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Research Article

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Borsuk–Ulam theorem for filtered spaces

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Abstract: Let X and Y be pathwise connected and paracompact Hausdorff spaces equipped with free involutions $T : X \to X$ and $S : Y \to Y$, respectively. Suppose that there exists a sequence

$$(X_i, T_i) \xrightarrow{h_i} (X_{i+1}, T_{i+1}) \quad \text{for } 1 \leq i \leq k,$$

where, for each i, X_i is a pathwise connected and paracompact Hausdorff space equipped with a free involution T_i, such that $X_{k+1} = X$, and $h_i : X_i \to X_{i+1}$ is an equivariant map, for all $1 \leq i \leq k$. To achieve Borsuk–Ulam-type theorems, in several results that appear in the literature, the involved spaces X in the statements are assumed to be cohomological n-acyclic spaces. In this paper, by considering a more wide class of topological spaces X (which are not necessarily cohomological n-acyclic spaces), we prove that there is no equivariant map $f : (X, T) \to (Y, S)$ and we present some interesting examples to illustrate our results.

Keywords: Borsuk–Ulam theorems, involutions, equivariant maps, filtered spaces

MSC 2010: Primary 55M20; secondary 55M35

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1 Introduction

Let X be a topological space. An *involution* on X is a homeomorphism $T : X \to X$ which is its own inverse. A classical example is the antipodal map $A : S^n \to S^n$, $A(x) = -x$, for all $x \in S^n$, where S^n denotes the n-sphere.

Suppose that X and Y are topological spaces equipped with involutions $T : X \to X$ and $S : Y \to Y$, respectively. A continuous map $f : X \to Y$ is called an *equivariant* map if $S \circ f = f \circ T$. A pair (X, T) will be referred to as a topological space X equipped with an involution $T : X \to X$, which is free, that is, $T(x) \neq x$, for any $x \in X$.

One formulation of the Borsuk–Ulam Theorem [3] is that there is no map from S^m to S^n equivariant with respect to the antipodal map, when $m > n$ (see, for example, [1, Section 7.2]).

In [7], it was proved that if X and Y are pathwise connected and paracompact Hausdorff spaces with free involutions $T : X \to X$ and $S : Y \to Y$ such that for some natural $n \geq 1$, $\check{H}^r(X, \mathbb{Z}_2) = 0$, for $1 \leq r \leq n$, and $\check{H}^{n+1}(Y/S, \mathbb{Z}_2) = 0$, where Y/S is the orbit space of Y by S, then there is no equivariant map $f : (X, T) \to (Y, S)$.

In [6], the previous result is generalized for the relative case (X, A), where A is a T-invariant subset of X (see [6, Theorem 1.1]).

The aim of this paper is to generalize these results for filtered spaces, as follows.

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Theorem 1. *Let X and Y be pathwise connected and paracompact Hausdorff spaces equipped with free involutions $T : X \to X$ and $S : Y \to Y$, respectively. Let us suppose that there exists a sequence*

$$(X_i, T_i) \xrightarrow{h_i} (X_{i+1}, T_{i+1}) \quad \text{for } 1 \le i \le k,$$

where, for each i, X_i is a pathwise connected and paracompact Hausdorff space equipped with a free involution $T_i : X_i \to X_i$ such that $X_{k+1} = X$ and h_i is an equivariant map. Assume that for some sequence of natural numbers $n_0 = 0 < n_1 \le n_2 \le \cdots \le n_k$ the following hold:
(i) $\check{H}^r(X_i; \mathbb{Z}_2) = 0$, *for* $n_{i-1} < r < n_i$, $1 \le i \le k$,
(ii) $h_i^* : \check{H}^{n_i}(X_{i+1}; \mathbb{Z}_2) \to \check{H}^{n_i}(X_i; \mathbb{Z}_2)$ *is the null homomorphism, for* $1 \le i \le k$,
(iii) $\check{H}^{n_k+1}(Y/S; \mathbb{Z}_2) = 0$,
where $\check{H}$ denotes the Čech cohomology. Then there is no equivariant map $f : (X, T) \to (Y, S)$.

Let us consider the case that (X, T) is a filtered pair with filtration

$$(A_1, T_1) \subset (A_2, T_2) \subset \cdots \subset (A_k, T_k) = (X, T),$$

that is, X is a filtered space with filtration

$$A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_k = X,$$

where each element A_i of the filtration is a pathwise connected and paracompact Hausdorff space and each involution $T_i : A_i \to A_i$ is the restriction of T to A_i.

