CHARACTERIZATION OF THE DUAL OF AN ORLICZ SPACE

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Our objective is to characterize the dual of an Orlicz space $\mathcal{L}_A(X,\Sigma,\mu)$, with the only hypothesis that (X,Σ,μ) is a measure space with no atoms of infinite measure.

This work originates from the reading of [7] and [8] of M.M.Rao, where we found some statements which were unclear to us. In particular the characterization in [8] seems to be incomplete, possibly for some fault in the fundamental definition. Here we present a new one.

§1. Preliminaries

If p is a nondecreasing function from $[0,\infty[$ to $[0,\infty]$ such that $0 < p(t_0) < \infty$ for some $t_0 \neq 0$, the function A defined on $[0,\infty[$ by the equality $A(u) = \int_0^u p(t)dt$ is called Young's function. The Young's complement of A is defined on $[0,\infty[$ as

$$\bar{A}(u) = \sup\{uv - A(v): \ v \in [0,\infty[\} .$$

In all that follows we will denote

$$a = \sup\{u \in [0,\infty[: A(u) = 0\} , \ \bar{a} = \sup\{u \in [0,\infty[: \bar{A}(u) = 0\} ,$$

$$b = \inf\{u \in [0,\infty[: A(u) = \infty\} , \ \bar{b} = \inf\{u \in [0,\infty[: \bar{A}(u) = \infty\} .$$

Moreover (X,Σ,μ) will be a fixed but arbitrary measure space with no atoms of infinite measure (i.e, if $E \in \Sigma$ and $\mu(E) = \infty$, then there exists $F \in \Sigma$, $F \subset E$ such that $0 < \mu(F) < \infty$). Unless otherwise stated, our functions will be from X on $\mathbb{R}$, and we shall employ the conventions that $0/0 = 1/\infty = 0.\infty = 0$ and $\inf \emptyset = \infty$.

The Orlicz space L_A is the space of all measurable functions, such that $\int_X A(k|f|)d\mu < \infty$, for some $k \in]0,\infty[$. This is a complete space with the seminorm $\|\cdot\|_A$, defined by

$$\|f\|_A = \inf\{k \in]0,\infty[: \int_X A(\frac{|f|}{k})d\mu \le 1\} .$$

(1.1) Proposition. The following assertions are true:

(i) A is nondecreasing and convex;

(ii) the right derivative A' of A exists and is finite-valued on $[0,b[$;

(iii) $uv \le A(u) + \bar{A}(v)$ for $u,v \in [0,\infty[$, the equality holding if, and only if, at least one of the relations

$u = \lim_{t\downarrow v} \bar{A}'(t)$ or $v = \lim_{t\downarrow u} A'(t)$ is satisfied by u and v;

(iv) for $E \in \Sigma$ we have $\mu(E) < \infty$ implies $\xi_E \in L_A$, the reciprocal holding if $a = 0$; besides, if $\mu(E) = \infty$ and $a > 0$, then $\|\xi_E\|_A = 1/a$;

(v) if $E \in \Sigma$ and $\xi_E \in L_A$, then $\|\xi_E\|_A \le \bar{b}\,\mu(E)$;

(vi) for $f \in L_A$ we have, $\int_X A(|f|)d\mu \le 1$ if and only if $\|f\|_A \le 1$;

(vii) if $\delta \in [0,\infty[$ and $f \in L_A$, then

$$\mu(\{x \in X: |f(x)| \ge \delta\|f\|_A\}).A(\delta) \le 1;$$

(viii) if $f \in L_A$, then $\|f\|_A \le \int_X A(|f|)d\mu + 1.$

Proof. Assertion (vii) is a consequence of (i) and (vi). The others may be found in [1] and [3].//

(1.2) Proposition. For f in L_A there is a sequence (s_n) of simple functions in L_A such that

(i) (s_n) is nondecreasing if $f \geq 0$;

(ii) $|f-s_n| \leq |f|$ and $|s_n| \leq |f|$ for all $n \in \mathbb{N}$;

(iii) (s_n) converges to f;

(iv) (s_n) converges in μ-measure to f;

(v) if $k \in]0,\infty[$ is such that $A(|f|/k) \in L_1$, then
$$\lim_{n\to\infty} \int_X A\left(\frac{|f-s_n|}{k}\right) d\mu = 0.$$

Proof. Discarding a trivial case, let $\|f\|_A \neq 0$. The measurability of f guarantees the existence of a sequence (s_n) of simple functions satisfying (i), (ii) and (iii). Moreover, if $E \in \Sigma$ is such that f is bounded on E, then $(s_n \xi_E)$ converges uniformly to $f\xi_E$. From (i) and (ii) it follows that each $s_n \in L_A$.

Suppose that $a > 0$ and let $B = \{x \in X: |f(x)| \geq 2a\|f\|_A\}$. Clearly $(s_n \xi_{B^c})$ converges uniformly to $f\xi_{B^c}$. By (1.1.vii) we have $\mu(B) < \infty$ and hence, by Egorov's Theorem, that $(s_n \xi_B)$ converges to $f\xi_B$ in measure. So (iv) holds.

For $a = 0$ we will find a subsequence (s_{n_k}) which converges in μ-measure to f.

Suppose $a=0$ and let $c \in]0,\infty[$ be such that $0 < A(c) < \infty$. It follows from (i), (ii) and (iii) that $\lim_{n\to\infty} \int_X A\left(\frac{|f-s_n|}{\|f\|_A}\right) d\mu = 0$, and so we can find a subsequence (s_{n_k}) such that $\int_X A\left(\frac{|f-s_{n_k}|}{\|f\|_A}\right) d\mu < \frac{1}{2^k} A\left(\frac{c}{2^k}\right)$.

From this we see that $\mu(\{x \in X: |f(x)-s_{n_k}(x)| > \frac{c\|f\|_A}{2^k}\}) < \frac{1}{2^k}$ for all $k \in \mathbb{N}$, and so (s_{n_k}) converges to f in μ-measure. Replace (s_n) by (s_{n_k}).

To establish (v) it suffices to recall (ii) and (iii) and apply the Lebesgue's Convergence Theorem.//

§2. Characterization of the dual space $(M_A)^*$.

Denote by M_A the closed subspace of L_A spanned by all simple functions.

