Silva provides a new, and shorter, proof of a characterisation of representation-finite algebras given in [33] adding to their list the fact that $(\mathrm{rad}_A^\infty)^2 = 0$. The fact that $(\mathrm{rad}_A^\infty)^2 = 0$ implies A representation-finite was first established in [45].

Danilo Dias da Silva is nowadays a researcher at the *Universidade Federal de Sergipe (UFS)* and he has working nowadays in some connection between representation theory and algebraic geometry after a post-doctorate in UNICAMP.

11 Generalized path algebras

As we have seen above, any algebra can be seen as the quotient of a path algebra (Theorem 1.1). In [44], in a joint work with S. X. Liu, we studied a generalization of such construction. Instead of assigning the base field to each vertex of a given quiver Q, we assigned an algebra. The multiplication for this case will be given not only by the concatenation of paths but also by the those of the involved algebras at the vertices. Our interest there was more of ring-theoretic nature and we then analised when a generalized path algebra was noetherian or prime.

However, it is clear that such a construction can be also very useful from the point of view of representation theory. With Viktor Chust [34, 35], we started to look at this and advanced in two directions which we will now comment. We shall avoid the technicalities here and refer to [34, 35, 44] for details, but we believe the definition of generalized path algebra should be clear, intuitively at least. One first point was to consider not only generalized path algebras but also their quotients, which we call generalized bound path algebras.

Any algebra A can be naturally realized as a generalized bound path algebra in two ways. For one hand, as the usual quotient of a path algebra, but also using a quiver with a sole vertex and no arrows and the algebra itself assigned to it. In fact, for most of the algebras these are the only possibilities. Our first aim was to describe the algebras which can be seen as a generalized bound path algebra in a different way from these two above (we call it a non-trivial simplification of A). This is important once one aims to look at properties of a given algebra from smaller ones.

We deal with this discussion in [34]. In the second paper [35], we looked at the representations of the generalized bound path algebras and studied it with respect to some homological properties.

Viktor Chust has recently started his Ph.D.'s studies at the IME-USP under my supervision.

12 History and mathematical education

Marcia Aguiar was my MSc. student and her dissertation was on the existence of multiplicative bases for triangular schurian algebras (2005). She obtained her Ph.D. in Education in the *Faculdade de Educação da Universidade de São Paulo (FE-USP)* in 2014 and, since 2015, she is a researcher at the *Universidade Federal do ABC (UFABC).*

During my sabbatical year at the *Instituto de Estudos Avançados da USP (IEA-USP)* in 2016, I worked in a project which was based in the History of Algebra and its connections with Algebra Education. At that moment, Aguiar and I wrote a paper which was published in the prestigious *Revista Estudos Avançados do IEA*, see [39]. We quote from the abstract of this paper:

The teaching of Algebra has been restricted to technical and operational issues, often leaving aside the development of concepts and so-called algebraic thinking. We believe that this approach underlies the deficiencies diagnosed in various government surveys and assessments. In this text, we present how concepts that were relevant to the development of Algebra over the centuries can and should participate in the process of teaching Algebra.

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