In this particular case, in which h_i is the inclusion map, for each $1 \le i \le k$, one has the next result, that follows from Theorem 1.

Corollary 1. *Let X and Y be pathwise connected and paracompact Hausdorff spaces, equipped with free involutions $T : X \to X$ and $S : Y \to Y$, respectively. Let us consider X a filtered space with filtration*

$$A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_k = X,$$

where each element of the filtration is a pathwise connected and paracompact Hausdorff space. Suppose that for some sequence of natural numbers $n_0 = 0 < n_1 \le n_2 \le \cdots \le n_k$ the following hold:
(i) $\check{H}^r(A_i; \mathbb{Z}_2) = 0$, *for* $n_{i-1} < r < n_i$, $1 \le i \le k$,
(ii) $j_i^* : \check{H}^{n_i}(A_{i+1}; \mathbb{Z}_2) \to \check{H}^{n_i}(A_i; \mathbb{Z}_2)$ *is the null homomorphism, for* $1 \le i \le k$, *where* $j_i : A_i \hookrightarrow A_{i+1}$ *is the natural inclusion,*
(iii) $\check{H}^{n_k+1}(Y/S; \mathbb{Z}_2) = 0$,
where $\check{H}$ denotes the Čech cohomology. Then there is no an equivariant map $f : (X, T) \to (Y, S)$.

2 Preliminaries

We start by introducing some basic notions and notations. We assume that all spaces under consideration are pathwise connected and paracompact Hausdorff spaces. The symbol $\cong$ denotes the appropriate isomorphism between algebraic objects. Throughout this paper, $\check{H}$ will always denote the Čech cohomology with $\mathbb{Z}_2$ coefficients, unless otherwise indicated.

If G is a compact Lie group which acts freely on a paracompact Hausdorff space X, then $X \to X/G$ is a principal G-bundle [4, Chapter II, Theorem 5.8]. Let $G \hookrightarrow EG \hookrightarrow BG$ be the universal G-bundle, where BG is the classifying space of the group G. Then we can take a classifying map $X/G \to BG$ for the principal G-bundle $X \to X/G$. The group G acts diagonally on the space $X \times EG$ with orbit space $X_G = (X \times EG)/G$. The projection $q : X \times EG \to EG$ is a G-equivariant map, that is, $q(ax) = aq(x)$, for all $a \in G$ and for all $x \in X$, and gives a fibration $X \hookrightarrow X_G \to BG$. This construction is originally due to Borel [2, Chapter IV]. The case of main interest for us in this work is $G = \mathbb{Z}_2$. We recall that $B\mathbb{Z}_2 = \mathbb{R}P^\infty$. Let us observe that a free involution $T : X \to X$ determines a $\mathbb{Z}_2$ free action on X.

Let $B\mathbb{Z}_2$ be the classifying space for $\mathbb{Z}_2$ and denote by $\alpha \in \check{H}^1(B\mathbb{Z}_2)$ the Euler class of the universal principal $\mathbb{Z}_2$-bundle over $B\mathbb{Z}_2$. Since X is a paracompact Hausdorff space, one can take a classifying map $g : X/T \to B\mathbb{Z}_2$ for the principal $\mathbb{Z}_2$-bundle $X \to X/T$. From $g^* : \check{H}^1(B\mathbb{Z}_2) \to \check{H}^1(X/T)$ one gets the Euler class

$$e_X = g^*(\alpha) \in \check{H}^1(X/T) \tag{2.1}$$

of $X \to X/T$.

Moreover, from [4, Section 3.10 (Sequence 10.5 of p. 161)], one has the Smith–Gysin exact sequence

$$\begin{array}{c} 0 \longrightarrow \check{H}^0(X/T) \xrightarrow{p^*} \check{H}^0(X) \xrightarrow{\tau} \check{H}^0(X/T) \xrightarrow{\smile e_X} \check{H}^1(X/T) \longrightarrow \cdots \\ \cdots \longrightarrow \check{H}^r(X/T) \xrightarrow{p^*} \check{H}^r(X) \xrightarrow{\tau} \check{H}^r(X/T) \xrightarrow{\smile e_X} \check{H}^{r+1}(X/T) \longrightarrow \cdots, \end{array}$$

where $p : X \to X/T$ is the projection, $\tau : \check{H}^r(X) \to \check{H}^r(X/T)$ is the transfer homomorphism and $\smile_{e_X}$ is the cup product with the Euler classes given in (2.1).

We will use the following well-known lemma to prove the main result.