In [8] the author gives a representation for $x^* \in (M_A)^*$ as an integral, in the sense of Dunford and Schwartz [2], relatively to a measure G defined as $G(E) = x^*(\xi_E)$. Clearly this defines G(X) only if $\xi_X \in L_A$. Since this does not occur when $a = 0$ and $\mu(X) = \infty$, in this case, G is not defined on Σ, and so we cannot apply the theory of [2]. Thus in [4] we introduce the concept of integration relative to an extended real-valued finitely additive measure G, defined on an ideal H of an algebra A.

To facilitate the reading, we transcribe from [4] some definitions and one proposition, supposing A, H and G are as above.

We shall write that a ring H of subsets of X is an ideal of A, if for $E \in A$ and $F \in H$ one has $E \cap F \in H$.

(2.1) <u>Definition</u>. A function s is H-simple if there is a pairwise disjoint, finite sequence $(E_1, E_2, \ldots, E_n)$ in H and there exists $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ such that $s = \sum_{i=1}^{n} c_i \xi_{E_i}$. If $v(G; s^{-1}(\{c_i\})) < \infty$ for $c_i \neq 0$, we shall say that s is G-integrable and for $E \in A$ we define $\int_E f dG = \sum_{i=1}^{n} c_i G(E \cap E_i)$, where $v(G,.)$ is as in (2.2.1) below.

(2.2) Definition. A function f is G-integrable if there is a sequence (s_n) of G-integrable simple functions such that

(i) (s_n) converges in G-measure to f, i.e. for every $\delta \in]0,\infty[$ we have

$$\lim_{n\to\infty}(\inf\{v(G,F): F\in H \text{ and } \{x\in X: |f_n(x)-f(x)| > \delta\}\subset F\}) = 0,$$

where $v(G,.)$ is the total variation of G, which is obtained replacing Σ by H in [2-III.1.4];

(ii) $$\lim_{m,n\to\infty} \int_X |s_n - s_m| dv(G,.) = 0.$$

If f is G-integrable and $E \in A$ we define

$$\int_E f dG = \lim_{n\to\infty} \int_E s_n dG .$$

(2.3) Proposition (Vitali's Theorem). Let be f a function and (s_n) a sequence of G-integrable functions, such that

(i) (s_n) converges in G-measure to f;

(ii) $$\lim_{v(G,E)\to 0} \int_E |s_n| dv(G,.) = 0$$ uniformly in $n \in \mathbb{N}$;

(iii) for each $\varepsilon \in]0,\infty[$ there is a set F_ε in H with $v(G,F_\varepsilon) < \infty$ and such that

$$\int_{F_\varepsilon^c} |s_n| dv(G,.) < \varepsilon , \quad \text{for all} \quad n \in \mathbb{N} .$$

Then f is G-integrable and for all $E \in A$ we have

$$\int_E f dG = \lim_{n\to\infty} \int_E s_n dG.$$

Next we take up our characterization of $(M_A)^*$.

From here on we agree that $A = \Sigma$, $\Sigma_1 = \{E \in \Sigma : \mu(E) < \infty\}$, $H = \Sigma_1$ if $a = 0$ and $H = \Sigma$ if $a > 0$.

(2.4) Definition. We shall say that $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$ if G is a real-valued, finitely additive, measure defined on H such that

(i) $G \ll \mu$, i.e. if $E \in \Sigma$ and $\mu(E) = 0$, then $G(E) = 0$;

(ii) $0 \leq v(G,E) < \infty$ for all $E \in H$;

(iii) $\alpha_G^{\bar{A}} = \inf\{k \in]0,\infty[: I_{\bar{A}}(\frac{G}{k},X) \leq 1\} < \infty$, where

$$I_{\bar{A}}(\frac{G}{k},E) = \sup\{\sum_{i=1}^{n} \bar{A}(\frac{|G(E_i)|}{k\mu(E_i)})\mu(E_i):(E_1,E_2,\ldots,E_n)\in D(\Sigma_1,E)\},$$

$E \in \Sigma$ and $D(\Sigma_1,E)$ is the set of all pairwise disjoint finite sequences $(E_1,E_2,\ldots,E_n)$ in H, such that $\bigcup_{i=1}^{n} E_i \subset E$.

Unless otherwise stated, G will denote a real-valued, finitely additive measure, defined on H such that $G \ll \mu$.

(2.5) Remarks. (i) We replace the domain of G and a condition (i) of [8 - Definition 3 - page 563], respectively, by H and (2.4.ii).

(ii) For $k \in]0,\infty[$, the function $I_{\bar{A}}(\frac{G}{k},.)$ is a real-valued finitely additive measure defined on Σ.

(iii) If $0 < \alpha_G^{\bar{A}} < \infty$, then $I_{\bar{A}}(\frac{G}{\alpha_G^{\bar{A}}},X) \leq 1$.

(iv) If $\alpha_G^{\bar{A}} = 0$, then $G(E) = 0$ for all $E \in \Sigma_1$.

(v) The function $\alpha_{(.)}^{\bar{A}}$ is a seminorm on the vector space $G_{\bar{A}}(X,\Sigma,\mu,H)$ and a norm on $G_{\bar{A}}(X,\Sigma,\mu,\Sigma_1)$.

(vi) If $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$, then by (1.1.iv) and (2.4.ii) every simple function in L_A is H-simple and G-integrable.

In (2.8) below, for a fixed $f \in L_A$ we shall find a sequence (s_n) of simple functions in L_A such that

$$\int_E f dG = \lim_{n\to\infty} \int_E s_n dG \text{ for every } E \in \Sigma \text{ and all } G \in G_{\overline{A}}(X,\Sigma,\mu,H).$$

First some propositions.

(2.6) <u>Proposition</u>. If $b = \infty$ and G is such that $\alpha_G^{\overline{A}} < \infty$, then G is μ-continuous, i.e. $\lim_{\mu(A)\to 0} G(A) = 0$.

Proof. [7 - Lemma 6].//

(2.7) <u>Proposition</u>. Let $G \in G_{\overline{A}}(X,\Sigma,\mu,H)$. If $s \in L_A$ is a simple, nonnegative function, then

$$\int_X s dv(G,.) \le c\|s\|_A, \text{ where } c = 2\alpha_G^{\overline{A}} \text{ if } a = 0$$

or $\mu(X) < \infty$, and $c = 2\alpha_G^{\overline{A}} + av(G,X)$ if $a > 0$ and $\mu(X) = \infty$.

Proof. Eliminating a trivial case, suppose $\|s\|_A \ne 0$.

Let $s = \sum_{i=1}^{n} c_i \xi_{E_i}$ be as in (2.1) with $c_i \ge 0$.

First we observe that if $(F_1,F_2,\ldots,F_\ell)$ is a pair - wise disjoint finite sequence in Σ_1 and $\alpha_G^{\overline{A}} \ne 0$, then by (1.1.iii) we have

$$\sum_{j=1}^{\ell} c_i|G(F_j)| \le \alpha_G^{\overline{A}}\|s\|_A[\sum_{j=1}^{\ell}(\overline{A}(\frac{|G(F_j)|}{\alpha_G^{\overline{A}}\mu(F_j)})+A(\frac{c_i}{\|s\|_A}))\mu(F_j)]. \quad (1)$$

Let $I_1 = \{i \in \mathbb{N}: 1 \le i \le n \text{ and } E_i \in \Sigma_1\}$. Using (1.1.vi), (2.5.ii), (2.5.iii), (1) and observing (2.5.iv) it follows that

$$\sum_{i\in I_1} c_i v(G,E_i) \le 2\alpha_G^{\overline{A}}\|s\|_A. \quad (2)$$

By (1.1.iv) we obtain

$$\sum_{i\notin I_1} c_i v(G,E_i) = a \sum_{i\notin I_1} c_i \|\xi_{E_i}\|_A v(G,E_i) \le a\|s\|_A v(G,X). \qquad (3)$$

From (2), (3) and (1.1.iv) we obtain the desired result.//

(2.8) <u>Proposition</u>. For f in L_A there is a sequence (s_n) of simple functions in L_A satisfying (1.2.i), (1.2.ii), (1.2.iii) and such that for all $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$ we have

(i) (s_n) converges in G-measure to f;

(ii) f is G-integrable and $\int_E f dG = \lim_{n\to\infty} \int_E s_n dG$, for $E \in \Sigma$;

(iii) $|\int_X f dG| \le$ ~~$\int_X |f| dv(G,.)$~~ $\le c\|f\|_A$, with c as in (2.7).

Proof. We may suppose $f \ge 0$, for (i) and (ii).

If $b = \infty$, let (s_n) as in (1.2). From (2.6) we obtain that (s_n) converges in G-measure to f.

It $b < \infty$, we have that $|f| \le 2b\|f\|_A$ μ-a.e. (1.1.vii) and since f is measurable we can take a sequence (s_n) of simple functions satisfying (1.2.i), (1.2.iii) and converging uniformly to f except on a set $E \in \Sigma$ with $\mu(E)=0$. Clearly $s_n \in L_A$ for $n \in \mathbb{N}$, $v(G,.) \ll \mu$ and (s_n) converges in G-measure to f.

Now we will prove that the sequence (s_n) above also satisfys (2.3.ii) and (2.3.iii), and thus we will obtain (ii).

For all $n \in \mathbb{N}$, let $y_n = \int_X s_n dv(G,.)$. Since (y_n) is bounded (2.7), and nondecreasing, given $\varepsilon \in]0,\infty[$ there is an $n_o \in \mathbb{N}$ such that

$$\int_E s_n dv(G,.) = \int_X s_n dv(G,.) - \int_{E^c} s_n dv(G,.) < \frac{\varepsilon}{2} + \int_E s_{n_o} dv(G,.) \quad ,$$

for all $n \ge n_o$ and $E \in \Sigma$.

If $\delta = \varepsilon/(2(\|s_{n_o}\|_\infty+1))$ and $F_\varepsilon = \{x\in X: s_{n_o}(x)\neq 0\}$, we have that $F_\varepsilon \in H$, and (s_n) satisfys (2.3.ii) and (2.3.iii).

Assertion (iii) is a consequence of (ii), ~~applied to~~ ~~[illegible]~~ and of (2.7).//

From (2.8) it follows that if $f \in L_A$, $\gamma \in \mathbb{R}$ and $G_1, G_2 \in G_{\overline{A}}(X,\Sigma,\mu,H)$ then $\int_X f d(\gamma G_1+G_2) = \gamma\int_X fdG_1 + \int_X fdG_2$.

Moreover since simple functions in L_A are in M_A, one can establish (2.9).

(2.9) <u>Proposition</u>. The function defined by

$$\|G\|_{\overline{A}} = \sup\{|\int_X fdG| : f \in L_A \text{ and } \|f\|_A \leq 1\},$$

is a norm on $G_{\overline{A}}(X,\Sigma,\mu,H)$, and

$$\|G\|_{\overline{A}} = \sup\{|\int_X fdG| : f \in M_A \text{ and } \|f\|_A \leq 1\}.$$

(2.10) <u>Proposition</u>. If $G \in G_{\overline{A}}(X,\Sigma,\mu,H)$, then the function x^* defined on M_A [or L_A] by $x^*(f) = \int_X fdG$, belongs to $(M_A)^*$ [to $(L_A)^*$] and $\|x^*\| = \|G\|_{\overline{A}}$.

(2.11) <u>Proposition</u>. Let $x^* \in (M_A)^*$ and let G be the function defined on H by $G(E) = x^*(\xi_E)$. Then $G \in G_{\overline{A}}(X,\Sigma,\mu,H)$ and the following holds:

(i) $x^*(f) = \int_X fdG$ for all $f \in M_A$, and $\|x^*\| = \|G\|_{\overline{A}}$.

Proof. We may suppose $\|x^*\| \neq 0$. It is clear that G is a real-valued, finitely additive measure, and $G \ll \mu$.

To prove that $v(G,E) < \infty$ for all $E \in \mathcal{H}$, it is enough to observe that, if $E \in \mathcal{H}$ and $(E_1,E_2,\dots,E_n)$ is a pairwise disjoint sequence in $\mathcal{H}$ with $\bigsqcup_{i=1}^{n} E_i \subset E$ then taking $I_1 = \{i \in \mathbb{N}: 1 \le i \le n \text{ and } x^*(\xi_{E_i}) \ge 0\}$, we have

$$\sum_{i=1}^{n} |G(E_i)| = x^*(\xi_{\bigsqcup_{i\in I_1} E_i}) - x^*(\xi_{\bigsqcup_{i\notin I_1} E_i}) \le 2\|x^*\| \|\xi_E\|_A .$$

Now we will prove that $\alpha_G^{\overline{A}} \le \|x^*\| < \infty$.