Lemma 1. *Let us suppose that X and Y are pathwise connected and paracompact Hausdorff spaces, equipped with free involutions $T : X \to X$ and $S : Y \to Y$, respectively. Let $e_X \in \check{H}^1(X/T)$ and $e_Y \in \check{H}^1(Y/S)$ be the Euler classes of the principal $\mathbb{Z}_2$-bundles $X \to X/T$ and $Y \to Y/S$, respectively. If for some natural number $n \geq 1$, we have $e_X^{n+1} \neq 0$ and $e_Y^{n+1} = 0$, then there is no an equivariant map $f : (X, T) \to (Y, S)$.*

Now, for a pair (X, T), let us suppose that there exists a sequence

$$(X_i, T_i) \xrightarrow{h_i} (X_{i+1}, T_{i+1}) \quad \text{for } 1 \leq i \leq k,$$

where, for each i, X_i is a pathwise connected and paracompact Hausdorff space equipped with a free involution $T_i : X_i \to X_i$ such that $X_{k+1} = X$ and $h_i : X_i \to X_{i+1}$ is an equivariant map, for each $1 \leq i \leq k$.

Let us consider the following diagram, where all the horizontal sequences are Smith–Gysin sequences of the spaces under consideration and all vertical morphisms are induced by the maps $h_i : X_i \to X_{i+1}$. For simplicity of notation, in Diagram 1, $\smile_e$ will denote the cup product with the Euler classes of the spaces under discussion. Moreover, the spaces X_i/T_i will be denoted by X_i^*.

Diagram 1. We have

$$\begin{array}{ccccccccccccccccc}
0 \to & H^0(X^*) & \xrightarrow{p^*} & H^0(X) & \xrightarrow{\tau} & H^0(X^*) & \xrightarrow{\smile_e} & H^1(X^*) & \to \cdots \to & H^r(X^*) & \xrightarrow{p^*} & H^r(X) & \xrightarrow{\tau} & H^r(X^*) & \xrightarrow{\smile_e} & H^{r+1}(X^*) & \to \cdots \\
& \downarrow \bar{h}_k^* & & \downarrow h_k^* & & \downarrow \bar{h}_k^* & & \downarrow \bar{h}_k^* & & \downarrow \bar{h}_k^* & & \downarrow h_k^* & & \downarrow \bar{h}_k^* & & \downarrow \bar{h}_k^* & \\
0 \to & H^0(X_k^*) & \xrightarrow{p^*} & H^0(X_k) & \xrightarrow{\tau} & H^0(X_k^*) & \xrightarrow{\smile_e} & H^1(X_k^*) & \to \cdots \to & H^r(X_k^*) & \xrightarrow{p^*} & H^r(X_k) & \xrightarrow{\tau} & H^r(X_k^*) & \xrightarrow{\smile_e} & H^{r+1}(X_k^*) & \to \cdots \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 \to & H^0(X_{j+1}^*) & \xrightarrow{p^*} & H^0(X_{j+1}) & \xrightarrow{\tau} & H^0(X_{j+1}^*) & \xrightarrow{\smile_e} & H^1(X_{j+1}^*) & \to \cdots \to & H^r(X_{j+1}^*) & \xrightarrow{p^*} & H^r(X_{j+1}) & \xrightarrow{\tau} & H^r(X_{j+1}^*) & \xrightarrow{\smile_e} & H^{r+1}(X_{j+1}^*) & \to \cdots \\
& \downarrow \bar{h}_j^* & & \downarrow h_j^* & & \downarrow \bar{h}_j^* & & \downarrow \bar{h}_j^* & & \downarrow \bar{h}_j^* & & \downarrow h_j^* & & \downarrow \bar{h}_j^* & & \downarrow \bar{h}_j^* & \\
0 \to & H^0(X_j^*) & \xrightarrow{p^*} & H^0(X_j) & \xrightarrow{\tau} & H^0(X_j^*) & \xrightarrow{\smile_e} & H^1(X_j^*) & \to \cdots \to & H^r(X_j^*) & \xrightarrow{p^*} & H^r(X_j) & \xrightarrow{\tau} & H^r(X_j^*) & \xrightarrow{\smile_e} & H^{r+1}(X_j^*) & \to \cdots \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 \to & H^0(X_2^*) & \xrightarrow{p^*} & H^0(X_2) & \xrightarrow{\tau} & H^0(X_2^*) & \xrightarrow{\smile_e} & H^1(X_2^*) & \to \cdots \to & H^r(X_2^*) & \xrightarrow{p^*} & H^r(X_2) & \xrightarrow{\tau} & H^r(X_2^*) & \xrightarrow{\smile_e} & H^{r+1}(X_2^*) & \to \cdots \\
& \downarrow \bar{h}_1^* & & \downarrow h_1^* & & \downarrow \bar{h}_1^* & & \downarrow \bar{h}_1^* & & \downarrow \bar{h}_1^* & & \downarrow h_1^* & & \downarrow \bar{h}_1^* & & \downarrow \bar{h}_1^* & \\
0 \to & H^0(X_1^*) & \xrightarrow{p^*} & H^0(X_1) & \xrightarrow{\tau} & H^0(X_1^*) & \xrightarrow{\smile_e} & H^1(X_1^*) & \to \cdots \to & H^r(X_1^*) & \xrightarrow{p^*} & H^r(X_1) & \xrightarrow{\tau} & H^r(X_1^*) & \xrightarrow{\smile_e} & H^{r+1}(X_1^*) & \to \cdots.
\end{array}$$