Let $\bar{c} \in]0,1[$ and we take $(E_1,E_2,\dots,E_n)$ a arbitrary pairwise disjoint sequence in Σ_1. If $u_i = \bar{c}|G(E_i)|/(\|x^*\|\mu(E_i))$ belongs to $[0,\bar{b}[$ (1.1.v), $c_i = \lim_{t\downarrow u_i} \overline{A}'(t)$ for $i \in \{1,2,\dots,n\}$ and $f = \sum_{i=1}^{n} c_i \operatorname{sgn}(G(E_i))\xi_{E_i}$, then $f \in L_A$ and by (1.1.iii) and (1.1.viii) we have

$$\|f\|_A \ge \frac{\bar{c}|x^*(f)|}{\|x^*\|} = \sum_{i=1}^{n} [A(c_i) + \overline{A}(\frac{\bar{c}\ |G(E_i)|}{\|x^*\|\mu(E_i)})]\mu(E_i)$$

$$\ge \|f\|_A - 1 + \sum_{i=1}^{n} \overline{A}(\frac{\bar{c}|G(E_i)|}{\|x^*\|\mu(E_i)})\mu(E_i) .$$

This relation it is easy to verify that $\alpha_G^{\overline{A}} \le \|x^*\|/\bar{c}$, and since $\bar{c} \in]0,1[$ is arbitrary, we conclude that $\alpha_G^{\overline{A}} \le \|x^*\|$. Thus $G \in G_{\overline{A}}(X,\Sigma,\mu,\mathcal{H})$.

To obtain (i) it is enough to observe that (2.10) is true and that $x^*(s) = \int_X s\,dG$ for a simple function in L_A.//

From (2.10) and (2.11) we obtain in (2.12) a characterization of $(M_A)^*$; it will follow that $G_{\overline{A}}(X,\Sigma,\mu,\mathcal{H})$ is a

Banach space.

(2.12) Theorem. There exists an isometric isomorphism of $(M_A)^*$ onto the $G_{\bar{A}}(X,\Sigma,\mu,H)$, given by the mapping $x^* \mapsto G$ where $G(E) = x^*(\xi_E)$ for $E \in H$, and the following holds:

(i) $x^*(f) = \int_X f dG$ for all $f \in M_A$ and $\|x^*\| = \|G\|_{\bar{A}}$.

§3. Characterization of the dual space $(N_A)^*$

Let $N_A = L_A/M_A$, write $\hat{f} = f+M_A$ for any $f \in L_A$, and denote by d(.) the quotient norm on M_A. One can easily show that

$$d(\hat{f}) = \inf\{ \|f+s\|_A : \ s \text{ is a simple function in } L_A\}.$$

(3.1) Definition. Let $\hat{f},\hat{g} \in N_A$. We define $\hat{f} \le \hat{g}$ if there exist $f_1 \in \hat{f}$ and $g_1 \in \hat{g}$ such that $f_1 \le g_1$.

In [8] it is proved that the above relation is a partial ordering on N_A. Two results which are used in this proof and are not obvious to us are established in (3.4) and (3.5). First two propositions.

(3.2) Proposition. Let $a > 0$ or $\mu(X) < \infty$. If f is a measurable function, bounded μ-a.e, then $f \in M_A$.

Proof. We may suppose f to be bounded. Since there exist $\bar{x} \in]0,\infty[$ such that $A(\bar{x}) \le 1/(\mu(X) + 1)$, and a sequence (s_n)

of simple functions, converging uniformly to f, it is easy to verify that $\lim_{n\to\infty}\|f-s_n\|_A = 0$. From $a > 0$ or $\mu(X) < \infty$ it follows that $f, s_n \in L_A$ for all $n \in \mathbb{N}$. Thus $f \in M_A$ .//

(3.3) Proposition. Let $J_A = \{f \in L_A: \int_X A(k|f|)d\mu < \infty \text{ for all } k \in [0,\infty[\}$. Then J_A is a vector space and the following assertions are true:

(i) $J_A \subset M_A \subset L_A$;

(ii) if $a = 0$ and $b = \infty$, then $J_A = M_A$;

(iii) if $b < \infty$ and $a > 0$, or if $b < \infty$ and $\mu(X) < \infty$, then $L_A = M_A$.

Proof. [8 - Lemma 2]. //

(3.4) Proposition. If $g \in M_A$ and f is a measurable function such that $|f| \le |g|$ μ-a.e, then $f \in M_A$.

Proof. We may suppose $|f| \le |g|$. Let (s_n) be a sequence of simple functions in L_A such that $\lim_{n\to\infty}\|g-s_n\|_A = 0$.

First we observe that for $k \in]0,\infty[$, if $n \in \mathbb{N}$ is such that $2\|g-s_n\|_A \le k$, then by (1.1.i) and (1.1.vi), for all $H \in \Sigma$ we have

$$\int_H A\left(\frac{|f|}{k}\right)d\mu \le \int_H A\left(\frac{|g|}{k}\right)d\mu \le \frac{1}{2} + \frac{1}{2}\int_H A\left(\frac{2|s_n|}{k}\right)d\mu . \qquad (1)$$

Suppose $b = \infty$ and $a > 0$. From $\lim_{n\to\infty}\|g-s_n\|_A = 0$ there exists an $n_o \in \mathbb{N}$ such that $\int_X A(|g-s_{n_o}|)d\mu \le 1$.

Let $B = \{x \in X: |g(x)-s_{n_o}(x)| \ge 2a\}$. Then $\mu(B) < \infty$ (1.1.vii), and the inequalities $|f| \le |g| \le |g-s_{n_o}| + |s_{n_o}|$ together with (3.2) implie that $f\xi_{B^c} \in M_A$. Moreover it follows from (1) that $f\xi_B \in J_A \subset M_A$. So $f \in M_A$ in this case.