3 Proof of the main theorem and examples

3.1 Proof of Theorem 1

The idea is to show that $e_X^{n_k+1} \neq 0$, by using Diagram 1, and then apply Lemma 1. An inductive construction will be done, as follows.

Firstly, considering only the space X_1, we have by hypothesis that $\check{H}^r(X_1) = 0$, for all $1 \leq r < n_1$, for some $n_1 \in \mathbb{N}$. To show that $e_{X_1}^{n_1} \neq 0$, let us first consider the last row of Diagram 1:

$$\begin{aligned} 0 \longrightarrow \check{H}^0(X_1/T_1) \xrightarrow{p^*} \check{H}^0(X_1) \xrightarrow{\tau_1} \check{H}^0(X_1/T_1) \xrightarrow{\smile e_{X_1}} \check{H}^1(X_1/T_1) \longrightarrow \cdots \\ \cdots \longrightarrow \check{H}^r(X_1) \xrightarrow{\tau_1} \check{H}^r(X_1/T_1) \xrightarrow{\smile e_{X_1}} \check{H}^{r+1}(X_1/T_1) \longrightarrow \cdots . \end{aligned} \tag{3.1}$$

By the connectivity, $p^* : \check{H}^0(X_1/T_1) \to \check{H}^0(X_1)$ is an isomorphism, and therefore the transfer homomorphism $\tau_1 : \check{H}^0(X_1) \to \check{H}^0(X_1/T_1)$ is the null homomorphism. Then $\smile e_{X_1}$ is a monomorphism.

Therefore, $e_{X_1} = 1 \smile e_{X_1} \neq 0$.

For each r, $1 \leq r < n_1$, we have by hypothesis that $\check{H}^r(X_1) = 0$ and since the rows in Diagram 1 are exact, this implies that

$$\smile e_{X_1} : \check{H}^r(X_1/T_1) \to \check{H}^{r+1}(X_1/T_1)$$

is injective and thus $e_{X_1}^{r+1} = e_{X_1}^r \smile e_{X_1} \neq 0$. In particular,

$$e_{X_1}^{n_1} = e_{X_1}^{n_1-1} \smile e_{X_1} \neq 0.$$

Now, we will show that $e_{X_2}^{n_1+1} \neq 0$. For this, let us consider the principal $\mathbb{Z}_2$-bundles

$$X_1 \to X_1/T_1 \quad \text{and} \quad X_2 \to X_2/T_2.$$

Denoting by

$$\overline{h}_1 : X_1/T_1 \to X_2/T_2,$$

the map induced by $h_1 : X_1 \to X_2$ and considering

$$g_2 : X_2/T_2 \to B\mathbb{Z}_2,$$

a classifying map for the principal $\mathbb{Z}_2$-bundle $X_2 \to X_2/T_2$, we have the following diagram:

$$\begin{array}{ccccc} X_1 & \xrightarrow{h_1} & X_2 & & \\ \downarrow & & \downarrow & & \\ X_1/T_1 & \xrightarrow{\overline{h}_1} & X_2/T_2 & \xrightarrow{g_2} & B\mathbb{Z}_2. \end{array} \tag{3.2}$$

Since h_1 is an equivariant map, the square in (3.2) is commutative, and therefore $g_2 \circ \overline{h}_1$ is a classifying map for the principal $\mathbb{Z}_2$-bundle $X_1 \to X_1/T_1$. Let $\alpha \in \check{H}^1(B\mathbb{Z}_2)$ be the Euler class of the universal $\mathbb{Z}_2$-bundle $E\mathbb{Z}_2 \to B\mathbb{Z}_2$. Then

$$e_{X_2} = g_2^*(\alpha) \quad \text{and} \quad e_{X_1} = \overline{h}_1^*(g_2^*(\alpha)) = \overline{h}_1^*(e_{X_2}).$$

Therefore, we have $\overline{h}_1^*(e_{X_2}^{n_1}) = e_{X_1}^{n_1} \neq 0$, that is, $e_{X_2}^{n_1}$ cannot be zero.