Let $b < \infty$, $a = 0$ and $E_n = \{x \in X: s_n(x) \neq 0\}$ for $n \in \mathbb{N}$. Then $|f| \le 2b\|f\|_A$ μ-a.e. (1.1.vii), $\mu(E_n) < \infty$ (1.1.iv) and $f\xi_{E_n} \in M_A$ (3.2), for all $n \in \mathbb{N}$. Using (1) we obtain that $\lim_{n\to\infty}\|f-f\xi_{E_n}\|_A = 0$, and since M_A is closed in L_A, we conclude that $f \in M_A$ also in this case.

The remaining cases are easy (3.3). //

(3.5) Proposition. If $f,g \in L_A$ and $|f| \le |g|$ μ-a.e, then $d(\bar{f}) \le d(\bar{g})$.

Proof. We may suppose $|f| \le |g|$ and, by (3.4) that $g \notin M_A$.

It suffices to prove that for all $\varepsilon \in]0,\infty[$ we have

$$d(\bar{f}) \le d(\bar{g}) + \varepsilon \; . \tag{1}$$

For a fixed $\varepsilon \in]0,\infty[$, let $s \in L_A$ be a simple function such that $0 < d(\bar{g}) \le \|g+s\|_A < d(\bar{g}) + \varepsilon$ and let $H_1 = \{x \in X: s(x) \neq 0\}$.

If $b < \infty$ and $a = 0$, then $\mu(H_1) < \infty$ (1.1.iv) and $|f| \le 2b\|f\|_A$ μ-a.e. (1.1.vii). Since $f\xi_{H_1} \in M_A$ (3.2) and

$$\|f\xi_{H_1^c}\|_A \le \|g\,\xi_{H_1^c}\|_A \le \|(g+s)\xi_{H_1^c}\|_A < d(\bar{g}) + \varepsilon ,$$

we conclude (1).

If $b < \infty$ and $a > 0$, then $d(\bar{f}) = d(\bar{g}) = 0$ (3.3iii).

For the remaining cases, we first note that if $H \in \Sigma$ and $A\left(|s|\,((\alpha-1)\,\|g+s\|_A)^{-1}\xi_H\right) \in L_1$ for all $\alpha \in]1,\infty[$, then

$$d(f\xi_H)^{\hat{}} \le d(\bar{g}) + \varepsilon . \tag{2}$$

In fact, by (1.1.i) and (1.1.vi) it is true that

$$\int_H A\Big(\frac{|f|}{\alpha\|g+s\|_A}\Big)d\mu \le \int_H A\Big(\frac{|g|}{\alpha\|g+s\|_A}\Big)d\mu \le \frac{1}{\alpha} + \Big(\frac{\alpha-1}{\alpha}\Big)\int_H A\Big(\frac{|s|}{(\alpha-1)\|g+s\|_A}\Big)d\mu$$

and hence $A((|f|(\alpha\|g+s\|_A)^{-1})\xi_H) \in L_1$. From (1.2) there exists a simple function $s_o \in L_A$ such that $\|(f-s_o)\xi_H\|_A \le \alpha\|g+s\|_A$. Thus $d(f\xi_H)^\wedge \le \|(f-s_o)\xi_H\|_A \le \alpha\|g+s\|_A < \alpha(d(\bar{g})+\varepsilon)$ and, since $\alpha \in]1,\infty[$ is arbitrary, (2) holds.

If $b = \infty$ and $a = 0$, then $\mu(H_1) < \infty$ (1.1.iv) and we obtain (1) replacing H by X in (2).

If $b = \infty$ and $a > 0$, let $H_2 = \{x \in X: |g(x)+s(x)| \ge 2a\|g+s\|_A\}$. Then (1.1.vii) tells us that $\mu(H_2) < \infty$ and we may replace H by H_2 in (2). Moreover, it follows from $|f| \le |g| \le |g+s| + |s|$ that $f\xi_{H_2^c}$ is bounded, and thus $f\xi_{H_2^c} \in M_A$ (3.2). Therefore (1) holds.//

Let E be a pseudonormed vector lattice, $x \in E$ and $z^* \in E^*$. Then we will set as usual $x_+ = x \vee o$, $x_- = (-x)\vee o$ and $|x| = x_+ + x_-$. Also, we will write that $z^* \ge 0$ if $z^*(y) \ge 0$ whenever $y \ge 0$.

(3.6) Proposition. The following assertions are true:

(i) N_A is a vector lattice, and if $\bar{f},\bar{g} \in N_A$, then $\bar{f}\vee\bar{g} = (\max\{f,g\})^\wedge$, $\bar{f}\wedge\bar{g} = (\min\{f,g\})^\wedge$ and $(|f|)^\wedge = |\bar{f}|$;

(ii) N_A and $(N_A)^*$ are Banach lattices;

(iii) if $x^* \in (N_A)^*$ and $x^* \ge 0$, then $\|x^*\| = \sup\{x^*(\bar{f}): f \ge 0$ and $\int_X A(f)d\mu < \infty\}$;

(iv) if $x^* \in (N_A)^*$, then $\|x^*\| = \|(x^*)_+\| + \|(x^*)_-\|$;

(v) if $x^*, y^* \in (N_A)^*$ and $x^*, y^* \ge 0$, then $\|x^*+y^*\| = \|x^*\| + \|y^*\|$;

(vi) if $\bar{f},\bar{g} \in N_A$ and $\bar{f},\bar{g} \ge 0$, then $d(\bar{f}\vee\bar{g}) = \max\{d(\bar{f}),d(\bar{g})\}$.

Proof. Except for (iii), these assertions are proved in [8]. Assertion (iii) follows from [8 - Lemma 6 - (2.12)], observing that $|\hat{f}| = (|f|)^\wedge$ for $f \in L_A$. //

From (3.6.ii) and (3.6.v), and from (3.6.ii) and (3.6.vi) we have, respectively, that $(N_A)^*$ is an L-space and that N_A is an M-space.

From here on ν will denote a real-valued finitely additive, bounded measure defined on Σ, $\nu_1=(\nu+v(\nu,.))/2$ and $\nu_2=(v(\nu,.)-\nu)/2$. Also we will denote by P the set of all the pairwise disjoint finite sequences $(E_1,E_2,\ldots,E_n)$ in Σ, such that $\bigcup_{i=1}^{n} E_i = X$.

(3.7) Definition. Let $f \in L_A$. If f is nonnegative, then we define

$$\int_X f d\nu_j = \inf\{ \sum_{i=1}^{n} d(f\xi_{E_i})^\wedge \nu_j(E_i) : (E_1,E_2,\ldots,E_n) \in P\},$$

for $j \in \{1,2\}$.

In the general case we define

$$\int_X f d\nu = (\int_X f_+ d\nu_1 - \int_X f_+ d\nu_2) - (\int_X f_- d\nu_1 - \int_X f_- d\nu_2).$$

(3.8) Remarks (i) From (3.7) and (3.5), we conclude that

$$|\int_X f d\nu| \le \int_X |f| d\nu_1 + \int_X |f| d\nu_2 \le d(|f|^\wedge)(\nu_1+\nu_2)(X) = d(\hat{f})v(\nu,X).$$

(ii) From [8 - Lemma 10] and (i), it follows that if $x^*(\hat{f}) = \int_X f d\nu$ for $\hat{f} \in N_A$, then $x^* \in (N_A)^*$ and $\|x^*\| \le v(\nu,X)$.

(iii) If ν is nonnegative, $E \in \Sigma$ and ν_E is defined on Σ by $\nu_E(F) = \nu(E \cap F)$, then $\int_X f\xi_E d\nu = \int_X f d\nu_E$ for all

$0 \leq f \in L_A$.

(iv) If $\hat{f} \in N_A$ and $\hat{f} \geq \hat{0}$, then there exists a nonnegative representative of $\hat{f}$.

Next we take up our characterization of $(N_A)^*$.

(3.9) Definition. We shall say that $\nu \in V_{\bar{A}}(X,\Sigma,\mu)$ if $\nu << \mu$ and there exists $\hat{f} \in N_A$, $\hat{f} \geq \hat{0}$ with $d(\hat{f}) \leq 1$, for which the following holds:

(i) if $E \in \Sigma$ and $v(\nu,E) \neq 0$, then $d(f\xi_E)^\wedge = 1$.

It is easy to see, using (3.6.vi), that $V_{\bar{A}}(X,\Sigma,\mu)$ is a vector space.

(3.10) Comment. In [8 - page 573], in place of $V_{\bar{A}}(X,\Sigma,\mu)$, the author uses $B_{\bar{A}}(\mu)$, the space of all $\nu << \mu$ such that there exists $f \in L_A$ with $\int_X A(|f|)d\mu < \infty$, $f \notin M_A$ and

[R] support of ν (i.e., "the sets E for which $v(\nu,E) > 0$") lies in the support of f.

The author claims that if $\nu \in B_{\bar{A}}(\mu)$ is nonnegative and $x^*(\hat{f}) = \int_X f d\nu$ for all $\hat{f} \in N_A$, then $\|x^*\| = \nu(X)$. To prove this, he considers $\pi = (E_1,E_2,\ldots,E_n) \in P$ with $\nu(E_i) > 0$ and defines $f = f_1 + f_2 + \ldots + f_n$ where $\int_X A(|f_i|)d\mu < \infty$, $d(\hat{f}_i) = 1$ and the support of f_i is E_i, for $i \in \{1,2,\ldots,n\}$. Then he writes:

"Thus $\sum_{i=1}^{n} d(f\xi_{E_i})^\wedge \nu(E_i) = \nu(X)$ and refining the partition π on the left yields $x^*(\hat{f}) = \nu(X)$". This last assertion is not clear to us, for f depends on the partition π. However we observe that if there is an f satisfying (3.9.i), then it is easy to see that $x^*(\hat{f}) = \nu(X)$. Thus we have replaced [R] by (3.9.i).

(3.11) <u>Notation</u>. If $E \in \Sigma$ and $x^* \in (N_A)^*$, we will denote by x_E^* the function defined on N_A by $x_E^*(\bar{f}) = x^*(f\xi_E)^\frown$.

(3.12) <u>Proposition</u>. Let $0 \le x^* \in (N_A)^*$. Then there exists $g \in L_A$, $g \ge 0$, with $d(\bar{g}) \le 1$, such that

(i) $\|x_E^*\| = x^*(g\xi_E)^\frown$ for all $E \in \Sigma$;

(ii) if $E \in \Sigma$ and $\|x_E^*\| \ne 0$, then $d(g\xi_E)^\frown = 1$.

Proof. By (3.6.iii), for $n \in \mathbb{N}$ there exists a nonnegative $f_n \in L_A$ such that $\int_X A(f_n)d\mu < \infty$ and $x^*(\bar{f}_n) > \|x^*\| - \frac{1}{n}$.
Moreover from (1.2) we know that there is a nonnegative $h_n \in L_A$ such that $\int_X A(h_n)d\mu < \frac{1}{2^n}$ and $\hat{h}_n = \hat{f}_n$.

Let $h = \lim_{n\to\infty}(\max(h_1,\dots,h_n))$ and observe that if $E = \{x \in X: h(x) = \infty\}$, then $\mu(E) = 0$. In fact, if $\mu(E)>0$, there exists $F \in \Sigma_1$, $F \subset E$ such that $\mu(F) > 0$. By (1.1.iii) and (1.1.vi) we have

$$\infty\cdot\mu(F) = \int_X \xi_F h d\mu \le \|\xi_F\|_{\bar{A}} \lim_{k\to\infty}\left[\int_X \bar{A}\left(\frac{\xi_F}{\|\xi_F\|_{\bar{A}}}\right)d\mu + \sum_{m=1}^{k}\int_X A(h_m)d\mu\right] < \infty ,$$

and so $\mu(F) = 0$, which is a contradiction.