To show that $e_{X_2}^{n_1+1} \neq 0$, let us consider the part of the last two rows of Diagram 1, as follows:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^{n_1}(X_2) & \xrightarrow{\tau_2} & H^{n_1}(X_2/T_2) & \xrightarrow{\smile e_{X_2}} & H^{n_1+1}(X_2/T_2) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle h_1^*} & & \downarrow{\scriptstyle \overline{h}_1^*} & & \downarrow{\scriptstyle \overline{h}_1^*} & & \\ \cdots & \longrightarrow & H^{n_1}(X_1) & \xrightarrow{\tau_1} & H^{n_1}(X_1/T_1) & \xrightarrow{\smile e_{X_1}} & H^{n_1+1}(X_1/T_1) & \longrightarrow & \cdots. \end{array}$$

If $e_{X_2}^{n_1+1} = 0$, then $e_{X_2}^{n_1} \in \operatorname{Ker}(\smile_{e_{X_2}})$ and by exactness, $e_{X_2}^{n_1} = \tau_2(\beta_{n_1})$, for some $\beta_{n_1} \in H^{n_1}(X_2)$. But by hypothesis, $h_1^* : H^{n_1}(X_2) \to H^{n_1}(X_1)$ is the null homomorphism, consequently,

$$0 = \tau_1 \circ h_1^*(\beta_{n_1}) = \overline{h}_1^*(\tau_2(\beta_{n_1})) = \overline{h}_1^*(e_{X_2}^{n_1}) = e_{X_1}^{n_1} \neq 0.$$

This implies that $e_{X_2}^{n_1+1} \neq 0$, as we wanted to show.

Now let us suppose that $e_{X_s}^{n_{s-1}+1} \neq 0$, for some $1 \leq s < k$. To prove that $e_{X_{s+1}}^{n_s+1} \neq 0$, firstly we will show that $e_{X_s}^{n_s} \neq 0$.

Consider the following row of Diagram 1 corresponding to X_s:

$$\cdots \longrightarrow \check{H}^r(X_s) \xrightarrow{\tau_s} \check{H}^r(X_s/T_s) \xrightarrow{\smile e_{X_s}} \check{H}^{r+1}(X_s/T_s) \longrightarrow \cdots .$$

By hypothesis, $\check{H}^r(X_s) = 0$, for $n_s + 1 \le r < n_{s+1}$. Thus $\smile_{e_{X_s}}$ is injective, for $n_s + 1 \le r < n_{s+1}$ and then $e_{X_s}^{n_s} \neq 0$.

Let us consider $X_s \to X_s/T_s$ and $X_{s+1} \to X_{s+1}/T_{s+1}$ the principal $\mathbb{Z}_2$-bundles, with the induced map $\overline{h}_s : X_s/T_s \to X_{s+1}/T_{s+1}$. Let us take $g_{s+1} : X_{s+1}/T_{s+1} \to B\mathbb{Z}_2$ a classifying map for the principal $\mathbb{Z}_2$-bundle $X_{s+1} \to X_{s+1}/T_{s+1}$, illustrated in the following commutative diagram:

$$\begin{array}{ccccc} X_s & \xrightarrow{h_s} & X_{s+1} & & \\ \downarrow & & \downarrow & & \\ X_s/T_s & \xrightarrow{\overline{h}_s} & X_{s+1}/T_{s+1} & \xrightarrow{g_{s+1}} & B\mathbb{Z}_2. \end{array}$$

Then the map $g_{s+1} \circ \overline{h}_s : X_s/T_s \to B\mathbb{Z}_2$ is a classifying map for the principal $\mathbb{Z}_2$-bundle $X_s \to X_s/T_s$.