Let $g = h\xi_{E^c}$. Then ~~$d(\bar{g}) \le \|g\|_A$~~ $\int_X A(g)d\mu \le \sum_{n=1}^{\infty}\int_X A(h_n)d\mu \le 1$ and since $x^*(\bar{g}) \ge x^*(\hat{h}_n) = x^*(\bar{f}_n) > \|x^*\| - \frac{1}{n}$, for all $n \in \mathbb{N}$, we conclude that $\|x^*\| = x^*(\bar{g})$. If $E \in \Sigma$, using (3.6.v) we obtain the following relations

$$0 \le \|x_E^*\| - x^*(g\xi_E)^\frown = x^*(g\xi_{E^c})^\frown - \|x^*_{E^c}\| \le 0 ,$$

$$\|x_E^*\| = x^*(g\xi_E)^\frown \le \|x_E^*\|\, d(g\xi_E)^\frown \le \|x_E^*\| .$$

This proves (ii). //

(3.13) <u>Theorem</u>. Let $0 \le x^* \in (N_A)^*$. Then there exists a unique $\nu \in V_{\bar{A}}(X,\Sigma,\mu)$, defined by $\nu(E) = \|x_E^*\|$, such that

(i) $x^*(\hat{f}) = \int_X f d\nu$ for all $\hat{f} \in N_A$, and $\|x^*\| = \nu(X)$.

Proof. Let $\nu(E) = \|x_E^*\|$ for $E \in \Sigma$. To show that (i) holds define $z^*(\hat{f}) = \int_X f d\nu$ for $\hat{f} \in N_A$; we shall prove that $\|z^* - x^*\| = 0$.

For $\hat{0} \leq \hat{f} \in N_A$ and $\pi = (E_1, E_2, \ldots, E_n) \in P$ we have

$$\sum_{i=1}^{n} d(f\xi_{E_i})^\wedge \nu(E_i) = \sum_{i=1}^{n} d(f\xi_{E_i})^\wedge \|x_{E_i}^*\| \geq \sum_{i=1}^{n} x^*(f\xi_{E_i})^\wedge = x^*(\hat{f}) ,$$

and from this it is clear that $z^* \geq x^*$. Thus $\|z^*\| = \|x^*\|$, for $\|z^*\| \leq \nu(X) = \|x^*\|$ (3.8.ii), and also $\|(z^*-x^*)+x^*\| = \|z^*-x^*\| + \|x^*\|$ (3.6.v). Hence $\|z^*-x^*\| = 0$.

To prove the uniqueness of ν suppose that $\tilde{\nu} \in V_{\bar{A}}(X,\Sigma,\mu)$ and $x^*(\hat{f}) = \int_X f d\tilde{\nu}$ for all $\hat{f} \in N_A$. Then by (3.8.iii) and (3.8.ii) we have that

$$x_E^*(\hat{f}) = x^*(f\xi_E)^\wedge = \int_X f\xi_E d\tilde{\nu} \leq d(\hat{f})\tilde{\nu}(E) ,$$

for $\hat{f} \in N_A$ and $E \in \Sigma$. So $\tilde{\nu} - \nu$ is nonnegative, belongs to $V_{\bar{A}}(X,\Sigma,\mu)$ and $\int_X f d(\tilde{\nu}-\nu) = 0$ for all $f \in L_A$.

Since $\tilde{\nu}-\nu \in V_{\bar{A}}(X,\Sigma,\mu)$, there exists $\hat{0} \leq \hat{g} \in N_A$ such that $d(g\xi_E)^\wedge = 1$ for $E \in \Sigma$ with $(\tilde{\nu}-\nu)(E) \neq 0$. Thus

$\int_X g d(\tilde{\nu}-\nu) = (\tilde{\nu}-\nu)(X)$, and $(\tilde{\nu}-\nu)(X) = 0$. //

(3.14) <u>Theorem</u>. There exists an isometric isomorphism of $(N_A)^*$ onto $V_{\bar{A}}(X,\Sigma,\mu)$, given by the mapping $x^* \mapsto \nu$, such that the following holds:

(i) $x^*(\hat{f}) = \int_X f d\nu$ for all $\hat{f} \in N_A$ and $\|x^*\| = v(\nu, X)$.

Proof. Let $x^* \in (N_A)^*$, $y^* = (x^*)_+$, $z^* = (x^*)_-$ and ν_1, ν_2 be defined on Σ by $\nu_1(E) = \|y_E^*\|$ and $\nu_2(E) = \|z_E^*\|$. Then

$\nu = \nu_1 - \nu_2 \in V_{\bar{A}}(X,\Sigma,\mu)$, $x^*(\hat{f}) = \int_X f d\nu$ for all $\hat{f} \in N_A$ and $\|x^*\| = \|y^*\| + \|z^*\| = \nu_1(X) + \nu_2(X)$ ((3.13) and (3.6.iv)). Since $\|x^*\| \le v(\nu,X)$ (3.8.ii), and $v(\nu,X) \le \nu_1(X) + \nu_2(X)$, it is clear that $\|x^*\| = v(\nu,X)$.

To prove that $x^* \mapsto \nu$ is onto, consider $\tilde{\nu} \in V_{\bar{A}}(X,\Sigma,\mu)$, $\tilde{\nu}_1 = (\tilde{\nu} + v(\tilde{\nu},.))/2$, $\tilde{\nu}_2 = (v(\tilde{\nu},.) - \tilde{\nu})/2$ and $y^*(\hat{f}) = \int_X f d\tilde{\nu}$ for $\hat{f} \in N_A$.

By what we proved above, there exists $\nu \in V_{\bar{A}}(X,\Sigma,\mu)$ such that $y^*(\hat{f}) = \int_X f d\nu$ for $\hat{f} \in N_A$.

Let $\nu_1 = (\nu+v(\nu,.))/2$ and $\nu_2 = (v(\nu,.) - \nu)/2$. Then $\int_X f d(\tilde{\nu}_1 + \nu_2) = \int_X f d(\nu_1 + \tilde{\nu}_2)$ for $\hat{f} \in N_A$. Since $\tilde{\nu} \in V_{\bar{A}}(X,\Sigma,\mu)$, it is easy to verify that $\tilde{\nu}_1,\tilde{\nu}_2 \in V_{\bar{A}}(X,\Sigma,\mu)$ and thus $\tilde{\nu}_1 + \nu_2 = \nu_1 + \tilde{\nu}_2$ (3.13). Hence $\nu = \tilde{\nu}$.

Finally observing that $x^* \mapsto \nu$ has a linear inverse, we conclude this proof. //

It is immediate, by (3.14), that $V_{\bar{A}}(X,\Sigma,\mu)$ is a Banach space.

§4. Characterization of the dual space $(L_A)^*$

As in §2, set $H = \Sigma$ if $a > 0$, and $H = \Sigma_1$, if $a = 0$.