For $\alpha \in \check{H}^1(B\mathbb{Z}_2)$ the Euler class of the universal $\mathbb{Z}_2$-bundle, we have

$$e_{X_{s+1}} = g_{s+1}^*(\alpha) \quad \text{and} \quad e_{X_s} = \overline{h}_s^*(g_{s+1}^*(\alpha)) = \overline{h}_s^*(e_{X_{s+1}}).$$

Thus, $\overline{h}_s^*(e_{X_{s+1}}^{n_s}) = e_{X_s}^{n_s} \neq 0$, which implies $e_{X_{s+1}}^{n_s} \neq 0$.

Now, let us consider the part of Diagram 1 related to X_s and X_{s+1}, respectively:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^{n_s}(X_{s+1}) & \xrightarrow{\tau_{s+1}} & H^{n_s}(X_{s+1}/T_{s+1}) & \xrightarrow{\smile e_{X_{s+1}}} & H^{n_s+1}(X_{s+1}/T_{s+1}) & \longrightarrow & \cdots \\ & & \downarrow h_s^* & & \downarrow \overline{h}_s^* & & \downarrow \overline{h}_s^* & & \\ \cdots & \longrightarrow & H^{n_s}(X_s) & \xrightarrow{\tau_s} & H^{n_s}(X_s/T_s) & \xrightarrow{\smile e_{X_s}} & H^{n_s+1}(X_s/T_s) & \longrightarrow & \cdots. \end{array}$$

Suppose that $e_{X_{s+1}}^{n_s+1} = 0$. Then $e_{X_{s+1}}^{n_s} \in \mathrm{Ker}(\smile_{e_{X_{s+1}}})$. As previously, $e_{X_{s+1}}^{n_s} = \tau_{s+1}(\beta_{n_s})$, for some $\beta_{n_s} \in H^{n_s}(X_{s+1})$. But since by hypothesis $h_s^* : H^{n_s}(X_{s+1}) \to H^{n_s}(X_s)$ is the null homomorphism, we have

$$0 = \tau_s \circ h_s^*(\beta_{n_s}) = \overline{h}_s^*(\tau_{s+1}(\beta_{n_s})) = \overline{h}_s^*(e_{X_{s+1}}^{n_s}) = e_{X_s}^{n_s} \neq 0.$$

Consequently, $e_{X_{s+1}}^{n_s+1} \neq 0$. This concludes the inductive construction.

Finally, to finish the proof, since $e_X^{n_k+1} \neq 0$ and by hypothesis $e_Y^{n_k+1} = 0$, it follows from Lemma 1 that there is no equivariant map $f : (X, T) \to (Y, S)$.

3.2 Examples

In this subsection, we will present examples which show the relevance of our results. First, in Example 1, we will construct spaces equipped with involutions and a map $(X_1, T_1) \xrightarrow{h} (X, T)$ which satisfy conditions (i) and (ii) of Theorem 1, while the image $h(X_1) \subset X$ does not satisfy conditions (i) and (ii) of Corollary 1.

In Example 2, we will show that Theorem 1 (Corollary 1) can be applied in the context of manifolds. By using cobordism theory, we will construct a pair (M, T), in which M is a manifold, $T : M \to M$ is a free involution, with a natural filtration for the pair (M, T), satisfying all the hypotheses of Theorem 1 (Corollary 1). We emphasize that for such class of manifolds, the classical results about Borsuk–Ulam Theorems cannot be applied, since in general, such results are valid for n-acyclic spaces, that is, spaces with zero cohomology at r-levels, with $1 \le r \le n$.

Example 1. Given Z a topological space and $A \subset Z$ a closed subspace equipped with free involution $\phi : A \to A$, firstly we will construct a topological space X with a free involution ψ such that $A \subset X$ and the involution ψ restricted to A is the given one.

Let $X_0 = Z \times \{0\}$ and $X_1 = Z \times \{1\}$ be two copies of Z. Let $X = (X_0 \sqcup X_1)/\sim$ be the identification space obtained by identifying (x, i) with $(\phi(x), 1 - i)$, $i = 0, 1$, whenever $x \in A$.

Let $\psi : X \to X$ be the map induced by $[(x, i)] \mapsto [(x, 1 - i)]$, $i = 0, 1$. Note that the map ψ is a well-defined free involution, which coincides with ϕ in the copy of A in X, that is,

$$\psi[(x, i)] = [(x, 1 - i)] = [(\phi(x), 1 - (1 - i))] = [(\phi(x), i)].$$

This construction gives us a topological space X equipped with a free involution ψ, where $A \subset X$ and $\psi|_A = \phi$.

Now, we will construct a pathwise connected and paracompact Hausdorff space X equipped with a free involution $T : X \to X$ and a map $h : (X_1, T_1) \to (X, T)$, where (X_1, T_1) is a pair constituted by a pathwise connected and paracompact Hausdorff space X_1 and by a free involution T_1 on X_1 in such a way that $(X_1, T_1) \xrightarrow{h} (X, T)$ satisfies hypotheses (i) and (ii) of Theorem 1.

Let m and n be positive integers, with $m < n$, and let X be the connected sum of two copies of $S^{m+1} \times S^{n-m}$.

To define an involution ψ on X, we will consider a specific construction to X. Let Z be the resultant space after removing an open disk from $S^{m+1} \times S^{n-m}$ and let S be the boundary of the disk. In S, one can define an involution $\phi : S \to S$ induced by the antipodal map on the sphere.

Applying the previous construction by considering Z as a copy of $S^{m+1} \times S^{n-m}$ with an open disk removed, $A = S$ and $\phi = T$, the result of this construction is a pair (X', ψ), in which X' is a topological space (equipped with a free involution ψ) which is a connected sum of two copies of $S^{m+1} \times S^{n-m}$ and such that X' is homeomorphic to X.

Now, we shall define the desired map h.

Let us consider $(X_1, T_1) := (S^n, A)$, where S^n is the n-dimensional sphere and A is the antipodal map. Let E be the space obtained from S^n contracting the boundaries of two symmetrical and non-connected disks, homeomorphic to the wedge of three spheres $S^n_1 \vee S^n_2 \vee S^n_3$, with wedge points in the North and South poles of S^n_2. Let D be the space obtained from E identifying two opposites hemispheres in the subspaces corresponding to S^n_1 and S^n_3, in such a way that the wedge points are preserved. This means that D is homeomorphic to the wedge $D^n_1 \vee S^n_2 \vee D^n_3$, where D^n_1 and D^n_3 are n-dimensional disks, the wedge points are the South and North poles and they are on the boundaries of the disks, as illustrated in Figure 1.

The next step is to define an equivariant map $f_1 : D \to Y$ in which the restriction $f_1 : S^n_2 \to S$ is an equivariant homeomorphism between S^n_2 and $S \subset X$, and $f_1(D^n_1)$ is a copy of $S^{m+1} \times \{P\}$. Moreover, f_1 maps D^n_3 to the symmetrical opposite $(f_1(D^n_3) = \psi(f_1(D^n_1)))$.

A way to define a map f_1 satisfying the above condition over D^n_1 and D^n_3 is to take an intermediary step, contracting the disks D^n_1 and D^n_3 into an $(m + 1)$-dimensional disks D^{m+1}_1 and D^{m+1}_3, respectively, and then identifying the respective boundaries. Then we obtain an intermediary space W which is homeomorphic to $S^{m+1}_1 \vee S^n_2 \vee S^{m+1}_3$.

Now, we can send S^{m+1}_1 to $S^{m+1} \times \{P\}$ and S^{m+1}_2 to $\psi(S^{m+1} \times \{P\})$ in straightforward way by a map f_2. The composite map of f_2 with the projection is the desired map f_1 .

The composite of the projection $S^n \to D$ with f_1 is the desired equivariant map $h : (S^n, A) \to (X, \psi)$.

Let us observe that since the disks D^n_1 and D^n_3 are contractible, $h : (S^n, A) \to (X, \psi)$ is homotopic to a map $g : S^n \to X$ with image $g(S^n) = S$. This means that the induced map on cohomology $h^* : \check{H}^j(X) \to \check{H}^j(S^n)$ is the null homomorphism, for all $j > 0$.

Therefore, the pairs (S^n, A) and (X, ψ), along with the equivariant map $h : (S^n, A) \to (X, \psi)$, constitute an example of a sequence

$$(X_i, T_i) \xrightarrow{h_i} (X_{i+1}, T_{i+1})$$

satisfying conditions (i) and (ii) of the Theorem 1. However, the pair $(h(S^n), \psi)$ does not.

Theorem 1 holds for $S^n \xrightarrow{h} X$ on the previous example and for any space Y equipped with a free involution $S : Y \to Y$ satisfying condition (iii) of Theorem 1. For example, $(Y, S) = (S^n, A)$.

Example 2. First, we will consider the following general case.

Let A be a pathwise connected and paracompact Hausdorff space, equipped with a free involution $T : A \to A$. If A is a closed subspace of a topological space Z, one can create a new topological space X with a free involution $\phi : X \to X$, with $A \subset X$ and such that the restriction $\phi|_A = T$ in the following way.