(4.1) <u>Theorem</u>. Let $x^* \in (L_A)^*$. Then there exist a unique $G \in G_{\bar{A}}(X,\Sigma,\mu,H)$, defined by $G(E) = x^*(\xi_E)$ for $E \in H$, and a unique $z^* \in (M_A)^{\perp}$, where
$(M_A)^{\perp} = \{z^* \in (L_A)^*: z^*(f) = 0$ for all $f \in M_A\}$, such that

(i) $x^*(f) = \int_X f\,dG + z^*(\bar{f})$, for all $f \in L_A$.

Proof. The same as in Proposition 2 of [8], noting that (2.10), (2.11) and (2.12) hold.//

(4.2) <u>Proposition</u>. Let $x^* \in (L_A)^*$, G and z^* as in (4.1), and $y^*(f) = \int_X f\,dG$ for all $f \in L_A$. The following assertions are true:

(i) if $x^* \geq 0$, then $y^* \geq 0$ and $z^* \geq 0$;

(ii) if $x^* \geq 0$, then $\|x^*\| = \|y^*\| + \|z^*\|$;

(iii) $|x^*| = |y^*| + |z^*|$ and $|y^*| \wedge |z^*| = 0$.

Proof. For (i) we only have to prove that $x^* \geq y^*$. For this let $0 \leq f \in L_A$ and take (s_n) as in (2.8). Since $z^* \in (M_A)^\perp$ and $x^* \geq 0$ we have

$$y^*(f) = \lim_{n\to\infty} \int_X s_n\,dG = \lim_{n\to\infty} x^*(s_n) \leq x^*(f).$$

For (ii), let $\varepsilon \in]0,\infty[$. Observing that $y^*, z^* \geq 0$, there exist $f,g \in L_A$, nonnegative, with $\|f\|_A \leq 1$, $\|g\|_A \leq 1$ and such that

$$\|y^*\| < y^*(f) + \frac{\varepsilon}{2} \quad \text{and} \quad \|z^*\| < z^*(g) + \frac{\varepsilon}{2} .$$

From (1.2), for $k \in \mathbb{N}$ fixed there exists $s \in M_A$ with $0 \leq s \leq g$ and such that

$$\int_X A(g-s)\,d\mu < (1 + \frac{1}{k}) - \int_X A(f)\,d\mu .$$

Let $h = \max(f,(g-s))$. Then $h \in L_A$ and

$$\int_X A(\frac{k\ h}{k+1})\,d\mu \leq \frac{k}{k+1} \int_X A(h)\,d\mu$$

$$\leq \frac{k}{k+1} \left[\int_X A(f)\,d\mu + \int_X A(g-s)\,d\mu\right] \leq 1.$$

Thus $\|h\|_A \le (k+1)/k$.

Hence we have that

$$\|y^*\| + \|z^*\| < y^*(f)+z^*(g)+\varepsilon = y^*(f)+z^*(g-s) + \varepsilon$$

$$\le y^*(h)+z^*(h) + \varepsilon = x^*(h) + \varepsilon$$

$$\le \|x^*\| \, \|h\|_A + \varepsilon \le \|x^*\| \left(\frac{k+1}{k}\right) + \varepsilon .$$

Since ε and k are arbitrary we conclude that (ii) holds.

Assertion (iii) may be found in [6-page 40].//

(4.3) Proposition. There is an isometric isomorphism of $(M_A)^{\perp}$ onto $(N_A)^*$ given by the mapping $j: x^* \mapsto z^*$, where z^* is defined by $z^*(\hat{f}) = x^*(f)$.

In the next theorem we present our characterization of $(L_A)^*$.

(4.4) Theorem. There is an isometric isomorphism of $(L_A)^*$ onto the Banach space $G_{\bar{A}}(X,\Sigma,\mu,H) \times V_{\bar{A}}(X,\Sigma,\mu)$ given by the mapping $x^* \mapsto (G,\nu)$ such that the following hold:

(i) $\quad x^*(f) = \int_X f dG + \int_X f d\nu \quad$ for all $f \in L_A$;

(ii) $\quad \|x^*\| = \|G\|_{\bar{A}} + v(\nu,X)$,

the first integral being as in (2.2) and the second, as in (3.7).

Proof. It is a consequence of (4.1), (4.3), (3.14) and (4.2), observing that $\|x^*\| = \| \, |x^*| \, \|$ for $x^* \in (L_A)^*$ [5-24.3]. //

This work is part of our Master's Dissertation written under the guidance of Dr. Iracema Martin Bund.

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THE CAUSALITY PROBLEM FOR LINEAR VOLTERRA INTEGRAL EQUATIONS

by CHAIM SAMUEL HONIG

We present here a problem with a very simple and elementary formulation: let E be a Banach space of numerical functions defined on [0,1]. For instance $E = \mathcal{C}([0,1])$; or $E = G([0,1])$ the space of regulated functions, i.e., functions that have only discontinuities of the first kind, endowed with the norm $\|x\| = \sup_{0 \le t \le 1} |x(t)|$; or $E = L_p([0,1])$, $1 \le p \le \infty$. Let

$$H: \Gamma = \{(t,s) \in [0,1] \times [0,1] \mid 0 \le s \le t\} \to \mathbb{R}$$

be a kernel such that

i) For every $x \in E$ we have $\mathcal{H} x \in E$, where

$$(\mathcal{H} x)(t) = \int_0^t H(t,s)x(s)ds, \quad 0 \le t \le 1$$

and the integral is the Lebesgue one.

ii) For every $f \in E$ the linear Volterra integral equation

$$\text{(H)} \quad x(t) - \int_0^t H(t,s)x(s)ds = f(t), \quad 0 \le t \le 1,$$

has one and one solution $x_f \in E$.

PROBLEM: <u>the operator</u> $f \in E \mapsto x_f \in E$ is <u>necessarely causal</u>?

(i.e., if $f \equiv 0$ on some interval $[0,c] \subset [0,1]$ does it follow that $x_f \equiv 0$ on the same interval?)

Remarks - 1) If $E = \mathcal{C}([0,1])$ and the kernel H is a continuous function, or, if $E = L_2([0,1]$ and $H \in L_2(\Gamma)$, it is well known that the answer is positive; see [2]; see also [3], p. 79, exerc. 3.15