Let us consider the disjoint union $Z \times \{0\} \sqcup Z \times \{1\}$ of two copies of Z. The new topological space X will be the quotient space obtained by identifying $(x, 1)$ with $(T(x), 0)$, whenever $x \in A$.

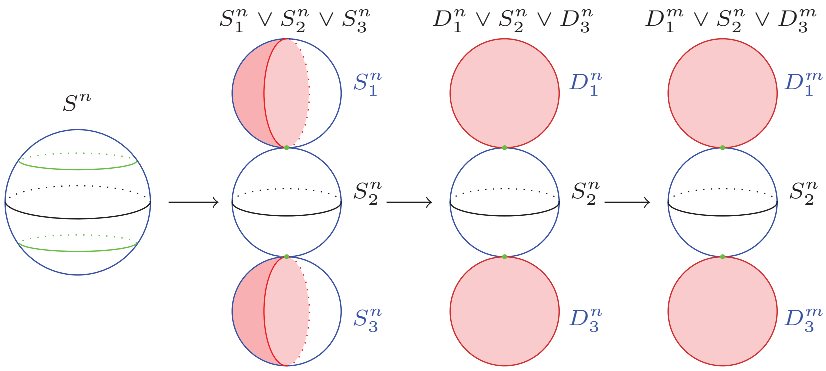

Figure 1

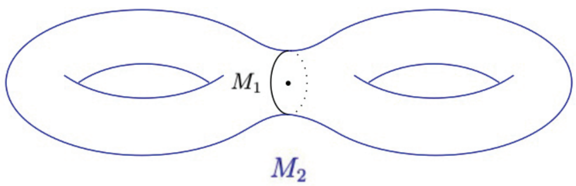

Figure 2

Let $\phi : X \to X$ be the induced map given by $[(x, i)] \mapsto [(x, 1 - i)]$. Note that ϕ is a well-defined involution which coincides with T on A, since $\phi[(x, 0)] = [(x, 1)] = [(T(x), 0)]$.

Now, let us consider the previous situation in the context of manifolds with free involutions.

Let $A = M_1$ be a closed manifold which admits a free involution T_1. By Conner and Floyd [5], we have that M_1 bounds, i.e., there exists a manifold $Z = W_2$ such that $\partial W_2 = M_1$ and $\dim W_2 = \dim M_1 + 1$. Therefore, $X = M_2 = 2W_2$ the double of M_1 is a closed manifold which admits a free involution T_2. If we replay this step n-times, we obtain the sequence

$$(M_1, T_1) \subset (M_2, T_2) \subset (M_3, T_3) \subset \cdots \subset (M = M_n, T_n),$$

with $\dim M_n = \dim M_1 + n - 1$. See Figure 2.

Since each M_i is the boundary of W_i, the inclusion $M_i \hookrightarrow M_{i+1}$ induces the null homomorphism for all nonzero dimensions. Hence

$$(M_1, T_1) \subset (M_2, T_2) \subset (M_3, T_3) \subset \cdots \subset (M_n, T_n)$$

satisfies conditions (i) and (ii) of Theorem 1.

As a specific example, let W_2 be the torus with an open disk removed and $\partial W_2 = M_1$ identified with the circle S^1. Also, let $T_1 : M_1 \to M_1$ be the antipodal map. Applying the previous construction, we obtain the bitorus M_2 equipped with the free involution T_2.

The bitorus is a 2-dimensional closed manifold. Thus, there exists a 3-dimensional compact manifold W_3 such that $\partial W_3 = M_2$. Repeating the construction for W_3 and (M_2, T_2), we obtain (M_3, T_3). If we replay this step n-times, we obtain the sequence

$$(M_1, T_1) \subset (M_2, T_2) \subset (M_3, T_3) \subset \cdots \subset (M_n, T_n).$$

Since each M_i is the boundary of W_i, the inclusion $M_i \hookrightarrow M_{i+1}$ induces the null homomorphism for all nonzero dimensions. Hence

$$(M_1, T_1) \subset (M_2, T_2) \subset (M_3, T_3) \subset \cdots \subset (M_n, T_n)$$

satisfies the conditions conditions (i) and (ii) of Theorem 1 (Corollary 1).

Let us note that the manifold $M = M_n$ has nonzero cohomology group at levels k, $1 \le k \le \dim M_1 + n - 1 = \dim M_n$. In this case, the classical Borsuk–Ulam-type theorems cannot be applied.

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